GROMOV HYPERBOLIC GROUPS AND THE MACAEV NORM

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Let $\Gamma$ be a Gromov hyperbolic group with a finite set $A$ of generators. We prove that $h_{\text{top}}(\Sigma(\infty)) \leq k_{\infty}^- (\lambda_A) \leq \text{gr}(\Gamma, A)$, where $\text{gr}(\Gamma, A)$ is the growth entropy, $h_{\text{top}}(\Sigma(\infty))$ is the Coornaert–Papadopoulos topological entropy of the subshift $\Sigma(\infty)$ associated with $(\Gamma, A)$, and $k_{\infty}^- (\lambda_A)$ is Voiculescu’s numerical invariant, which is an obstruction to the existence of quasicentral approximate units relative to the Macaev norm for a tuple of unitary operators $\lambda_A = (\lambda_a)_{a \in A}$ in the left regular representation of $\Gamma$. We also prove that these three quantities are equal for a hyperbolic group splitting over a finite group.

1. Introduction

Let $\Gamma$ be a finitely generated group with a finite generating set $A$. We consider the family $\lambda_A = (\lambda_a)_{a \in A}$ of left translation operators on $\ell^2(\Gamma)$, specifically the value of Voiculescu’s numerical invariant $k_{\infty}^-$ for this family. Voiculescu introduced this invariant $k_{\infty}^-(\tau)$, for a tuple $\tau$ of Hilbert space operators, in a remarkable series of papers [1979; 1981; 1990; David and Voiculescu 1990] to deal with perturbation problems.

For the case of free groups, Voiculescu gave an estimate for $k_{\infty}^- (\lambda_A)$; we obtain its exact value. For the case of certain amalgamated free product groups, we proved in [Okayasu 2004] that $k_{\infty}^- (\lambda_A)$ equals the growth entropy $\text{gr}(\Gamma, A)$ of $\Gamma$ with respect to $A$. These groups are Gromov hyperbolic groups in the sense of [Gromov 1987]. In [Okayasu 2004], we showed that if a subshift $\Sigma$ satisfies a certain condition, then $k_{\infty}^-(\tau) = h_{\text{top}}(\Sigma)$ for the family $\tau$ of creation operators on the Fock space associated with $\Sigma$, which is used to define the Matsumoto algebra [1997] associated to $\Sigma$. (Here $h_{\text{top}}(\Sigma)$ is the topological entropy of $\Sigma$.) This equation holds for every shift of finite type.

M. Coornaert and A. Papadopoulos [2001] have shown the following: Let $X$ be a proper geodesic metric space that is $\delta$-hyperbolic. The class of functions on $X$ called horofunctions (a generalization of Busemann functions) gives a description of the boundary at infinity $\partial X$. When $X$ is the Cayley graph of a hyperbolic group


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\(\Gamma\), the space of cocycles associated with horofunctions that take integral values on the vertices is a shift of finite type \(\Sigma(\infty)\). (See also [Gromov 1987].)

Continuing this line of investigation, we first determine (Theorem 1.1) a lower bound for \(k^-_\infty(\lambda_A)\) in terms of the topological entropy \(h_{\text{top}}(\Sigma(\infty))\), for arbitrary hyperbolic groups. We therefore have

\[
h_{\text{top}}(\Sigma(\infty)) \leq k^-_\infty(\lambda_A) \leq \text{gr}(\Gamma, A),
\]

since the upper bound was already given in [Okayasu 2004]. We also show here that if a given hyperbolic group \(\Gamma\) splits over a finite group, the equation \(h_{\text{top}}(\Sigma(\infty)) = \text{gr}(\Gamma, A)\) holds for a certain finite generating set \(A\) of \(\Gamma\) (Corollary 1.2). As a consequence, the inequalities turn into an equalities for such groups:

\[
h_{\text{top}}(\Sigma(\infty)) = k^-_\infty(\lambda_A) = \text{gr}(\Gamma, A).
\]

It was already known from [Voiculescu 1993] that \(k^-_\infty(\lambda_A) \neq 0\) for every nonelementary hyperbolic group \(\Gamma\), because \(\Gamma\) is nonamenable.

**Notation.** We denote by \(\Sigma(\infty)\) the shift of finite type relative to \((\Gamma, A)\), constructed in [Coornaert and Papadopoulos 2001].

**Theorem 1.1.** Let \(\Gamma\) is a Gromov hyperbolic group with a finite generating set \(A\) and \(\lambda\) its left regular representation. Set \(\lambda_A = (\lambda_a)_{a \in A}\). Then we have

\[
h_{\text{top}}(\Sigma(\infty)) \leq k^-_\infty(\lambda_A) \leq \text{gr}(\Gamma, A).
\]

**Corollary 1.2.** Let \(\Gamma\) is a nonelementary hyperbolic group with a finite generating set \(A\), \(\lambda\) its left regular representation and \(\lambda_A = (\lambda_a)_{a \in A}\). Suppose that either

1. \(\Gamma\) can be written nontrivially as a free product \(G_1 \ast G_2\) and \(A = F_1 \cup F_2\) for some finite generating sets \(F_1, F_2\) of \(G_1, G_2\); or

2. \(\Gamma\) has a form of a free product \(G_1 \ast_H G_2\) with finite amalgamated subgroup \(H\), which is properly contained in both factors and of index greater than 2 in at least one factor, and \(A = F_1 \cup F_2\) for some finite generating sets \(F_1, F_2\) of \(G_1, G_2\), containing \(H\); or

3. \(\Gamma\) is an HNN extension

\[
G \ast_H \theta = \langle G, x \mid hx = x\theta(h) \text{ for } h \in H \rangle,
\]

where \(H\) is a proper finite subgroup of \(G\) and \(A = F \cup \{x, x^{-1}\}\) for some finite generating set \(F\) of \(G\), which contains both \(H\) and \(\theta(H)\).

Then \(k^-_\infty(\lambda_A) = \text{gr}(\Gamma, A) = h_{\text{top}}(\Sigma(\infty))\).
2. Preliminaries

Voiculescu’s perturbation theory. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and let $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ denote, respectively, the spaces of bounded linear operators and compact operators on $\mathcal{H}$. A symmetrically normed ideal $(\mathcal{G}, \| \cdot \|_\mathcal{G})$ is an ideal $\mathcal{G}$ of $\mathcal{K}(\mathcal{H})$ which is a Banach space endowed with the norm $\| \cdot \|_\mathcal{G}$ satisfying

$$
\| XTY \|_\mathcal{G} \leq \| X \| \cdot \| T \| \cdot \| Y \|
$$

for $T \in \mathcal{G}$ and $X, Y \in \mathcal{B}(\mathcal{H})$, where $\| \cdot \|$ is the operator norm on $\mathcal{B}(\mathcal{H})$.

It is well-known that the Schatten $p$-classes $\mathcal{C}_p(\mathcal{H})$ are symmetrically normed ideals. So are the ideals $\mathcal{C}_p^-(\mathcal{H})$ defined for $1 \leq p \leq \infty$ by the norm

$$
\| T \|_p^- = \sum_{j=1}^{\infty} \lambda_j j^{-1+1/p}
$$

(where $\lambda_1 \geq \lambda_2 \geq \cdots$ are the eigenvalues of $(T^*T)^{1/2}$); they are important for perturbation theory. The particular case $\mathcal{C}_1^-(\mathcal{H})$ is also known as the Macaev ideal. Note that $\mathcal{C}_1^-(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})$ but

$$
\mathcal{C}_p^-(\mathcal{H}) \subset \mathcal{C}_q^-(\mathcal{H}) \subset \mathcal{C}_q(\mathcal{H}) \quad \text{if } 1 < p < q.
$$

The dual $\mathcal{G}^*$, where the duality is given by the bilinear form $(X, Y) \mapsto \text{Tr}(XY)$, is again a normed ideal. We have $\mathcal{C}_p^*(\mathcal{H}) = \mathcal{C}_p(\mathcal{H})$, where $p > 1$ and $1/p + 1/q = 1$. Moreover $\mathcal{C}_p^-(\mathcal{H})^* = \mathcal{C}_q^+(\mathcal{H})$, where $\mathcal{C}_q^+(\mathcal{H})$ consists of all $T \in \mathcal{K}(\mathcal{H})$ such that

$$
\| T \|_q^+ = \sup_k \frac{\sum_{j=1}^{k} \lambda_j}{\sum_{j=1}^{k} j^{-1/q}} < \infty.
$$

Let $\mathcal{G}$ be a symmetrically normed ideal of $\mathcal{K}(\mathcal{H})$. For an $N$-tuple $\tau = (T_1, \ldots, T_N)$ of bounded linear operators on $\mathcal{H}$, we define

$$
k_{\mathcal{G}}(\tau) = \liminf_{A \in \mathcal{F}(H)_1^+} \max_{1 \leq i \leq N} \| [A, T_i] \|_\mathcal{G},
$$

where the inferior limit is taken with respect to the natural order on

$$
\mathcal{F}(H)_1^+ = \{ T \in \mathcal{K}(\mathcal{H}) \mid T : \text{finite rank, } 0 \leq T \leq I \}
$$

and $[A, B] = AB - BA$. We write $k_{\mathcal{G}}^-(\tau)$ when $\mathcal{G} = \mathcal{C}_p^-(\mathcal{H})$.

We see from the definition that $k_{\mathcal{G}}(\tau)$ measures the obstruction to the existence of a sequence $(A_n)_{n=1}^\infty \subseteq \mathcal{F}(\mathcal{H})_1^+$ such that $A_n \not \rightarrow I$ and $\lim_{n \rightarrow \infty} \| [A_n, T_i] \|_\mathcal{G} = 0$ for $1 \leq i \leq N$. If such a sequence exists, it is called a quasicentral approximate unit for $\tau$ relative to $\mathcal{G}$. 
Proposition 2.1 [Voiculescu 1990, Proposition 2.1]. Let \( \tau = (T_1, \ldots, T_N) \in \mathbb{B}(\mathcal{H})^N \) and \( X_i \in \mathcal{C}_1^+(\mathcal{H}) \) for \( i = 1, \ldots, N \). If
\[
\sum_{i=1}^N [X_i, T_i] \in \mathcal{C}_1(\mathcal{H}) + \mathbb{B}(\mathcal{H})_+,
\]
then
\[
\left| \text{Tr} \left( \sum_{i=1}^N [X_i, T_i] \right) \right| \leq k_\infty^-(\tau) \sum_{a=1}^N \| X_i \|_{1,1}^+,
\]
where \( \| X_i \|_{1,1}^+ = \inf_{Y \in \mathcal{F}(\mathcal{H})} \| X_i - Y \|_{1,1}^+ \).

Proposition 2.2 [Gohberg and Krein 1969, Theorem 14.1]. For \( T \in \mathcal{C}_1^+(\mathcal{H}) \), we have
\[
\| T \|_{1,1}^+ = \limsup_{n \to \infty} \frac{\sum_{j=1}^n s_j(T)}{\sum_{j=1}^n 1/j}.
\]

Subshifts. We briefly define the necessary concepts from symbolic dynamics; see [Lind and Marcus 1995] for a more leisurely introduction.

Let \( \mathcal{A} \) be a finite alphabet and \( \mathcal{A}^\infty \) the one-sided infinite product space \( \prod_{i=0}^\infty \mathcal{A} \) with the product topology (of discrete topologies). The shift map \( \sigma \) on \( \mathcal{A}^\infty \) is given by \( (\sigma(x))_i = x_{i+1} \) for \( i \in \mathbb{N} \). A word over \( \mathcal{A} \) is a finite sequence \( w = (a_1, \ldots, a_n) \) with \( a_i \in \mathcal{A} \). For \( x \in \mathcal{A}^\infty \) and a word \( w = (a_1, \ldots, a_n) \), we say that \( w \) occurs in \( x \) if there is an index \( i \) such that \( x_i = a_1, \ldots, x_{i+n-1} = a_n \). For a collection \( \mathcal{F} \) of words over \( \mathcal{A}^\infty \), we define the (one-sided) subshift \( X = X_\mathcal{F} \) to be the subset of sequences in \( \mathcal{A}^\infty \) in which no word in \( \mathcal{F} \) occurs.

Let \( X \) be a subshift of \( \mathcal{A}^\infty \). We denote by \( W_n(X) \) the set of all words with length \( n \) that occur in \( X \) and we set
\[
W(X) = \bigcup_{n=0}^\infty W_n(X).
\]
Let \( \varphi : W_{m+n+1}(X) \to \mathcal{A} \) be a map, which we call a block map. The extension of \( \varphi \) from \( X \) to \( \mathcal{A}^\infty \) is defined by \( (x_i)_{i \in \mathbb{N}} \mapsto (y_i)_{i \in \mathbb{N}} \), where
\[
y_i = \varphi((x_{i-m}, x_{i-m+1}, \ldots, x_{i+n})).
\]
We also denote this extension by \( \varphi \) and call it a sliding block code.

The topological entropy of a subshift \( X \) is defined by
\[
h_{\text{top}}(X) = \lim_{n \to \infty} \frac{1}{n} \log \text{card } W_n(X).
\]
A simple class of subshifts is that of shifts of finite type (SFT), those that can be described by a finite set of forbidden words. Let \( M = [M(a, b)]_{a, b \in \mathcal{A}} \) be a 0–1
matrix. Then
\[ \Sigma_M := \{ (x_i)_{i=0}^{\infty} \in \mathbb{R}^N \mid M(x_i, x_{i+1}) = 1 \} \]
is called the one-sided topological Markov shift by \( M \) and it is a shift of finite type.

**Gromov hyperbolic groups.** For basic facts about Gromov hyperbolic spaces and groups, see [Gromov 1987] and [Coornaert and Papadopoulos 1993].

Let \((X, | |)\) be a metric space which is proper, geodesic and \( \delta \)-hyperbolic for some \( \delta \geq 0 \). A function \( f : X \to \mathbb{R} \) is \( \varepsilon \)-convex, where \( \varepsilon \geq 0 \), if for any geodesic segment \([x_0, x_1]\) in \( X \) and any \( t \in [0, 1] \), we have
\[ f(x_t) \leq (1 - t)f(x_0) + tf(x_1) + \varepsilon, \]
where \( x_t \) is the point on \([x_0, x_1]\) satisfying \( |x_0 - x_t| = t|x_0 - x_1| \).

**Definition 2.3.** Let \( \varepsilon \geq 0 \). An \( \varepsilon \)-horofunction on \( X \) is an \( \varepsilon \)-convex function \( h : X \to \mathbb{R} \) satisfying \( h(x) - \lambda = \text{dist}(x, h^{-1}(\lambda)) \) for every \( x \in X \) and \( \lambda \in \mathbb{R} \) such that \( h(x) \geq \lambda \).

**Definition 2.4.** Let \( r : [0, \infty) \to X \) be a geodesic ray. The associated Busemann function \( h_r : X \to \mathbb{R} \) is defined by
\[ h_r(x) = \lim_{t \to \infty} |x - r(t)| - t. \]

A Busemann function on a \( \delta \)-hyperbolic \( X \) is a \( 4\delta \)-horofunction [Coornaert and Papadopoulos 2001, Proposition 2.5]. Thus Busemann functions form an important class of horofunctions.

**Definition 2.5.** A function \( \varphi : X \times X \to \mathbb{R} \) is called an \( \varepsilon \)-cocycle if there is an \( \varepsilon \)-horofunction \( h : X \to \mathbb{R} \) such that
\[ \varphi(x, y) = h(x) - h(y) \]
for every \( x, y \in X \). We call such a function \( h \) a primitive for \( \varphi \). (If \( h \) is a primitive for \( \varphi \), so is \( h + c \) for any constant \( c \).)

**Proposition 2.6** [Coornaert and Papadopoulos 2001, Proposition 2.7]. Let \( \varphi \) be a cocycle on \( X \). For \( x, y, z \) and \( w \in X \), we have
\begin{align*}
(1) \quad & \varphi(x, x) = 0, \\
(2) \quad & \varphi(x, y) = -\varphi(y, x), \\
(3) \quad & \varphi(x, y) = \varphi(x, z) + \varphi(z, y), \\
(4) \quad & |\varphi(x, y)| \leq |x - y|, \\
(5) \quad & |\varphi(x, y) - \varphi(z, w)| \leq |x - z| + |y - w|. 
\end{align*}
Let \( \gamma \) be an isometry of \( X \), \( h : X \to \mathbb{R} \) an \( \varepsilon \)-horofunction and \( \varphi : X \times X \to \mathbb{R} \) an \( \varepsilon \)-cocycle. The functions \( \gamma h \) and \( \gamma \varphi \) defined by
\[
\gamma h(x) = h(\gamma^{-1} x), \quad \gamma \varphi(x, y) = \varphi(\gamma^{-1} x, \gamma^{-1} y),
\]
for \( x, y \in X \), are an \( \varepsilon \)-horofunction and an \( \varepsilon \)-cocycle, respectively. If \( \varphi \) is the cocycle of \( h \), then \( \gamma \varphi \) is the cocycle of \( \gamma h \). Let \( \Phi \) be the set of \( \varepsilon \)-cocycles on \( X \) for all possible values of \( \varepsilon \geq 0 \). We equip \( \Phi \) with the topology of uniform convergence on compact sets.

**Definition 2.7.** Let \( \varphi \) be a cocycle on \( X \). A \( \varphi \)-gradient arc is a path \( g : I \to X \), parameterized by arclength, satisfying
\[
\varphi(g(t), g(t')) = t' - t
\]
for every \( t, t' \in I \). If \( I = \mathbb{R} \) or \( I = [0, \infty) \), we say that \( g \) is a \( \varphi \)-gradient line or ray, respectively. If \( g(0) = x \), we say that \( g \) starts at \( x \).

**Lemma 2.8** [Coornaert and Papadopoulos 2001, Lemma 2.9]. Let \( \varphi \) be a cocycle on \( X \) and \( I \subseteq \mathbb{R} \) an interval with \( a \in I \), \( I_1 = I \cap (-\infty, a] \) and \( I_2 = I \cap [a, \infty) \). If \( g : I \to X \) is a path whose restrictions to \( I_1 \) and \( I_2 \) are \( \varphi \)-gradient arcs, then \( g \) is itself a \( \varphi \)-gradient arc.

**Proposition 2.9** [Coornaert and Papadopoulos 2001, Proposition 2.10]. Let \( \varphi \) be a cocycle on \( X \).

1. Any \( \varphi \)-gradient arc \( g : I \to X \) is a geodesic.
2. If \( x, y \in X \) satisfying \( \varphi(x, y) = |x - y| \), and if \( g : [a, b] \to X \) is a geodesic joining \( x \) and \( y \), then \( g \) is a \( \varphi \)-gradient arc.

**Proposition 2.10** [Coornaert and Papadopoulos 2001, Proposition 2.13]. For every cocycle \( \varphi \) on \( X \) and for every \( x \in X \), there is a \( \varphi \)-gradient ray \( g : [0, \infty) \to X \) starting at \( x \).

Let \( \varphi \) be a cocycle on \( X \) and \( g : [0, \infty) \to X \) a \( \varphi \)-gradient ray. By Proposition 2.9, part (1), \( g \) is a geodesic and so converges to a well-defined point \( g(\infty) \in \partial X \).

**Proposition 2.11** [Coornaert and Papadopoulos 2001, Proposition 3.1]. Let \( \varphi \) be a cocycle on \( X \) and let \( g, g' : [0, \infty) \to X \) be \( \varphi \)-gradient rays. Then \( g(\infty) = g'(\infty) \).

**Definition 2.12.** We define a map \( \pi : \Phi \to \partial X \) by setting \( \Phi(\varphi) = g(\infty) \in \partial X \), where \( g : [0, \infty) \to X \) is a \( \varphi \)-gradient ray.

Let \( \text{Isom}(X) \) denote the group of isometries of \( X \). The action of \( \text{Isom}(X) \) on \( \Phi \) defined by \( (\gamma, \varphi) \mapsto \gamma \varphi \) is continuous.

**Proposition 2.13** [Coornaert and Papadopoulos 2001, Proposition 3.3]. The map \( \pi : \Phi \to \partial X \) is continuous, surjective, and commutes with the actions of \( \text{Isom}(X) \) on \( \Phi \) and \( \partial X \).
For any cocycle \( \varphi \), any geodesic ray \( r : [0, \infty) \to X \) satisfying \( r(\infty) = \pi(\varphi) \), and any \( t \geq 0 \), we set

\[
R_{\varphi,t} = \{ x \in X \mid \varphi(x, r(t)) = 0 \} \cap B(r(t), 16\delta).
\]

**Proposition 2.14** [Coornaert and Papadopoulos 2001, Proposition 3.4]. For \( \varphi \in \Phi \), let \( r : [0, \infty) \to X \) be a geodesic ray such that \( r(\infty) = \pi(\varphi) \). For all \( x \in X \) and \( t \in \mathbb{R} \) satisfying \( t > |x - r(0)| + 16\delta \), we have

\[
\varphi(x, r(t)) = \text{dist}(x, R_{\varphi,t}).
\]

In all that follows, \( \Gamma \) is a \( \delta \)-hyperbolic group with respect to a finite set of generators \( A \) and \( X \) is the Cayley graph associated to the pair \( (\Gamma, A) \). We denote by \( X^0 = \Gamma \) the set of vertices and by \( X^1 \) the set of edges of \( X \). For \( x \in \Gamma \), we denote by \( |x| \) the word length of \( x \) with respect to \( A \).

**Definition 2.15.** A horofunction \( h : X \to \mathbb{R} \) is said to be integral if \( h(x) \in \mathbb{Z} \) for every \( x \in X^0 \). A cocycle having an integral horofunction as a primitive is called an integral cocycle.

Every integral cocycle is completely determined by its values on \( \Gamma \times \Gamma \), by [Coornaert and Papadopoulos 2001, Corollary 4.4]. Thus we can regard an integral cocycle on \( X \) as a function from \( \Gamma \times \Gamma \) to \( \mathbb{Z} \). Let \( \Phi_0 \subseteq \Phi \) be the space of integral cocycles on \( X \). The topology induced on \( \Phi_0 \) by \( \Phi \) is the topology of pointwise convergence on \( \Gamma \times \Gamma \). For simplicity, we denote by \( \pi : \Phi_0 \to \partial \Gamma \) the restriction of the map \( \pi : \Phi \to \partial \Gamma \).

**Proposition 2.16** [Coornaert and Papadopoulos 2001, Proposition 4.5]. The map \( \pi : \Phi_0 \to \partial \Gamma \) is continuous, \( \Gamma \)-equivalent, surjective and uniformly finite to one. In fact, for every \( \xi \in \partial \Gamma \) we have

\[
\text{card} \{ \varphi \in \Phi_0 \mid \pi(\varphi) = \xi \} \leq (2N_0 + 1)^{N_1},
\]

where \( N_0 \) is the integral part of \( 16\delta + 1 \) and \( N_1 \) is the number of elements in \( \Gamma \) contained in the closed ball of radius \( N_0 \) centered at the identity.

**Lemma 2.17** [Coornaert and Papadopoulos 2001, Lemma 5.1]. For every \( \varphi \in \Phi_0 \) and \( x \in X^0 \), there is \( a \in A \) such that \( \varphi(x, xa) = 1 \).

Now we fix a total order relation on the finite generating set \( A \). Let \( \varphi \in \Phi_0 \) and \( x \in X^0 \). The lexicographic order on \( A^{N} \) induces a total order on the set of \( \varphi \)-gradient rays starting at \( x \).

**Proposition 2.18** [Coornaert and Papadopoulos 2001, Proposition 5.2]. Let \( \varphi \in \Phi_0 \) and \( x \in X^0 \). The set of \( \varphi \)-gradient rays starting at \( x \) has a smallest element.

**Definition 2.19.** We define a map \( \alpha : \Phi_0 \to \Phi_0 \) by \( \alpha(\varphi) = a^{-1}\varphi \), where \( \varphi \in \Phi_0 \) and \( a \) is the smallest element in \( A \) satisfying \( \varphi(e, a) = 1 \).
Proposition 2.20 [Coornaert and Papadopoulos 2001, Proposition 5.6]. The map \( \alpha : \Phi_0 \rightarrow \Phi_0 \) is continuous.

Proposition 2.21 [Coornaert and Papadopoulos 2001, Proposition 5.7]. Let \( \varphi \in \Phi_0 \) and \( g : [0, \infty) \rightarrow X \) be the smallest \( \varphi \)-gradient ray starting at \( e \). For \( n \in \mathbb{N} \), let \( a_n \in A \) be the label of the oriented edge from \( g(n) \) to \( g(n+1) \) and \( g_n : [0, \infty) \rightarrow X \) the smallest \( \alpha^n(\varphi) \)-gradient ray starting at \( e \).

1. \( \alpha^n(\varphi) = g(n)^{-1}\varphi \).
2. \( g_n(t) = g(n)^{-1}g(t+n) \) for any \( t \in [0, \infty) \).
3. For every \( k \in \mathbb{N} \), the label of oriented edge from \( g_n(k) \) to \( g_n(k+1) \) is \( a_{k+n} \).

Next we introduce the shift of finite type \((\Sigma(\infty), T)\) and the conjugacy \( P \) from \((\Phi_0, \alpha) \) to \((\Sigma(\infty), T)\). We take integers \( R \geq 100\delta + 1 \) and \( L \geq 2R + 32\delta + 1 \). For a subset \( Y \subseteq X \) and \( \epsilon \geq 0 \), we set \( N(Y, \epsilon) = \{ x \in X \mid \text{dist}(x, Y) \leq \epsilon \} \).

For \( \varphi \in \Phi_0 \), let \( g : [0, \infty) \rightarrow X \) be the smallest \( \varphi \)-gradient ray starting at \( e \). Set \( V(\varphi) = N(g([0, L]), R) \).

\( V(\varphi) \) is contained in the closed ball \( B(e, L + R) \) of radius \( L + R \) centered at \( e \).

For each \( \varphi \in \Phi_0 \), we define a function \( \rho(\varphi) : V(\varphi) \rightarrow \mathbb{R} \) by

\[
\rho(\varphi)(x) = \varphi(x, e)
\]

for \( x \in V(\varphi) \). Note that \( \rho(\varphi) \) is the restriction to \( V(\varphi) \) of the primitive \( h \) of \( \varphi \) with \( h(e) = 0 \). We set

\[
S = \{ \rho(\varphi) : V(\varphi) \rightarrow \mathbb{R} \mid \varphi \in \Phi_0 \}.
\]

Lemma 2.22 [Coornaert and Papadopoulos 2001, Lemma 6.2]. The set \( S \) is finite.

Definition 2.23. Let \( \Sigma \) be the set of sequences \( (\sigma_n)_{n \geq 0} \) with \( \sigma_n \in S \) for \( n \geq 0 \), and give it the product topology (of discrete topologies on copies of \( S \)). The map \( T : \Sigma \rightarrow \Sigma \) is the shift map. Define a map \( P : \Phi_0 \rightarrow \Sigma \) by

\[
\Phi_0 \ni \varphi \mapsto (\sigma_n)_{n \geq 0} \in \Sigma,
\]

where \( \sigma_n = \rho(\alpha^n(\varphi)) \) for \( n \geq 0 \).

Let \( s \in S \). We denote by \( V(s) \) the domain of the function \( s \). Since \( R \geq 1 \), the domain \( V(s) \) contains the closed unit ball \( B(e, 1) \). Hence the value \( s(a) \) is well-defined for all \( a \in A \). Since the finite generating set \( A \) is equipped with a fixed total order relation, we can define \( w(s) \) to be the smallest element \( a \in A \) satisfying \( s(a) = -1 \). (Such an \( a \) exists because of Lemma 2.17.)
Let $\sigma = (\sigma_n)_{n \geq 0} \in \Sigma$. We define a sequence $(\gamma_n(\sigma))_{n \geq 0}$ by setting $\gamma_0(\sigma) = e$, $\gamma_n(\sigma) = w(\sigma_0) \cdots w(\sigma_{n-1})$ for $n \geq 1$.

For $n \geq 0$, we set $V_n(\sigma) = \gamma_n(\sigma)V(\sigma_n)$. This depends only on the first $n + 1$ coordinates of $\sigma$. We also define functions $f_n(\sigma) : V_n(\sigma) \to \mathbb{R}$ by $f_n(\sigma)(x) = \gamma_n(\sigma)^{-1}x - n$ for $x \in V_n(\sigma)$.

Lemma 2.24 [Coornaert and Papadopoulos 2001, Lemma 6.5]. For $\varphi \in \Phi_0$, take $\sigma = P(\varphi)$ and let $g : [0, \infty) \to X$ be the smallest $\varphi$-gradient ray starting at $e$. Assume $n \geq 0$.

1. $\gamma_n(\sigma) = g(n)$.
2. $V_n(\sigma) = N(\varphi([n, n + L]), R)$.
3. $f_n(\sigma)$ is the restriction to $V_n(\sigma)$ of the primitive $h$ of $\varphi$ with $h(e) = 0$.

Definition 2.25. Let $\sigma \in \Sigma$. We say that $\sigma$ is consistent if for all $i, j \geq 0$, we have $f_i(\sigma)(x) = f_j(\sigma)(x)$ for all $x \in V_i(\sigma) \cap V_j(\sigma)$. We denote by $\Sigma(\infty)$ the set of all consistent sequences.

Lemma 2.26 [Coornaert and Papadopoulos 2001, Lemma 6.8]. $P(\Phi_0) \subseteq \Sigma(\infty)$.

Theorem 2.27 [Coornaert and Papadopoulos 2001, Theorem 7.18]. The set of consistent sequences $\Sigma(\infty)$ is a shift of finite type. Moreover $(\Phi_0, \alpha)$ and $(\Sigma(\infty), T)$ are conjugate via the map $P$.

3. The topological entropy of $\Sigma(\infty)$

Let $\Gamma$ be a Gromov hyperbolic group with a finite generating set $A$ on which we fix a total order relation. Let $\Sigma(\infty)$ the corresponding SFT.

For $n \in \mathbb{N}$, we denote by $W_n$ the set of all words with length $n$ that occur in $\Sigma(\infty)$ and by $A_n$ the set of all elements in $\Gamma$ with word length $n$ with respect to the finite generating set $A$ (as a particular case, $A_0 = \{e\}$). We set $D_n = \bigcup_{1 \leq k \leq n} W_k$ and $B_n = \bigcup_{0 \leq k \leq n} A_k$. For each $s \in S$, we set $W_n(s) = \{ (\sigma_0, \ldots, \sigma_{n-1}) \in W_n \mid \sigma_0 = s \}$, and for each $a \in A$, $A_n(a) = \{ a \gamma \in A_n \mid \gamma \in A_{n-1} \}$.

We write $D_n(s) = \bigcup_{1 \leq k \leq n} W_k(s)$ and $B_n(a) = \bigcup_{1 \leq k \leq n} A_n(a)$.

We denote by $\text{gr}(\Gamma, A)$ the growth entropy of $\Gamma$ with respect to $A$:

$$\text{gr}(\Gamma, A) = \lim_{n \to \infty} \frac{1}{n} \log \text{card} A_n.$$
We also define
\[
\overline{A}_n = \{ \gamma \in A_n \mid \gamma = w(\sigma_0) \cdots w(\sigma_{n-1}) \text{ for some } (\sigma_0, \ldots, \sigma_{n-1}) \in W_n \},
\]
\[
\overline{B}_n = \bigcup_{1 \leq k \leq n} \overline{A}_k.
\]
\[
\overline{A}_n(w(s)) = \{ \gamma \in A_n(w(s)) \mid \gamma = w(\sigma_0) \cdots w(\sigma_{n-1}) \text{ for some } (\sigma_0, \ldots, \sigma_{n-1}) \in W_n(s) \},
\]
\[
\overline{B}_n(w(s)) = \bigcup_{1 \leq k \leq n} \overline{A}_n(w(s)).
\]

**Lemma 3.1.** There is a constant $K > 0$ such that
\[
\text{card} \{(\sigma_0, \ldots, \sigma_{n-1}) \in W_n \mid w(\sigma_0) \cdots w(\sigma_{n-1}) = \gamma\} \leq K,
\]
for every $n \geq 1$ and every $\gamma \in A_n$.

**Proof.** Let $\varphi, \varphi' \in \Phi_0$ and $g, g'$ their smallest gradient rays starting at $e$ such that $g(n) = g'(n) = \gamma \in A_n$. Note that $\varphi(e, \gamma) = \varphi'(e, \gamma) = -n$. We denote $\sigma = P(\varphi)$ and $\sigma' = P(\varphi')$. By Lemma 2.24, we have $\gamma_n(\sigma) = \gamma_n(\sigma') = \gamma$.

We first claim that $g = g'$ on $[0, n]$. We now assume that $g \neq g'$ on $[0, n]$. We may assume that $g' < g$ in the lexicographic order on $A^0$. Without loss of generality, we may assume that $g' < g$ in the lexicographic order on $A^0$. Therefore $g$ would be not the smallest $\varphi$-gradient ray.

Hence we have $g = g'$ on $[0, n]$.

Let $h, h'$ be primitives for $\varphi, \varphi'$ satisfying $h(e) = h'(e) = 0$, respectively. We set $B = B(\gamma, L + R)$.

We secondly claim that if $h = h'$ on $B$, then $h = h'$ on $N(g([0, n + L]), R)$. Notice that $R > 16\delta$ and $L > 2R$. Let $k \in [0, n]$ satisfying $n - k \leq 2R$. Since $N(g([k, n + L]), R) \subseteq B$, we have $h = h'$ on $N(g([k, n + L]), R)$. Next let $k \geq 0$ satisfying $n - k > 2R$. For $x \in B(g(k), R)$, we have
\[
n = |g(0) - g(n)| = |g(0) - g(k)| + |g(k) - g(n)|
\leq |g(0) - x| - |x - g(k)| + |g(k) - g(n)|
\geq |g(0) - x| - R + (n - k)
\geq |g(0) - x| + R > |g(0) - x| + 16\delta.
\]

By Proposition 2.14, we have $\varphi(x, g(n)) = \text{dist}(x, R_{\varphi, n})$. Recall that
\[
R_{\varphi, n} = \{ x \in X \mid \varphi(x, g(n)) = 0 \} \cap B(g(n), 16\delta).
\]
Hence \( h(x) + n = \text{dist}(x, R_{g,n}) \). This shows that the value of \( h(x) \) depends only on the restriction of \( h \) on \( B(g(n), 16\delta) \subseteq B \). Namely we obtain our claim.

We now assume that \( h = h' \) on \( B \). In this case, we remark that \( g = g' \) on \([0, n+L] \). By Proposition 2.21, we have \( V(\alpha^k(\varphi)) = N(g(k)^{-1}g([k, k+L]), R) = V(\alpha^k(\varphi')) \) for \( 0 \leq k \leq n \). For each \( x \in V(\alpha^k(\varphi)) \), since

\[
N(g(k)^{-1}g([k, k+L]), R) = g(k)^{-1}N(g([k, k+L]), R),
\]

there is \( y \in N(g([k, k+L]), R) \) such that \( x = g(k)^{-1}y \). Then

\[
\rho(\alpha^k(\varphi))(x) = g(k)^{-1}\varphi(x, e) = \varphi(y, g(k)) = h(y) - h(g(k)) = h(y) + k.
\]

Similarly we also obtain \( \rho(\alpha^k(\varphi))(x) = h'(y) + k \). Hence if \( h = h' \) on \( B \), then it follows from the second claim that

\[
\rho(\alpha^k(\varphi))(x) = h(y) + k = h'(y) + k = \rho(\alpha^k(\varphi')).
\]

Therefore \( \rho(\alpha^k(\varphi)) = \rho(\alpha^k(\varphi')) \); that is, \( \sigma_k = \sigma_k' \) for all \( 0 \leq k \leq n \).

Hence it suffices to set \( K = (2(L+R)+1)^b \), where \( b = \text{card } B = \text{card } B(e, L+R) \). Indeed, for every \( x \in B \) we have, using Proposition 2.6,

\[
|h(x) + n| = |h(x) - h(y)| = |\varphi(x, y)| = |x - y| \leq L + R.
\]

This easily leads to the assertion. \( \square \)

**Corollary 3.2.** \( h_{\text{top}}(\Sigma(\infty)) \leq \text{gr}(\Gamma, A) \).

**Proof.** For each \( n \geq 0 \), the map \( W_n \ni (\sigma_0, \ldots, \sigma_{n-1}) \mapsto w(\sigma_0) \cdots w(\sigma_{n-1}) \in A_n \) is uniformly finite-to-one by Lemma 3.1. Thus

\[
\text{card } W_n \leq K \text{card } A_n.
\]

The assertion follows immediately. \( \square \)

**Remark 3.3.** A fundamental theorem of J. Stallings [1971] shows that a finitely generated group \( \Gamma \) has infinitely many ends if and only if it has a form of either (2) or (3) of Corollary 1.2. In particular, a torsion-free group has the form (1).

### 4. Proof of main results

**Proof of Corollary 1.2.** In view of Corollary 3.2, we just need to show that \( h_{\text{top}}(\Sigma(\infty)) \geq \text{gr}(\Gamma, A) \) if one of the conditions (1)–(3) of Corollary 1.2 is satisfied. Remark 3.3 shows that it suffices to check cases (2) and (3); but we check case (1) explicitly as well because it is very simple.

**Case (1):** It suffices to show that the map \((\sigma_0, \ldots, \sigma_{n-1}) \mapsto w(\sigma_0) \cdots w(\sigma_{n-1})\) from \( W_n \) to \( A_n \) is surjective. Let \( \gamma \in A_n \). There is the smallest geodesic segment \( r : [0, n] \to X \) from \( e \) to \( \gamma \). We can take \( g \) to be a geodesic ray extending \( r \), meaning
that \( r(k) = g(k) \) for all \( 0 \leq k \leq n \). Indeed, by assumption, we have \( \Gamma = G_1 \ast G_2 \).

Then \( \gamma \) is written as a reduced word \( g_1 \cdots g_m \), where \( g_k \in G_{i_k} \) with \( i_k \neq i_{k+1} \) for \( 1 \leq k \leq m - 1 \). Hence for \( l \geq 1 \), it is enough to set

\[
g(n + 2l) = \gamma \cdot ab \cdots ab, \quad g(n + 2l - 1) = \gamma \cdot ab \cdots ba,
\]

for some \( a \in F_i \) and \( b \in F_{i_m} \), with \( i \neq i_m \) and \( a, b \neq e \).

We consider the cocycle \( \varphi_g \) having the Busemann function \( h_g \) as a primitive. It is clear that \( g \) is a \( \varphi_g \)-gradient ray. Moreover by definition, \( g \), is in fact, the smallest \( \varphi_g \)-gradient ray starting at \( e \). Hence \( \gamma = w(P(\varphi_g)) \cdots w(P(\varphi_{g-1})) \).

It follows that the map above is surjective. Thus \( \text{card } A_n \leq \text{card } W_n \), and \( \text{gr}(\Gamma, A) \leq h_{\text{top}}(\Sigma(\infty)) \) as needed.

**Case (2):** Now we assume that \( \Gamma = G_1 \ast H G_2 \). Let \( \gamma \in A_n \) with \( n \geq 2 \). We express the element \( \gamma \) by the reduced word \( g_1 \cdots g_m \), where \( g_k \in G_{i_k} \setminus H \) with \( i_k \neq i_{k+1} \) for \( 1 \leq k \leq m - 1 \). We take a sequence \( (g_{m+1}, g_{m+2}, \ldots) \) such that \( g_k \in F_{i_k} \setminus H \) with \( i_{k-1} \neq i_k \) for all \( k \geq m + 1 \). We define a sequence \( (g(k))_{k=1}^\infty \) in \( X \) by \( g(k) = g_1 \cdots g_k \) for \( k \geq 1 \).

Let \( (y, z) = \frac{1}{2}(|y| + |z| - |y - z|) \) be the Gromov product based at \( e \). For \( l \geq k \geq m \),

\[
2(g(k), g(l)) = |g(k)| + |g(l)| - |g(k)^{-1}g(l)| \geq k + l - |g(k)^{-1}g(l)| = k + l - |g_{k+1} \cdots g_l|
\]

\[
\geq k + l - (l - k) = 2k
\]

tends to \( \infty \) with \( k \); thus there exists \( \xi \in \partial X \) such that the sequence \( (g(k))_{k=1}^\infty \) converges to \( \xi \). Let \( r : [0, \infty) \to X \) be a geodesic ray starting at \( e \) with \( r(\infty) = \xi \). We denote by \( \varphi_r \) the cocycle with respect to the Busemann function \( h_r \). Let \( g' : [0, \infty) \to X \) be the smallest \( \varphi_r \)-gradient ray starting at \( e \). Because \( r \) is also a \( \varphi_r \)-gradient ray, it follows from Proposition 2.11 that \( g'(\infty) = \xi \). We can express \( g' \) by the infinite reduced word \( (g'_1, g'_2, \ldots) \) with \( g'_k \in G_{j_k} \setminus H \) and \( j_k \neq j_{k+1} \) for \( k \geq 1 \). Since \( g'(\infty) = \xi \), we have \( i_k = j_k \) for all \( k \geq 1 \). Moreover we obtain \( \gamma = g(m) = g_1 \cdots g_m = g'_1 \cdots g'_m h \) for some \( h \in H \). Let \( k_m \geq 1 \) such that \( g'(k_m) = g'_1 \cdots g'_m \). Then we have \( |g(m) - g'(k_m)| \leq 1 \). Note that \( n - 1 \leq k_m \leq n + 1 \).

Hence we have proved that for any \( \gamma \in A_n \), there is \( \gamma' \in A_n \) such that \( \gamma' \in B(\gamma, 2) \) and \( \gamma' = w(\sigma_0) \cdots w(\sigma_{n-1}) \) for some \( (\sigma_0, \ldots, \sigma_{n-1}) \in W_n \). Therefore \( \text{card } A_n \leq \text{card } B(e, 2) \cdot \text{card } W_n \), and the assertion follows.

**Case (3):** We assume that \( \Gamma = G \ast_H \theta \). Let \( \gamma \in A_n \). The element \( \gamma \) can be represented by either (i) \( g_0 \in G \) or (ii) a reduced word \( g_0x^{\epsilon_0} \cdots g_{m-1}x^{\epsilon_{m-2}}g_m \), where \( g_k \in G \) and \( \epsilon_k \in \{1, -1\} \) for all \( 0 \leq k \leq m \). In case (i), we set \( g_k = e \) for \( k \geq 1 \) and \( \epsilon_k = 1 \) for \( k \geq 0 \). In case (ii), we set \( g_k = e \) for \( k \geq m + 1 \) and \( \epsilon_k = \epsilon_{m-1} \).
for $k \geq m$. Then we define the sequence $(g(k))_{k=0}^{\infty}$ in $X$ by $g(k) = g_0 x^{\varepsilon_0} \cdots g_k x^{\varepsilon_k}$ for all $k \geq 0$. Again, for $l \geq k \geq m$,

$$2\langle g(k), g(l) \rangle = |g(k)| + |g(l)| - |g(k)^{-1}g(l)| \geq k + l - |x^{\varepsilon_{k+1}} \cdots x^{\varepsilon_l}| = 2k$$

goes to $\infty$ with $k$; hence $(g(k))_{k=0}^{\infty}$ converges to some $\xi \in \partial \Gamma$. Let $r : [0, \infty) \to X$ be a geodesic ray with $r(0) = e$ and $r(\infty) = \xi$. We denote by $\varphi_r$ the cocycle with respect to the Busemann function $h_r$. Let $g' : [0, \infty) \to X$ be the smallest $\varphi_r$-gradient ray starting at $e$. We can also represent the geodesic ray $g'$ as the infinite reduced word $(g_0' x^{\varepsilon_0'}, g_1' x^{\varepsilon_1'}, \ldots)$. Since $g'(\infty) = \xi$, we have $\varepsilon_i = \delta_i$ for all $i \geq 0$. Moreover we obtain $\gamma = g_0 x^{\varepsilon_0} \cdots g_m = g_0' x^{\varepsilon_0} \cdots g_m'$, for some either $g \in H$ if \( \varepsilon_m = 1 \), or $g \in \theta(H)$ if $\varepsilon_m = -1$. Let $k_m \geq 1$ such that $g'(k_m) = g_0' x^{\varepsilon_0} \cdots g_m'$. Then we have $|\gamma - g(k_m)| \leq 1$. Note that $n - 1 \leq k_m \leq n + 1$. Hence we have shown that for each $\gamma \in A_n$, there is $\gamma' \in A_n$ such that $\gamma' \in B(\gamma, 2)$ and $\gamma' = w(\sigma_0) \cdots w(\sigma_{n-1})$ for some $(\sigma_0, \ldots, \sigma_{n-1}) \in W_n$. Therefore card $A_n \leq \text{card } B(e, 2) \cdot \text{card } W_n$, and $h_{\text{top}}(\Sigma(\infty)) \leq \text{gr}(\Gamma, A)$ as needed.

□

**Remark 4.1.** It is easy to check that the topological entropy $h_{\text{top}}(\Sigma(\infty))$ does not depend on the choice of total order relations on $A$.

**Proof of Theorem 1.1.** It suffices to show that $h_{\text{top}}(\Sigma(\infty)) \leq k_{\infty}(\lambda_A)$, because the inequality $k_{\infty}(\lambda_A) \leq \text{gr}(\Gamma, A)$ has been proved in [Okayasu 2004, Proposition 4.1]. Let $\lambda_{w(S)} = \{\lambda_{w(s)} \mid s \in S\}$. Note that $k_{\infty}(\lambda_{w(S)}) \leq k_{\infty}(\lambda_A)$.

Since $\Sigma(\infty)$ is an SFT, there are $N \in \mathbb{N}$ and $W \subseteq \mathbb{N}^{N+1}$ such that

$$\Sigma(\infty) = \{(\sigma_n)_{n \geq 0} \in \Sigma \mid (\sigma_n, \ldots, \sigma_{n+N}) \in W \text{ for any } n \geq 0\}.$$

Let $I = \mathbb{N}^N$ and $\beta_N : \Sigma(\infty) \to I^\mathbb{N}$ be the $N$-th higher block code. Then the subshift $\beta_N(\Sigma(\infty))$ is the Markov shift $\Sigma_M$ for some matrix $M = [M(i, j)]_{i, j \in I}$. Let $\mu$ be the maximal measure on $\Sigma(\infty)$, i.e., $h_{\text{top}}(\Sigma(\infty)) = h_\mu(T|_{\Sigma(\infty)})$. For simplicity, we denote by $h$ the topological entropy of $\Sigma(\infty)$. We denote by $[\sigma_0, \ldots, \sigma_{n-1}]$ the cylinder set at $0$-th coordinate. For $(\sigma_0, \ldots, \sigma_{n-1}) \in W_n$ with $n \geq N$, we have

$$\mu([\sigma_0, \ldots, \sigma_{n-1}]) = \frac{l_i r_j}{e^{(n-N)h}},$$

where $i = (\sigma_0, \ldots, \sigma_{N-1})$, $j = (\sigma_{n-N}, \ldots, \sigma_{n-1}) \in I$ and $l_i, r_j$ are the left and right Perron vectors of $M$ with $\sum_{i,j} l_i r_j = 1$ (see [Kitchens 1998]).

For each $n \geq 0$, denote by $P_n$ the projection onto the subspace

$$\text{span}\{\delta_\gamma \in \ell^2(\Gamma) \mid |\gamma| = n\}.$$

For $a \in A$, define the partial isometry $T_a \in \mathcal{B}(\ell^2(\Gamma))$ [Okayasu 2002; 2004] by

$$T_a = \sum_{n \geq 0} P_{n+1} \lambda_a P_n.$$
For each \( s \in S \), we define \( X_s \) by
\[
\sum_{n \geq 1} \sum_{(\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) \in W_n(s)} \mu([\sigma_0, \sigma_1, \ldots, \sigma_{n-1}]) T_{w(\sigma_1)} \cdots T_{w(\sigma_{n-1})} P_0 T_{w(\sigma_{n-1})}^* \cdots T_{w(\sigma_1)}^* T_{w(\sigma_0)}^*.
\]
Then \( \sum_{s \in S} [X_s, \lambda(w(s))] = P_0 \), because
\[
\sum_{s \in S} \lambda(w(s)) X_s
\]
\[= \sum_{n \geq 1} \sum_{s \in S} \sum_{(\sigma_0, \ldots, \sigma_{n-1}) \in W_n(s)} \mu([\sigma_0, \ldots, \sigma_{n-1}]) T_{w(\sigma_1)} \cdots T_{w(\sigma_{n-1})} P_0 T_{w(\sigma_{n-1})}^* \cdots T_{w(\sigma_1)}^* T_{w(\sigma_0)}^*.
\]
and
\[
\sum_{s \in S} X_s \lambda(w(s))
\]
\[= \sum_{n \geq 1} \sum_{s \in S} \sum_{(\sigma_0, \ldots, \sigma_{n-1}) \in W_n(s)} \mu([\sigma_0, \ldots, \sigma_{n-1}]) T_{w(\sigma_1)} \cdots T_{w(\sigma_{n-1})} P_0 T_{w(\sigma_{n-1})}^* \cdots T_{w(\sigma_1)}^* T_{w(\sigma_0)}^*.
\]
Next we give an estimate of \( \|X_s\|_1^+ \). For \( n \in \mathbb{N} \) and \( \gamma \in \tilde{A}_n(w(s)) \), we define
\[
s_{\gamma} = \sum_{(\sigma_0, \ldots, \sigma_{n-1}) \in W_n(s)} \mu([\sigma_0, \ldots, \sigma_{n-1}])
\]
This sum is uniformly finite by Lemma 3.1. Thus
\[
C_1 e^{-nh} \leq s_{\gamma} \leq C_2 e^{-nh}
\]
for constants \( C_1, C_2 > 0 \), independent of \( n \) and \( \gamma \).
Let \( s_1 \geq s_2 \geq \cdots \) be the eigenvalues of \((X_s^* X_s)^{1/2}\). For each \( j \in \mathbb{N} \), there is \( \gamma_j \in \tilde{A}_{n_j}(w(s)) \) such that \( s_j = s_{\gamma_j} \).
Let \( \varepsilon > 0 \). Recall that
\[
\|X_s\|_1^+ = \inf_{Y \in \mathcal{F}(\Gamma)_1^+} \|X_s - Y\|_1^+.
\]
By doing finite rank perturbations if necessary, we may assume that for all \( j \geq 1 \),
\[
e^{-n_j(h+\varepsilon)} \leq s_j \leq e^{-n_j(h-\varepsilon)}.
\]
Let \( N \in \mathbb{N} \) with \( e^{-N\varepsilon} \leq C_1 \) and \( n \geq N \). If there is \( m > n \) such that \( j \leq \text{card } \overline{B}_n(w(s)) \) and \( \gamma_j \in \overline{A}_m(w(s)) \), we have
\[
e^{-m(h-\varepsilon)} \geq e^{-n(h+\varepsilon)}.
\]

For otherwise we would have
\[
s_j \leq e^{n(h-\varepsilon)} < e^{-n(h+\varepsilon)} \leq e^{-n\varepsilon} \frac{S_\gamma}{C_1} \leq s_n
\]
for all \( \gamma \in \overline{B}_n(w(s)) \) and this is a contradiction. Therefore \( e^{-m(h-\varepsilon)} \geq e^{-n(h+\varepsilon)} \), namely
\[
m \leq n \frac{h + \varepsilon}{h - \varepsilon}.
\]

We put
\[
k = \max \left\{ m \in \mathbb{N} \mid m \leq n \frac{h + \varepsilon}{h - \varepsilon} \right\}.
\]

Since
\[
\mu(\{s\}) = \sum_{(\sigma_0, \ldots, \sigma_{n-1}) \in W_n(s)} \mu(\{\sigma_0, \ldots, \sigma_{n-1}\}) \leq \text{card } W_n(s) \cdot C e^{-nh},
\]
for some \( C > 0 \), we obtain
\[
\frac{\mu(\{s\}) e^{nh}}{C} \leq \text{card } W_n(s).
\]

Hence
\[
\|X_s\|_1^\perp \leq \limsup_{n \to \infty} \frac{\sum_{j=1}^{\text{card } \overline{B}_n(w(s))} s_j}{\sum_{j=1}^{\text{card } \overline{B}_n(w(s))} j^{-1}} \leq \limsup_{n \to \infty} \frac{\sum_{j=1}^{k} \sum_{\gamma \in \overline{A}_j(w(s))} \sum_{w(\sigma_0) \ldots w(\sigma_{j-1}) = \gamma} \mu(\{\sigma_0, \ldots, \sigma_{j-1}\}) \log \text{card } \overline{B}_n(w(s))}{\log \text{card } \overline{B}_n(w(s))} \leq \limsup_{n \to \infty} \frac{n}{\log \text{card } \overline{A}_n(w(s))} \frac{h + \varepsilon}{h - \varepsilon} \mu(\{s\}) \leq \limsup_{n \to \infty} \frac{n}{\log \text{card } W_n(s) - \log K} \frac{h + \varepsilon}{h - \varepsilon} \mu(\{s\}) \leq \frac{h + \varepsilon}{h(h - \varepsilon)} \mu(\{s\}).
\]
Here we have used that \( \text{card } W_n(s) \leq K \text{card } \overline{A}_n(w(s)) \) (Lemma 3.1). Since \( \varepsilon > 0 \) is arbitrary, we have
\[
\| X_s \|_{\tilde{1}} \leq \frac{1}{h} \mu([s]).
\]
Thanks to Proposition 2.1, we obtain
\[
h = h_{\text{top}}(\Sigma(\infty)) \leq k_\infty^-(\lambda_{w(S)}) \leq k_\infty^-(\lambda_{A}) \leq \text{gr}(\Gamma, A).
\]

\[\square\]

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