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Let Γ be a finite discrete group acting smoothly on a compact manifold X , and let D be a first-order elliptic self-adjoint Γ -equivariant differential operator acting on sections of some Γ -equivariant Hermitian vector bundle over X . We use these data to define Toeplitz operators with symbols in the transformation group C^* -algebra $C(X) \rtimes \Gamma$. If the symbol of such a Toeplitz operator is invertible, then the operator is Fredholm. In the case where X is a spin manifold and D is the Dirac operator, we give a geometric-topological formula for the index.

Let X be a smooth compact manifold without boundary, let V be a Hermitian vector bundle over X , and suppose D is a first-order elliptic self-adjoint differential operator acting on sections of V . Let P be the positive spectral projection of D . Then P is an order zero pseudodifferential operator, and it follows from standard facts about pseudodifferential operators on compact manifolds that given a smooth function f on X , the pointwise multiplication operator M_f acting on square-integrable sections $L^2(V)$ of V commutes with P modulo the ideal of compact operators. From this fact it is easy to show that if f is invertible, then the Toeplitz operator $PM_f : \text{Ran } P \rightarrow \text{Ran } P$ is a Fredholm operator. Furthermore, the index of PM_f can be computed using the Atiyah–Singer Index Theorem; see [Baum and Douglas 1982].

Now suppose that a discrete group Γ acts smoothly on both X and V in a compatible way, and suppose that D commutes with the action of Γ on sections of V . Then P also commutes with this action. In addition, there is a natural action ρ of the transformation group C^* -algebra $C(X) \rtimes \Gamma$ on $L^2(V)$, and P commutes with the elements of $C(X) \rtimes \Gamma$ modulo the compacts. Therefore, whenever $F \in C(X) \rtimes \Gamma$ is invertible, $T_F := P\rho(F)$ is a Fredholm operator, and it is natural to ask what the index of this operator is.

Let X be an odd-dimensional oriented spin manifold and let Γ be a finite group acting on X by isometries that preserve the orientation and spin structure of X . In [Park 2002], the case of free actions was considered; in this paper we consider

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general actions. We use the Lefschetz theorem in [Fang 2005] to prove a theorem that computes the Fredholm index of T_F in terms of the geometry and topology of X and a “Chern character” form explicitly constructed from F .

We begin by more precisely defining the objects under discussion. Let X be a smooth compact manifold, and let Γ be a discrete group acting smoothly on X from the right. Let V be a Γ -equivariant complex vector bundle over X , and equip V with a Γ -invariant Hermitian structure. Then Γ acts on the left on both the smooth sections $C^\infty(V)$ and the square-integrable sections $L^2(V)$ of V :

$$(\gamma \cdot s)(x) = s(x\gamma)\gamma^{-1}.$$

If $C(X)$ acts on $L^2(V)$ by pointwise multiplication, we have a covariant representation of $(C(X), \Gamma)$ on $L^2(V)$, and hence a representation ρ of $C(X) \rtimes \Gamma$ on $L^2(V)$.

Let D be a first-order, Γ -equivariant, elliptic self-adjoint differential operator acting on sections of V , and let $P = \chi_{[0, \infty)}(D)$ denote the positive spectral projection of D ; this operator is also Γ -equivariant. For each F in $C(X) \rtimes \Gamma$, define the Toeplitz operator $T_F \in \mathcal{B}(L^2(V))$ to be $T_F = P\rho(F)P + I - P$. More generally, for each positive integer n , let $P_n \in \mathcal{B}((L^2(V))^n)$ be the matrix with P as each diagonal entry and all other entries zero, and let ρ_n denote the obvious representation of $M(n, C(X) \rtimes \Gamma)$ on $(L^2(V))^n$ determined by ρ . Then for each F in $M(n, C(X) \rtimes \Gamma)$, let $T_F = P_n\rho_n(F)P_n + I - P_n$. We note that while Toeplitz operators are typically defined as operators on the range of P_n , we have opted to extend our Toeplitz operators to all of $(L^2(V))^n$ in a way that does not affect their index theory.

Proposition 1. *If F is in $GL(n, C(X) \rtimes \Gamma)$, then T_F is Fredholm.*

Proof. It suffices to show that P commutes with elements of $C(X) \rtimes \Gamma$ modulo the compacts, for then it follows easily that $T_F T_{F^{-1}} = I = T_{F^{-1}} T_F \pmod{\mathcal{K}((L^2(V))^n)}$. The operator P is Γ -equivariant and therefore commutes with $\rho(\gamma)$ for each γ in Γ . On the other hand, for all f in $C(X)$, the commutator $[P, \rho(f)]$ is in $\mathcal{K}(L^2(V))$, by [Baum and Douglas 1982, Lemma 2.10]. These elements are dense in $C(X) \rtimes \Gamma$, so the desired conclusion follows. \square

Our goal is to find a geometric-topological formula for the index of T_F that can be computed directly from F . To this end, we will show that the Fredholm index of T_F is equal to the Γ -invariant index of a certain Γ -equivariant operator.

Definition 2. Let V be a Γ -equivariant complex vector bundle over X , and let R be a Γ -equivariant elliptic pseudodifferential operator acting on sections of V . The Γ -invariant index of R is denoted by $\text{Ind}_{\Gamma\text{-inv}}(R)$, and is the dimension of the Γ -invariant subspace of $\ker R$ minus the dimension of the Γ -invariant subspace of $\ker R^*$.

Proposition 3. *Let R be a Γ -equivariant elliptic operator. Then*

$$\text{Ind}_{\Gamma\text{-inv}}(R) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Ind}_{\gamma}(R),$$

where $\text{Ind}_{\gamma}(R)$ is the trace of the Γ -equivariant index of R evaluated at γ .

Proof. Let σ be a representation of Γ on a finite-dimensional complex vector space W , and decompose σ as

$$\sigma = n_0 \mathbf{1} + \sum_{i=1}^k n_i \sigma_i,$$

where the σ_i are irreducible and distinct, and $\mathbf{1}$ denotes the trivial representation. Let W^{Γ} be the subspace of W that is fixed by Γ . The fact that each of the σ_i fixes only 0 implies that $n_0 = \dim W^{\Gamma}$. Then, by [Serre 1977, Exercise 2.5],

$$\dim W^{\Gamma} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Tr}(\sigma(\gamma)),$$

whence the result follows. □

For each natural number n , let $\text{Map}(\Gamma, \mathbf{M}(n, C(X)))$ denote the C^* -algebra of all functions from the group Γ to $\mathbf{M}(n, C(X))$. The algebra $\mathbf{M}(n, C(X))$ acts on $\text{Map}(\Gamma, \mathbf{M}(n, C(X)))$ via pointwise multiplication; let $\mathcal{L}(\text{Map}(\Gamma, \mathbf{M}(n, C(X))))$ be the algebra of $\mathbf{M}(n, C(X))$ -linear maps on $\text{Map}(\Gamma, \mathbf{M}(n, C(X)))$. We define a homomorphism

$$\mu : \mathbf{M}(n, C(X)) \rtimes \Gamma \longrightarrow \mathcal{L}(\text{Map}(\Gamma, \mathbf{M}(n, C(X))))$$

as follows: for all ψ in $\text{Map}(\Gamma, \mathbf{M}(n, C(X)))$, all functions f in $\mathbf{M}(n, C(X))$, and all elements α and γ in Γ , set $(\mu(f)\psi)(\alpha) = \mu(\alpha \cdot f)\psi(\alpha)$ and $(\mu(\gamma))(\alpha) = \psi(\gamma^{-1}\alpha)$, and then extend μ to all of $\mathbf{M}(n, C(X)) \rtimes \Gamma$ by stipulating that μ be an algebra homomorphism. If $\mathbf{M}(n, C(X))$ acts on $(L^2(V))^n$ by pointwise multiplication and if λ denotes the left regular representation of $\mathbf{M}(n, C(X))$ on the Hilbert space $\text{Map}(\Gamma, (L^2(V))^n)$, then $\lambda(F)$ is matrix multiplication by $\mu(F)$ for every F in $\mathbf{M}(n, C(X)) \rtimes \Gamma$.

The group Γ acts on $\text{Map}(\Gamma, (L^2(V))^n)$ by the formula

$$(\gamma \cdot \psi)(\alpha) = \gamma \cdot (\psi(\gamma^{-1}\alpha)),$$

and the subspace $\text{Map}(\Gamma, (L^2(V))^n)^{\Gamma}$ of elements fixed by Γ contains precisely the ψ for which $\gamma \cdot (\psi(e)) = \psi(\gamma)$ for all γ in Γ ; here e denotes the identity element of Γ . Thus the elements of $\text{Map}(\Gamma, (L^2(V))^n)^{\Gamma}$ are determined by their value at e . Conversely, specifying a value at e uniquely determines an element of $\text{Map}(\Gamma, (L^2(V))^n)^{\Gamma}$.

Define $U : (L^2(V))^n \rightarrow \text{Map}(\Gamma, (L^2(V))^n)^\Gamma$ as

$$(U(s_1, s_2, \dots, s_n))(\alpha) = \frac{1}{\sqrt{|\Gamma|}}(\alpha \cdot s_1, \alpha \cdot s_2, \dots, \alpha \cdot s_n).$$

Because the inner product on $L^2(V)$ has been chosen to be Γ -invariant, U is a unitary operator, and $U^*\psi = \sqrt{|\Gamma|}\psi(e)$ for every ψ .

Define \tilde{D} on $\text{Map}(\Gamma, (L^2(V))^n)$ by the formula $(\tilde{D}\psi)(\alpha) = D(\psi(\alpha))$. Let \tilde{P} be the positive spectral projection of \tilde{D} , and for each F in $M(n, C(X) \rtimes \Gamma)$, define

$$\tilde{T}_F = \tilde{P}\lambda(F)\tilde{P} + I - \tilde{P}.$$

We can express the Fredholm index of Toeplitz operators T_F in terms of the \tilde{T}_F :

Proposition 4. *For every F in $\text{GL}(n, C(X) \rtimes \Gamma)$,*

$$\text{Ind } T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Ind}_\gamma \tilde{T}_F.$$

Proof. For each F , the operator \tilde{T}_F is Γ -equivariant with symbol $\mu(F)$. Furthermore, when \tilde{T}_F is restricted to the Hilbert space $\text{Map}(\Gamma, (L^2(V))^n)^\Gamma$, we have $\tilde{T}_F = UT_FU^*$. Thus the result follows immediately from Propositions 1 and 3. \square

For the remainder of the paper, we will assume that X is a $(2m+1)$ -dimensional spin manifold with spinor bundle S and Dirac operator D , and that Γ acts on X by orientation-preserving isometries that preserve the spin structure. In this case, we can combine Proposition 4 with the index theorem in [Fang 2005] to get a geometric-topological index formula for the index of T_F .

Theorem 5. *For each γ in Γ , let $X_1^\gamma, X_2^\gamma, \dots, X_{K_\gamma}^\gamma$ be the connected components of the fixed point set of γ , and for each $1 \leq k \leq K_\gamma$, let NX_k^γ denote the normal bundle of X_k^γ in X . Then for all F in $\text{GL}(n, C^\infty(X) \rtimes \Gamma)$,*

$$\text{Ind } T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{k=1}^{K_\gamma} \left(\frac{-i}{2\pi}\right)^{(1+\dim X_k^\gamma)/2} \int_{X_k^\gamma} \widehat{A}(X_k^\gamma) \text{ch}(\mu(F)) \Lambda^{-1},$$

where

$$\text{ch}(\mu(F)) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \text{Tr}((\mu(F))^{-1} d\mu(F))^{2k+1})$$

and

$$\Lambda = \text{Pf}\left(2 \sin\left(\frac{i}{2}(R(NX_k^\gamma) + \ln J(X_k^\gamma))\right)\right);$$

here Pf is the Pfaffian, $R(NX_k^\gamma)$ is the curvature of NX_k^γ , and $J(X_k^\gamma)$ is the Jacobian matrix of the action of γ on NX_k^γ .

For this theorem to be useful to us, we need to be able to express $\text{ch}(\mu(F))$ in terms of F .

The action of F on X induces an action on the algebra $\Omega^*(X)$ of smooth differential forms, and we can extend our map μ to an algebra homomorphism

$$\mu : M(n, \Omega^*(X)) \rtimes \Gamma \longrightarrow \mathcal{L}(\text{Map}(\Gamma, M(n, \Omega^*(X)))).$$

Take $\sum_{\gamma \in \Gamma} \omega_\gamma \gamma$ in $M(n, \Omega^*(X)) \rtimes \Gamma$. Then for all ψ in $\text{Map}(\Gamma, M(n, \Omega^*(X)))$,

$$\left(\mu \left(\sum_{\gamma \in \Gamma} \omega_\gamma \gamma \right) \psi \right) (\alpha) = \sum_{\gamma \in \Gamma} (\alpha \cdot \omega_\gamma) \psi(\alpha \gamma).$$

For any element B of $\mathcal{L}(\text{Map}(\Gamma, M(n, \Omega^*(X))))$, its trace is computed by the formula

$$\text{Tr}(B) = \sum_{\alpha \in \Gamma} (B \delta_\alpha)(\alpha),$$

where δ_α is the constant function 1 when evaluated at $\alpha \in \Gamma$, and is otherwise zero. Thus

$$\begin{aligned} \text{Tr} \left(\mu \left(\sum_{\gamma \in \Gamma} \omega_\gamma \gamma \right) \right) &= \sum_{\alpha, \gamma \in \Gamma} (\mu(\omega_\gamma \gamma) \delta_\alpha)(\alpha) \\ &= \sum_{\alpha, \gamma \in \Gamma} (\alpha \cdot \omega_\gamma) \delta_\alpha(\alpha \gamma) \\ &= \sum_{\alpha \in \Gamma} \alpha \cdot \omega_e. \end{aligned}$$

Definition 6. Let $\nu : M(n, \Omega^*(X)) \rtimes \Gamma \rightarrow \Omega^*(X)$ be given by the formula

$$\nu \left(\sum_{\gamma \in \Gamma} \omega_\gamma \gamma \right) = \sum_{\alpha \in \Gamma} \alpha \cdot \omega_e,$$

and for all F in $\text{GL}(n, C^\infty(X) \rtimes \Gamma)$, define

$$\widehat{\text{ch}}(F) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \nu((F^{-1} dF)^{2k+1}),$$

where the exterior derivative d is extended to $M(n, \Omega^*(X)) \rtimes \Gamma$ by applying d entrywise in $M(n, C^\infty(X))$ and setting

$$d \left(\sum_{\gamma \in \Gamma} \omega_\gamma \gamma \right) = \sum_{\gamma \in \Gamma} (d\omega_\gamma) \gamma.$$

Combining Theorem 5 and Definition 6, we have:

Theorem 7. For all F in $GL(n, C^\infty(X) \rtimes \Gamma)$,

$$\text{Ind } T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{k=1}^{K_\gamma} \left(\frac{-i}{2\pi}\right)^{(1+\dim X_k^\gamma)/2} \int_{X_k^\gamma} \widehat{A}(X_k^\gamma) \widehat{\text{ch}}(F) \Lambda^{-1}.$$

This formula looks rather daunting, but in many cases it simplifies considerably.

Example 8. Let $SO(2m + 2)$ act on \mathbb{R}^{2m+2} in the usual way, let Γ be a finite subgroup of $SO(2m + 2)$, and let S^{2m+1} be the unit sphere in \mathbb{R}^{2m+2} . The action of Γ on \mathbb{R}^{2m+2} restricts to an action on S^{2m+1} , and because spheres have unique spin structures [Lawson and Michelsohn 1989], the action of Γ on S^{2m+1} trivially preserves the spin structure. Now, each γ in Γ fixes a subspace of \mathbb{R}^{2m+2} , and so the fixed point set X^γ of γ acting on S^{2m+1} is an equatorial sphere, and in particular, the fixed point set is connected. Furthermore, all spheres have stably trivial tangent bundles, so $\widehat{A}(X^\gamma) = 1$ for all γ in Γ . Finally, a straightforward computation shows that the normal bundle of an equatorial sphere has curvature zero, and so

$$\text{Ind } T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left(\frac{-i}{2\pi}\right)^{(1+\dim X^\gamma)/2} \int_{X^\gamma} \widehat{\text{ch}}(F) (\text{Pf}(2 \sin(\frac{1}{2} \ln J(X^\gamma))))^{-1}.$$

Given γ in Γ , there exists an orthonormal frame of NX^γ for which the action of γ on NX^γ decomposes into blocks

$$\begin{pmatrix} \cos \theta_j^\gamma & \sin \theta_j^\gamma \\ -\sin \theta_j^\gamma & \cos \theta_j^\gamma \end{pmatrix},$$

with $0 < \theta_j^\gamma < 2\pi$ and $j = 1, 2, \dots, L^\gamma = m - \frac{1}{2}(\dim X^\gamma + 1)$ (see [Lawson and Michelsohn 1989]). Incorporating this into our formula we obtain

$$\text{Ind } T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \left(\frac{-i}{2\pi}\right)^{(1+\dim X^\gamma)/2} \frac{1}{\sin(\theta_1^\gamma/2) \cdots \sin(\theta_{L^\gamma}^\gamma/2)} \int_{X^\gamma} \widehat{\text{ch}}(F).$$

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