INDEX THEORY OF TOEPLITZ OPERATORS ASSOCIATED TO TRANSFORMATION GROUP C*-ALGEBRAS

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Let $\Gamma$ be a finite discrete group acting smoothly on a compact manifold $X$, and let $D$ be a first-order elliptic self-adjoint $\Gamma$-equivariant differential operator acting on sections of some $\Gamma$-equivariant Hermitian vector bundle over $X$. We use these data to define Toeplitz operators with symbols in the transformation group $C^*$-algebra $C(X) \rtimes \Gamma$. If the symbol of such a Toeplitz operator is invertible, then the operator is Fredholm. In the case where $X$ is a spin manifold and $D$ is the Dirac operator, we give a geometric-topological formula for the index.

Let $X$ be a smooth compact manifold without boundary, let $V$ be a Hermitian vector bundle over $X$, and suppose $D$ is a first-order elliptic self-adjoint differential operator acting on sections of $V$. Let $P$ be the positive spectral projection of $D$. Then $P$ is an order zero pseudodifferential operator, and it follows from standard facts about pseudodifferential operators on compact manifolds that given a smooth function $f$ on $X$, the pointwise multiplication operator $M_f$ acting on square-integrable sections $L^2(V)$ of $V$ commutes with $P$ modulo the ideal of compact operators. From this fact it is easy to show that if $f$ is invertible, then the Toeplitz operator $PM_f : \text{Ran} P \to \text{Ran} P$ is a Fredholm operator. Furthermore, the index of $PM_f$ can be computed using the Atiyah–Singer Index Theorem; see [Baum and Douglas 1982].

Now suppose that a discrete group $\Gamma$ acts smoothly on both $X$ and $V$ in a compatible way, and suppose that $D$ commutes with the action of $\Gamma$ on sections of $V$. Then $P$ also commutes with this action. In addition, there is a natural action $\rho$ of the transformation group $C^*$-algebra $C(X) \rtimes \Gamma$ on $L^2(V)$, and $P$ commutes with the elements of $C(X) \rtimes \Gamma$ modulo the compacts. Therefore, whenever $F \in C(X) \rtimes \Gamma$ is invertible, $T_F := P\rho(F)$ is a Fredholm operator, and it is natural to ask what the index of this operator is.

Let $X$ be an odd-dimensional oriented spin manifold and let $\Gamma$ be a finite group acting on $X$ by isometries that preserve the orientation and spin structure of $X$. In [Park 2002], the case of free actions was considered; in this paper we consider


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general actions. We use the Lefschetz theorem in [Fang 2005] to prove a theorem that computes the Fredholm index of $T_F$ in terms of the geometry and topology of $X$ and a “Chern character” form explicitly constructed from $F$.

We begin by more precisely defining the objects under discussion. Let $X$ be a smooth compact manifold, and let $\Gamma$ be a discrete group acting smoothly on $X$ from the right. Let $V$ be a $\Gamma$-equivariant complex vector bundle over $X$, and equip $V$ with a $\Gamma$-invariant Hermitian structure. Then $\Gamma$ acts on the left on both the smooth sections $C^\infty(V)$ and the square-integrable sections $L^2(V)$ of $V$:

$$(\gamma \cdot s)(x) = s(x\gamma)\gamma^{-1}.$$  

If $C(X)$ acts on $L^2(V)$ by pointwise multiplication, we have a covariant representation of $(C(X), \Gamma)$ on $L^2(V)$, and hence a representation $\rho$ of $C(X) \rtimes \Gamma$ on $L^2(V)$.

Let $D$ be a first-order, $\Gamma$-equivariant, elliptic self-adjoint differential operator acting on sections of $V$, and let $P = \chi_{[0, \infty)}(D)$ denote the positive spectral projection of $D$; this operator is also $\Gamma$-equivariant. For each $F$ in $C(X) \rtimes \Gamma$, define the Toeplitz operator $T_F \in \mathfrak{B}(L^2(V))$ to be $T_F = P \rho(F) P + I - P$. More generally, for each positive integer $n$, let $P_n \in \mathfrak{B}((L^2(V))^n)$ be the matrix with $P$ as each diagonal entry and all other entries zero, and let $\rho_n$ denote the obvious representation of $M(n, C(X) \rtimes \Gamma)$ on $(L^2(V))^n$ determined by $\rho$. Then for each $F$ in $M(n, C(X) \rtimes \Gamma)$, let $T_F = P_n \rho_n(F) P_n + I - P_n$. We note that while Toeplitz operators are typically defined as operators on the range of $P_n$, we have opted to extend our Toeplitz operators to all of $(L^2(V))^n$ in a way that does not affect their index theory.

**Definition 2.** Let $V$ be a $\Gamma$-equivariant complex vector bundle over $X$, and let $R$ be a $\Gamma$-equivariant elliptic pseudodifferential operator acting on sections of $V$. The $\Gamma$-invariant subspace of ker $R$ minus the dimension of the $\Gamma$-invariant subspace of ker $R^\ast$.

**Proposition 1.** If $F$ is in $\text{GL}(n, C(X) \rtimes \Gamma)$, then $T_F$ is Fredholm.

**Proof.** It suffices to show that $P$ commutes with elements of $C(X) \rtimes \Gamma$ modulo the compacts, for then it follows easily that $T_F T_{F^{-1}} = I = T_{F^{-1}} T_F \mod \mathfrak{B}((L^2(V))^n)$. The operator $P$ is $\Gamma$-equivariant and therefore commutes with $\rho(\gamma)$ for each $\gamma$ in $\Gamma$. On the other hand, for all $f$ in $C(X)$, the commutator $[P, \rho(f)]$ is in $\mathfrak{B}(L^2(V))$, by [Baum and Douglas 1982, Lemma 2.10]. These elements are dense in $C(X) \rtimes \Gamma$, so the desired conclusion follows. \hfill \Box

Our goal is to find a geometric-topological formula for the index of $T_F$ that can be computed directly from $F$. To this end, we will show that the Fredholm index of $T_F$ is equal to the $\Gamma$-invariant index of a certain $\Gamma$-equivariant operator.
Proposition 3. Let $R$ be a $\Gamma$-equivariant elliptic operator. Then

$$\text{Ind}_{\Gamma-\text{inv}}(R) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Ind}_\gamma(R),$$

where $\text{Ind}_\gamma(R)$ is the trace of the $\Gamma$-equivariant index of $R$ evaluated at $\gamma$.

Proof. Let $\sigma$ be a representation of $\Gamma$ on a finite-dimensional complex vector space $W$, and decompose $\sigma$ as

$$\sigma = n_0 \mathbf{1} + \sum_{i=1}^k n_i \sigma_i,$$

where the $\sigma_i$ are irreducible and distinct, and $\mathbf{1}$ denotes the trivial representation. Let $W^\Gamma$ be the subspace of $W$ that is fixed by $\Gamma$. The fact that each of the $\sigma_i$ fixes only 0 implies that $n_0 = \dim W^\Gamma$. Then, by [Serre 1977, Exercise 2.5],

$$\dim W^\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(\gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \text{Tr}(\sigma(\gamma)),$$

whence the result follows. \qed

For each natural number $n$, let $\text{Map}(\Gamma, M(n, C(X)))$ denote the $C^*$-algebra of all functions from the group $\Gamma$ to $M(n, C(X))$. The algebra $M(n, C(X))$ acts on $\text{Map}(\Gamma, M(n, C(X)))$ via pointwise multiplication; let $\mathcal{L}(\text{Map}(\Gamma, M(n, C(X))))$ be the algebra of $M(n, C(X))$-linear maps on $\text{Map}(\Gamma, M(n, C(X)))$. We define a homomorphism

$$\mu : M(n, C(X)) \rtimes \Gamma \longrightarrow \mathcal{L}(\text{Map}(\Gamma, M(n, C(X))))$$

as follows: for all $\psi$ in $\text{Map}(\Gamma, M(n, C(X)))$, all functions $f$ in $M(n, C(X))$, and all elements $\alpha$ and $\gamma$ in $\Gamma$, set $(\mu(f)\psi)(\alpha) = \mu(\alpha \cdot f)\psi(\alpha)$ and $(\mu(\gamma))\psi(\alpha) = \psi(\gamma^{-1}\alpha)$, and then extend $\mu$ to all of $M(n, C(X)) \rtimes \Gamma$ by stipulating that $\mu$ be an algebra homomorphism. If $M(n, C(X))$ acts on $(L^2(V))^n$ by pointwise multiplication and if $\lambda$ denotes the left regular representation of $M(n, C(X))$ on the Hilbert space $\text{Map}(\Gamma, (L^2(V))^n)$, then $\lambda(F)$ is matrix multiplication by $\mu(F)$ for every $F$ in $\text{Map}(\Gamma, (L^2(V))^n)$.

The group $\Gamma$ acts on $\text{Map}(\Gamma, (L^2(V))^n)$ by the formula

$$(\gamma \cdot \psi)(\alpha) = \gamma \cdot (\psi(\gamma^{-1}\alpha)),$$

and the subspace $\text{Map}(\Gamma, (L^2(V))^n)\Gamma$ of elements fixed by $\Gamma$ contains precisely the $\psi$ for which $\gamma \cdot (\psi(e)) = \psi(\gamma)$ for all $\gamma$ in $\Gamma$; here $e$ denotes the identity element of $\Gamma$. Thus the elements of $\text{Map}(\Gamma, (L^2(V))^n)\Gamma$ are determined by their value at $e$. Conversely, specifying a value at $e$ uniquely determines an element of $\text{Map}(\Gamma, (L^2(V))^n)\Gamma$. 


Define \( U : (L^2(V))^n \to \text{Map}(\Gamma, (L^2(V))^n) \) as
\[
(U(s_1, s_2, \ldots, s_n))(\alpha) = \frac{1}{\sqrt{\lvert \Gamma \rvert}}(\alpha \cdot s_1, \alpha \cdot s_2, \ldots, \alpha \cdot s_n).
\]

Because the inner product on \( L^2(V) \) has been chosen to be \( \Gamma \)-invariant, \( U \) is a unitary operator, and \( U^* \psi = \sqrt{\lvert \Gamma \rvert} \psi (e) \) for every \( \psi \).

Define \( \hat{\mathcal{D}} \) on \( \text{Map}(\Gamma, (L^2(V))^n) \) by the formula \( (\hat{\mathcal{D}} \psi)(\alpha) = D(\psi(\alpha)) \). Let \( \hat{\mathcal{P}} \) be the positive spectral projection of \( \hat{\mathcal{D}} \), and for each \( F \) in \( \mathcal{M}(n, C(X) \rtimes \Gamma) \), define
\[
\tilde{T}_F = \hat{\mathcal{P}} \lambda(F) \hat{\mathcal{P}} + I - \hat{\mathcal{P}}.
\]

We can express the Fredholm index of Toeplitz operators \( T_F \) in terms of the \( \tilde{T}_F \):

**Proposition 4.** For every \( F \) in \( \text{GL}(n, C(X) \rtimes \Gamma) \),
\[
\text{Ind } T_F = \frac{1}{\lvert \Gamma \rvert} \sum_{\gamma \in \Gamma} \text{Ind}_\gamma \tilde{T}_F.
\]

**Proof.** For each \( F \), the operator \( \tilde{T}_F \) is \( \Gamma \)-equivariant with symbol \( \mu(F) \). Furthermore, when \( \tilde{T}_F \) is restricted to the Hilbert space \( \text{Map}(\Gamma, (L^2(V))^n)^\Gamma \), we have \( \tilde{T}_F = UT_F U^* \). Thus the result follows immediately from Propositions 1 and 3. \( \square \)

For the remainder of the paper, we will assume that \( X \) is a \((2m+1)\)-dimensional spin manifold with spinor bundle \( S \) and Dirac operator \( D \), and that \( \Gamma \) acts on \( X \) by orientation-preserving isometries that preserve the spin structure. In this case, we can combine Proposition 4 with the index theorem in [Fang 2005] to get a geometric-topological index formula for the index of \( T_F \).

**Theorem 5.** For each \( \gamma \) in \( \Gamma \), let \( X^\gamma_1, X^\gamma_2, \ldots, X^\gamma_{K_\gamma} \) be the connected components of the fixed point set of \( \gamma \), and for each \( 1 \leq k \leq K_\gamma \), let \( N X^\gamma_k \) denote the normal bundle of \( X^\gamma_k \) in \( X \). Then for all \( F \) in \( \text{GL}(n, C^\infty(X) \rtimes \Gamma) \),
\[
\text{Ind } T_F = \frac{1}{\lvert \Gamma \rvert} \sum_{\gamma \in \Gamma} \sum_{k=1}^{K_\gamma} \left( \frac{-i}{2\pi} \right)^{(1+\dim X^\gamma_k)/2} \int_{X^\gamma_k} \hat{A}(X^\gamma_k) \text{ch}(\mu(F)) \Lambda^{-1},
\]
where
\[
\text{ch}(\mu(F)) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \text{Tr}((\mu(F))^{-1} d\mu(F))^{2k+1}
\]
and
\[
\Lambda = \text{Pf} \left( 2 \sin \left( \frac{i}{2} (R(N X^\gamma_k) + \ln J(X^\gamma_k)) \right) \right);
\]
here \( \text{Pf} \) is the Pfaffian, \( R(N X^\gamma_k) \) is the curvature of \( N X^\gamma_k \), and \( J(X^\gamma_k) \) is the Jacobian matrix of the action of \( \gamma \) on \( N X^\gamma_k \).
For this theorem to be useful to us, we need to be able to express \( \text{ch}(\mu(F)) \) in terms of \( F \).

The action of \( F \) on \( X \) induces an action on the algebra \( \Omega^*(X) \) of smooth differential forms, and we can extend our map \( \mu \) to an algebra homomorphism

\[
\mu : M(n, \Omega^*(X)) \rtimes \Gamma \to \mathcal{L}(\text{Map}(\Gamma, M(n, \Omega^*(X)))).
\]

Take \( \sum_{\gamma \in \Gamma} \omega_{\gamma} \) in \( M(n, \Omega^*(X)) \rtimes \Gamma \). Then for all \( \psi \) in \( \text{Map}(\Gamma, M(n, \Omega^*(X))) \),

\[
\left( \mu \left( \sum_{\gamma \in \Gamma} \omega_{\gamma} \right) \right)(\alpha) = \sum_{\gamma \in \Gamma} (\alpha \cdot \omega_{\gamma}) \psi(\alpha \gamma).
\]

For any element \( B \) of \( \mathcal{L}(\text{Map}(\Gamma, M(n, \Omega^*(X)))) \), its trace is computed by the formula

\[
\text{Tr}(B) = \sum_{\alpha \in \Gamma} (B \delta_{\alpha})(\alpha),
\]

where \( \delta_{\alpha} \) is the constant function 1 when evaluated at \( \alpha \in \Gamma \), and is otherwise zero. Thus

\[
\text{Tr} \left( \mu \left( \sum_{\gamma \in \Gamma} \omega_{\gamma} \right) \right) = \sum_{\alpha, \gamma \in \Gamma} (\mu(\omega_{\gamma}) \delta_{\alpha})(\alpha) = \sum_{\alpha, \gamma \in \Gamma} (\alpha \cdot \omega_{\gamma}) \delta_{\alpha}(\alpha \gamma) = \sum_{\alpha \in \Gamma} \alpha \cdot \omega_{e}.
\]

**Definition 6.** Let \( \nu : M(n, \Omega^*(X)) \rtimes \Gamma \to \Omega^*(X) \) be given by the formula

\[
\nu \left( \sum_{\gamma \in \Gamma} \omega_{\gamma} \right) = \sum_{\alpha \in \Gamma} \alpha \cdot \omega_{e},
\]

and for all \( F \) in \( \text{GL}(n, C^\infty(X)) \rtimes \Gamma \), define

\[
\hat{\text{ch}}(F) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \nu((F^{-1}dF)^{2k+1}),
\]

where the exterior derivative \( d \) is extended to \( M(n, \Omega^*(X)) \rtimes \Gamma \) by applying \( d \) entrywise in \( M(n, C^\infty(X)) \) and setting

\[
d \left( \sum_{\gamma \in \Gamma} \omega_{\gamma} \right) = \sum_{\gamma \in \Gamma} (d \omega_{\gamma}) \gamma.
\]

Combining Theorem 5 and Definition 6, we have:
Theorem 7. For all \( F \) in \( \text{GL}(n, C^\infty(X) \times \Gamma) \),

\[
\text{Ind} T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \sum_{k=1}^{K_{\gamma}} (-i \frac{1}{2\pi})^{(1+\dim X_\gamma')/2} \int_{X_\gamma'} \hat{A}(X_\gamma') \hat{\text{ch}}(F) \Lambda^{-1}.
\]

This formula looks rather daunting, but in many cases it simplifies considerably.

Example 8. Let \( \text{SO}(2m + 2) \) act on \( \mathbb{R}^{2m+2} \) in the usual way, let \( \Gamma \) be a finite subgroup of \( \text{SO}(2m + 2) \), and let \( S^{2m+1} \) be the unit sphere in \( \mathbb{R}^{2m+2} \). The action of \( \Gamma \) on \( \mathbb{R}^{2m+2} \) restricts to an action on \( S^{2m+1} \), and because spheres have unique spin structures [Lawson and Michelsohn 1989], the action of \( \Gamma \) on \( S^{2m+1} \) trivially preserves the spin structure. Now, each \( \gamma \) in \( \Gamma \) fixes a subspace of \( \mathbb{R}^{2m+2} \), and so the fixed point set \( X' \) of \( \gamma \) acting on \( S^{2m+1} \) is an equatorial sphere, and in particular, the fixed point set is connected. Furthermore, all spheres have stably trivial tangent bundles, so \( \hat{A}(X') = 1 \) for all \( \gamma \) in \( \Gamma \). Finally, a straightforward computation shows that the normal bundle of an equatorial sphere has curvature zero, and so

\[
\text{Ind} T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (-i \frac{1}{2\pi})^{(1+\dim X')/2} \int_{X'} \hat{\text{ch}}(F) (\text{Pf}(2 \sin(\frac{1}{2} \ln J(X'))))^{-1}.
\]

Given \( \gamma \) in \( \Gamma \), there exists an orthonormal frame of \( N X' \) for which the action of \( \gamma \) on \( N X' \) decomposes into blocks

\[
\begin{pmatrix}
\cos \theta_{j}' & \sin \theta_{j}' \\
-\sin \theta_{j}' & \cos \theta_{j}'
\end{pmatrix},
\]

with \( 0 < \theta_{j}' < 2\pi \) and \( j = 1, 2, \ldots, L' = m - \frac{1}{2} (\dim X' + 1) \) (see [Lawson and Michelsohn 1989]). Incorporating this into our formula we obtain

\[
\text{Ind} T_F = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} (-i \frac{1}{2\pi})^{(1+\dim X')/2} \frac{1}{\sin(\theta_{1}'/2) \cdots \sin(\theta_{L'}'/2)} \int_{X'} \hat{\text{ch}}(F).
\]

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References


EFTON PARK  
DEPARTMENT OF MATHEMATICS  
TEXAS CHRISTIAN UNIVERSITY  
BOX 298900  
FORT WORTH, TX 76129  
e.park@tcu.edu