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A SURFACE OF GENERAL TYPE WITH $p_g = q = 2$ AND $K_X^2 = 5$

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We give an example of a minimal complex surface of general type with $p_g = q = 2$ and $K_X^2 = 5$.

1. Introduction

Recently, there has been considerable interest in understanding the geometry of irregular complex projective surfaces with $\chi(X, \omega_X) = 1 + p_g(X) - q(X) = 1$, and in particular of surfaces with $p_g = q = 2$. Let X be a smooth minimal complex surface of general type. If $\chi(X, \omega_X) = 1$, then one has the bound $1 \le K_X^2 \le 9$. If, in addition, the surface is irregular, that is, $q(X) = h^0(X, \Omega_X^1) > 0$, then $K_X^2 \ge 2p_g(X)$ and so $p_g(X) \le 4$. In [Debarre 1982], it is shown that the case $p_g = q = 4$ corresponds to the product of two curves of genus 2. In [Hacon and Pardini 2002] and [Pirola 2002], surfaces with $p_g = q = 3$ are completely classified. When $K_X^2 = 2p_g(X) = 6$ they are symmetric products of curves of genus 3 and when $K_X^2 = 8$ they admit an irrational pencil. The case $p_g = q = 2$ seems far more delicate. At any rate Catanese suggests that, analogously to the $p_g = q = 3$ case, a surface of general type with $p_g = q = 2$ and with no fibration over an elliptic curve is a degree 2 covering of a principally polarized abelian surface (A, Θ) branched along a divisor in the linear series $|2\Theta|$.

Zucconi [2003] has classified surfaces of general type with $p_g = q = 2$ which admit an irrational pencil. Manetti [2003] showed that a minimal surface of general type with K_X ample and $K_X^2 = 4$, is a degree 2 covering of a principally polarized abelian surface (A, Θ) branched along a divisor $D \in |2\Theta|$. Ciliberto and Mendes Lopes [2002] conjecture that this should be the case for any minimal surface of general type with $p_g = q = 2$ and $K_X^2 = 4$.

Here we give a counterexample to Catanese's conjecture above. The example we construct is birational to a triple cover of an abelian surface. Its canonical divisor K_X is ample, $p_g = q = 2$ and $K_X^2 = 5$. The construction is motivated in Section 3, where we obtain restrictions on the structure of the sheaf $\mathrm{alb}_{X,*}(\omega_X)$.

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2. Construction and verification

We will need some results from the theory of Mukai transforms. Let $\hat{A}$ be the dual abelian variety of A and $\mathcal{P}$ be the normalized Poincaré line bundle on $A \times \hat{A}$. Following [Mukai 1981], define the functor $\hat{\mathcal{S}}$ of $\mathcal{O}_A$-modules into the category of $\mathcal{O}_{\hat{A}}$-modules by

$$\hat{\mathcal{S}}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$

The derived functor $R\hat{\mathcal{S}}$ of $\hat{\mathcal{S}}$ then induces an equivalence of categories between the two derived categories $D(A)$ and $D(\hat{A})$. More precisely, from [Mukai 1981] we know that *there are isomorphisms of functors*

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-g] \quad \text{and} \quad R\hat{\mathcal{S}} \circ R\mathcal{S} \cong (-1_{\hat{A}})^*[-g],$$

where $[-g]$ denotes "shift the complex g places to the right". The Weak Index Theorem (WIT) holds for a coherent sheaf $\mathcal{F}$ on A if there exists an integer $i(\mathcal{F})$ such that for all $j \neq i(\mathcal{F})$, one has $R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$. The coherent sheaf $R^{i(\mathcal{F})}\hat{\mathcal{S}}(\mathcal{F})$ is denoted simply by $\hat{\mathcal{F}}$.

Now consider (A, M), a simple polarized abelian surface of type $(1, 2)$. Assume M is symmetric, i.e., $(-1)^*M \cong M$. The linear series $|M|$ has 4 isolated base points $\{o, p, q, r\}$. We may assume that o is the identity of the abelian surface and that p, q, r are 2-torsion elements with $r = p+q$ (see [Barth 1987], for instance). Each divisor $D \in |M|$ is either a nonsingular curve of genus 3 or a singular curve with a simple node distinct from the base points. $M^\vee$ satisfies the WIT of index 2. Let

$$\mathcal{F} = \widehat{M^\vee} := R^2\hat{\mathcal{S}}(M^\vee)$$

be the Fourier–Mukai transform of $M^\vee$. The vector bundle $\mathcal{F}$ has rank 2. Let $\mathcal{E} = \mathcal{F}^\vee$. One can check that

$$\dim \operatorname{Hom}\big(S^3\mathcal{E}, \textstyle\bigwedge^2\mathcal{E}\big) = h^0\big(\hat{A}, (S^3\mathcal{E})^\vee \otimes \textstyle\bigwedge^2\mathcal{E}\big) = 2.$$

By Miranda's triple covering construction [1985], there is a 2-dimensional family of triple coverings $\hat{f} : \hat{X} \to \hat{A}$ with

$$\hat{f}_*\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{A}} \oplus \mathcal{E}.$$

The idea is this: to construct a triple covering $\hat{f} : \hat{X} \to \hat{A}$ over $\hat{A}$ with *Tschirnhausen module* $\mathcal{E}$ [Miranda 1985], we first construct a triple covering $f : X \to A$ with Tschirnhausen module $\phi_{M^\vee}^*\mathcal{E}$. In Claim 1 we identify those coverings of this type that descend to a triple covering $f : \hat{X} \to \hat{A}$. In Claim 2 we verify that for a general such covering the singularities of X are rational. It follows that the singularities of $\hat{X}$ are also rational. Finally, we compute the invariants of $\hat{X}$ via the invariants of X.

Let $\phi_{M^\vee} : A \to \hat{A}$ be the isogeny defined by $M^\vee$. We have the commutative diagram

$$\begin{tikzcd} X = A \times_{\hat{A}} \hat{X} \arrow[r, "\phi"] \arrow[d, "f"'] & \hat{X} \arrow[d, "\hat{f}"] \\ A \arrow[r, "\phi_{M^\vee}"] & \hat{A} \end{tikzcd}$$

where $\phi : X \to \hat{X}$ is a $4:1$ étale covering and $f : X \to A$ is a triple covering determined by a section of

$$\phi_{M^\vee}^* \operatorname{Hom}\bigl(S^3\mathscr{E}, \textstyle\bigwedge^2\mathscr{E}\bigr) \subset \operatorname{Hom}\bigl(S^3\phi_{M^\vee}^*\mathscr{E}, \textstyle\bigwedge^2\phi_{M^\vee}^*\mathscr{E}\bigr).$$

By [Mukai 1981], $\phi_{M^\vee}^*\mathscr{E} \cong M^\vee \oplus M^\vee$. Thus

$$\operatorname{Hom}\bigl(S^3\phi_{M^\vee}^*\mathscr{E}, \textstyle\bigwedge^2\phi_{M^\vee}^*\mathscr{E}\bigr) \cong H^0(A, M)^{\oplus 4}.$$

To determine the corresponding 2-dimensional subspace, we consider the Heisenberg group action on $H^0(A, M)$. The Heisenberg group can be identified with

$$\mathscr{G}(\delta) := \{(\alpha, t, l) \mid \alpha \in k^*,\ t \in \mathbb{Z}_2,\ l \in \hat{\mathbb{Z}}_2\}$$

with group law $(\alpha, t, l)(\alpha, t', l') = (\alpha\alpha' l'(t), t{+}t', l{+}l')$. Moreover, $H^0(A, M)$ corresponds to $\operatorname{Hom}(\mathbb{Z}_2, k)$. The action of $\mathscr{G}(\delta)$ on $\operatorname{Hom}(\mathbb{Z}_2, k)$ is given by

$$(\alpha, t, l) f(x) = \alpha l(x) f(t + x).$$

Let X, Y be the sections in $H^0(A, M)$ corresponding to the characteristic functions of 0, 1 in $\operatorname{Hom}(\mathbb{Z}_2, k)$ respectively.

Claim 1. *The* 2*-dimensional subspace is determined as*

$$\phi_{M^\vee}^* \operatorname{Hom}\bigl(S^3\mathscr{E}, \textstyle\bigwedge^2\mathscr{E}\bigr) \cong \{(sX, tY, -tX, -sY) \mid s, t \in k\} \subset H^0(A, M)^{\oplus 4}.$$

Grant this for the time being. Following [Miranda 1985], we can then construct a triple covering $f : X \to A$ by using the data $a = sX$, $b = tY$, $c = -tX$, $d = -sY$. Over an affine open subset U of A, the triple covering can be described in $U \times \mathbb{A}^2$ as the covering by the 2×2 minors of

$$\begin{pmatrix} Z + a & W - 2d & c \\ b & Z - 2a & W + d \end{pmatrix}$$

where Z, W are coordinates for $\mathbb{A}^2$.

Following [Miranda 1985, §4], we have $A = s^2X^2 + stY^2$, $B = (t^2 - s^2)XY$ and $C = s^2Y^2 + stX^2$. The branch locus is defined by $D = B^2 - 4AC \in H^0(M)^{\otimes 4}$ and one can see that it corresponds to a divisor $D_1 + D_2 + D_3 + D_4$ with $D_i \in |M|$. For a general choice of s, t, the D_i are all distinct and nonsingular. It is easy to check, as in [Miranda 1985, §5], that for general choices of s, t, the only possible singularities of X lie over the 4 base points $\{p, q, r, o\}$. We remark that f is totally ramified only over these 4 base points.

Let $x \in X$ be a point lying over one of $\{p, q, r, o\}$.

Claim 2. *For general s, t, the singularity of X at x is locally isomorphic to a cone over a twisted cubic.*

Therefore, X has only rational singularities and so does $\hat{X}$. A resolution $\hat{\mu} : \hat{X}' \to \hat{X}$ can be obtained by blowing up along the singularity. The corresponding resolution $\mu : X' \to X$ is the blow up of X along the 4 points lying over $\{p, q, r, o\}$. Let $\{E_i\}_{i=1,\dots,4}$ be the exceptional divisors and $\{R_i\}_{i=1,\dots,4}$ the proper transform of the D_i. Then

$$K_{X'} = \sum_{i=1,\dots,4} R_i + \sum_{i=1,\dots,4} E_i.$$

Note that $R_i \cdot R_j = 0$, $R_i \cdot E_j = 1$ for all i, j and $E_i^2 = -3$, $E_i \cdot E_j = 0$ for $i \neq j$. Thus we have $K_{X'}^2 = 20$, and

$$\begin{aligned} p_g(X') &= h^0(X', \omega_{X'}) = h^2(X', \mathcal{O}_{X'}) = h^2(X, \mathcal{O}_X) \\ &= h^2(A, \mathcal{O}_A) + 2h^2(A, M^\vee) = 5. \end{aligned}$$

Similarly, $q(X') = 2$ and $\chi(X', \omega_{X'}) = 4$. One can also check that $K_{X'}$ is ample.

Since $X' \to \hat{X}'$ is an étale cover of degree 4, one has

$$\chi(\hat{X}', \omega_{\hat{X}'}) = 1, \quad (K_{\hat{X}'})^2 = 5,$$

and $K_{\hat{X}'}$ is ample. $\hat{X}$ has only rational singularity. It is easy to see that $q(\hat{X}') = 2$ and hence $p_g(\hat{X}') = 2$. Therefore $\hat{X}'$ is a surface of general type with $p_g = q = 2$ and $K^2 = 5$.

Proof of Claim 1. We follow [Mumford 1970]. Let $H(M^\vee)$ be the kernel of $\phi_{M^\vee} : A \to \hat{A}$, i.e., the set of points $x \in A$ such that $T_x^* M^\vee \cong M^\vee$. Then $H(M) = H(M^\vee)$. Let $\mathcal{G}(M)$ be the set of pairs (x, φ) such that $x \in H(M)$ and φ is an isomorphism $\varphi : M \to T_x^* M$. Then $\mathcal{G}(M)$ is a group sitting in the exact sequence

$$0 \to k^* \to \mathcal{G}(M) \to H(M) \to 0.$$

There is an isomorphism of groups $\mathcal{G}(M) \cong \mathcal{G}(\delta)$. Under this identification, the representation of $\mathcal{G}(M)$ on $H^0(A, M)$ corresponds to the unique representation of $\mathcal{G}(\delta)$ on $V = V(\delta) := \mathrm{Hom}(\mathbb{Z}_2, k)$, which is defined by

$$((\alpha, t, l) f)(x) = \alpha \cdot l(x) \cdot f(t + x).$$

With respect to the ordered basis formed by the characteristic functions of 0 and 1, this representation is induced by

$$(1,1,1) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1,1,0) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1,0,1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding $\mathscr{G}(\delta)$ representation on $S^3 V^\vee \otimes \bigwedge^2 V \otimes V$ can easily be computed. By [Mukai 1981, Proposition 3.11] we have

$$\phi_{M^\vee}^* \mathscr{F} \cong H^2(A, M) \otimes M \cong (H^0(A, M) \otimes M^\vee)^\vee \cong \phi_{M^\vee}^* \mathscr{E}^\vee.$$

One sees that

$$H^0(A, S^3 \phi_{M^\vee}^* \mathscr{E}^\vee \otimes \textstyle\bigwedge^2 \phi_{M^\vee}^* \mathscr{E}) \cong S^3 H^0(A, M)^\vee \otimes \textstyle\bigwedge^2 H^0(A, M) \otimes H^0(A, M).$$

This vector space is in turn isomorphic to $\bigoplus_{i=1}^4 H^0(A, M)$. We can now compute the corresponding $\mathscr{G}(M)$ representation in terms of the above $\mathscr{G}(\delta)$ representation.

Let p_i, $i = 1, \dots, 4$ denote the projection onto the i-th factor. With respect to the ordered basis

$$\begin{aligned} \{e_1, e_2, \dots, e_8\} & \\ &= \{p_1^* X, p_1^* Y, \dots, p_4^* X, p_4^* Y\} \\ &= \{\hat{X}^3 \otimes X \wedge Y \otimes X, \hat{X}^3 \otimes X \wedge Y \otimes Y, \hat{X}^2 \hat{Y} \otimes X \wedge Y \otimes X, \hat{X}^2 \hat{Y} \otimes X \wedge Y \otimes Y, \\ &\qquad \hat{X} \hat{Y}^2 \otimes X \wedge Y \otimes X, \hat{X} \hat{Y}^2 \otimes X \wedge Y \otimes Y, \hat{Y}^3 \otimes X \wedge Y \otimes X, \hat{Y}^3 \otimes X \wedge Y \otimes Y\}, \end{aligned}$$

the element $(1, 1, 1)$ maps to $R \in M_8(k)$ defined by $R_{i,j} = 0$ if $i + j \neq 8$ and $R_{i,8-i} = \{-1, 1, 1, -1, -1, 1, 1, -1\}$, and $(1, 1, 0)$ maps to $M \in M_8(k)$ defined by $M_{i,j} = 0$ if $i + j \neq 8$ and $M_{i,8-i} = \{1, 1, 1, 1, 1, 1, 1, 1\}$. In particular,

$$R^2 = M^2 = 1 \quad \text{and} \quad RM = MR.$$

There is an induced representation of $H(M^\vee) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-invariant elements form the subspace

$$\{s(e_1 - e_8) + t(e_4 - e_5) \mid s, t \in k\} = \{(sX, tY, -tX, -sY) \mid s, t \in k\}.$$

These invariant elements correspond to the subspace $\phi_{M^\vee}^* \operatorname{Hom}(S^3 \mathscr{E}, \bigwedge^2 \mathscr{E})$. $\square$

Proof of Claim 2. On a neighborhood of one of the base loci o, p, q, r we may assume that X, Y (or any two distinct sections of $H^0(A, M)$) are local coordinates. By [Harris 1992, p. 14, exercise 1.25], the 2×2 minors mentioned above define a twisted cubic if and only if for all $[u : v] \in \mathbb{P}^1$ the linear forms

$$u(Z + sX) - vtY, \; u(W + 2sY) - v(Z - 2sX), \; -utX - v(W - sY)$$

are linearly independent: in other words, if and only if the matrix

$$\begin{pmatrix} us & -vt & u & 0 \\ 2vs & 2us & -v & u \\ -ut & vs & 0 & -v \end{pmatrix}$$

has a nonzero 3×3 minor for every u, v. By inspection one sees that this is the case for general s, t (more precisely for $t \neq 0$ and $t^2 \neq 9s^2$). □

3. Computation of $\mathbf{alb}_{X,*}(\omega_X)$

Using the techniques of [Hacon and Pardini 2002], we now find restrictions on the structure of the coherent sheaf $\mathrm{alb}_{X,*}(\omega_X)$. It was this computation that suggested to us the possibility of constructing the example of Section 2.

Proposition 3.1 [Ciliberto and Mendes Lopes 2002, Proposition 2.3]. *Let X be a minimal surface of general type with $p_g = q = 2$. Then $a := \mathrm{alb}_X : X \to \mathrm{Alb}(X) =: A$ is not surjective if and only if $B := a(X)$ is a curve of genus 2 and $a : X \to B$ has smooth connected fibers of genus 2 with constant modulus and $K_X^2 = 8$.*

We now therefore consider the situation where $a : X \to \mathrm{Alb}(X) =: A$ is surjective. For any coherent sheaf $\mathcal{F}$ on X, define

$$V^i(X, \mathcal{F}) := \{P \in \mathrm{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes P) \neq 0\}.$$

Since a is generically finite, $R^i a_* \omega_X = 0$ for all $i > 0$ and so $V^i(X, \omega_X) = V^i(A, a_*\omega_X)$ for all i.

Lemma 3.2. *Let X be a minimal surface with $p_g = q = 2$ and surjective Albanese map. If* $\dim V^1(X, \omega_X) \geq 1$, *there exists an elliptic pencil $X \to E$ with $g(E) = 1$.*

Proof. By the generic vanishing theorems of Green and Lazarsfeld, we have $\dim V^1(X, \omega_X) < 2$, and if T is a component of $V^1(X, \omega_X)$ of dimension 1, then T is a translate of an elliptic curve $T_0 \subset \mathrm{Pic}^0(X)$. The pencil $X \to E$ is induced by $a : X \to \mathrm{Alb}(X)$ composed with the dual map of abelian varieties $\mathrm{Alb}(X) \to E := T_0^\vee$. □

Corollary 3.3. *Let X be a minimal surface of general type with $p_g = q = 2$ without irrational pencils. Then $a : X \to A$ is surjective with $V^1(A, a_*\omega_X)$ supported on finitely many points.* □

A vector bundle U on an abelian variety A is unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \cong \mathcal{O}_A$. A vector bundle is homogeneous if and only if it is isomorphic to $\bigoplus_{i=1}^n (P_i \otimes U_i)$ with $P_i \in \mathrm{Pic}^0(A)$ and U_i unipotent vector bundles. By [Mukai 1981], there is an one-to-one correspondence between sheaves supported on finitely many points and homogeneous vector bundles via the Fourier–Mukai transform.

Lemma 3.4. *Let $\mathcal{F}$ be a coherent sheaf on an abelian surface. Then*

$$R^i\mathcal{S}R^j\hat{\mathcal{S}}(\mathcal{F}) = 0 \quad \textit{for } (i, j) \in \{(1, 2), (2, 2), (0, 0), (1, 0)\}.$$

Moreover, there exist an injection $d : R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F}$ and a surjection $d' : R^0\mathcal{S}R^2\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F})$. In particular, $R^0\hat{\mathcal{S}}\mathcal{F}$ and $R^2\hat{\mathcal{S}}\mathcal{F}$ satisfy the WIT of index 2 and 0, respectively.

Proof. As mentioned above, by [Mukai 1981], there is an isomorphism of functors

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-2].$$

In particular there is a spectral sequence $E_2^{p,q} = R^p\mathcal{S}R^q\hat{\mathcal{S}}\mathcal{F}$ with $E_\infty^{p,q} = 0$ if $p + q \neq 2$. The only possibly nonvanishing differentials d_2 are

$$d : R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \quad \textit{and} \quad d' : R^0\mathcal{S}R^2\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}.$$

One sees that $E_2^{p,q} = E_\infty^{p,q} = 0$ for $(p, q) \in \{(1, 2), (2, 2), (0, 0), (1, 0)\}$. Moreover, $\ker d = E_3^{0,1} = E_\infty^{0,1} = 0$, so d is an injection. Similarly d' is a surjection. □

Theorem 3.5. *Let X be a minimal surface of general type with $p_g = q = 2$ without any irrational pencil. Then there exist homogeneous vector bundles $\mathcal{H}$, and a negative definite line bundle L on $\hat{A} = \mathrm{Pic}^0(A)$ (i.e. $L^\vee$ is ample) such that $a_*\omega_X$ fits into the exact sequences*

$$0 \to \mathcal{O}_A \to a_*\omega_X \to \mathcal{F} \to 0,$$
$$0 \to \mathcal{H} \to \hat{L} \to (-1_A)^*\mathcal{F} \to 0.$$

Proof. Notice that $\omega_A = \mathcal{O}_A$. By assumption, X has no irrational pencils; thus $a : X \to A$ is surjective and $\dim V^1(X, \omega_X) = 0$, hence $V^1(X, \omega_X) = \{\mathcal{O}_X, P_1, \ldots, P_n\}$. Let $\mathcal{F}$ be the coherent sheaf defined by the short exact sequence

$$0 \to \mathcal{O}_A \to a_*\omega_X \to \mathcal{F} \to 0.$$

Since $R^i a_*\omega_X = 0$ for $i > 0$, one sees that for $i \geq 0$,

$$H^i(A, a_*\omega_X) \cong H^i(X, \omega_X) \cong H^i(A, \omega_A)$$

and therefore $h^1(\mathcal{F}) = h^2(\mathcal{F}) = 0$. Moreover, for all $\mathcal{O}_X \neq P \in \mathrm{Pic}^0(A)$, one has $h^i(A, \mathcal{F} \otimes P) = h^i(X, \omega_X \otimes P)$ for all i. In particular $V^2(A, \mathcal{F}) = \varnothing$ and $V^1(A, \mathcal{F}) = \{P_1, \ldots, P_n\}$. We have $R^2\hat{\mathcal{S}}\mathcal{F} = 0$ and $R^1\hat{\mathcal{S}}\mathcal{F} = \bigoplus B_i$, where the sheaves B_i are supported at the points P_i (and are artinian $\mathcal{O}_{\hat{A},P_i}$-modules; see [Mukai 1981, Example 2.9]). In particular, $R^1\hat{\mathcal{S}}\mathcal{F}$ satisfies the WIT of index 0.

Now consider the spectral sequence of the proof of Lemma 3.4. The only nonzero E_2 terms are $E_2^{0,1}$ and $E_2^{2,0}$. Therefore, one has the exact sequence

$$0 \to R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to (-1_A)^*\mathcal{F} \to 0.$$

First note that $R^1\hat{\mathcal{S}}\mathcal{F}$ is supported on finitely many points. It follows that $R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} = R\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}$ is a homogeneous vector bundle; call it $\mathcal{H}$. It suffices to show that $R^0\hat{\mathcal{S}}\mathcal{F}$ is a negative line bundle.

Let $U = \mathrm{Pic}^0(A) - \{\mathcal{O}_A, P_1, \ldots, P_n\}$. Then, for all $P \in U$,

$$h^0(A, \mathcal{F} \otimes P) = h^0(A, a_*\omega_X \otimes P) = \chi(X, \omega_X) = 1.$$

Thus $R^0\hat{\mathcal{S}}\mathcal{F}|_U$ is locally free of rank 1. Let $L = (R^0\hat{\mathcal{S}}\mathcal{F})^{\vee\vee}$. Then L is a reflexive sheaf of rank 1 on a nonsingular surface and hence a line bundle. Since $R^0\hat{\mathcal{S}}\mathcal{F} = R^0\hat{\mathcal{S}}a_*\omega_X$ is torsion-free, we have an exact sequence of coherent sheaves on $\hat{A}$:

$$0 \to R^0\hat{\mathcal{S}}\mathcal{F} \to L \to Q \to 0$$

where Q is supported at most on the points P_i.

We claim that $Q = 0$. Suppose on the contrary that $Q \neq 0$. By Lemma 3.4, $R^i\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} = 0$ for $i = 0, 1$, hence $R^0L \cong R^0Q$. So for general $P \in A = \mathrm{Pic}^0(\hat{A})$ one has

$$h^0(L \otimes P) = h^0(Q \otimes P) \neq 0,$$

since $Q \neq 0$ is supported on points. It follows that L is an ample line bundle and therefore satisfying I.T of index 0. In particular, $R^2\mathcal{S}L = 0$. On the other hand, since Q is supported on points, we have that $R^1\mathcal{S}Q = 0$. The exact sequence

$$R^1\mathcal{S}Q \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}L$$

yields $R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} = 0$. It follows that $\mathcal{F} = 0$, since there is a surjection

$$R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to (-1)^*\mathcal{F}.$$

One concludes that $\mathcal{O}_A = a_*\omega_X$, and so that $X \to A$ is birational, which is the required contradiction.

We may therefore assume that $Q = 0$ and hence $L = R^0\hat{\mathcal{S}}\mathcal{F}$ is a line bundle. By Lemma 3.4, L satisfies the WIT of index 2, hence it is a negative definite line bundle. □

Remark. It follows that if $X \to A$ has degree 2, then $\mathcal{F}$ has rank 1. The only possibility is that $a_*\omega_X = \mathcal{O}_A \oplus \mathcal{O}_A(-\Theta)$, where Θ is a principal polarization. This is a $2:1$ covering branched along a divisor $D \in |2\Theta|$.

We have given an example with $a_*\omega_X = \mathcal{O}_A \oplus \hat{L}$, where $L^\vee$ is an ample line bundle of type $(1, 2)$. We have not been able to rule out the cases in which $\mathcal{H} \neq 0$. For example, is it possible to have $a_*\omega_X = \mathcal{O}_A \oplus \mathcal{F}$, with $\mathcal{F}$ as follows?

Example. Let (A, L) be a general polarized abelian surface of type $(1, 3)$ and $x \in A$ a closed point. Then $h^i(A, L \otimes \mathcal{I}_x \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$. Let $\mathcal{E} = \widehat{L \otimes \mathcal{I}_x}$ and $\mathcal{F} = \mathcal{E}^\vee$. Then we have an exact sequence

$$0 \to P_x^\vee \to \hat{L}^\vee \to \mathcal{F} \to 0.$$

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