A SURFACE OF GENERAL TYPE WITH $p_g = q = 2$ AND $K_X^2 = 5$

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We give an example of a minimal complex surface of general type with $p_g = q = 2$ and $K_X^2 = 5$.

1. Introduction

Recently, there has been considerable interest in understanding the geometry of irregular complex projective surfaces with $\chi(X, \omega_X) = 1 + p_g(X) - q(X) = 1$, and in particular of surfaces with $p_g = q = 2$. Let $X$ be a smooth minimal complex surface of general type. If $\chi(X, \omega_X) = 1$, then one has the bound $1 \leq K_X^2 \leq 9$. If, in addition, the surface is irregular, that is, $q(X) = h^0(X, \Omega_X^1) > 0$, then $K_X^2 \geq 2p_g(X)$ and so $p_g(X) \leq 4$. In [Debarre 1982], it is shown that the case $p_g = q = 4$ corresponds to the product of two curves of genus 2. In [Hacon and Pardini 2002] and [Pirola 2002], surfaces with $p_g = q = 3$ are completely classified. When $K_X^2 = 2p_g(X) = 6$ they are symmetric products of curves of genus 3 and when $K_X^2 = 8$ they admit an irrational pencil. The case $p_g = q = 2$ seems far more delicate. At any rate Catanese suggests that, analogously to the $p_g = q = 3$ case, a surface of general type with $p_g = q = 2$ and with no fibration over an elliptic curve is a degree 2 covering of a principally polarized abelian surface $(A, \Theta)$ branched along a divisor in the linear series $|2\Theta|$.

Zucconi [2003] has classified surfaces of general type with $p_g = q = 2$ which admit an irrational pencil. Manetti [2003] showed that a minimal surface of general type with $K_X$ ample and $K_X^2 = 4$, is a degree 2 covering of a principally polarized abelian surface $(A, \Theta)$ branched along a divisor $D \in |2\Theta|$. Ciliberto and Mendes Lopes [2002] conjecture that this should be the case for any minimal surface of general type with $p_g = q = 2$ and $K_X^2 = 4$.

Here we give a counterexample to Catanese’s conjecture above. The example we construct is birational to a triple cover of an abelian surface. Its canonical divisor $K_X$ is ample, $p_g = q = 2$ and $K_X^2 = 5$. The construction is motivated in Section 3, where we obtain restrictions on the structure of the sheaf $\text{alb}_{X,s}(\omega_X)$.  

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2. Construction and verification

We will need some results from the theory of Mukai transforms. Let \( \hat{A} \) be the dual abelian variety of \( A \) and \( \hat{\mathcal{P}} \) be the normalized Poincaré line bundle on \( A \times \hat{A} \). Following [Mukai 1981], define the functor \( \hat{\mathcal{F}} \) of \( \hat{A} \)-modules into the category of \( \hat{A} \)-modules by

\[
\hat{\mathcal{F}}(M) = \pi_{\hat{A},*}(\hat{\mathcal{P}} \otimes \pi_A^*M).
\]

The derived functor \( R\hat{\mathcal{F}} \) of \( \hat{\mathcal{F}} \) then induces an equivalence of categories between the two derived categories \( D(A) \) and \( D(\hat{A}) \). More precisely, from [Mukai 1981] we know that there are isomorphisms of functors

\[
R\hat{\mathcal{F}} \circ R\hat{\mathcal{F}} \cong (-1)^*[-g] \quad \text{and} \quad R\hat{\mathcal{F}} \circ R\hat{\mathcal{F}} \cong (-1)^*[-g],
\]

where \([−g]\) denotes “shift the complex \( g \) places to the right”. The Weak Index Theorem (WIT) holds for a coherent sheaf \( \hat{\mathcal{F}} \) on \( A \) if there exists an integer \( i(\hat{\mathcal{F}}) \) such that for all \( j \neq i(\mathcal{F}) \), one has \( R^j\hat{\mathcal{F}}(\mathcal{F}) = 0 \). The coherent sheaf \( R^i(\hat{\mathcal{F}})(\hat{\mathcal{F}}) \) is denoted simply by \( \hat{\mathcal{F}} \).

Now consider \((A, M)\), a simple polarized abelian surface of type \((1, 2)\). Assume \( M \) is symmetric, i.e., \((-1)^*M \cong M\). The linear series \(|M|\) has 4 isolated base points \( \{o, p, q, r\} \). We may assume that \( o \) is the identity of the abelian surface and that \( p, q, r \) are 2-torsion elements with \( r = p + q \) (see [Barth 1987], for instance). Each divisor \( D \in |M| \) is either a nonsingular curve of genus 3 or a singular curve with a simple node distinct from the base points. \( M^\vee \) satisfies the WIT of index 2. Let

\[
\hat{\mathcal{F}} = M^\vee := R^2\hat{\mathcal{F}}(M^\vee)
\]

be the Fourier–Mukai transform of \( M^\vee \). The vector bundle \( \hat{\mathcal{F}} \) has rank 2. Let \( \mathcal{E} = \hat{\mathcal{F}}^\vee \). One can check that

\[
\dim \text{Hom}(S^3\mathcal{E}, \wedge^2\mathcal{E}) = h^0(\hat{A}, (S^3\mathcal{E})^\vee \otimes \wedge^2\mathcal{E}) = 2.
\]

By Miranda’s triple covering construction [1985], there is a 2-dimensional family of triple coverings \( \hat{f} : \hat{X} \rightarrow \hat{A} \) with

\[
\hat{f}_*\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{A}} \oplus \mathcal{E}.
\]

The idea is this: to construct a triple covering \( \hat{f} : \hat{X} \rightarrow \hat{A} \) over \( \hat{A} \) with Tschirnhausen module \( \mathcal{E} \) [Miranda 1985], we first construct a triple covering \( f : X \rightarrow A \) with Tschirnhausen module \( \phi_{M^\vee}^\vee \mathcal{E} \). In Claim 1 we identify those coverings of this type that descend to a triple covering \( \hat{f} : \hat{X} \rightarrow \hat{A} \). In Claim 2 we verify that for a general such covering the singularities of \( X \) are rational. It follows that the singularities of \( \hat{X} \) are also rational. Finally, we compute the invariants of \( \hat{X} \) via the invariants of \( X \).
Let $\phi_{M^\vee} : A \to \hat{A}$ be the isogeny defined by $M^\vee$. We have the commutative diagram

$$
\begin{array}{ccc}
X = A \times \hat{A} & \xrightarrow{\phi} & \hat{X} \\
\downarrow f & & \downarrow \hat{f} \\
A & \xrightarrow{\phi_{M^\vee}} & \hat{A}
\end{array}
$$

where $\phi : X \to \hat{X}$ is a 4 : 1 étale covering and $f : X \to A$ is a triple covering determined by a section of

$$
\phi_{M^\vee}^* \text{Hom}(S^3\mathcal{E}, \wedge^2\mathcal{E}) \subset \text{Hom}(S^3\phi_{M^\vee}^*\mathcal{E}, \wedge^2\phi_{M^\vee}^*\mathcal{E}).
$$

By [Mukai 1981], $\phi_{M^\vee}^*\mathcal{E} \cong M^\vee \oplus M^\vee$. Thus

$$
\text{Hom}(S^3\phi_{M^\vee}^*\mathcal{E}, \wedge^2\phi_{M^\vee}^*\mathcal{E}) \cong H^0(A, M)^{\oplus 4}.
$$

To determine the corresponding 2-dimensional subspace, we consider the Heisenberg group action on $H^0(A, M)$. The Heisenberg group can be identified with

$$
\mathcal{G}(\delta) := \{(a, t, l) \mid a \in k^*, t \in \mathbb{Z}_2, l \in \hat{\mathbb{Z}}_2\}
$$

with group law $(a, t, l)(a', t', l') = (a a' t(t), t + t', l + l')$. Moreover, $H^0(A, M)$ corresponds to $\text{Hom}(\mathbb{Z}_2, k)$. The action of $\mathcal{G}(\delta)$ on $\text{Hom}(\mathbb{Z}_2, k)$ is given by

$$(a, t, l) f(x) = a f(t x) f(t + x).$$

Let $X, Y$ be the sections in $H^0(A, M)$ corresponding to the characteristic functions of 0, 1 in $\text{Hom}(\mathbb{Z}_2, k)$ respectively.

**Claim 1.** The 2-dimensional subspace is determined as

$$
\phi_{M^\vee}^* \text{Hom}(S^3\mathcal{E}, \wedge^2\mathcal{E}) \cong \{(s X, t Y, -t X, -s Y) \mid s, t \in k\} \subset H^0(A, M)^{\oplus 4}.
$$

Grant this for the time being. Following [Miranda 1985], we can then construct a triple covering $f : X \to A$ by using the data $a = s X, b = t Y, c = -t X, d = -s Y$. Over an affine open subset $U$ of $A$, the triple covering can be described in $U \times \mathbb{A}^2$ as the covering by the 2 × 2 minors of

$$
\begin{pmatrix}
Z + a & W - 2d & c \\
b & Z - 2a & W + d
\end{pmatrix}
$$

where $Z, W$ are coordinates for $\mathbb{A}^2$.

Following [Miranda 1985, §4], we have $A = s^2 X^2 + s t Y^2, B = (t^2 - s^2) X Y$ and $C = s^2 Y^2 + s t X^2$. The branch locus is defined by $D = B^2 - 4 A C \in H^0(M)^{\oplus 4}$ and one can see that it corresponds to a divisor $D_1 + D_2 + D_3 + D_4$ with $D_i \in |M|$. For a general choice of $s, t$, the $D_i$ are all distinct and nonsingular. It is easy to
There is an isomorphism of groups $\phi$. Let $K$ follow [Mumford 1970]. Let $K$ and hence $p$. Similarly, $q$.

Thus we have $K = \sum_{i=1}^{4} R_i + \sum_{i=1}^{4} E_i$. Note that $R_i \cdot R_j = 0$, $R_i \cdot E_j = 1$ for all $i$, $j$ and $E_i^2 = -3$, $E_i \cdot E_j = 0$ for $i \neq j$. Thus we have $K^2 = 20$, and

$$p_g(X') = h^0(X', \omega_{X'}) = h^2(X', \mathcal{O}_{X'}) = h^2(X, \mathcal{O}_X) = h^2(A, \mathcal{O}_A) + 2h^2(A, M^\vee) = 5.$$ 

Similarly, $q(X') = 2$ and $\chi(X', \omega_{X'}) = 4$. One can also check that $K_{X'}$ is ample.

Since $X' \to \hat{X}'$ is an étale cover of degree 4, one has

$$\chi(\hat{X}', \omega_{\hat{X}'}) = 1, \quad (K_{\hat{X}'})^2 = 5,$$

and $K_{\hat{X}'}$ is ample. $\hat{X}$ has only rational singularity. It is easy to see that $q(\hat{X}') = 2$ and hence $p_g(\hat{X}') = 2$. Therefore $\hat{X}'$ is a surface of general type with $p_g = q = 2$ and $K^2 = 5$.

**Proof of Claim 1.** We follow [Mumford 1970]. Let $H(M^\vee)$ be the kernel of $\phi_{M^\vee} : A \to \hat{A}$, i.e., the set of points $x \in A$ such that $T_x^* M^\vee \cong M^\vee$. Then $H(M) = H(M^\vee)$. Let $\mathfrak{g}(M)$ be the set of pairs $(x, \varphi)$ such that $x \in H(M)$ and $\varphi$ is an isomorphism $\varphi : M \to T_x^* M$. Then $\mathfrak{g}(M)$ is a group sitting in the exact sequence

$$0 \to k^* \to \mathfrak{g}(M) \to H(M) \to 0.$$ 

There is an isomorphism of groups $\mathfrak{g}(M) \cong \mathfrak{g}(\delta)$. Under this identification, the representation of $\mathfrak{g}(M)$ on $H^0(A, M)$ corresponds to the unique representation of $\mathfrak{g}(\delta)$ on $V = V(\delta) := \text{Hom}(\mathbb{Z}_2, k)$, which is defined by

$$((a, t, l)f)(x) = a \cdot l(x) \cdot f(t + x).$$
With respect to the ordered basis formed by the characteristic functions of 0 and 1, this representation is induced by

\[(1, 1, 1) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1, 1, 0) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1, 0, 1) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

The corresponding \(\mathcal{E}(\delta)\) representation on \(S^3 V^\vee \otimes \bigwedge^2 V \otimes V\) can easily be computed. By [Mukai 1981, Proposition 3.11] we have

\[\phi^*_M \mathcal{F} \cong H^2(A, M) \otimes M \cong (H^0(A, M) \otimes M^\vee)^\vee \cong \phi^*_M \mathcal{E}^\vee.\]

One sees that

\[H^0(A, S^3 \phi^*_M \mathcal{E}^\vee \otimes \bigwedge^2 \phi^*_M \mathcal{E}) \cong S^3 H^0(A, M)^\vee \otimes \bigwedge^2 H^0(A, M) \otimes H^0(A, M).\]

This vector space is in turn isomorphic to \(\bigoplus_{i=1}^4 H^0(A, M)\). We can now compute the corresponding \(\mathcal{E}(M)\) representation in terms of the above \(\mathcal{E}(\delta)\) representation.

Let \(p_i, i = 1, \ldots, 4\) denote the projection onto the \(i\)-th factor. With respect to the ordered basis

\[\{e_1, e_2, \ldots, e_8\} = \{p_1^* X, p_1^* Y, \ldots, p_4^* X, p_4^* Y\} \]

\[= \{\hat{X}^3 \otimes X \wedge Y \otimes X, \hat{X}^3 \otimes X \wedge Y \otimes Y, \hat{X}^2 \hat{Y} \otimes X \wedge Y \otimes X, \hat{X}^2 \hat{Y} \otimes X \wedge Y \otimes Y, \hat{X} \hat{Y} \otimes X \wedge Y \otimes Y, \hat{X} \hat{Y} \otimes X \wedge Y \otimes X, \hat{X} \hat{Y} \otimes X \wedge Y \otimes Y, \hat{X} \hat{Y} \otimes X \wedge Y \otimes X\},\]

the element \((1, 1, 1)\) maps to \(R \in M_8(k)\) defined by \(R_{i,j} = 0\) if \(i + j \neq 8\) and \(R_{i,8-i} = \{-1, 1, 1, -1, -1, 1, 1, -1\}\), and \((1, 1, 0)\) maps to \(M \in M_8(k)\) defined by \(M_{i,j} = 0\) if \(i + j \neq 8\) and \(M_{i,8-i} = \{1, 1, 1, 1, 1, 1, 1, 1\}\). In particular,

\[R^2 = M^2 = 1 \quad \text{and} \quad RM = MR.\]

There is an induced representation of \(H(M^\vee) \cong \mathbb{Z}_2 \times \mathbb{Z}_2\). It is easy to see that the \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-invariant elements form the subspace

\[\{s(e_1 - e_8) + t(e_4 - e_5) \mid s, t \in k\} = \{(sX, tY, -tX, -sY) \mid s, t \in k\}.
\]

These invariant elements correspond to the subspace \(\phi^*_M \mathcal{H}(S^3 \mathcal{E}, \bigwedge^2 \mathcal{E})\).

**Proof of Claim 2.** On a neighborhood of one of the base loci \(a, p, q, r\) we may assume that \(X, Y\) (or any two distinct sections of \(H^0(A, M)\)) are local coordinates. By [Harris 1992, p. 14, exercise 1.25], the \(2 \times 2\) minors mentioned above define a twisted cubic if and only if for all \([u : v] \in \mathbb{P}^1\) the linear forms

\[u(Z + sX) - vY, \ u(W + 2sY) - v(Z - 2sX), \ -utX - v(W - sY)\]
are linearly independent: in other words, if and only if the matrix
\[
\begin{pmatrix}
us & -vt & u & 0 \\
2us & 2us & -v & u \\
-ut & vs & 0 & -v
\end{pmatrix}
\]
has a nonzero $3 \times 3$ minor for every $u, v$. By inspection one sees that this is the case for general $s, t$ (more precisely for $t \neq 0$ and $t^2 \neq 9s^2$).

3. Computation of $\text{alb}_{X,*}(\omega_X)$

Using the techniques of [Hacon and Pardini 2002], we now find restrictions on the structure of the coherent sheaf $\text{alb}_{X,*}(\omega_X)$. It was this computation that suggested to us the possibility of constructing the example of Section 2.

**Proposition 3.1** [Ciliberto and Mendes Lopes 2002, Proposition 2.3]. Let $X$ be a minimal surface of general type with $p_g = q = 2$. Then $a := \text{alb}_X : X \to \text{Alb}(X) =: A$ is not surjective if and only if $B := a(X)$ is a curve of genus 2 and $a : X \to B$ has smooth connected fibers of genus 2 with constant modulus and $K_X^2 = 8$.

We now therefore consider the situation where $a : X \to \text{Alb}(X) =: A$ is surjective. For any coherent sheaf $\mathcal{F}$ on $X$, define
\[
V^i(X, \mathcal{F}) := \{ P \in \text{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes P) \neq 0 \}.
\]
Since $a$ is generically finite, $R^i a_* \omega_X = 0$ for all $i > 0$ and so $V^i(X, \omega_X) = V^i(A, a_* \omega_X)$ for all $i$.

**Lemma 3.2.** Let $X$ be a minimal surface with $p_g = q = 2$ and surjective Albanese map. If $\dim V^1(X, \omega_X) \geq 1$, there exists an elliptic pencil $X \to E$ with $g(E) = 1$.

**Proof.** By the generic vanishing theorems of Green and Lazarsfeld, we have $\dim V^1(X, \omega_X) < 2$, and if $T$ is a component of $V^1(X, \omega_X)$ of dimension 1, then $T$ is a translate of an elliptic curve $T_0 \subset \text{Pic}^0(X)$. The pencil $X \to E$ is induced by $a : X \to \text{Alb}(X)$ composed with the dual map of abelian varieties $\text{Alb}(X) \to E := T_0^\vee$.

**Corollary 3.3.** Let $X$ be a minimal surface of general type with $p_g = q = 2$ without irrational pencils. Then $a : X \to A$ is surjective with $V^1(A, a_* \omega_X)$ supported on finitely many points.

A vector bundle $U$ on an abelian variety $A$ is unipotent if it has a filtration
\[
0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U
\]
such that $U_i / U_{i-1} \cong \mathcal{O}_A$. A vector bundle is homogeneous if and only if it is isomorphic to $\bigoplus_{i=1}^n (P_i \otimes U_i)$ with $P_i \in \text{Pic}^0(A)$ and $U_i$ unipotent vector bundles. By [Mukai 1981], there is an one-to-one correspondence between sheaves supported
on finitely many points and homogeneous vector bundles via the Fourier–Mukai transform.

**Lemma 3.4.** Let $\mathcal{F}$ be a coherent sheaf on an abelian surface. Then

$$R^i\mathcal{F} R^j\hat{\mathcal{F}}(\mathcal{F}) = 0 \quad \text{for } (i, j) \in \{(1, 2), (2, 2), (0, 0), (1, 0)\}.$$  

Moreover, there exist an injection $d : R^0\mathcal{F} R^1\hat{\mathcal{F}} \to R^2\mathcal{F} R^0\hat{\mathcal{F}}$ and a surjection $d' : R^0\mathcal{F} R^2\hat{\mathcal{F}} \to R^2\mathcal{F} R^1\hat{\mathcal{F}}$. In particular, $R^0\mathcal{F}$ and $R^2\hat{\mathcal{F}}$ satisfy the WIT of index 2 and 0, respectively.

**Proof.** As mentioned above, by [Mukai 1981], there is an isomorphism of functors

$$R^\mathcal{F} \circ R^{\hat{\mathcal{F}}} \cong (-1)^*[-2].$$

In particular there is a spectral sequence $E_2^{p,q} = R^p\mathcal{F} R^q\hat{\mathcal{F}}$ with $E_\infty^{0,0} = 0$ if $p + q \neq 2$. The only possibly nonvanishing differentials $d_2$ are

$$d : R^0\mathcal{F} R^1\hat{\mathcal{F}} \to R^2\mathcal{F} R^0\hat{\mathcal{F}} \quad \text{and} \quad d' : R^0\mathcal{F} R^2\hat{\mathcal{F}} \to R^2\mathcal{F} R^1\hat{\mathcal{F}}.$$  

One sees that $E_2^{0,0} = E_\infty^{0,0} = 0$ for $(p, q) \in \{(1, 2), (2, 2), (0, 0), (1, 0)\}$. Moreover, ker $d = E_3^{0,1} = E_\infty^{0,1} = 0$, so $d$ is an injection. Similarly $d'$ is a surjection. □

**Theorem 3.5.** Let $X$ be a minimal surface of general type with $p_g = q = 2$ without any irrational pencil. Then there exist homogeneous vector bundles $\mathcal{H}$, and a negative definite line bundle $L$ on $\hat{\mathcal{A}} = \text{Pic}^0(A)$ (i.e. $L^r$ is ample) such that $a_s\omega_X$ fits into the exact sequences

$$0 \to \mathcal{O}_A \to a_s\omega_X \to \mathcal{F} \to 0,$$

$$0 \to \mathcal{H} \to \hat{\mathcal{L}} \to (-1)_A^*\mathcal{F} \to 0.$$

**Proof.** Notice that $\omega_A = \mathcal{O}_A$. By assumption, $X$ has no irrational pencils; thus $a : X \to A$ is surjective and dim $V^1(X, \omega_X) = 0$, hence $V^1(X, \omega_X) = \{\mathcal{O}_X, P_1, \ldots, P_n\}$. Let $\mathcal{F}$ be the coherent sheaf defined by the short exact sequence

$$0 \to \mathcal{O}_A \to a_s\omega_X \to \mathcal{F} \to 0.$$  

Since $R^i a_s\omega_X = 0$ for $i > 0$, one sees that for $i \geq 0$,

$$H^i(A, a_s\omega_X) \cong H^i(X, \omega_X) \cong H^i(A, \omega_A)$$

and therefore $h^1(\mathcal{F}) = h^2(\mathcal{F}) = 0$. Moreover, for all $\mathcal{O}_X \neq P \in \text{Pic}^0(A)$, one has $h^i(A, \mathcal{F} \otimes P) = h^i(X, \omega_X \otimes P)$ for all $i$. In particular $V^2(A, \mathcal{F}) = \emptyset$ and $V^1(A, \mathcal{F}) = \{P_1, \ldots, P_n\}$. We have $R^2\hat{\mathcal{F}} = 0$ and $R^1\hat{\mathcal{F}} = \bigoplus B_i$, where the sheaves $B_i$ are supported at the points $P_i$ (and are artinian $\mathcal{O}_{\hat{\mathcal{A}}, P}$-modules; see [Mukai 1981, Example 2.9]). In particular, $R^1\hat{\mathcal{F}}$ satisfies the WIT of index 0.
Now consider the spectral sequence of the proof of Lemma 3.4. The only nonzero $E_2$ terms are $E_2^{0,1}$ and $E_2^{2,0}$. Therefore, one has the exact sequence

$$0 \to R^0\mathcal{F} R^1\mathcal{F} \to R^2\mathcal{F} R^0\mathcal{F} \to (-1)^* \mathcal{F} \to 0.$$  

First note that $R^1\mathcal{F}$ is supported on finitely many points. It follows that $R^0\mathcal{F} R^1\mathcal{F} = R\mathcal{F} R^1\mathcal{F}$ is a homogeneous vector bundle; call it $\mathcal{H}$. It suffices to show that $R^0\mathcal{H}$ is a negative line bundle.

Let $U = \text{Pic}^0(\hat{A}) - \{0_A, P_1, \ldots, P_n\}$. Then, for all $P \in U$,

$$h^0(A, \mathcal{H} \otimes P) = h^0(A, a_\ast \omega_X \otimes P) = \chi(X, \omega_X) = 1.$$  

Thus $R^0\mathcal{H}|_U$ is locally free of rank 1. Let $L = (R^0\mathcal{F})^\vee$. Then $L$ is a reflexive sheaf of rank 1 on a nonsingular surface and hence a line bundle. Since $R^0\mathcal{F} = R^0\mathcal{F} a_\ast \omega_X$ is torsion-free, we have an exact sequence of coherent sheaves on $\hat{A}$:

$$0 \to R^0\mathcal{F} \to L \to Q \to 0$$

where $Q$ is supported at most on the points $P_i$.

We claim that $Q = 0$. Suppose on the contrary that $Q \neq 0$. By Lemma 3.4, $R^i\mathcal{F} R^0\mathcal{F} = 0$ for $i = 0, 1$, hence $R^0\mathcal{F} \cong R^0 Q$. So for general $P \in A = \text{Pic}^0(\hat{A})$ one has

$$h^0(L \otimes P) = h^0(Q \otimes P) \neq 0,$$

since $Q \neq 0$ is supported on points. It follows that $L$ is an ample line bundle and therefore satisfying I.T of index 0. In particular, $R^2\mathcal{F}L = 0$. On the other hand, since $Q$ is supported on points, we have that $R^1\mathcal{F}Q = 0$. The exact sequence

$$R^1\mathcal{F}Q \to R^2\mathcal{F} R^0\mathcal{F} \to R^2\mathcal{F}L$$

yields $R^2\mathcal{F} R^0\mathcal{F} = 0$. It follows that $\mathcal{F} = 0$, since there is a surjection

$$R^2\mathcal{F} R^0\mathcal{F} \to (-1)^* \mathcal{F}.$$  

One concludes that $0_A = a_\ast \omega_X$, and so that $X \to A$ is birational, which is the required contradiction.

We may therefore assume that $Q = 0$ and hence $L = R^0\mathcal{F}$ is a line bundle. By Lemma 3.4, $L$ satisfies the WIT of index 2, hence it is a negative definite line bundle. □

Remark. It follows that if $X \to A$ has degree 2, then $\mathcal{F}$ has rank 1. The only possibility is that $a_\ast \omega_X = 0_A \oplus 0_A(-\Theta)$, where $\Theta$ is a principal polarization. This is a $2:1$ covering branched along a divisor $D \in |2\Theta|$. 
We have given an example with $a_*o_X = \mathcal{O}_A \oplus \hat{L}$, where $L'$ is an ample line bundle of type $(1, 2)$. We have not been able to rule out the cases in which $\mathcal{H} \neq 0$. For example, is it possible to have $a_*o_X = \mathcal{O}_A \oplus \mathcal{F}$, with $\mathcal{F}$ as follows?

**Example.** Let $(A, L)$ be a general polarized abelian surface of type $(1, 3)$ and $x \in A$ a closed point. Then $h^i(A, L \otimes \mathcal{F}_x \otimes P) = 0$ for all $i > 0$ and all $P \in \text{Pic}^0(A)$. Let $\mathcal{E} = L \otimes \mathcal{F}_x$ and $\mathcal{F} = \mathcal{E}'$. Then we have an exact sequence

$$0 \to P_x' \to \hat{L} \to \mathcal{F} \to 0.$$ 

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**References**


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