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We treat the heat equation with singular drift terms and its generalization: the linearized Navier–Stokes system. In the first case, we obtain boundedness of weak solutions for highly singular, "supercritical" data. In the second case, we obtain regularity results for weak solutions with mildly singular data (those in the Kato class). This not only extends some of the classical regularity theory from the case of elliptic and heat equations to that of linearized Navier–Stokes equations but also proves an unexpected gradient estimate, which extends the recent interesting boundedness result of O'Leary.

1. Introduction

We will prove local boundedness and other regularity properties for weak solutions to the two parabolic equations

$$\Delta u(x,t) - b(x,t)\nabla u(x,t) - \partial_t u(x,t) = 0, \quad (x,t) \in \Omega \subset \mathbb{R}^n \times \mathbb{R} \tag{1}$$

and

$$\begin{cases} \Delta u(x,t) - b(x,t)\nabla u(x,t) + \nabla P(x,t) - \partial_t u(x,t) = 0, & (x,t) \in \Omega \subset \mathbb{R}^3 \times \mathbb{R}, \\ \operatorname{div} u = 0,\ \operatorname{div} b = 0,\ b(\cdot\,,t) \in L^2_{\mathrm{loc}}. \end{cases} \tag{2}$$

Here Δ is the standard Laplacian and $b = b(x,t)$ is a given L^2_{loc} singular vector field to be specified later. Ω is a domain.

There has been a mature theory of existence and regularity for equation (1); see [Ladyženskaja et al. 1967] and [Lieberman 1996], for example. When $b = b(x)$ and $|b| \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $p > n$, weak solutions to (1) are locally bounded and Hölder-continuous. This condition is sharp in general. Here is an example, taken from [Han and Lin 1997, p. 108]. The function $u = \ln\ln|x|^{-1} - \ln\ln R^{-1}$ is an unbounded weak solution of

$$\Delta u + b\nabla u = 0$$

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in the ball $B(0, R)$ in $\mathbb{R}^2$, for $R < 1$. Here

$$b = \nabla u = -\frac{\nabla |x|}{|x| \ln |x|^{-1}}$$

and hence $b \in L^2_{\text{loc}}$ with $n = 2$.

The first goal of the paper is to show that the simple condition $\operatorname{div} b \leq 0$ will ensure that weak solutions of (1) are locally bounded when the data b is almost twice as singular as before. This will be made precise in Theorem 1.2 and Remark 1.3. Thus one has achieved a leap in the boundedness condition rather than a marginal improvement.

Clearly a strong impetus still exists for the study of parabolic equations with very singular coefficients. In the study of nonlinear equations with gradient structure such as the Navier–Stokes equations and harmonic maps, highly singular functions occur naturally. So, it is very important to investigate a possible gain of regularity in the presence of a singular drift term b. This line of research has been followed in several papers, including [Chen and Zhao 1995; Chen and Song 2003] and others that we now discuss briefly. Under the condition $|b| \in L^n(\mathbb{R}^n)$, Stampacchia [1965] proved that bounded solutions of $\Delta u + b\nabla u = 0$ are Hölder-continuous. Cranston and Zhao [1987] proved that solutions to this equation are continuous when b is in a suitable Kato class, that is, when

$$\lim_{r \to 0} \sup_x \int_{|x-y| \leq r} \frac{|b(y)|}{|x-y|^{n-1}} dy = 0.$$

Kovalenko and Semenov [1990] proved the Hölder continuity of solutions to (1), when $|b|^2$ is independent of time and is sufficiently small in the form sense, i.e., for a sufficiently small $\epsilon > 0$,

$$\int_{\mathbb{R}^n} |b|^2(x)\phi^2(x)\,dx \leq \epsilon \int_{\mathbb{R}^n} |\nabla\phi|^2(x)\,dx, \quad \phi \in C_0^\infty(\mathbb{R}^n).$$

It is a well known fact that the form boundedness condition provides a more general class of singular functions than the corresponding L^p class, Morrey–Campanato class and Kato class. This result was generalized in [Semenov 1999] to equations with leading term in divergence form. Osada [1987] proved that the fundamental solution of (1) has global Gaussian upper and lower bound when b is the derivative of bounded functions (in the distribution sense) and $\operatorname{div} b = 0$. In [Liskevich and Zhang 2004], Hölder continuity of solutions to (1) was established when $b = b(x)$, $|b|^2$ is form bounded and $\operatorname{div} b = 0$. Most recently, in [Zhang 2004b], we considered (1) with time-dependent functions $b = b(x, t)$, proving that weak solutions to (1) are locally bounded provided that $\operatorname{div} b = 0$ and $|b|^m$ is form bounded for a fixed $m > 1$. That is, for any $\phi \in C^\infty(\mathbb{R}^n \times (0, \infty))$ with compact support in the spatial direction,

$$\iint_{\mathbb{R}^n} |b|^m \phi^2 \, dx \, dt \le k \iint_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \, dt,$$

where k is independent of ϕ. The key improvement over the previous result is that the power on b drops from 2 to any number greater than 1.

This class of data b contains the velocity function in the three-dimensional Navier–Stokes equations. As a result we gave a different proof of the *local* boundedness of velocity in the two-dimensional case. Moreover assuming a *local* bound in the pressure, we prove boundedness of velocity in the three-dimensional case.

The first goal of the paper is to treat the end-point case of the condition above, namely, $m = 1$. We will prove that weak solutions to (1) are locally bounded provided that $|b|\big(\ln(1+|b|)\big)^2$ is form bounded and $\operatorname{div} b \le 0$.

Many authors have also studied the regularity property of the related heat equation $\Delta u + Vu - u_t = 0$. Here V is a singular potential. We refer the reader to [Aizenman and Simon 1982; Simon 1982] and the references therein. The function V is allowed in the Kato class which is a little more singular than the corresponding L^p class. It remains a challenging problem to push this theory to a broader class of functions.

In this paper we use the following definition of weak solutions.

Definition 1.1. Let $D \subseteq \mathbb{R}^n$ be a domain and $T \in (0, \infty]$. A function u such that $u, |\nabla u| \in L^2_{\text{loc}}(D \times [0, T])$ is a weak solution to (1) if, for any $\phi \in C_0^\infty(D \times (-T, T))$, there holds

$$\int_0^T \int_D (u \partial_t \phi - \nabla u \nabla \phi) \, dx \, dt - \int_0^T \int_D b \nabla u \, \phi \, dx \, dt = - \int_D u_0(x) \phi(x, 0) \, dx.$$

Theorem 1.2. *Suppose that* $\operatorname{div} b \le 0$ *in the weak sense, that* $b \in L^2_{\text{loc}}$, *and that* $|b|\big(\ln(1+|b|)\big)^2$ *is form bounded, in the sense that there exists* $k > 0$ *such that for any* $\phi \in C^\infty(\mathbb{R}^n \times (0, \infty))$ *with compact support in the spatial direction,*

$$\iint_{\mathbb{R}^n} |b|\big(\ln(1+|b|)\big)^2 \phi^2 \, dx \, dt \le k \iint_{\mathbb{R}^n} |\nabla \phi|^2 \, dx \, dt. \tag{3}$$

Then weak solutions to equation (1) are locally bounded.

Remark 1.3. In the special case that b is independent of time and $b \in L^p(\mathbb{R}^n)$ with $p > n/2$, then it is easy to check that (3) is satisfied. Recall that the standard theory essentially only allows functions in L^p with $p > n$. The strength of the theorem comes from the fact that weak solutions are locally bounded in any domain regardless of its value on the parabolic boundary. If the domain is $\mathbb{R}^n \times (0, \infty)$ or if the initial Dirichlet boundary condition is imposed, then using a Nash type estimate, one can show that solutions are locally bounded when $t > 0$ as long as the fundamental solution is well defined. In this case one can choose b to be as singular as any L^2 functions. Actually the presence of b is totally irrelevant except for the purpose of making the integrals in the definition of a weak solution finite. Here is a sketch of the proof. Let u solve

$$\begin{cases} \Delta u - b\nabla u - u_t = 0 & \text{in } D \times (0,\infty), \\ u(x,t) = 0 & (x,t) \in \partial D \times (0,\infty), \\ u(x,0) = u_0(x). \end{cases}$$

Let $G(x,t;y,t)$ be the fundamental solution with initial Dirichlet boundary condition. If $\operatorname{div} b = 0$, differentiating in time shows that $\int_D G(x,t;y,0)\,dy \le 1$. By the Nash inequality, one has

$$\begin{aligned} \frac{d}{dt}\int_D G(x,t;y,t)^2 dy &= -2\int_D |\nabla G(x,t;y,t)|^2 dy \\ &\le -c\,\frac{\left(\int_D G(x,t;y,t)^2 dy\right)^{1+(2/n)}}{\left(\int_D G(x,t;y,0)\,dy\right)^{4/n}}. \end{aligned}$$

Hence, $G(x,t;y,0) \le c/t^{n/2}$. Therefore

$$u(x,t) = \int_D G(x,t;y)u_0(y)\,dy$$

is bounded as soon as $t > 0$ and u_0 is in $L^1(D)$.

However this does not imply local boundedness of weak solutions unconditionally. It would be interesting to establish existence and more regularity results for (1) with the singular data in Theorem 1.2.

In the time-independent elliptic case, there was already a strong indication that standard regularity theory can be improved in the presence of divergence-free data. In the important papers [Frehse and Ruzicka 1994, 1995], local boundedness of Green's function (away from the singularity) of the operator $-\Delta - b\nabla$ with Dirichlet boundary conditions was proved, under the conditions $n = 5$, $|\nabla^2 b| \in L^{4/3}$, $b = 0$ on the boundary and $\operatorname{div} b = 0$ (see [Frehse and Ruzicka 1994, Lemmas 1.48 and Lemma 1.11]). The upshot is that the bounds on the Green's function is independent of the norm $|\nabla^2 b| \in L^{4/3}$. The proof uses essentially the fact that the Green's function vanishes on the boundary. So the drift term is integrated out. In contrast we do not have the benefit of a zero boundary.

In the three-dimensional case, we derive a further regularity result:

Corollary 1.4. *Assume* $|b| \in L^\infty([0,T], L^2(\mathbb{R}^3)) \cap L^q(\mathbb{R}^3 \times [0,T])$ *with* $q > 3$ *and* $\operatorname{div} b = 0$. *If* u *is a weak solution of* (1) *in* $\mathbb{R}^3 \times [0,T]$ *with* $\int_{\mathbb{R}^3} u^2(x,0)\,dx < \infty$, *then* u *is locally bounded and for almost every* t, $u(\cdot,t)$ *are Hölder-continuous.*

Proof. We know from [Zhang 2004b, Corollary 1] that b satisfies the conditions of Theorem 1.2.

Next, denote by G_0 the Gaussian heat kernel of the heat equation. Then, for $t > t_0$,

$$u(x,t) = \int_{\mathbb{R}^3} G_0(x,t;y,t_0)u(y,t_0)\,dy - \int_{t_0}^t \int_{\mathbb{R}^3} G_0(x,t;y,s)b\nabla u(y,s)\,dy\,ds.$$

Since b is divergence free, we have

$$u(x,t) = \int_{\mathbb{R}^3} G_0(x,t;y,t_0)u(y,t_0)\,dy + \int_{t_0}^t \int_{\mathbb{R}^3} \nabla_y G_0(x,t;y,s)bu(y,s)\,dy\,ds.$$

Therefore, in the weak sense,

$$\begin{aligned}\nabla_x u(x,t) &\\ &= \int_{\mathbb{R}^3} \nabla_x G_0(x,t;y,t_0)u(y,t_0)\,dy + \int_{t_0}^t \int_{\mathbb{R}^3} \nabla_x\nabla_y G_0(x,t;y,s)bu(y,s)\,dy\,ds \\ &\equiv I_1(x,t) + I_2(x,t).\end{aligned}$$

It is well known that

$$\nabla_x\nabla_y G_0(x,t;y,t_0) = -\nabla_x\nabla_x G_0(x,t;y,t_0)$$

is a parabolic Calderón–Zygmund kernel (see [Lieberman 1996], for example). Hence by our assumption on b and the fact that u is bounded in $\mathbb{R}^3 \times [t_0, T]$, the second term I_2 in the last integral is in $L^q(\mathbb{R}^3 \times [t_0, T])$, $q > 3$. It follows that $|I_2(\cdot\,,t)| \in L^q(\mathbb{R}^3)$, with $q > 3$, for a.e. t. The Sobolev imbedding theorem then shows that $u(\cdot\,,t)$ is Hölder continuous for a.e. t. □

Next we turn to equation (2), which is the first step in tackling the full Navier–Stokes equations. When $b = 0$, equation (2) is just the Stokes equations, which have been studied for long time. Our focus is on how to allow b to be as singular as possible while retaining the boundedness of weak solutions. As far as equation (2) is concerned, our result does not improve the standard theory as dramatically as it does for equation (1). We have to restrict the data b in a suitable Kato class for (2). Nevertheless, Theorem 1.7 still generalizes the key part of the important work of [Aizenman and Simon 1982; Cranston and Zhao 1987] on the elliptic equations to the case of linearized Navier–Stokes system. Moreover, we even obtain gradient estimates for solutions of (2) while only continuity was expected.

As pointed out in [Aizenman and Simon 1982; Simon 1982], Kato class functions are quite natural objects in studying elliptic and parabolic equations with singular lower-order terms. Roughly speaking, a function is in a Kato class with respect to an equation if its convolutions with certain kernel functions are small in some sense. The kernel function usually is related to the fundamental solution of the principal term of the equation. For instance, for the equation $\Delta u(x) + V(x)u(x) = 0$ in $\mathbb{R}^n$, $n \geq 3$, the function V is in the Kato class if

$$\lim_{r\to 0} \sup_x \int_{B(x,r)} \frac{|V(y)|}{|x-y|^{n-2}}\, dy = 0.$$

In [Aizenman and Simon 1982], it was proved that weak solutions to $\Delta u + Vu = 0$ are continuous and satisfy a Harnack inequality when V is in the above Kato class. Numerous papers have been written on this subject in the last thirty years, mainly in the context of elliptic and heat equations.

In the context of Navier–Stokes equations, the corresponding time-dependent Kato class was defined recently in [Zhang 2004a], which mirrors those for the heat equation [Zhang 1997]. Normally, with data in the Kato class, weak solutions of elliptic equations are just continuous, as proved in [Aizenman and Simon 1982] and [Cranston and Zhao 1987]. It was proved in [Zhang 2004a] that weak solutions to (2) are bounded when b is in the Kato class. Here we prove that the *spatial gradient* of solutions to (2) are bounded provided that b is in the Kato class locally. One can use the idea of the Kato class to recover some (but not all) the decay estimates in the interesting papers [Schonbek 1985; 1986] and to prove some pointwise decay estimates; see [Zhang 2004a].

To make our statement precise, we introduce some notation. Henceforth, we set

$$(4) \qquad K_1(x,t;y,s) = \begin{cases} \left(|x-y| + \sqrt{t-s}\right)^{-(n+1)} & \text{if } t \geq s,\ x \neq y, \\ 0 & \text{if } s < t. \end{cases}$$

We write $Q_r(x,t) = B(x,r) \times [t-r^2, t]$.

Definition 1.5. A vector-valued function $b = b(x,t) \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ is in class K_1 if it satisfies the condition

$$(5)\ \lim_{h\to 0} \sup_{(x,t)\in\mathbb{R}^{n+1}} \int_{t-h}^{t} \int_{\mathbb{R}^n} \big(K_1(x,t;y,s) + K_1(x,s;y,t-h)\big)\, |b(y,s)|\, dy\, ds = 0.$$

For clarity of presentation, given $t > l$, we introduce the quantity

$$(6) \qquad B(b,l,t) = \sup_x \int_l^t \int_{\mathbb{R}^n} \big(K_1(x,t;y,s) + K_1(x,s;y,l)\big)\, |b(y,s)|\, dy\, ds.$$

From [Zhang 2004a, Remark 1.2], we see that the function class K_1 permits solutions which are very singular. In case the spatial dimension is 3, a function in this class can have an apparent singularity of certain type that is not L^p_{loc} for any $p > 1$ and of dimension 1. One can also construct time-dependent functions in K_1 with quite singular behaviors. The class K_1 also contains the space $L^{p,q}$ with $n/p + 2/q < 1$, which sometimes is referred to as the Prodi–Serrin class. For the nonlinear Navier–Stokes equation, if a weak solution is known to be in this class, then it is actually smooth. As for the linearized equation (2), following the argument in [Serrin 1962], it is clear that weak solutions are bounded if b is in the above $L^{p,q}$ class. Now we are able to prove that the spatial gradient of weak solutions are bounded *automatically*, without resorting to the nonlinear structure. Using Hölder's inequality, one can see that the class K_1 also contains the Morrey-type space introduced in [O'Leary 2003], where boundedness of weak solutions in that space are proved.

One can also define a slightly bigger Kato class by requiring the limit in (5) to be a small positive number rather than 0. We will not seek such generality this time. The appearance of two kernel functions is due to the asymmetry of the equation in time direction.

Let D for a domain in $\mathbb{R}^3$ and $T > 0$. Following standard practice, we will use this definition for solutions of (2) throughout the paper.

Definition 1.6. A divergence-free vector field

$$u \in L^\infty(0, T; L^2(D)) \cap L^2(0, T; W^{1,2}(D))$$

is called a (weak) solution of (2) if, for any vector-valued $\phi \in C^\infty(D \times [0, T])$ with $\operatorname{div} \phi = 0$ and $\phi = 0$ on $\partial D \times [0, T]$, the vector field u satisfies

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^n} \langle u, \partial_t\phi + \Delta\phi\rangle \, dx\, dt - \int_{t_1}^{t_2}\int_{\mathbb{R}^n} \langle b\nabla u, \phi\rangle \, dx\, dt = -\int_{\mathbb{R}^n} \langle u(x,t), \phi(x,t)\rangle \Big|_{t_1}^{t_2} \, dx.$$

Next we state the theorem on equation (2), the linearized Navier–Stokes equations.

Theorem 1.7. *Let u be a solution of (2) in a domain $\Omega \subset \mathbb{R}^3 \times \mathbb{R}$. Suppose $Q_{4r}(x,t) \subset \Omega$, $\operatorname{div} b = 0$ and that $b|_{Q_{2r}(x,t)}$ is in class K_1 and $b \in L^2_{\text{loc}}$. Then both u and $|\nabla u|$ are bounded functions in $Q_{2r}(x,t)$.*

Moreover, for some positive constants $C = C(b)$ and r_0, depending on the size of the Kato norm of b, there hold, when $0 < r < r_0$,

$$\begin{aligned} |u(x,t)| &\le \frac{C}{r^5}\int_{Q_{2r}(x,t)} |u(y,s)|\, dy\, ds, \\ |\nabla u(x,t)| &\le \frac{C}{r^5}\int_{Q_{2r}(x,t)} |\nabla u(y,s)|\, dy\, ds + \frac{C}{r^6}\int_{Q_{2r}(x,t)} |u(y,s) - \bar{u}_{Q_{2r}}|\, dy\, ds. \end{aligned} \tag{7}$$

Here $\bar{u}_{Q_{2r}}$ is the average of u in $Q_{2r}(x,t)$.

If in addition $u(\cdot, t) \in W_0^{1,2}(B(x,2r))$ for a.e. t, then

$$|\nabla u(x,t)| \le \frac{C}{r^5}\int_{Q_{2r}(x,t)} |\nabla u(y,s)|\, dy\, ds. \tag{8}$$

Remark 1.8. One might think that the last term on the right-hand side of the gradient estimate (7) is too singular to be natural, especially when $r \to 0$. However, both (7) and equation (2) are actually scaling invariant under the scaling: $u_\lambda = u(\lambda x)$, $b_\lambda = \lambda b(\lambda x)$. So (7) is in the right setting. Moreover, the gradient estimate in (7) immediately simplifies to (8) when u vanishes on the lateral boundary or when u enjoys a suitable extension property in $W^{1,2}$ space.

Remark 1.9. The reader may wonder whether any extra regularity in the time direction is possible. The answer is no as indicated in the example in [Serrin 1962], restated in [O'Leary 2003]. Let ϕ be a harmonic function in $\mathbb{R}^3$ and $a = a(t)$ be an integrable function. Then $u = a\nabla\phi$ is a weak solution of the Navier–Stokes equation. Obviously in the time direction u is no more regular than a.

Theorem 1.2 will be proved in Section 2, and Theorem 1.7 in Section 3. The idea for the proof of Theorem 1.7 is to combine a recent localization argument in [O'Leary 2003] with a refined iteration. Using the idea in the proof of Theorem 1.2, in Section 4 we introduce a sufficient condition on the velocity that implies boundedness of weak solutions of three-dimensional Navier–Stokes equations. The main improvement is that no absolute value of the velocity is involved.

2. Proof of Theorem 1.2

Since the drift term b in (1) can be much more singular than those allowed by the standard theory, the existence and uniqueness of weak solutions of (1) can not be taken for granted. In order to proceed first we need some approximation results whose proof can be found in [Zhang 2004b], requiring only small modifications.

Proposition 2.1. *Suppose that u is a weak solution of equation (1) in the cube $Q = D \times [0, T]$, where b satisfies the condition in Theorem 1.2. Here D is a domain in $\mathbb{R}^n$. Then u is the L^1_{loc} limit of functions $\{u_k\}$. Here $\{u_k\}$ is a weak solution of (1) in which b is replaced by smooth b_k such that* $\operatorname{div} b_k \le 0$ *and $b_k \to b$ strongly in $L^2(Q)$, $k \to \infty$.*

Proof. The proof is almost identical to that of [Zhang 2004b, Proposition 2.4]. The only difference is that we are assuming $\operatorname{div} b \le 0$ instead of $\operatorname{div} b = 0$. Let $G = G(x, t; y, s)$ be the fundamental solution of (1) with $\operatorname{div} b \le 0$ and b smooth. Then it is easy to show by differentiation that

$$\int_{\mathbb{R}^n} G(x, t; y, s)\, dx \le 1.$$

The rest of the argument is the same. □

By this approximation result, we can and do assume that the vector field b is smooth and we will differentiate it freely.

The theorem will be derived from the following local "mean value" property.

Claim. *Let u be a nonnegative solution of (1) in the parabolic cube*

$$Q_{\sigma_0 r} = B(x, \sigma_0 r) \times [t - (\sigma_0 r)^2, t].$$

Here $x \in \mathbb{R}^n$, $r > 0$, $t > 0$ and σ_0 is a suitable number greater than 1. Suppose b satisfies (3) in $Q_{\sigma_0 r}$. Then there exists $C = C(r, b) > 0$ such that

$$\sup_{Q_r} u^2 \leq C(r, b) \frac{1}{|Q_{\sigma_0 r}|} \int_{Q_{\sigma_0 r}} u^2 \, dy \, ds. \tag{9}$$

Proof. Pick a solution u of (1) in the parabolic cube $Q_{\sigma r} = B(x, \sigma r) \times [t - (\sigma r)^2, t]$, where $x \in \mathbb{R}^n$, $\sigma > 1$, $r > 0$ and $t > 0$. By direct computation, for any rational number $p \geq 1$ that can be written as the quotient of two integers with the denominator being odd, one has

$$\Delta u^p - b\nabla u^p - \partial_t u^p = p(p-1)|\nabla u|^2 u^{p-2}. \tag{10}$$

Here the condition on p is to ensure that u^p makes sense when u changes sign. Actually u^2 is a subsolution to (1). Hence one can also assume that u is a nonnegative subsolution to (1) by working with u^2.

Choose a refined cutoff function ψ as follows. First take $\eta = \eta(s)$ with values in $[0, 1]$, supported in $[t - (\sigma r)^2, t]$, and satisfying $|\eta'| \leq 2/((\sigma - 1)r)^2$ and $\eta(s) = 1$ for $s \in [t - r^2, t]$. Also choose $\phi = \phi(y)$ taking values in $[0, 1]$, supported in $B(x, \sigma r)$, and satisfying $\phi(y) = 1$ for $y \in B(x, r)$ and

$$\frac{|\nabla \phi|}{\phi} \leq \frac{A}{(\sigma - 1)r} |\ln \phi|^{3/2}, \quad \text{where } A > 0.$$

By modifying the function

$$\exp\left(-\frac{\sigma^2}{\sigma^2 - |x - y|^2}\right)^k,$$

for k sufficiently large, it is easy to show that such a ϕ exists. Now set $\psi = \eta\phi$.

Set $w = u^p$ and use $w\psi^2$ as a test function in (10). This yields

$$\int_{Q_{\sigma r}} (\Delta w - b\nabla w - \partial_s w) w \psi^2 \, dy \, ds = p(p-1) \int_{Q_{\sigma r}} |\nabla u|^2 w^2 u^{-2} \geq 0.$$

Using integration by parts, one deduces

$$\int_{Q_{\sigma r}} \nabla(w\psi^2) \nabla w \, dy \, ds \leq -\int_{Q_{\sigma r}} b\nabla w (w\psi^2) \, dy \, ds - \int_{Q_{\sigma r}} (\partial_s w) w \psi^2 \, dy \, ds. \tag{11}$$

By direct calculation,

$$\begin{aligned}\int_{Q_{\sigma r}} \nabla(w\psi^2)\nabla w\, dy\, ds &= \int_{Q_{\sigma r}} \nabla((w\psi)\psi)\nabla w\, dy\, ds\\ &= \int_{Q_{\sigma r}} \big(\nabla(w\psi)(\, \nabla(w\psi) - (\nabla\psi)w) + w\psi\nabla\psi\nabla w\big)\, dy\, ds\\ &= \int_{Q_{\sigma r}} \big(|\nabla(w\psi)|^2 - |\nabla\psi|^2 w^2\big)\, dy\, ds.\end{aligned}$$

Substituting into (11), we obtain

$$\begin{aligned}(12)\quad &\int_{Q_{\sigma r}} |\nabla(w\psi)|^2\, dy\, ds\\ &\le -\int_{Q_{\sigma r}} b\nabla w(w\psi^2)\, dy\, ds - \int_{Q_{\sigma r}} (\partial_s w)w\psi^2\, dy\, ds + \int_{Q_{\sigma r}} |\nabla\psi|^2 w^2\, dy\, ds.\end{aligned}$$

Next, notice that

$$\begin{aligned}\int_{Q_{\sigma r}} (\partial_s w)w\psi^2\, dy\, ds &= \frac{1}{2}\int_{Q_{\sigma r}} (\partial_s w^2)\psi^2\, dy\, ds\\ &= -\int_{Q_{\sigma r}} w^2\phi^2\eta\partial_s\eta\, dy\, ds + \frac{1}{2}\int_{B(x,\sigma r)} w^2(y,t)\phi^2(y)\, dy.\end{aligned}$$

Combining this with (12), we see that

$$\begin{aligned}(13)\quad \int_{Q_{\sigma r}} |\nabla(w\psi)|^2\, dy\, ds &+ \frac{1}{2}\int_{B(x,\sigma r)} w^2(y,t)\phi^2(y)\, dy\\ &\le \int_{Q_{\sigma r}} (|\nabla\psi|^2 + \eta\partial_s\eta)\; w^2\, dy\, ds - \int_{Q_{\sigma r}} b(\nabla w)(w\psi^2)\, dy\, ds\\ &\equiv T_1 + T_2.\end{aligned}$$

The first term on the right-hand side is already in good shape. We estimate the second as follows:

$$\begin{aligned}T_2 &= -\int_{Q_{\sigma r}} b(\nabla w)(w\psi^2)\, dy\, ds\\ &= -\frac{1}{2}\int_{Q_{\sigma r}} b\psi^2\nabla w^2\, dy\, ds = \frac{1}{2}\int_{Q_{\sigma r}} \operatorname{div}(b\psi^2)w^2\, dy\, ds\\ &= \frac{1}{2}\int_{Q_{\sigma r}} \operatorname{div} b(\psi w)^2\, dy\, ds + \frac{1}{2}\int_{Q_{\sigma r}} b\nabla(\psi^2)w^2\, dy\, ds\\ &= \frac{1}{2}\int_{Q_{\sigma r}} \operatorname{div} b(\psi w)^2\, dy\, ds + \int_{Q_{\sigma r}} b(\nabla\psi)\psi w^2\, dy\, ds\\ &\le \int_{Q_{\sigma r}} b(\nabla\psi)\psi w^2\, dy\, ds.\end{aligned}$$

Here we just used the assumption that $\operatorname{div} b \le 0$.

The next paragraph contains a key argument of the paper.

Let $D > 0$ be a number to be chosen later. We have

$$\begin{aligned}
T_2 &\le \int_{Q_{\sigma r}} |b|\,|\nabla\psi|\,\psi w^2\,dy\,ds \\
&\le \int_{|b|\ge D} |b|\,|\nabla\psi|\,\psi w^2\,dy\,ds + \int_{|b|\le D} |b|\,|\nabla\psi|\,\psi w^2\,dy\,ds \\
&\le \int_{|b|\ge D} |b|\,|\nabla\psi|\,\psi w^2\,dy\,ds + \frac{CD}{(\sigma-1)r}\int_{Q_{\sigma r}} w^2\,dy\,ds.
\end{aligned}$$

Using the relations $\psi=\phi\eta$ and $|\nabla\phi| \le \frac{A}{(\sigma-1)r}\phi|\ln\phi|^{3/2}$, we get

$$\begin{aligned}
T_2 \le{}& \iint_{\substack{|b|\ge 1/\phi\\ |b|\ge D}} |b||\nabla\phi|\phi w^2\,dy\,\eta^2\,ds \\
&\qquad + \iint_{|b|\le 1/\phi} |b|\,|\nabla\phi|\,\phi w^2\,dy\,\eta^2\,ds + \frac{CD}{(\sigma-1)r}\int_{Q_{\sigma r}} w^2\,dy\,ds \\
\le{}& \frac{A}{(\sigma-1)r}\iint_{\substack{|b|\ge 1/\phi\\ |b|\ge D}} |b|\,|\ln\phi|^{3/2}(\phi w)^2\,dy\,\eta^2\,ds \\
&\qquad + \frac{A}{(\sigma-1)r}\iint_{\substack{B(x,\sigma r)\\ |b|\le 1/\phi}} |b|\,|\ln\phi|^{3/2}(\phi w)^2\,dy\,\eta^2\,ds + \frac{CD}{(\sigma-1)r}\int_{Q_{\sigma r}} w^2\,dy\,ds \\
\le{}& \frac{A}{(\sigma-1)r}\int_{|b|\ge D} |b|(\ln|b|)^{3/2}(\psi w)^2\,dy\,ds \\
&\qquad + \frac{A}{(\sigma-1)r}\int_{Q_{\sigma r}} \frac{1}{\phi}|\ln\phi|^{3/2}(\phi w)^2\eta^2\,dy\,ds + \frac{CD}{(\sigma-1)r}\int_{Q} srw^2\,dy\,ds \\
\le{}& \frac{A}{(\sigma-1)r(\ln D)^{1/2}}\int_{Q_{\sigma r}} |b|(\ln|b|)^2(\psi w)^2\,dy\,ds \\
&\qquad + \frac{A}{(\sigma-1)r}\int_{Q_{\sigma r}} \phi|\ln\phi|^{3/2}w^2\eta^2\,dy\,ds + \frac{CD}{(\sigma-1)r}\int_{Q_{\sigma r}} w^2\,dy\,ds.
\end{aligned}$$

By our assumptions on b,

$$\int_{Q_{\sigma r}} |b|(\ln|b|)^2(\psi w)^2\,dy\,ds \le k\int_{Q_{\sigma r}} |\nabla(\psi w)|^2\,dy\,ds,$$

and using the boundedness that $\phi|\ln\phi|^{3/2}$, we deduce

$$T_2 \le \frac{A}{(\sigma-1)r(\ln D)^{1/2}}\int_{Q_{\sigma r}} |\nabla(\psi w)|^2\,dy\,ds + \frac{C_1 D}{(\sigma-1)r}\int_{Q_{\sigma r}} w^2\,dy\,ds.$$

Now we set $D = e^{(2A/((\sigma-1)r))^2}$, so that $A = \frac{1}{2}(\sigma-1)r(\ln D)^{1/2}$. Then

$$T_2 \le \frac{1}{2}\int_{Q_{\sigma r}} |\nabla(\psi w)|^2\,dy\,ds + c_0 e^{(c_1/((\sigma-1)r))^2}\int_{Q_{\sigma r}} w^2\,dy\,ds, \tag{14}$$

where c_0 and c_1 are positive constants independent of r and σ.

Combining (13) with (14), we reach

$$\int_{Q_{\sigma r}} |\nabla(w\psi)|^2\,dy\,ds + \int_{B(x,\sigma r)} w^2(y,t)\phi^2(y)\,dy \le c_0 e^{(c_1/((\sigma-1)r))^2}\int_{Q_{\sigma r}} w^2\,dy\,ds. \tag{15}$$

By modifying Moser's iteration somewhat, we deduce from this the L^2–L^∞ estimate (9). Indeed, by Hölder's inequality,

$$\int_{\mathbb{R}^n} (\phi w)^{2(1+(2/n))} = \int_{\mathbb{R}^n} (\phi w)^2\ (\phi w)^{4/n} \le \left(\int_{\mathbb{R}^n} (\phi w)^{2n/(n-2)}\right)^{(n-2)/n}\left(\int_{\mathbb{R}^n} (\phi w)^2\right)^{2/n}.$$

Using the Sobolev inequality, one obtains

$$\int_{\mathbb{R}^n} (\phi w)^{2(1+(2/n))} \le C\left(\int (\phi w)^2\right)^{2/n}\left(\int_{\mathbb{R}^n} |\nabla(\phi w)|^2\right).$$

Together with (15), this last inequality implies, for some $C_1 > 0$,

$$\int_{Q_{\sigma' r}(x,t)} u^{2p\theta} \le \left(c_0 e^{(c_1/((\sigma-\sigma')r))^2}\int_{Q_{\sigma r}(x,t)} u^{2p}\right)^{\theta},$$

where $\theta = 1 + (2/n)$ and $\sigma' < \sigma$.

Take a number $\rho > 1$ such that $\rho^2 < \theta$. Set $\tau_i = \rho^{-i}$, $\sigma_0 = 1/(1-\rho^{-1})$, $\sigma_i = \sigma_{i-1} - \tau_i = \sigma_0 - \sum_1^i \tau_j$, $p = \theta^i$, $i = 1, 2, \ldots$. We then have, for some $c_2, c_3 > 0$,

$$\int_{Q_{\sigma_{i+1} r}(x,t)} u^{2\theta^{i+1}} \le c_3\left(c_2^{i+1} e^{c_1^2 r^{-2}\rho^{2i}}\int_{Q_{\sigma_i r}(x,t)} u^{2\theta^i}\right)^{\theta}.$$

After iteration this implies, for some $c_4 > 0$,

$$\left(\int_{Q_{\sigma_{i+1} r}(x,t)} u^{2\theta^{i+1}}\right)^{\theta^{-i-1}} \le \exp\bigl(c_4 \textstyle\sum_1^i j\theta^{-j}\bigr)\ \exp\bigl(c_1 r^{-2}\textstyle\sum_1^i \rho^{2j}\theta^{-j}\bigr)\displaystyle\int_{Q_{\sigma_0 r}(x,t)} u^2.$$

Observe that $\rho^2/\theta < 1$. Letting i go to ∞ and observing that $\sigma_i \to 1$ as $i \to \infty$, we obtain

$$\sup_{Q_r} u^2 \le C(r, b) \int_{Q_{\sigma_0 r}} u^2.$$

This completes the proof of the claim, and thus of Theorem 1.2. □

3. Proof of Theorem 1.7

We will need a short lemma concerning the kernel function K_1 defined in (4). For the proof, see [Zhang 2004a]. We start by recalling from (6) the notation

$$B(b, 0, t) \equiv \sup_{x\in\mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} \big(K_1(x, t; y, s) + K_1(x, s; y, 0)\big)|b(y, s)|\, dy\, ds.$$

Lemma 3.1 [Zhang 2004a]. *For all $x, y, z \in \mathbb{R}^n$ and $t > \tau > 0$,*

$$\begin{aligned} K_1 * bK_1 &\equiv \int_0^t \int_{\mathbb{R}^n} \frac{1}{\big(|x-z| + \sqrt{t-\tau}\big)^{n+1}} \frac{|b(z, \tau)|}{\big(|z-y| + \sqrt{\tau}\big)^{n+1}}\, dz\, d\tau \\ &\le C B(b, 0, t) K_1(x, t; y, 0). \end{aligned}$$

Next we state and prove a representation formula for solutions of (2) and their spatial gradient, following and extending the idea in [O'Leary 2003]. The formula for solutions is contained in that paper. However, we will outline the proof since it is useful in the proof of the formula for the gradient, which is a new contribution of this paper.

Remark 3.2. The representation formula (17) below for the gradient is understood as a comparison of two L^1_{loc} functions in space-time. This is legal for two reasons. First we assumed that ∇u is an L^2 function a priori. Second, it is easy to check $K_1(\cdot, \cdot; y, s)$ is L^1_{loc} and $b\nabla u$ is L^1_{loc} by the assumption that b is L^2. Therefore the last function on the right-hand side of (17) is a L^1_{loc} function.

Lemma 3.3 (Mean value inequality). (a) *Let u be a solution of (2) in the region Ω. Suppose $Q_{2r}(x, t) \subset \Omega$. Then there exists a constant λ such that*

$$\begin{aligned} |u(x, t)| \le \lambda \frac{1}{r^5} \int_{Q_r(x,t) - Q_{r/2}(x,t)} |u(y, s)|\, dy\, ds \\ + \lambda \int_{Q_r(x,t)} K_1(x, t; y, s)|b(y, s)|\, |u(y, s)|\, dy\, ds. \end{aligned} \tag{16}$$

(b) *Under the same assumptions as in* (*a*), *there exists a constant* λ *such that*

$$
\begin{aligned}
(17)\quad |\nabla u(x,t)| \le \frac{\lambda}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla u(y,s)|\,dy\,ds \\
+\frac{\lambda}{r^6}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y,s)|\,dy\,ds \\
+\lambda\int_{Q_r(x,t)} K_1(x,t;y,s)|b(y,s)|\,|\nabla u(y,s)|\,dy\,ds.
\end{aligned}
$$

Proof. (a) Let $E = E(x,t;y,s)$ be the fundamental solution (matrix) of the Stokes system in $\mathbb{R}^3\times(0,\infty)$ and E_k be the k-th column of E. This function has been studied for a long time. All of its basic properties used below can be found in [Solonnikov 1964] and [Fabes et al. 1972]. Fixing (x,t), we construct a standard cutoff function η such that $\eta(y,s)=1$ in $Q_{r/2}(x,t)$, $\eta(y,s)=0$ outside of $Q_r(x,t)$, $0\le\eta\le 1$ and $|\nabla\eta|^2+|\Delta\eta|+|\partial_s\eta|\le c/r^2$.

Given x and t, we define a vector-valued function

$$
(3.7)\qquad \Phi_k(y,s)=\frac{1}{4\pi}\,\mathrm{curl}\left(\eta(y,s)\int_{\mathbb{R}^3}\frac{\mathrm{curl}\,E_k(x,t;z,s)}{|z-y|}dz\right).
$$

It is clear that when $t>s$, Φ_k is a valid test function for equation (2) since Φ_k is smooth, compactly supported and divergence-free. Using Φ_k as a test function on (2), by Definition 1.6, we obtain

$$
\int_0^t\int u(y,s)\big(\Delta\Phi_k+b\nabla\Phi_k+\partial_s\Phi_k\big)\,dy\,ds=\lim_{s\to t}\int u(y,s)\Phi_k(y,s)\,dy.
$$

Since E_k is divergence-free, $\mathrm{curl}\,\mathrm{curl}\,E_k=-\Delta E_k$. Thus

$$
\begin{aligned}
(18)\quad \Phi_k(y,s)&=\eta(y,s)E_k(x,t;y,s)+\frac{1}{4\pi}\nabla\eta(y,s)\times\int_{\mathbb{R}^3}\frac{\mathrm{curl}\,E_k(x,t;z,s)}{|z-y|}dz\\
&\equiv \eta E_k+\vec{Z}.
\end{aligned}
$$

Using the property of the fundamental matrix E and the fact that $\vec{Z}$ is a lower-order term, it is easy to see that

$$
\lim_{s\to t}\int u(y,s)\Phi_k(y,s)\,dy=\eta(y,t)u_k(x,t)=u_k(x,t),
$$

where u_k is the k-th component of u. Hence (setting $ub\nabla\Phi_k=u\cdot\sum_{i=1}^3 b_i\partial_i\Phi_k$), we have

$$
\begin{aligned}
u_k(x,t)=\int_{Q_r(x,t)} u\big(\Delta(\eta E_k)+\partial_s(\eta E_k)\big)\,dy\,ds+\int_{Q_r(x,t)} u\big(\Delta\vec{Z}+\partial_s\vec{Z}\big)\,dy\,ds\\
+\int_{Q_r(x,t)} ub\nabla\Phi_k\,dy\,ds.
\end{aligned}
$$

Note that $\Delta E_k + \partial_s E_k$ vanishes when $s < t$. Therefore

$$
\begin{aligned}
(19)\quad u_k(x,t) = \int_{Q_r(x,t)} u(y,s)\big(E_k(x,t;y,s)(\Delta\eta+\partial_s\eta)(y,s) \\
+ 2\nabla\eta(y,s)\nabla_y E_k(x,t;y,s)\big)\,dy\,ds \\
+ \int_{Q_r(x,t)} u(y,s)\big(\Delta\vec{Z} + \partial_s\vec{Z}\big)(y,s)\,dy\,ds \\
+ \int_{Q_r(x,t)} ub\nabla_y\Phi_k(y,s)\,dy\,ds \\
\equiv J_1 + J_2 + J_3.
\end{aligned}
$$

We estimate J_1, J_2, J_3 separately. From estimates on E (see [Solonnikov 1964] or [Fabes et al. 1972], for example), we have, for $(y,s) \in Q_r(x,t) - Q_{r/2}(x,t)$,

$$
\begin{aligned}
|E_k(x,t;y,s)| &\le \frac{c}{(|x-y|+\sqrt{t-s})^3} \le \frac{c}{r^3},\\
|\nabla_y E_k(x,t;y,s)| &\le \frac{c}{(|x-y|+\sqrt{t-s})^4} \le \frac{c}{r^4}.
\end{aligned}
$$

Using these and the properties of η, we see that

$$
(20)\qquad |J_1| \le \frac{C}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y,s)|\,dy\,ds.
$$

Next we give an estimate for J_2, closely modeled on [O'Leary 2003, p. 626]. Recall that

$$
E_k(x,t;y,s) = G(x,t;y,s)e_k + \frac{1}{4\pi}\nabla\partial_k\int_{\mathbb{R}^3}\frac{G(x,t;z,s)}{|y-z|}dz,
$$

where G is the fundamental solution of the heat equation and $\{e_1, e_2, e_3\}$ is the standard orthonormal basis of $\mathbb{R}^3$. Since $\operatorname{curl}\nabla = 0$, the vector field $\vec{Z}$ defined in (18) takes the very simple form

$$
(21)\qquad \vec{Z} = \frac{1}{4\pi}\nabla_y\eta(y,s) \times \left(\nabla_y\left(\int_{\mathbb{R}^3}\frac{G(x,t;z,s)}{|z-y|}dz\right) \times e_k\right).
$$

Plugging this into the expression for J_2, we obtain by direct computation

$$
(22)\qquad |J_2| \le \frac{C}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y,s)|\,dy\,ds,
$$

where we have used the estimate

$$
(23)\qquad \left|D_s^l D_y^m \int_{\mathbb{R}^3}\frac{G(x,t;z,s)}{|z-y|}dz\right| \le \frac{C_{m,l}}{(|x-y|+\sqrt{t-s})^{1+m+2l}}
$$

from [Solonnikov 1964, Chapter 2, Section 5], and the fact that $|x-y| \ge r/2$ here.

Finally, direct computation using (23) shows that

$$|J_3| \le \lambda \int_{Q_r(x,t)} K_1(x,t;y,s)|b(y,s)|\,|u(y,s)|\,dy\,ds. \tag{24}$$

Substituting (20), (22) and (24) into (19), we finish the proof of part (a) of the lemma.

(b) Our next task is to prove the representation formula for ∇u. In (19) the cutoff function apparently depends on (x,t). In order to prove the gradient estimate, we need modify it a little. For $(w,l) \in Q_{r/4}(x,t)$, we take

$$\Phi_k(y,s) = \frac{1}{4\pi}\,\mathrm{curl}\left(\eta(y,s)\int_{\mathbb{R}^3} \frac{\mathrm{curl}\, E_k(w,l;z,s)}{|z-y|}\,dz\right)$$

as a test function for (2). Since $\eta(w,l)=1$ in this case, following the computation before (19), we obtain

$$\begin{aligned} u_k(w,l) = \int_{Q_r(x,t)} u(y,s)\big(E_k(w,l;y,s)(\Delta\eta+\partial_s\eta)(y,s) \\ + 2\nabla\eta(y,s)\nabla_y E_k(w,l;y,s)\big)\,dy\,ds \\ + \int_{Q_r(x,t)} u(y,s)\big(\Delta\vec{Z}+\partial_s\vec{Z}\big)(y,s)\,dy\,ds \\ + \int_{Q_r(x,t)} ub\nabla_y\Phi_k(y,s)\,dy\,ds. \end{aligned} \tag{25}$$

Here $\vec{Z}$ is defined as in (18) except that $E_k(x,t;y,s)$ is replaced by $E_k(w,l;y,s)$. We would like to differentiate (25) in the spatial variables. However, since ∇u is only known as an L^2 function, we have to consider the weak derivatives.

Let $\rho=\rho(w)$ be a smooth cutoff function supported in $B(x,r/4)$. Then (25) implies, for $i=1,2,3$ and a.e. l,

$$\int_{\mathbb{R}^3} u_k(w,l)\partial_{w_i}\rho(w)\,dw = -\int_{\mathbb{R}^3}(M_1+M_2+M_3)\rho(w)\,dw, \tag{26}$$

where

$$\begin{aligned} M_1 &= \int_{Q_r(x,t)} u(y,s)\partial_{w_i}\big(E_k(w,l;y,s)(\Delta\eta+\partial_s\eta)(y,s) \\ &\qquad + 2\nabla\eta(y,s)\nabla_y E_k(w,l;y,s)\big)\,dy\,ds, \\ M_2 &= \int_{Q_r(x,t)} u(y,s)\partial_{w_i}\big(\Delta\vec{Z}+\partial_s\vec{Z}\big)(y,s)\,dy\,ds, \\ M_3 &= \int_{Q_r(x,t)} ub\nabla_y\partial_{w_i}\Phi_k(y,s)\,dy\,ds. \end{aligned}$$

Here we have used integration by parts, which is legitimate since we will show that M_1, M_2 are bounded functions and M_3 is L^1_{loc}. Note also we should have integrated in the time direction as well, since ∇u is only known to L^2 in space time. However, since all the estimates below are uniform for $l \in [t-(r/4)^2, t]$, our estimates are valid.

We estimate M_1 first. Noting that $\partial_{w_i} E_k(w,l;y,s) = -\partial_{y_i} E_k(w,l;y,s)$, we deduce

$$M_1 = -\int_{Q_r(x,t)} u(y,s)\big(\partial_{y_i} E_k(w,l;y,s)(\Delta\eta+\partial_s\eta)(y,s) \\ + 2\nabla\eta(y,s)\nabla_y\partial_{y_i} E_k(w,l;y,s)\big)\,dy\,ds.$$

Using integration by parts, we obtain

$$M_1 = \int_{Q_r(x,t)} \partial_{y_i} u(y,s)\big(E_k(w,l;y,s)(\Delta\eta+\partial_s\eta)(y,s) \\ + 2\nabla\eta(y,s)\nabla_y E_k(w,l;y,s)\big)\,dy\,ds \\ + \int_{Q_r(x,t)} u(y,s)\big(E_k(w,l;y,s)\partial_{y_i}(\Delta\eta+\partial_s\eta)(y,s) \\ + 2(\partial_{y_i}\nabla\eta(y,s))\nabla_y E_k(w,l;y,s)\big)\,dy\,ds.$$

By standard properties of E and the bounds on η and its derivatives, we have

$$|M_1| \le \frac{c}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla u(y,s)|\,dy\,ds \\ + \frac{c}{r^6}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y,s)|\,dy\,ds. \tag{27}$$

Here we have also used the fact that the arguments (w,l) and (y,s) have a parabolic distance of at least $r/4$.

Next we need to find an upper bound for M_2. Recall from (21) that

$$\vec{Z} = \frac{1}{4\pi}\nabla_y\eta(y,s)\times\left(\nabla_y\Big(\int_{\mathbb{R}^3}\frac{G(w,l;z,s)}{|z-y|}dz\Big)\times e_k\right) \\ = \frac{1}{4\pi}\nabla_y\eta(y,s)\times\left(\nabla_y\Big(\int_{\mathbb{R}^3}\frac{G(w,l;z+y,s)}{|z|}dz\Big)\times e_k\right).$$

Using the vector identity $F\times(G\times H) = (F\cdot H)G - H(F\cdot G)$, we get

$$\vec{Z} = \frac{1}{4\pi}\partial_{y_k}\eta\nabla_y\int_{\mathbb{R}^3}\frac{G(w,l;z+y,s)}{|z|}dz - \frac{1}{4\pi}\left(\sum_{j=1}^{3}\partial_{y_j}\eta\partial_{y_j}\int_{\mathbb{R}^3}\frac{G(w,l;z+y,s)}{|z|}dz\right)e_k.$$

Since $\partial_{w_i} G(w,l;\,y+z,s) = -\partial_{y_i} G(w,l;\,y+z,s)$, the above shows

$$\partial_{w_i}\vec{Z} = \frac{1}{4\pi}\partial_{y_k}\eta\nabla_y \int_{\mathbb{R}^3} \frac{\partial_{w_i} G(w,l;\,z+y,s)}{|z|}dz \\ -\frac{1}{4\pi}\left(\sum_{j=1}^{3}\partial_{y_j}\eta\partial_{y_j}\int_{\mathbb{R}^3}\frac{\partial_{w_i} G(w,l;\,z+y,s)}{|z|}dz\right)e_k.$$

Hence

$$\partial_{w_i}\vec{Z} = -\frac{1}{4\pi}\partial_{y_k}\eta\nabla_y \int_{\mathbb{R}^3} \frac{\partial_{y_i} G(w,l;\,z+y,s)}{|z|}dz \\ -\frac{1}{4\pi}\left(\sum_{j=1}^{3}\partial_{y_j}\eta\partial_{y_j}\int_{\mathbb{R}^3}\frac{\partial_{y_i} G(w,l;\,z+y,s)}{|z|}dz\right)e_k. \tag{28}$$

Substituting into the defining formula for M_2 (page 382) and integrating by parts, we obtain

$$M_2 = \int_{Q_r(x,t)} \partial_{y_i} u(y,s)\big(\Delta\vec{Z} + \partial_s\vec{Z}\big)(y,s)\,dy\,ds \\ + \frac{1}{4\pi}\int_{Q_r(x,t)} u(y,s)(\Delta+\partial_s)\left(\partial_{y_i}\partial_{y_k}\eta\nabla_y\int_{\mathbb{R}^3}\frac{G(w,l;\,z+y,s)}{|z|}\right)dz\,dy\,ds \\ + \frac{1}{4\pi}\int_{Q_r(x,t)} u(y,s)(\Delta+\partial_s)\left(\sum_{j=1}^{3}\partial_{y_i}(\partial_{y_j}\eta)\partial_{y_j}\int_{\mathbb{R}^3}\frac{G(w,l;\,z+y,s)}{|z|}dz\right)dy\,ds.$$

As in the estimate of J_2 (see previous page), we have

$$|\Delta\vec{Z} + \partial_s\vec{Z}| \le c/r^5,$$

where, as before, we have used the fact that the parabolic distance between (w,l) and (y,s) is a least $r/4$.

For the rest of the terms, using (23) and the same argument as in the estimate of K_1, we deduce

$$|M_2| \le \frac{c}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla u(y,s)|\,dy\,ds \\ + \frac{c}{r^6}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y,s)|\,dy\,ds. \tag{29}$$

Finally we bound M_3. Using integration by parts on the y variable and using the assumption that div $b=0$, we can write

$$M_3 = \int_{Q_r(x,t)} ub\nabla_y\partial_{w_i}\Phi_k(y,s)\,dy\,ds = -\int_{Q_r(x,t)}\partial_{w_i}\Phi_k(y,s)b\nabla_y u\,dy\,ds \\ = -\int_{Q_r(x,t)}(\partial_{w_i}(\eta E_k + \vec{Z}))b\nabla u\,dy\,ds.$$

The last step follows from the counterpart of (18), with (w, l) replacing (x, t) there. From well known properties of the Stokes system,

$$|\partial_{w_i} E_k(w, l; y, s)| \le C K_1(w, l; y, s).$$

This, together with (28) and (23), yields

$$|\partial_{w_i}(\eta E_k + \vec{Z})| \le C K_1(w, l; y, s).$$

Therefore

$$|M_3| \le C \int_{Q_r(x,t)} K_1(w, l; y, s)|b\nabla u|\, dy\, ds. \tag{30}$$

Recall from Remark 3.2 that the integral on the right-hand side of (30) is a L^1_{loc} function of space-time.

Combining (26)–(30), we have, for $(w, l) \in Q_{r/4}(x, t)$, a.e.

$$\begin{aligned}\left|\int_{\mathbb{R}^3} u(w, l)\partial_{w_i}\rho(w)\, dw\right| &\le \frac{c}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla u(y, s)|\, dy\, ds + \frac{c}{r^6}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y, s)|\, dy\, ds \\ &\quad + c\int_{\mathbb{R}^3}\int_{Q_r(x,t)} K_1(w, l; y, s)|b|\, |\nabla u|\, dy\, ds\, \rho(w)\, dw.\end{aligned}$$

Since ρ is arbitrary and ∇u and M_3 are L^1_{loc}, we deduce, for $(w, l) \in Q_{r/4}(x, t)$,

$$\begin{aligned}|\partial_{w_i} u(w, l)| &\le \frac{c}{r^5}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |\nabla u(y, s)|\, dy\, ds + \frac{c}{r^6}\int_{Q_r(x,t)-Q_{r/2}(x,t)} |u(y, s)|\, dy\, ds \\ &\quad + \int_{Q_r(x,t)} K_1(w, l; y, s)|b||\nabla u|\, dy\, ds.\end{aligned}$$

This proves the lemma. □

Proof of Theorem 1.7.

(a) *Proof of (7), the bound on $u(x, t)$.* The idea is to iterate (16) in a special manner, since simple iteration will double the domain of integration in each step and will not yield a local formula. The key is to cut in half the size of the cube in (16) after each iteration. Here are the details.

To simplify the presentation, we will use capital letters to denote points in space-time. Thus $X = (x, t)$, $Y = (y, s)$, $Z = (z, \tau)$, etc. From (16), we have

$$|u(X)| \le \lambda \frac{1}{r^5}\int_{Q_r(X)} |u(Y)|\, dY + \lambda \int_{Q_r(X)} K_1(X; Y)|b(Y)||u(Y)|\, dY. \tag{3.19}$$

For $Y \in Q_r(X)$, we apply the representation formula for cubes of half the previous size to get

$$|u(Y)| \le \lambda \frac{2^5}{r^5} \int_{Q_{r/2}(Y)} |u(Z)|\, dZ + \lambda \int_{Q_{r/2}(Y)} K_1(Y; Z)|b(Z)|\,|u(Z)|\, dZ.$$

Combining the two preceding inequalities, we obtain

$$\begin{aligned}|u(X)| \le{}& \\ &\lambda \frac{1}{r^5} \int_{Q_r(X)} |u(Y)|\, dY + \lambda \int_{Q_r(X)} K_1(X; Y)|b(Y)|\, \lambda \frac{2^5}{r^5} \int_{Q_{r/2}(Y)} |u(Z)|\, dZ\, dY \\ &+ \lambda \int_{Q_r(X)} K_1(X; Y)|b(Y)| \lambda \int_{Q_{r/2}(Y)} K_1(Y; Z)|b(Z)|\,|u(Z)|\, dZ\, dY.\end{aligned}$$

Notice that $Q_{r/2}(Y) \subset Q_{3r/2}(X)$ when $Y \in Q_r(X)$. Thus we get

$$\begin{aligned}|u(X)| \le{}& \lambda \frac{1}{r^5} \int_{Q_r(X)} |u(Y)|\, dY + \lambda^2 \frac{2^5}{r^5} \|u\|_{L^1(Q_{2r}(X))} \int_{Q_r(X)} K_1(X; Y)|b(Y)|\, dY \\ &+ \lambda^2 \int_{Q_{3r/2}(X)} \int_{Q_r(X)} K_1(X; Y)\,|b(Y)|\,K_1(Y; Z)\, dY\, |b(Z)|\,|u(Z)|\, dZ.\end{aligned}$$

Applying Lemma 3.1, we deduce

$$\begin{aligned}|u(X)| \le{}& \lambda \frac{1}{r^5} \|u\|_{L^1(Q_{2r}(X))} + \lambda^2 \frac{2^5}{r^5} \|u\|_{L^1(Q_{2r}(X))} B(b) \\ &+ \lambda^2 c B(b) \int_{Q_{3r/2}(X)} K_1(X; Z)|b(Z)|\,|u(Z)|\, dZ,\end{aligned}$$

where $B(b) = B(b, t - (4r)^2, t)$; see (6).

For $u(Z)$, we use the representation formula in the cube $Q_{r/4}(Z)$:

$$|u(Z)| \le \lambda \frac{4^5}{r^5} \int_{Q_{r/4}(Z)} |u(W)|\, dW + \lambda \int_{Q_{r/4}(W)} K_1(Z; W)|b(W)|\,|u(W)|\, dW.$$

Note that for $Z \in Q_{3r/2}(X)$, we have $Q_{r/4}(Z) \subset Q_{7r/4}(X) \subset Q_{2r}(X)$. Hence

$$\begin{aligned}|u(X)| \le{}& \lambda \frac{1}{r^5} \|u\|_{L^1(Q_{2r}(X))} + \lambda^2 \frac{2^5}{r^5} \|u\|_{L^1(Q_{2r}(X))} B(b) \\ &+ \lambda^2 \frac{4^5}{r^5} c B(b) \int_{Q_{3r/2}(X)} K_1(X; Z)|b(Z)|\, dZ \\ &+ \lambda^3 c B(b) \int_{Q_{3r/2}(X)} K_1(X; Z)|b(Z)| \int_{Q_{r/4}(Z)} K_1(Z; W)|b(W)|\,|u(W)|\, dW\, dZ.\end{aligned}$$

Exchanging the integrals and using Lemma 3.1 again, we deduce that

$$|u(X)| \le \lambda \frac{1}{r^5}\|u\|_{L^1(Q_{2r}(X))} + \lambda^2 \frac{2^5}{r^5}\|u\|_{L^1(Q_{2r}(X))} B(b) + \lambda^2 \frac{4^5}{r^5} c B(b)^2 \\ + \lambda^3 c^2 B(b)^2 \int_{Q_{7r/4}(X)} K_1(X;W)|b(W)||u(W)|\,dW\,dZ.$$

We iterate this process, halving the size of cube each time. By induction, it is clear that for some $C, c_1 > 0$,

$$|u(X)| \le \frac{C}{r^5}\|u\|_{L^1(Q_{2r}(X))} \sum_{j=1}^{\infty} \big(2^5 c_1 B(b)\big)^j.$$

When $2^5 c_1 B(b) < 1$, this series converges to yield the mean value inequality for u. Since b is in the Kato class of Definition 1.6, we know that

$$2^5 c_1 B(b) = 2^5 c_1 B(b, t-(4r)^2, t) < 1$$

when r is sufficiently small. This proves the bound on u.

(b) *Proof of (8), the gradient bound.* Since $u + c$ is also a solution of (2) for any constant c, we will assume that $\bar{u}_{Q_{2r}}$, the average of u in $Q_{2r}(x,t)$, is 0.

The idea is to iterate (17) in the manner just described. From (17), using the same notation as in part (a), we have

$$|\nabla u(X)| \le m(X,r) + \lambda \int_{Q_r(X)} K_1(X;Y)|b(Y)||\nabla u(Y)|\,dY, \tag{31}$$

where

$$m(X,r) \equiv \frac{\lambda}{r^5}\int_{Q_r(X)} |\nabla u(Y)|\,dy\,ds + \frac{\lambda}{r^6}\int_{Q_r(X)} |u(Y)|\,dY.$$

Recall that both sides of (31) are L^1_{loc} functions and hence finite almost everywhere (see Remark 3.2).

Applying (31) to $\nabla u(Y)$ and $Q_{r/2}(Y)$, we obtain

$$|\nabla u(Y)| \le m(Y,r/2) + \lambda \int_{Q_{r/2}(Y)} K_1(Y;Z)|b(Z)||\nabla u(Z)|\,dZ.$$

For $Y \in Q_r(X)$, it is clear that there exists a $\mu > 0$, independent of r such that $m(Y,r/2) \le \mu m(X,2r)$. Hence

$$|\nabla u(Y)| \le \mu m(X,2r) + \lambda \int_{Q_{r/2}(Y)} K_1(Y;Z)|b(Z)||\nabla u(Z)|\,dZ. \tag{32}$$

Substituting (32) in (31) we have

$$|\nabla u(X)| \le m(X,r) + \mu\, m(X,2r)\lambda \int_{Q_r(X)} K_1(X;Y)|b(Y)|\,dY \\ + \lambda \int_{Q_r(X)} K_1(X;Y)|b(Y)|\lambda \int_{Q_{r/2}(Y)} K_1(Y;Z)|b(Z)|\,|\nabla u(Z)|\,dZ\,dY.$$

Therefore

$$|\nabla u(X)| \le m(X,r) + \mu\lambda\, m(X,2r)B(b) \\ + \lambda^2 \int_{Q_{3r/2}(X)} \int_{Q_{r/2}(Z)} K_1(X;Y)|b(Y)|K_1(Y;Z)\,|b(Z)|\,|\nabla u(Z)|\,dZ\,dY.$$

Lemma 3.1 then implies

$$|\nabla u(X)| \le m(X,r) + \mu\lambda\, m(X,2r)B(b) \\ + c\lambda^2 B(b) \int_{Q_{3r/2}(X)} K_1(X,Z)|b(Z)|\,|\nabla u(Z)|\,dZ.$$

Now, using (31) on $|\nabla u(Z)|$ and the cube $Q_{r/4}(Z)$ and repeating the above argument, halving the size of the cube at each step, we finally reach

$$|\nabla u(X)| \le C\, m(X,2r) \textstyle\sum_{k=1}^{\infty} \lambda^{k+1} \mu^{k+1} B(b)^k.$$

As before, this implies the desired gradient bound when $B(b)$ is small.

The last statement of Theorem 1.7 is a simple consequence of the gradient estimate and the Poincaré inequality. □

We end the section by showing that if the sum of the entries of the drift term b is zero, (2) on the torus has bounded solutions for many initial values, regardless of the singularity of b. To state the result rigorously, we will assume that b is bounded. However all coefficients are independent of the bounds of b.

Proposition 3.4. *Given bounded vector fields $b = (b_1(x,t), \dots, b_n(x,t))$, consider the linearized Navier–Stokes equation with periodic boundary condition on a torus* (*that is, with periodic boundary conditions*):

$$\begin{cases} \Delta u(x,t) - b(x,t)\nabla u(x,t) + \nabla P(x,t) - \partial_t u(x,t) = 0, & (x,t) \in D \times \mathbb{R}, \\ \operatorname{div} u = 0,\ \operatorname{div} b = 0,\ b(\cdot\,,t) \in L^\infty \\ u(x,0) = u_0(x). \end{cases} \tag{33}$$

Here $D = [0, 2\pi]^n$. The functions $b(\cdot\,,t)$, $u_0(\cdot)$ and $u(\cdot\,,t)$ have period 2π.

Suppose that $\sum_{j=1}^n b_j(x,t) = \lambda$ is a constant, and that u_0 is any finite linear combination of

$$\int_D e^{ik\sum_{j=1}^n (x_j - y_j)} f(y)\,dy,$$

where k is a positive integer and f is a bounded, divergence-free vector field with period 2π. Then there exists a constant c independent of b and λ such that

$$|u(x,t)| \le ce^{-t} C\big(\|u_0\|_\infty\big).$$

Proof. Under the assumption that b is bounded, the existence of solutions to (33) follows from the standard theory. Let $E = E(x,t;\,y,s)$ be the fundamental solution of (33). The existence of E is also standard.

First we just assume that u_0 has one term:

$$u_0(x) = \int_D e^{ik\sum_{j=1}^n (x_j - y_j)} f(y)\,dy.$$

Fixing (x,t), consider

$$I(s) \equiv \int_D E(x,t;\,y,s) u_0(y)\,dy. \tag{34}$$

Because the rows of E satisfy the conjugate equation of (33), we have

$$\begin{aligned} I'(s) &= \int_D \frac{d}{ds} E(x,t;\,y,s) u_0(y)\,dy \\ &= \int_D \big(-\Delta_y E(x,t;\,y,s) - b(y,s)\nabla_y E(x,t;\,y,s) + \nabla Y(y,s)\big) u_0(y)\,dy. \end{aligned}$$

Here

$$\nabla Y = \begin{pmatrix} \partial_1 P_1 & \cdots & \partial_n P_1 \\ \vdots & \ddots & \vdots \\ \partial_1 P_1 & \ldots & \partial_n P_n \end{pmatrix},$$

where $P_1, \ldots, P_n$ are scalar functions. Integrating by parts and using the fact that b is divergence-free, we deduce that

$$I'(s) = -\int_D E(x,t;\,y,s)\Delta u_0(y)\,dy + \int_D E(x,t;\,y,s) \sum_{j=1}^n b_j \partial_{y_j} u_0(y)\,dy.$$

Noticing that $\Delta u_0 = -nk^2 u_0$ and $\partial_{y_j} u_0(y) = iku_0$, we have

$$I'(s) = nk^2 \int_D E(x,t;\,y,s) u_0(y)\,dy + ik \int_D \bigg(\sum_{j=1}^n b_j\bigg) E(x,t;\,y,s) u_0(y)\,dy.$$

By our assumption that $\sum_{j=1}^n b_j = \lambda$, this shows that

$$I'(s) = (nk^2 + ik\lambda) I(s).$$

Hence

$$I(s) = e^{(nk^2 + ik\lambda)s} I(0).$$

From (34) and the fact that E is the fundamental solution to (33), we have $I(0) = u(x,t)$ and $I(t) = u_0(x)$. This shows

$$u(x,t) = e^{-(nk^2+ik\lambda)t} u_0(x). \tag{35}$$

Now let S be a set of finite positive integers and

$$u_0(x) = \sum_{l \in S} c_l \int_D e^{ik_l \sum_{j=1}^n (x_j - y_j)} f_l(y)\, dy.$$

Here k_l is a positive integer and f_l is a bounded, divergence-free vector field with period 2π. By (35) one has

$$u(x,t) = \sum_{l \in S} e^{-(nk_l^2+ik_l\lambda)t} c_l \int_D e^{ik_l \sum_{j=1}^n (x_j - y_j)} f_l(y)\, dy. \qquad \square$$

4. A regularity condition for Navier–Stokes equations not involving absolute values

In this section we introduce another sufficient condition on the velocity for boundedness of weak solutions of three-dimensional Navier–Stokes equations. The novelty is that no absolute value of u is involved. This is useful since it allows more cancellation effects to be taken into account. Throughout the years, various conditions on u that imply regularity have been proposed. One is the Prodi–Serrin condition, which requires that $u \in L^{p,q}$ with $3/p + 2/q \le 1$ for some $3 < p \le \infty$ and $q \ge 2$. See, for example, [Serrin 1962; Struwe 1988]. Recently it was shown in [Iskauriaza et al. 2003] that the condition $p = 3$ and $q = \infty$ also implies regularity. In another development, the Prodi–Serrin condition was improved in [Montgomery-Smith 2005] by a log factor, i.e., by requiring

$$\int_0^T \frac{\|u(\cdot,t)\|_p^q}{1 + \log^+ \|u(\cdot,t)\|_p} dt < \infty,$$

where $3/p + 2/q = 1$ and $3 < p < \infty$, $2 < q < \infty$.

Most recently in [Zhang 2004a], a form boundedness condition on velocity was introduced, which will imply the boundedness of weak solutions. The form boundedness condition, with its root in the perturbation theory of elliptic operators and mathematical physics, seems to be different from all previous conditions. It seems to be one of the most general conditions under the available tools (this has been well documented in the theory of linear elliptic equations; see [Simon 1982], for instance). Moreover, it contains the Prodi–Serrin condition except when p or q are infinite. It also includes suitable Morrey–Campanato type spaces. However we are not sure this condition contains the one in [Montgomery-Smith 2005].

More precisely:

Theorem 4.1 [Zhang 2004a, Theorem 5.1]. *Let u be a Leray–Hopf solution to the 3-dimensional Navier–Stokes equation in $\mathbb{R}^3 \times (0, \infty)$.*

$$\begin{cases} \Delta u(x,t) - u(x,t)\nabla u(x,t) + \nabla P(x,t) - \partial_t u(x,t) = 0, & (x,t) \in \Omega \subset \mathbb{R}^3 \times \mathbb{R}, \\ \operatorname{div} u = 0. \end{cases}$$

Suppose that for every $(x_0, t_0) \in \mathbb{R}^3 \times (0, \infty)$, there exists a cube

$$Q_r = B(x_0, r) \times [t_0 - r^2, t_0]$$

such that u satisfies the form bounded condition

$$\int_{Q_r} |u|^2 \phi^2 \, dy \, ds \leq \frac{1}{24} \left(\int_{Q_r} |\nabla \phi|^2 \, dy \, ds + \sup_{s \in [t_0 - r^2, t_0]} \int_{B(x_0, r)} \phi^2(y, s) \, dy \right) + B(\|\phi\|_{L^2(Q_r)}). \tag{36}$$

Here ϕ is any smooth function vanishing on the parabolic side of Q_r and $B = B(t)$ is any given locally bounded function of $t \in \mathbb{R}^1$. Then u is a classical solution when $t > 0$.

In this section we are able to extend the form boundedness condition further. The main improvement is that our new condition in u below, (37), does not involve the absolute value of u. This differs significantly from known conditions on u so far where $|u|$ is always present. By a simple integration by parts argument, it is clear that condition (37) is more general than condition (36).

Theorem 4.2. *Let u be a Leray–Hopf solution to the three-dimensional Navier–Stokes equation in $\mathbb{R}^3 \times (0, \infty)$. Suppose for every $(x_0, t_0) \in \mathbb{R}^3 \times (0, \infty)$, there exists a cube $Q_r = B(x_0, r) \times [t_0 - r^2, t_0]$ such that u satisfies the form boundedness condition: for a given $\delta > 0$,*

$$\int_{Q_r} \phi \nabla u \cdot \phi \, dy \, ds \leq \frac{1-\delta}{2} \left(\int_{Q_r} |\nabla \phi|^2 \, dy \, ds + \tfrac{1}{2} \sup_{s \in [t_0 - r^2, t_0]} \int_{B(x_0, r)} \phi^2(y, s) \, dy \right) + B\big(\|\phi\|_{L^2(Q_r)}\big). \tag{37}$$

Here ϕ is any smooth vector field vanishing on parabolic the side of Q_r and $B = B(t)$ is any given locally bounded function of $t \in \mathbb{R}^1$. Then u is a classical solution when $t > 0$.

Remark 4.3. Equation (37) is actually a condition on the strain tensor $\nabla u + (\nabla u)^T$. Theorem 4.2 immediately implies that weak solutions to the three-dimensional Navier–Stokes equations are locally bounded in any open subset of the region where the eigenvalues of the strain tensor are bounded from above by a mild function.

Proof of Theorem 4.2. Let t_0 be the first moment of singularity formation. We will reach a contradiction. It is clear that we only need to prove that u is bounded in $Q_{r/8} = Q_{r/8}(x_0, t_0)$ for some $r > 0$. In fact the number 8 is not essential. Any number greater than 1 would work.

Consider the equation for vorticity $w = \nabla \times u$. It is well known that, in the interior of Q_r, w is a classical solution to the parabolic system with singular coefficients

$$\Delta w - u\nabla w + w\nabla u - w_t = 0. \tag{38}$$

Let $\psi = \psi(y, s)$ be the refined cutoff function defined right after (10) such that $\psi = 1$ in $Q_{r/2}$, $\psi = 0$ in Q_r^c and such that $0 \le \psi \le 1$, $|\nabla \psi| \le C/r$ and $|\psi_t| \le C/r^2$. We can use $w\psi^2$ as a test function on (38) to obtain

$$\begin{aligned}(39)\quad \int_{Q_r} |\nabla(w\psi)|^2\,dy\,ds &+ \frac{1}{2}\int_{B(x_0,r)} |w\psi|^2(y, t_0)\,dy \\ &\le \frac{C}{r^2}\int_{Q_r} |w|^2\,dy\,ds - \int_{Q_r} u\nabla w\cdot w\psi^2\,dy\,ds + \int_{Q_r} w\nabla u\cdot w\psi^2\,dy\,ds \\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

The term I_1 is already in good shape. Next, using integration by parts and the divergence-free condition on u, we have

$$I_2 = \frac{1}{2}\int_{Q_r} u\cdot\nabla\psi\ \psi|w|^2\,dy\,ds.$$

Since $\nabla u \in L^2_{\text{loc}}(\mathbb{R}^3 \times [0, \infty))$ and $\|u(\cdot, t)\|_{\mathbb{R}^3}$ is nonincreasing in time, it is easy to prove by Sobolev imbedding and Hölder's inequality that $u|_{Q_r}$ satisfies the form boundedness condition (3). In fact this is [Zhang 2004b, Corollary 2]. Hence we can bound I_2 in exactly the same way as T_2 in (13) with b being chosen as u here. Following the argument between (13) and (14), we obtain, for any given $\delta > 0$,

$$I_2 \le \frac{\delta}{4}\int_{Q_r} |\nabla(\psi w)|^2\,dy\,ds + c_\delta e^{(c_1/r)^2}\int_{Q_r} w^2\,dy\,ds. \tag{40}$$

Note that in (14), δ was chosen as 2. However, a closer look at the proof shows that (40) is true.

Next we estimate I_3. From condition (37) we have

$$\begin{aligned}(41)\quad I_3 \le \frac{1-\delta}{2}\Bigg(\int_{Q_r} |\nabla(w\psi)|^2\,dy\,ds + \frac{1}{2}\sup_{s\in[t_0-r^2,t_0]}\int_{B(x_0,r)} (w\psi)^2(y, s)\,dy\Bigg)& \\ + B\big(\|w\psi\|_{L^2(Q_r)}\big).&\end{aligned}$$

Substituting (40) and (41) in (39) we obtain

$$\begin{aligned}\int_{Q_r} |\nabla(w\psi)|^2\,dy\,ds &+ \frac{1}{2}\int_{B(x_0,r)} |w\psi|^2(y,t_0)\,dy\\ &\le \frac{\delta}{4}\int_{Q_r} |\nabla(\psi w)|^2\,dy\,ds + c_\delta e^{(c_1/r)^2}\int_{Q_r} w^2\,dy\,ds\\ &\quad + \frac{1-\delta}{2}\bigg(\int_{Q_r} |\nabla(w\psi)|^2\,dy\,ds + \tfrac{1}{2}\sup_{s\in[t_0-r^2,t_0]}\int_{B(x_0,r)} (w\psi)^2(y,s)\,dy\bigg)\\ &\qquad + CB\big(\|w\|_{L^2(Q_r)}\big).\end{aligned}$$

Repeating the process above, but restricting the integrals to $Q_r \cap \{(y,s) \mid s < t\}$ with $t < t_0$, we obtain, for any $t \in [t_0 - r^2, t_0]$,

$$\begin{aligned}\frac{1}{2}\int_{B(x_0,r)} &|w\psi|^2(y,t)\,dy\\ &\le \frac{\delta}{4}\int_{Q_r} |\nabla(\psi w)|^2\,dy\,ds + c_\delta e^{(c_1/r)^2}\int_{Q_r} w^2\,dy\,ds\\ &\quad + \frac{1-\delta}{2}\bigg(\int_{Q_r} |\nabla(w\psi)|^2\,dy\,ds + \tfrac{1}{2}\sup_{s\in[t_0-r^2,t_0]}\int_{B(x_0,r)} (w\psi)^2(y,s)\,dy\bigg)\\ &\qquad + CB\big(\|w\|_{L^2(Q_r)}\big).\end{aligned}$$

Combining the last two estimates, we deduce

$$\begin{aligned}\int_{Q_r} |\nabla(w\psi)|^2\,dy\,ds &+ \sup_{t_0-r^2\le s\le t_0}\int_{B(x_0,r)} |w\psi|^2(y,s)\,dy\\ &\le \frac{C_1}{r^2}\|w\|_{L^2(Q_r)} + B_1\big(\|w\|_{L^2(Q_r)}\big).\end{aligned}\tag{42}$$

Here B_1 is a locally bounded, one-variable function.

Using standard results, we know that (42) implies that u is regular. Here is the proof.

From (42), it is clear that $\int_{Q_r} |\operatorname{curl}\,(w\psi)|^2\,dy\,ds \le C$. Hence, since $\psi = 1$ in $Q_{r/2}$,

$$\int_{Q_{r/2}} |\Delta u|^2\,dy\,ds \le C.$$

Let $\eta = \eta(y)$ be a cutoff function such that $\eta = 1$ in $B(x_0, r/4)$ and $\eta = 0$ in $B(x_0, r/2)^c$. Then for each $s \in [t_0 - (r/4)^2, t_0]$, we have, in the weak sense,

$$\Delta(u\eta) = \eta\Delta u + 2\nabla u\nabla\eta + u\Delta\eta \equiv f,$$

in $Q_{r/2}$. Using standard elliptic estimates and the fact that $u\eta$ vanishes on the boundary, we get

$$\|D^2u(\cdot,s)\|_{L^2(B(x_0,r/4))} \le C\|f(\cdot,s)\|_{L^2(B(x_0,r/2))}.$$

This shows that

$$\|D^2u\|_{L^2(Q_{r/4})} \le C\|\Delta u\|_{L^2(Q_{r/2})} + C\|u\|_{L^2(Q_{r/2})}.$$

By the Sobolev imbedding,

$$\nabla u \in L^{6,2}(Q_{r/4}). \tag{43}$$

Next, from [Temam 1984, p. 316, (1.22)],

$$\|u(\cdot,s)\eta\|_{W^{1,2}} \le C\big(\|u(\cdot,s)\eta\|_{L^2} + \|\operatorname{div}(u\eta)(\cdot,s)\|_{L^2} + \|\operatorname{curl}(u(\cdot,s)\eta)\|_{L^2}\big).$$

Here all norms are over the ball $B(x_0, r/2)$. Therefore

$$\begin{aligned}\|u\eta(\cdot,s)\|_{W^{1,2}} \\ \le C\big(\|u(\cdot,s)\eta\|_{L^2} + \|u\nabla\eta(\cdot,s)\|_{L^2} + \|\omega\eta(\cdot,s)\|_{L^2} + \||u(\cdot,s)||\nabla\eta|\|_{L^2}\big).\end{aligned}$$

It follows that

$$\|u(\cdot,s)\|_{W^{1,2}(B(x_0,r/4))} \le C.$$

From the Sobolev imbedding we know that

$$u \in L^{6,\infty}(Q_{r/4}). \tag{44}$$

We treat u and ∇u as coefficients in the vortex equation (38). By (43) and (44), the standard parabolic theory (see [Lieberman 1996], for instance) shows that ω is bounded and Hölder-continuous in $Q_{r/8}$. Here the bound depends only on the L^2 norm of ω in Q_r and r. This is so because of the relation $3/6 + 2/\infty < 1$ for the norm of u and $3/6 + 2/2 < 2$ for the norm of ∇u. Now a standard bootstrapping argument shows that u is smooth. □

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