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We compute the group of Morita self-equivalences (the Picard group) of a Poisson structure on an orientable surface, under the assumption that the degeneracies of the Poisson tensor are linear. The answer involves mapping class groups of surfaces, i.e., groups of isotopy classes of diffeomorphisms. We also show that the Picard group of these structures coincides with the group of outer Poisson automorphisms.

1. Introduction

There are many similarities between Poisson geometry and the theory of associative algebras (see [Cannas da Silva and Weinstein 1999], for example). Based on the notion of Morita equivalence for Poisson manifolds introduced in [Xu 1991a], Bursztyn and Weinstein [2004] have recently defined the *Picard group* $\mathrm{Pic}(P)$ of an integrable Poisson manifold P as the group of all Morita equivalences between P and itself (see Section 2.3 for a definition). The group operation is the relative tensor product $\otimes_P$ of bimodules [Xu 1991b], and the identity bimodule of (an integrable) Poisson manifold P is its source simply connected symplectic groupoid $\Gamma(P)$. The Picard group contains the group of outer Poisson automorphisms of P in a natural way, but it can in principle be strictly larger (such is the case for an open symplectic surface: see Section 4).

Here we present a complete computation of the Picard group for a certain class of Poisson structures on compact connected oriented surfaces. We consider Poisson structures (called *topologically stable structures*, or TSS for short) which are nondegenerate almost everywhere on the surface, except that they have linear degeneracies on a finite set of simple closed curves. Although TSS's are sufficiently generic (the set of these structures is open and dense in the vector space of all Poisson structures on a given surface), in many ways they resemble the symplectic structures. In [Radko 2002], the first author has obtained a complete description

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of the moduli space of isomorphism classes of TSS's by giving a complete set of explicit invariants. A complete set of criteria for Morita equivalence of two TSS's was found in [Bursztyn and Weinstein 2004]. The problem of computing the Picard group of a TSS was posed in that same paper.

Our main result is the following. Let π be a TSS on a surface P. Let Z be the zero set of this structure, consisting of a disjoint union of a finite number of simple closed curves $T_1, T_2, \dots T_n$ on P. A modular vector field [Weinstein 1997] of (P, π) restricts to a canonical nowhere zero vector field ξ along the zero set Z. Consider the group $D(\pi)$ of all diffeomorphisms $\phi : P \to P$ that

– preserve the zero set ($\phi(Z) = Z$), and

– preserve the restriction of a modular vector field ($\phi_* \xi = \xi$).

Let $D_0(\pi)$ be the subgroup of $D(\pi)$ consisting of diffeomorphisms fixing a neighborhood of Z pointwise. (This actually implies that $D_0(\pi)$ is precisely the subgroup of diffeomorphisms which preserve the leaf space of (P, π) pointwise). Then

$$\text{Pic}(P, \pi) \cong D(\pi)/\text{isotopy by elements of } D_0(\pi). \tag{1–1}$$

In other words, the Picard group can be identified with the set of equivalence classes of diffeomorphisms in $D(\pi)$, two diffeomorphisms ϕ_1 and ϕ_2 being equivalent if $\phi_1 \circ \phi_2^{-1}$ can be connected to the identity by a continuous path of diffeomorphisms lying in $D_0(\pi)$.

Our description of the Picard group of a TSS gives also an explicit formula for it, involving the mapping class groups. If $Z = \bigsqcup_{i \in I} T_i$ is the zero set of π, and $P \setminus Z = \bigsqcup_{j \in J} L_j$ is the decomposition of the complement of the zero set into the 2-dimensional symplectic leaves, then

$$\text{Pic}(P, \pi) \cong \Bigl(\prod_{i \in I} \mathbb{T} \times \prod_{j \in J} \mathcal{M}(L_j) \Bigr) \rtimes G. \tag{1–2}$$

Here, for each zero curve, $\mathbb{T}$ is the 1-torus of translations by the flow of the restriction of a modular vector field to this curve, while $\mathcal{M}(S)$ is the mapping class group of an open surface S — that is, letting $\sim$ stand for isotopy,

$$\mathcal{M}(S) = \{\text{diffeomorphisms of } S \text{ fixing pointwise a neighborhood of infinity}\}/\sim .$$

Mapping class groups of surfaces are well understood, and can be explicitly described in terms of generators and relations. However, their appearance in relation to Picard groups of TSS's was unexpected and surprising.

The group G in (1–2) is the (discrete) group of automorphisms of a labeled graph encoding the Morita equivalence class of (P, π). Automorphisms in G permute the leaves, and thus act naturally on the set I of zero-dimensional leaves and the set J of 2-dimensional leaves.

We finish the introduction with an outline of the proof of the main result. In order to prove the isomorphism (1–1), one needs to be able to prove that

- every class of an element of $D(\pi)$ gives rise to a Morita self-equivalence bimodule, and
- every Morita self-equivalence bimodule arises in this way.

The first part is relatively easy: if $\phi \in D(\pi)$, then a Moser-type argument (as in [Radko 2002]) implies that ϕ is isotopic to a Poisson automorphism ϕ_0. Then one simply assigns to the isotopy class of ϕ the bimodule $\Gamma_{\phi_0}(P)$ associated to this Poisson automorphism (see Section 2.3.1 for a definition). This results in a map $j : (D(\pi)/\text{isotopy}) \to \mathrm{Pic}(P, \pi)$.

The second part is more difficult. We need to show that the map j is onto. Clearly, it is sufficient to show that if X is a Morita self-equivalence bimodule over P, then for some bimodule X' in the image of j, the relative tensor product $X \otimes_P X'$ is isomorphic to the identity bimodule. In particular, it is necessary to be able to determine when a Morita equivalence bimodule is trivial (i.e., isomorphic to the identity bimodule).

In algebra, one meets a very simple criterion of triviality: an (A,A)-bimodule X is trivial if and only if it contains a vector v_0 which is bicyclic (i.e., $X = Av_0A$) and central (i.e., $av_0 = v_0a$ for all $a \in A$). In that case, the map $a \mapsto av_0$ gives an isomorphism between the identity bimodule A and X. In Poisson geometry the analogous notion is that of an *identity bisection* (Definition 1). The existence of an identity bisection gives a criterion for a bimodule over an integrable Poisson manifold to be trivial.

There are three obstructions to existence of an identity bisection for a Morita self-equivalence bimodule X over P:

O1 X may induce a nontrivial automorphism of the leaf space (see Section 2.3.2).

O2 The restriction of X to the open symplectic manifold $P \setminus Z$ may be nontrivial in the Picard group of $P \setminus Z$.

O3 Even if the restriction of X to $P \setminus Z$ is trivial and thus has an identity bisection defined on $P \setminus Z$, this bisection may fail to extend to all of P.

The first obstruction vanishes if and only if X is a *static* (P,P)-bimodule (i.e., it induces the identity map on the leaf space of P). One can easily check that for any bimodule $X \in \mathrm{Pic}(P, \pi)$, there is a bimodule X' in the image of j such that $X \otimes_P X'$ is static. Thus we can restrict our consideration to static bimodules.

If X is static, then we show that the restriction of X to a cylindrical neighborhood of a zero curve automatically has exactly one identity bisection on this neighborhood. This gives a *local identity bisection* (see Definition 2) of X in a neighborhood of the zero set Z. The existence and uniqueness of this local identity bisection has a lot to do with the presence of a very special local symmetry of the Poisson structure, given by the flow of the modular vector field [Weinstein 1997], and the fact that modular vector fields are "preserved" by Morita equivalences [Ginzburg and Lu 1992].

Fix a 2-dimensional symplectic leaf $L \subset P \setminus Z$. By a result of Bursztyn and Weinstein [Bursztyn and Weinstein 2004], $\mathrm{Pic}(L) \cong \mathrm{Out}(\pi_1(L))$. However, since any $X \in \mathrm{Pic}(P)$ has a local identity bisection near the zero set, the restriction of X to L must be "trivial near the boundary" of L. Thus it cannot give an arbitrary element in $\mathrm{Out}(\pi_1(L))$; it must give elements that "preserve peripheral structure" of L. (See Section 4).

By the Dehn–Nielsen–Baer theorem (Theorem 5), one can find a symplectomorphism defined on L and trivial near its boundary, which induces any element of $\mathrm{Out}(\pi_1(L))$, preserving peripheral structure. Thus given a bimodule $X \in \mathrm{Pic}(P, \pi)$, with vanishing O1, one can use the Dehn–Nielsen–Baer theorem to find a bimodule X' in the image of j such that $X \otimes_P X'$ has the property that both obstructions O1 and O2 vanish.

Assuming that obstructions O1 and O2 vanish, the last obstruction O3 is related to the behavior of the identity bisection(s) of the restriction of X to $P \setminus Z$ near Z. The issue is whether the identity bisection defined on a leaf $L \subset P \setminus Z$ matches the unique local identity bisection defined in a neighborhood of $\partial L \subset Z$. It turns out that the obstruction is integer-valued in a neighborhood of each zero curve. Moreover, the value of the obstruction associated to the Dehn twist near a zero curve is exactly 1. This implies that given $X \in \mathrm{Pic}(P, \pi)$, with O1 and O2 vanishing, it is possible to find X' associated to a combination of Dehn twists (and thus in the image of j) such that all obstructions O1, O2 and O3 vanish for $X \otimes_P X'$. These Dehn twists are special elements of mapping class groups $\mathcal{M}(L_j)$; see (1–2).

2. Preliminaries on Morita equivalence and Picard groups

2.1. *Symplectic groupoids and modules over Poisson manifolds.*

2.1.1. *Symplectic groupoids.* Let (P, π) be an integrable Poisson manifold and let $\Gamma(P)$ be its (source-connected and simply connected) *symplectic groupoid*. The elements of $\Gamma(P)$ can be thought of as classes of *cotangent paths* up to cotangent homotopy [Crainic and Fernandes 2002; 2003]. (The notion of cotangent homotopy is somewhat technical and is not explicitly needed here. See the first of the references just cited for details.)

A cotangent path is a pair (a, γ), where $a : [0, 1] \to T^*P$ is a path in the cotangent bundle and γ is a path $\gamma : [0, 1] \to P$ on the manifold (called the base path of a) such that

$$\operatorname{pr}(a(t)) = \gamma(t) \quad \text{and} \quad \frac{d\gamma}{dt} = \tilde{\pi}(a(t)).$$

Here $\operatorname{pr} : T^*P \to P$ is the natural projection and $\tilde{\pi} : T^*P \to TP$ is the bundle map associated to the Poisson structure. The symplectic structure on $\Gamma(P)$ is obtained from the natural symplectic structure on its Lie algebroid, T^*P. The source and target maps $s, t : \Gamma(P) \to P$ send the class of a cotangent path a over a base path γ to the beginning and end of γ, respectively.

2.1.2. *The symplectic groupoid of a symplectic manifold.* In the special case of a symplectic manifold S, the bundle map $\tilde{\pi} : T^*S \to TS$ is invertible and therefore a cotangent path is uniquely determined by its base path. In this case, cotangent homotopy is equivalent to the usual homotopy of base paths. In particular, as a groupoid, $\Gamma(S)$ can be identified with the fundamental groupoid of S; in other words, an element in $\Gamma(S)$ can be viewed as a class of a (not necessarily closed) path in S, considered up to a homotopy fixing the endpoints. The natural symplectic structure on $\Gamma(S)$ is such that $\Gamma(S) = (\tilde{S} \times \tilde{S}^{\mathrm{op}})/\pi_1(S)$, where the universal cover $\tilde{S}$ is taken with the pull-back of the original symplectic structure on S, and $\tilde{S}^{\mathrm{op}}$ denotes the same manifold with the negative of the symplectic structure. The *isotropy group* $\Gamma_p(S) = s^{-1}(p) \cap t^{-1}(p)$ of $\Gamma(S)$ at a point $p \in S$ is canonically isomorphic to the fundamental group $\pi_1(S, p)$ of S at p, and is therefore discrete.

2.1.3. *Modules over Poisson manifolds.* If one thinks of Poisson manifolds as semi-classical analogs of associative algebras, a symplectic manifold (X, Ω) should be thought of as an analog of the algebra $\operatorname{End}(V)$, where V is a vector space. The structure of a left (respectively, right) module over an algebra A on a vector space V is given by an algebra (anti) homomorphism from A to $\operatorname{End}(V)$.

The analogous notion in Poisson geometry is that of a *left (right) module* over a Poisson manifold P. This is defined as a complete (anti)-symplectic realization $J : (X, \Omega) \to (P, \pi)$, i.e., an (anti)-Poisson map from a symplectic manifold (X, Ω) to (P, π).

Any symplectic realization $J : (X, \Omega) \to (P, \pi)$ induces a canonical action of the Lie algebroid T^*P of the Poisson manifold on X. In other words, there is a Lie algebra homomorphism $\Phi_X : \Gamma(T^*P) \to \Gamma(TX)$ given by $\Phi_X(\alpha) = \tilde{\Omega}^{-1}(J^*\alpha)$ for $\alpha \in \Omega^1(P) = \Gamma(T^*P)$. If P is integrable and the symplectic realization $J : X \to P$ is complete, this Lie algebroid action integrates to an action of the symplectic groupoid $\Gamma(P)$ on X. In this case, the map J is called the moment map for the groupoid action. Thus, there is a correspondence between the modules over an integrable Poisson manifold and the actions of its symplectic groupoid.

2.2. ***Bimodules and Morita equivalence.*** Most of the definitions and constructions in this section can be found in [Bursztyn and Weinstein 2004]. We refer the reader to that paper for more details on Picard groups of Poisson manifolds and symplectic groupoids.

2.2.1. *Bimodules over Poisson manifolds.* For two Poisson manifolds, a P_1, P_2-*bimodule* is a symplectic manifold X which is a left P_1-module and a right P_2-module, such that the corresponding (left and right) Lie algebroid (or, equivalently, groupoid) actions commute. Thus, we have a pair $P_1 \stackrel{s_X}{\leftarrow} X \stackrel{t_X}{\rightarrow} P_2$ of maps such that

(1) s_X is a complete Poisson map and t_X is a complete anti-Poisson map;

(2) s_X and t_X are surjective submersions with connected and simply-connected fibers;

(3) $\{s_X^* C^\infty(P_1), t_X^* C^\infty(P_2)\} = 0$;

The last condition is equivalent to the requirement the s_X- and t_X- fibers are symplectically orthogonal to each other in X. We will frequently refer to X as a (P_1, P_2)-bimodule, implicitly denoting by s_X and t_X the associated bimodule maps.

Two bimodules are isomorphic if there is a symplectomorphism preserving the bimodule structure.

2.2.2. *The relative tensor product* $\otimes_P$. Let X be a (P_1, P_2)-bimodule and Y a (P_2, P_3)-bimodule. The relative tensor product $X \otimes_{P_2} Y$ (see [Hilsum and Skandalis 1987] for the case of bimodules over groupoids and [Xu 1991b] for the Poisson case) is defined as the orbit space

$$X \otimes_{P_2} Y = X \times_{P_2} Y / \Gamma(P_2),$$

where $X \times_{P_2} Y = \{(x, y) \in X \times Y \ : t_X(x) = s_Y(y)\}$ is the fibered product, and the action of the symplectic groupoid $\Gamma(P_2)$ is given by $g \cdot (x, y) = (xg,\ g^{-1}y)$ for $g \in \Gamma(P_2)$, $(x, y) \in X \times_{P_2} Y$.

If $X \otimes_{P_2} Y$ is a smooth manifold, it is automatically a (P_1, P_3)-bimodule. The symplectic structure of $X \otimes_{P_2} Y$ was first described in [Xu 1991b]. To ensure the smoothness, it is enough to assume that Y is a *left principal bimodule* [Landsman 2001; Bursztyn and Weinstein 2004]; in other words, that the action of $\Gamma(P_1)$ is free and transitive on the fibers of s_Y.

For a Poisson manifold P, the symplectic groupoid $\Gamma(P)$, considered as a (P, P)-bimodule using its source and target as the bimodule maps, plays the role of the identity bimodule: for a right P-module X and a left P-module Y, we have

$$X \otimes_P \Gamma(P) \cong X, \qquad \Gamma(P) \otimes_P Y \cong Y.$$

2.3. ***The Picard group.*** Two (integrable) Poisson manifolds P_1 and P_2 are called *Morita equivalent* if there exists a (P_1, P_2)-bimodule X with the property that it is *invertible*, i.e., there is a (P_2, P_1)-bimodule Y such that $X \otimes_{P_2} Y \cong \Gamma(P_1)$ and $Y \otimes_{P_1} X \cong \Gamma(P_2)$. By a result of Bursztyn and Weinstein [2004], invertibility is equivalent to the requirement that the actions of $\Gamma(P_1)$ and $\Gamma(P_2)$ on the fibers of t_X and s_X are free and transitive. If X is invertible, the inverse is necessarily isomorphic to the *opposite bimodule* X^{op} (obtained by switching the bimodule maps and changing the symplectic structure of X to its negative).

The *Picard group* of a Poisson manifold P is the group $\mathrm{Pic}(P)$ of all isomorphism classes of Morita equivalence (P, P)-bimodules, considered with the operation of the relative tensor product. In analogy with the algebraic case, the Picard group can be considered as a group of generalized automorphisms of the structure.

2.3.1. *Bimodules associated to Poisson isomorphisms.* Each Poisson diffeomorphism $\varphi \in \mathrm{Poiss}(P)$ gives rise to a self-equivalence bimodule $\Gamma_\varphi(P) \in \mathrm{Pic}(P)$. As a symplectic manifold, $\Gamma_\phi(P) = \Gamma(P)$. The bimodule structure of $\Gamma_\varphi(P)$ is given by

$$P \xleftarrow{s} \Gamma(P) \xrightarrow{\varphi \circ t} P.$$

In other words, the bimodule structure of $\Gamma_\phi(P)$ is obtained from that of $\Gamma(P)$ by twisting the target map t by ϕ.

It follows that there is a group homomorphism

$$j : \mathrm{Poiss}(P) \to \mathrm{Pic}(P)$$

given by $j(\varphi) = \Gamma_\varphi(P)$. It turns out [Bursztyn and Weinstein 2004] that the kernel of this homomorphism consists exactly of inner Poisson automorphisms, i.e., automorphisms that can be induced by the action of the group of lagrangian bisections of $\Gamma(P)$.

2.3.2. *The static Picard group.* Let $\mathscr{L}(P)$ be the leaf space of P, regarded as a topological space with the quotient topology. Let $\mathrm{Aut}(\mathscr{L}(P))$ be the group of its homeomorphisms.

A (P, P)-bimodule $P \xleftarrow{s_X} X \xrightarrow{t_X} P$ defines a homeomorphism $\phi_X \in \mathrm{Aut}(\mathscr{L}(P))$ by $\phi_X : L \mapsto t_X \circ s_X^{-1}(L)$. This gives a group homomorphism $h : \mathrm{Pic}(P) \to \mathrm{Aut}(\mathscr{L}(P))$. Its kernel forms a subgroup $\mathrm{StatPic}(P) \subseteq \mathrm{Pic}(P)$, called the *static Picard group* of P, which consists of self-equivalence bimodules fixing the leaf space pointwise. The computations of the Picard group can sometimes be simplified by using the exact sequence

$$1 \to \mathrm{StatPic}(P) \to \mathrm{Pic}(P) \xrightarrow{h} \mathrm{Aut}(\mathscr{L}(P)). \tag{2–1}$$

2.4. ***Identity bisections.*** Let X be a bimodule over a Poisson manifold P. Let $\Gamma(P)$ be a symplectic groupoid, with source and target maps $s, t : \Gamma(P) \to P$.

Definition 1. A map $\varepsilon : P \to X$ is called an *identity bisection* of X if the following conditions are satisfied:

(1) $s_X \circ \varepsilon = t_X \circ \varepsilon = \mathrm{id} \in \mathrm{Diff}(P)$;

(2) for any $\gamma \in \Gamma(P)$ with $s(\gamma) = p$, $t(\gamma) = q$, one has

$$\gamma \cdot \varepsilon(q) = \varepsilon(p) \cdot \gamma,$$

i.e. the actions of the symplectic groupoid commute with taking the bisection.

(3) $\varepsilon(P) \subset X$ is a lagrangian submanifold.

Conditions 1 and 2 are actually equivalent for a bisection $\varepsilon : P \to X$ defined on all of P (see [Coste et al. 1987]). However, we prefer to distinguish between them, since locally these conditions are not equivalent. Indeed, if $U \subset P$ is any subset that intersects every leaf of P, and $\varepsilon : U \to X|_U$ satisfies Condition 2 (i.e., for all $\gamma \in \Gamma(P)$ with $s(\gamma) = p \in U$, $t(\gamma) = q \in U$, one has $\gamma \cdot \varepsilon(q) = \varepsilon(p) \cdot \gamma$), then ε can be uniquely extended to all of P. On the other hand, Condition 1 may be satisfied on U without there being an identity bisection defined on all of P. For this reason we make the following definition:

Definition 2. Let $U \subset P$ be a subset and let X be a (P, P)-bimodule. We say that $\varepsilon : U \to X|_U$ is a *local identity bisection*, if

(1) $s_X \circ \varepsilon(p) = t_X \circ \varepsilon(p) = p$ for all $p \in U$;

(2) $\varepsilon(U) \subset X$ is a lagrangian submanifold.

Lemma 3. *A Morita self-equivalence (P, P)-bimodule X is isomorphic to the identity bimodule $\Gamma(P)$ if and only if X has an identity bisection.*

Proof. Assume that X has an identity bisection $\varepsilon : P \to X$. Then

$$\gamma \mapsto \gamma \cdot \varepsilon(t(\gamma)) = \varepsilon(s(\gamma)) \cdot \gamma$$

is a bimodule map from $\Gamma(P)$ to X. This map is injective and surjective, since the action of $\Gamma(P)$ on X is free and transitive on the fibers. Thus, we obtain a bimodule isomorphism. The condition that $\varepsilon(P)$ is lagrangian guarantees that this map is a symplectomorphism. Conversely, if X is isomorphic to $\Gamma(P)$, the image of the identity bisection of $\Gamma(P)$ under this isomorphism is an identity bisection in X. $\square$

3. Mapping class groups of surfaces

3.1. *The groups* Mod(S)*,* PMod(S) *and* $\mathcal{M}(S)$. Throughout this section, unless explicitly stated otherwise, let S be a connected oriented surface. For simplicity, we will assume that S is the interior of a surface $\bar{S}$ with boundary ∂S. We will be mostly concerned with the case that ∂S is nonempty.

We denote by $\mathrm{Diff}(S)$ the group of diffeomorphisms of S, and by $\mathrm{Diff}(S \text{ fix } \partial S)$ the subgroup of diffeomorphisms that preserve pointwise a neighborhood of the boundary of S. Diffeomorphisms in $\mathrm{Diff}(S \text{ fix } \partial S)$ automatically preserve orientation, as long as $\partial S \neq \varnothing$. Both $\mathrm{Diff}(S)$ and $\mathrm{Diff}(S \text{ fix } \partial S)$ are topological groups with the C^∞-topology. Mapping class groups arise from the consideration of the connected components of these groups. Following the notation of [Ivanov 2002], we consider:

(1) *The mapping class group* $\mathrm{Mod}(S) = \pi_0(\mathrm{Diff}(S))$, defined as the group of isotopy classes of all diffeomorphisms of S.

(2) *The pure mapping class group* $\mathrm{PMod}(S)$, defined as the subgroup of $\mathrm{Mod}(S)$ generated by diffeomorphisms that lie in $\mathrm{Diff}(S \text{ fix } \partial S)$.

(3) $\mathcal{M}(S) = \pi_0(\mathrm{Diff}(S \text{ fix } \partial S))$.

Remark 4. In the case of closed surfaces (i.e., $\partial S = \varnothing$), we have $\mathcal{M}(S) = \mathrm{PMod}(S) = \mathrm{Mod}(S)$.

Let $\varphi \in \mathrm{Diff}(S)$ be a diffeomorphism and $\varphi_* : \pi_1(S, p) \to \pi_1(S, \varphi(p))$ be the induced group homomorphism. This gives an automorphism of $\pi_1(S)$ defined up to an inner automorphism. Moreover, if φ_1 and φ_2 are isotopic, the corresponding automorphisms of $\pi_1(S)$ differ by an inner automorphism. Thus, there is a natural group homomorphism $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S))$.

The following theorem (due to Dehn, Nielsen [1927] and Baer [1928] for closed surfaces, and to Magnus [1934] and Zieschang [1981] for surfaces with boundary), allows to obtain the descriptions of the subgroups of outer automorphisms of the fundamental group corresponding to various mapping class groups under the map $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S))$ described above.

Theorem 5. (1) *If S is a closed connected orientable surface and is not a sphere, there is a group isomorphism* $\mathrm{Mod}(S) \cong \mathrm{Out}(\pi_1(S))$.

(2) *If S has nonempty boundary $\partial S = \bigcup_{i=1}^k T_i \neq \varnothing$ and $\chi(S) < 0$ (i.e., S is not a disc or a cylinder), then*

$$\mathrm{Mod}(S) \cong \big\{\alpha \in \mathrm{Out}(\pi_1(S)) : \exists \tilde{\alpha} \in \mathrm{Aut}(\pi_1(S)),\ [\tilde{\alpha}] = \alpha, \\ \textit{and } \forall i\ \ \exists j \textit{ s.t. } \tilde{\alpha}([T_i]) = [T_j]\big\},$$

i.e., the subgroup consisting of outer automorphisms which preserve the oriented peripheral structure. Moreover,

$$\mathrm{PMod}(S) \cong \big\{\alpha \in \mathrm{Out}(\pi_1(S)) : \forall i \;\; \exists \tilde{\alpha_i} \in \mathrm{Aut}(\pi_1(S)),$$
$$[\tilde{\alpha}_i] = \alpha \quad s.t.\ \tilde{\alpha}([T_i]) = [T_i]\}.\big\}$$

We will use the relation of mapping class groups with outer automorphisms of the fundamental group when dealing with the Picard group and its subgroups in the case of an open symplectic surface.

Remark 6. (1) If S is a sphere, then $\mathrm{Mod}(S) \cong \mathbb{Z}_2$ and is generated by an orientation-reversing diffeomorphism. However, $\mathrm{Out}(\pi_1(S)) = \{e\}$.

(2) If S is a disc, then $\mathrm{Mod}(S) \cong \mathbb{Z}_2$, but $\mathrm{Out}(\pi_1(S)) \cong \{e\} \cong \mathrm{PMod}(S)$.

(3) If $S = C = I \times S^1$ is a cylinder with coordinates (r, θ), $r \in I = [1, 2]$, then $\mathrm{PMod}(C) \cong \{e\}$ and $\mathrm{Out}(\pi_1(C)) \cong \mathbb{Z}_2$. The group $\mathrm{Mod}(C) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by the automorphisms $\Phi_1(r, \theta) = (r, -\theta)$ (which induces the unique nontrivial outer automorphism of $\pi_1(C)$) and $\Phi_2(r, \theta) = (3 - r, \theta)$ (which switches the boundary components of the cylinder).

Remark 7. If S is disconnected, one can still define the mapping class groups in a similar way. In particular, for a disconnected surface S with connected components S_i, $i \in I$ we have

$$\mathcal{M}(S) = \textstyle\prod_{i \in I} \mathcal{M}(S_i). \tag{3–1}$$

3.2. ***Fundamental groups of surfaces.*** Recall that an orientable surface of genus g with b boundary components can be obtained from a $4g$-gon by an identification of sides according to the word $(a_1 b_1 a_1^{-1} b_1^{-1}) \dots (a_g b_g a_g^{-1} b_g^{-1})$ and removal of b disjoint discs from its interior. This leads to the following presentation of $\pi_1(S)$:

$$\pi_1(S) = \langle a_1, b_1, \dots a_g, b_g,\ d_1, \dots, d_b : a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} d_1 \dots d_b = 1\rangle.$$

Note that if $\partial S \neq \varnothing$ (i.e., $b \neq 0$), we can eliminate a d_i from this presentation, and consider $\pi_1(S)$ as a free group $\mathbb{F}_{2g+b-1}$ on $(2g + b - 1)$ generators. In particular:

Lemma 8. *Let S be an orientable surface with a nontrivial boundary, and let $\gamma \in \pi_1(S)$ be the class of a connected component of ∂S. If $\alpha \in \pi_1(S)$ commutes with γ, then $\alpha = \gamma^k$ for some $k \in \mathbb{Z}$.*

Proof. In our identification of $\pi_1(S)$ with the free group $\mathbb{F}_{2g+b-1}$, the class γ corresponds to one of the generators d_i. If an element $w \in \mathbb{F}_{2g+b-1}$ commutes with d_i, it must be a power of that element. □

3.3. ***Dehn twists.*** Observe that if two diffeomorphisms in $\mathrm{Diff}(S \,\mathrm{fix}\, \partial S)$ are isotopic in $\mathrm{Diff}(S \,\mathrm{fix}\, \partial S)$ they are also isotopic in $\mathrm{Diff}(S)$. Thus, there is a natural surjective map $\mathcal{M}(S) \to \mathrm{PMod}(S)$. The kernel of this map can be described in terms of special elements in $\mathcal{M}(S)$, called the *Dehn twists around the boundary.*

We will first describe the Dehn twists of an annulus. Let $A = \{(r, \theta) : r \in [1, 2]\}$ be an annulus with the boundary $\partial A = T_1 \cup T_2$ consisting of two circles, $T_1 = \{r = 1\}$ and $T_2 = \{r = 2\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

- $f(x) = 0$ for $x \leq 1$;
- $f(x) = 2\pi$ for $x \geq 2$;
- $f'(x) \geq 0$.

The standard twist automorphism of A is defined by the formula

$$\Phi(r, \theta) = (r, \theta + f(r)).$$

Note that $\Phi|_{\partial A} = \mathrm{id}$ and the class of Φ up to isotopy fixed on the boundary is independent of the choice of f. This class $[\Phi]_{\mathcal{M}(S)}$ is called the (left) *Dehn twist* of the annulus (see [Dehn 1987] for an English translation of Dehn's original papers where these twists were introduced). Moreover, any class in $\mathcal{M}(A)$ is a power of the Dehn twist:

Lemma 9. *Let A be an annulus. The group $\mathcal{M}(A) = \pi_0(\mathrm{Diff}(A \,\mathrm{fix}\, \partial A))$ is isomorphic to the infinite cyclic group $\mathbb{Z}$ generated by the Dehn twist.*

Using orientation-preserving embeddings of the annulus into a surface S, one can transplant the standard twist diffeomorphism of the annulus to a diffeomorphism of the surface. If $e : A \to S$ is an orientation-preserving embedding, take the diffeomorphism $e \circ \Phi \circ e^{-1} : e(A) \to e(A)$ and extend it by the identity to a diffeomorphism $\Phi_e \in \mathrm{Diff}(S)$. Up to an isotopy fixed on the boundary the diffeomorphism Φ_e depends only on the isotopy class of the embedding, which in turn is determined by the isotopy class of the oriented image $e(a)$ of the axis $a = \{r = 3/2\}$ of the annulus. One can call the diffeomorphism Φ_e a *twist about the circle $T = e(a)$ supported on the annulus $e(A)$*. The Dehn twist about the circle T is the class of this diffeomorphism up to an isotopy by diffeomorphisms equal to the identity outside of the support of the diffeomorphism.

The main property of Dehn twists which we will need in Section 4 is the following

Lemma 10. *Let S be an open orientable surface with $\partial S = \bigcup_{i=1}^{k} T_i$. Let $D(\partial S) \cong \mathbb{Z}^k$ be the subgroup of $\mathcal{M}(S)$ generated by the Dehn twists around the curves parallel to the boundary components. Then there is a (split) exact sequence*

$$D(\partial S) \xrightarrow{i} \mathcal{M}(S) \xrightarrow{p} \mathrm{PMod}(S),$$

where i is the inclusion, and p is the natural projection, sending the class of a diffeomorphism fixing the boundary in $\mathcal{M}(S)$ to its class in $\mathrm{PMod}(S)$.

Remark 11. For S with $\partial S = \cup_{i=1}^{k} T_i$, the group $\mathcal{M}(S)$ has two natural subgroups: one is $D(\partial S) \cong \mathbb{Z}^k$ generated by the Dehn twists around the curves parallel to the boundary components, and the other is $G(S)$, generated by the Dehn twists around the nonseparating curves (see [Ivanov 2002], for example). In fact, $\mathcal{M}(S)$ is generated by these two subgroups. Moreover, $G(S)$ and $D(\partial S)$ commute, so $\mathcal{M}(S) \cong G(S) \oplus D(\partial S) \cong \mathrm{PMod}(S) \oplus \mathbb{Z}^k$.

3.4. ***Moser's argument.*** Finally, when we are dealing with a symplectic surface, it is useful to represent the classes in mapping class groups by symplectomorphisms. For a surface S with boundary $\partial S = \bigcup_{i=1}^{k} T_i$ and a symplectic structure ω, denote by $\mathrm{Symp}(S)$ the group of symplectomorphisms of S, and by $\mathrm{Symp}(S \text{ fix } \partial S)$ the subgroup of symplectomorphisms trivial on a neighborhood of the boundary. Moser's type argument [1965], extended to noncompact manifolds in [Greene and Shiohama 1979], implies the following (for details see of [Radko 2002, proof of Lemma 3, p. 533]):

Lemma 12. *Let (S, ω) be a symplectic surface with nontrivial boundary, such that any neighborhood of the boundary has an infinite volume. Let $\alpha \in \mathrm{Mod}(S)$ be a class in the mapping class group which can be represented by an orientation-preserving diffeomorphism.*

(1) *There is a symplectomorphism $\varphi \in \mathrm{Symp}(S)$ such that $[\varphi]_{\mathrm{Mod}(S)} = \alpha$.*

(2) *Moreover, if $\alpha \in \mathrm{PMod}(S)$, one can choose φ to be in $\mathrm{Symp}(S \text{ fix } \partial S)$.*

Remark 13. In general, $\mathrm{Mod}(S)$ also contains orientation-reversing diffeomorphisms, which are of course not isotopic to symplectomorphisms. In this case, the analog of Lemma 12 states that every element of $\mathrm{Mod}(S)$ is isotopic to a symplectomorphism or an antisymplectomorphism.

4. Symplectic manifolds

4.1. ***The Picard group.*** The Picard group of a symplectic manifold was computed in [Bursztyn and Weinstein 2004]. For convenience, we outline here one of the possible approaches to the computation of this group.

Let S be a symplectic manifold, $\Gamma(S)$ be its symplectic groupoid and $X \in \mathrm{Pic}(S)$ be an invertible (S,S)-bimodule. Let $p \in S$ be a point, and consider the set $X_p = s_X^{-1}(p) \cap t_X^{-1}(p)$, which we will call the isotropy of X at p. Since X is invertible, the isotropy group $\Gamma_p(S) = s^{-1}(p) \cap t^{-1}(p) \cong \pi_1(S, p)$ acts freely transitively on X_p on the left and on the right. Thus, X_p is a discrete set isomorphic to $\pi_1(S)$. Moreover, for any a fixed $x \in X_p$ and $\gamma \in \Gamma_p(S)$ there exists a unique element

$\mathrm{Hol}_x(\gamma) \in \Gamma_p(S) \cong \pi_1(S, p)$ such that

$$\gamma \cdot x = x \cdot \mathrm{Hol}_x(\gamma). \tag{4–1}$$

Thus, we obtain a map $X_p \to \mathrm{Aut}(\pi_1(S, p))$ given by $x \mapsto \mathrm{Hol}_x \in \mathrm{Aut}(\pi_1(S))$. It turns out that the class of the resulting automorphism Hol_x in the group of outer automorphisms of $\pi_1(S)$ is independent of the choices of $p \in S$ and $x \in X_p$. Thus, there is a map

$$\begin{aligned} \mathrm{Pic}(S) &\to \mathrm{Out}(\pi_1(S)), \\ X &\mapsto [\mathrm{Hol}_x]_{\mathrm{Out}(\pi_1(S))}. \end{aligned}$$

This map is actually a group isomorphism:

Theorem 14 [Bursztyn and Weinstein 2004]. *For a connected symplectic manifold S, the Picard group is isomorphic to the group of outer automorphisms of the fundamental group*,

$$\mathrm{Pic}(S) \cong \mathrm{Out}(\pi_1(S)). \tag{4–2}$$

In the remainder of this section, we will describe (certain subgroups of) the Picard group for symplectic surfaces in terms of mapping class group.

4.2. ***Closed symplectic surfaces.*** In the case that S is a connected closed surface and not a sphere, the Dehn–Nielsen–Baer theorem (Theorem 5) states that

$$\mathrm{Out}(\pi_1(S)) \cong \mathrm{Mod}(S).$$

Combining this with (4–2), we obtain

$$\mathrm{Pic}(S) \cong \mathrm{Mod}(S);$$

see also [Bursztyn and Weinstein 2004, Remark 6.4].

Using this isomorphism of the Picard group with a mapping class group, we can characterize the bimodules in the Picard group which come from Poisson diffeomorphisms. Let $j : \mathrm{Poiss}(S) \to \mathrm{Pic}(S)$ be the natural map given by $\varphi \mapsto \Gamma_\varphi(S)$. It is not hard to see that the composition of maps

$$\mathrm{Poiss}(S) \xrightarrow{j} \mathrm{Pic}(S) \to \mathrm{Out}(\pi_1(S)) \to \mathrm{Mod}(S)$$

simply takes a symplectomorphism $\varphi \in \mathrm{Symp}(S)$ to its isotopy class in $\mathrm{Mod}(S)$. Thus the image of $\mathrm{Poiss}(S)$ in $\mathrm{Mod}(S)$ consists exactly of those elements which can be represented by orientation-preserving diffeomorphisms. In the isomorphism $\mathrm{Mod}(S) \cong \mathrm{Out}(\pi_1(S))$, these correspond to the subgroup $\mathrm{Out}^+(\pi_1(S))$ of automorphisms of $\pi_1(S)$ that act trivially on the second cohomology group $H^2(\pi_1(S), \mathbb{Z}) \cong H^2(S, \mathbb{Z})$. Thus we have the following relations between the group of Poisson diffeomorphisms, the Picard group and outer automorphisms of the fundamental group in the case of a closed surface which is not a sphere:

$$j(\mathrm{Poiss}(S)) \cong \mathrm{Out}^+(\pi_1(S)) \subsetneq \mathrm{Out}(\pi_1(S)) \cong \mathrm{Mod}(S) \cong \mathrm{Pic}(S).$$

(For the sphere, $j(\mathrm{Poiss}(S)) \cong \mathrm{Out}(\pi_1(S)) = \{e\} \subset \mathbb{Z}_2 = \mathrm{Mod}(S) \cong \mathrm{Pic}(S)$.) On the other hand, since each class in $\mathrm{Mod}(S)$ for a closed surface can be represented by either a symplectomorphism or an antisymplectomorphism, it follows that every bimodule in the Picard group comes from either a symplectomorphism or an antisymplectomorphism.

4.3. ***Open surfaces: bimodules trivial near the boundary.*** Let S be a two-dimensional open symplectic manifold whose boundary is nontrivial and consists of a finite number of simple closed curves, $\partial S = \bigcup_{i=1}^k T_i$. Let C_i be an annular collar near T_i, such that $\partial C_i = T_i \cup T_i'$, where T_i' is a curve parallel to T_i. Set $C = \bigcup_{i=1}^k C_i$. Assume that the symplectic structure is such that each neighborhood of a boundary curve has an infinite symplectic volume (in particular, each C_i has an infinite volume). Since ∂S is nontrivial, every diffeomorphism is orientation-preserving, and, thus, by Moser's argument (see Lemma 12), is isotopic to a symplectomorphism.

In this subsection, we will characterize the image of the map $j : \mathrm{Poiss}(S) \to \mathrm{Pic}(S)$ for the case of an open symplectic surface. Fix once and for all points $p_i \in C_i$, $i = 1, \dots, k$. Let

$$\mathrm{Pic}(S, \partial S) = \{X \in \mathrm{Pic}(S) \ : \forall i \ \exists x_i \in X_{p_i} \text{ s.t. } \mathrm{Hol}_{x_i}([T_i]) = [T_i]\}$$

be the subgroup of the Picard group of S consisting of the bimodules which are "trivial near the boundary" (for each $i = 1, \dots, k$, the point $x_i \in X_{p_i}$ can be chosen in such a way that the induced holonomy automorphism Hol_{x_i} preserves the class of the corresponding boundary curve).

Proposition 15. *The subgroup of bimodules in the Picard group of S which come from Poisson automorphisms is isomorphic to the pure mapping class group*,

$$j(\mathrm{Poiss}(S)) = \mathrm{Pic}(S, \partial S) \cong \mathrm{PMod}(S).$$

Proof. Let $\partial S = \bigcup_{i=1}^k T_i$. Recall that $\mathrm{PMod}(S)$ is the subgroup of $\mathrm{Mod}(S)$ generated by the diffeomorphisms that preserve setwise the boundary components of S:

$$\mathrm{PMod}(S) = \left\{\alpha \in \mathrm{Mod}(S) : \forall T_i \ \exists \tilde{\alpha} \in \mathrm{Diff}(S) \text{ s.t. } [\tilde{\alpha}] = \alpha \text{ and } \tilde{\alpha}(T_i) = T_i\right\}.$$

Under the map $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S))$, this corresponds to the following description of the pure mapping class group:

$$\mathrm{PMod}(S) \cong \left\{\alpha \in \mathrm{Out}(\pi_1(S)) : \forall T_i \ \exists \tilde{\alpha} \in \mathrm{Aut}(\pi_1(S)) \text{ s.t. } \tilde{\alpha}([T_i]) = [T_i]\right\}.$$

(For surfaces with boundary having negative Euler characteristic this follows from the Dehn–Nielsen–Baer theorem, and for the disc and the cylinder it can be verified directly; see Remark 6).

Let $w : \mathrm{Pic}(S) \to \mathrm{Out}(\pi_1(S))$ be the isomorphism of Theorem 14, given by

$$w(X) = [\mathrm{Hol}_x]_{\mathrm{Out}(\pi_1(S))}, \quad x \in X \in \mathrm{Pic}(S).$$

We claim that the image of $\mathrm{Pic}(S, \partial S)$ under this map is isomorphic to $\mathrm{PMod}(S)$.

Let $X \in \mathrm{Pic}(S, \partial S)$ be a bimodule. Then for any $i = 1, \dots, k$, there exists x_i such that $s_X(x_i) = t_X(x_i) = p_i \in C_i$ and the induced holonomy action is trivial on the homotopy class $[T_i]$ of the curve T_i. Then $\mathrm{Hol}_{x_i} \in \mathrm{Aut}(\pi_1(S))$ is such that $[\mathrm{Hol}_{x_i}]_{\mathrm{Out}(\pi_1(S))} = w(X)$ and $\mathrm{Hol}_{x_i}([T_i]) = [T_i]$. Hence $w(X) \in \mathrm{PMod}(S)$.

Conversely, let $\alpha \in \mathrm{PMod}(S)$. Let $\varphi \in \mathrm{Symp}(S, \partial S)$ be a symplectomorphism trivial near the boundary and representing the class α, i.e., $\varphi|_{C_i} = \mathrm{id}$, $[\varphi]_{\mathrm{PMod}(S)} = \alpha$. Let X be the (S,S)-bimodule $\Gamma_\varphi(S)$ obtained from the symplectic groupoid $\Gamma(S)$ by composing the target map with the symplectomorphism φ. Thus $s_X = s$ and $t_X = \varphi \circ t$. Note that as left S-modules (and in particular, as sets) $X = \Gamma(S)$. Let $\varepsilon : S \to \Gamma(S)$ be the identity bisection of $\Gamma(S)$, so $\varepsilon(p)$ is the unit element of $\Gamma(S)$ over $p \in S$. Let $x_i = \varepsilon(p_i) \in X$, $i = 1, \dots, k$. Since $\varphi|_{C_i} = \mathrm{id}$, $s_X(x_i) = t_X(x_i) = p_i$.

We claim that $\mathrm{Hol}_{x_i}([T_i]) = [T_i]$. To see this, let $\tilde{T}_i \subset C_i$ be a curve parallel to T_i. Note that $\varepsilon(\tilde{T}_i)$ is a lift of $\tilde{T}_i$, lying in the isotropy of X. That is, $s_X(x) = t_X(x)$ for all $x \in \varepsilon(\tilde{T}_i)$. Since for each $p \in S$, the isotropy $X_p = s_X^{-1}(p) \cap t_X^{-1}(p)$ is discrete, it follows that $\varepsilon(\tilde{T}_i)$ must be a horizontal lift of $\tilde{T}_i$ for the connection on X that defines Hol_{x_i}. Thus $\mathrm{Hol}_{x_i}([T_i]) = [T_i]$ and $X \in \mathrm{Pic}(S, \partial S)$.

Lastly, we have seen that every bimodule in $\mathrm{Pic}(S, \partial S)$ is in the image of $j : \mathrm{Poiss}(S) \to \mathrm{Pic}(S)$. Conversely, every bimodule in the image of j must correspond to an element of $\mathrm{PMod}(S) \subset \mathrm{Out}(\pi_1(S))$ and thus lie in $\mathrm{Pic}(S, \partial S)$. Thus the image of j is exactly $\mathrm{Pic}(S, \partial S) \subset \mathrm{Pic}(S)$. □

Thus we have the following relations between the group of Poisson automorphisms, the Picard group and mapping class groups in the case of an open surface (which is not a disc or a cylinder):

$$j(\mathrm{Poiss}(S)) = \mathrm{Pic}(S, \partial S) \cong \mathrm{PMod}(S) \subsetneq \mathrm{Mod}(S) \subsetneq \mathrm{Out}(\pi_1(S)) \cong \mathrm{Pic}(S).$$

If S is a disc, the Picard group is trivial; if S is a cylinder,

$$j(\mathrm{Poiss}(S)) \cong \mathrm{PMod}(S) = \{e\} \subset \mathbb{Z}_2 \cong \mathrm{Out}(\pi_1(S)) \cong \mathrm{Pic}(S)).$$

4.4. ***The group of bimodules with chosen trivializations near the boundary.*** For the purposes of computation of the (static) Picard group of a TSS in Section 5.3, it is useful to consider the group $\mathcal{P}(S)$ of self-equivalence bimodules over an open symplectic surface which are trivial near the boundary, together with a chosen trivialization. By this we mean the following.

Let $X \in \mathrm{Pic}(S, \partial S)$ be a bimodule trivial near the boundary, and $x_i \in X_{p_i}$, $i = 1, \dots, k$, be the points as in the definition of $\mathrm{Pic}(S, \partial S)$. One can think of a pair $(X, (x_1, \dots, x_k))$ as a self-equivalence bimodule in $\mathrm{Pic}(S, \partial S)$ with a chosen trivialization, given by x_i's. We say that $(X, (x_1, \dots, x_k))$ and $(X', (x'_1, \dots, x'_k))$ are isomorphic if there is a bimodule isomorphism $f : X \to X'$ preserving the trivialization, $f(x_j) = x'_j$ for all $j = 1, \dots, k$.

Definition 16. Let $\mathcal{P}(S)$ be the set of isomorphism classes of pairs $(X, (x_1, \dots, x_k))$, where

(1) $X \in \mathrm{Pic}(S, \partial S)$;

(2) $x_i \in X_{p_i}$, where $p_i \in C_i$ are fixed points near the boundary.

(3) $\mathrm{Hol}_{x_i}([T_i]) = [T_i]$.

It is easy to verify that $\mathcal{P}(S)$ is a group with the following structure:

(a) Multiplication is given by the relative tensor product:

$$(X, (x_1, \dots, x_k)) \times (X', (x'_1, \dots, x'_k)) = \big(X \otimes_S X',\ ([(x_1, x'_1)], \dots, [(x_n, x'_n)])\big),$$

where $[(x_i, x'_i)]$ is the equivalence class of $(x_i, x'_i) \in X \times_S X'$ in $X \otimes_S X'$.

(b) Inversion is defined by

$$(X, (x_1, \dots, x_k))^{-1} = (X^{-1}, (x_1, \dots, x_k)).$$

(c) The identity of $\mathcal{P}(S)$ is the pair $(\Gamma(S), (\varepsilon(p_1), \dots, \varepsilon(p_k)))$, where ε is the identity bisection of the symplectic groupoid $\Gamma(S)$.

The main result of this section is that $\mathcal{P}(S)$ is isomorphic to the group $\mathcal{M}(S) = \pi_0(\mathrm{Diff}(S), \partial S)$ of diffeomorphisms of S fixing a neighborhood of the boundary up to isotopies:

Proposition 17. $\mathcal{P}(S) \cong \mathcal{M}(S)$.

Proof. Let $\alpha \in \mathcal{M}(S)$ be an isotopy class and $\varphi \in \mathrm{Symp}(S, \partial S)$ be a symplectomorphism trivial near the boundary which represents this class, $\varphi|_C = \mathrm{id}$, $[\varphi]_{\mathcal{M}(S)} = \alpha$. Consider the map $\theta : \mathcal{M}(S) \to \mathcal{P}(S)$ given by

$$[\varphi] \mapsto (\Gamma_\varphi(S),\ (\varepsilon(p_1), \dots, \varepsilon(p_n))). \qquad (4\text{–}3)$$

Claim 18. *The map θ is well-defined.*

Proof. Let $\alpha \in \mathcal{M}(S)$. We need to show that $\theta(\alpha)$ does not depend on the choice of φ such that $[\varphi] = \alpha$. Since θ clearly takes compositions of symplectomorphisms to products of the corresponding elements in $\mathcal{P}(S)$, it is sufficient to prove this for a representative of the class of the identity diffeomorphism.

Let $\varphi \in \mathrm{Symp}(S, \partial S)$ be a symplectomorphism such that $[\varphi] = \mathrm{id} \in \mathcal{M}(S)$. Thus there exists an isotopy $\varphi^{(\tau)}$, $\tau \in [0, 1]$, between the identity diffeomorphism $\mathrm{id} = \varphi^{(0)}$ and $\varphi = \varphi^{(1)}$. We may assume that $\varphi^{(\tau)} \in \mathrm{Symp}(S, \partial S)$ for all $\tau \in [0, 1]$. Let $X^{(\tau)}$ be the bimodule $X^{(\tau)} = \Gamma_{\varphi^{(\tau)}}(S)$.

Let $x \in X^{(0)} = \Gamma(S)$ be the homotopy class of a path γ such that $s(x) = \gamma(0)$ and $t(x) = \gamma(1)$. Let $y^{(\tau_0)} \in X^{(\tau_0)} = \Gamma_{\varphi^{(\tau_0)}}(S)$ be the homotopy class of the path $\beta^{(\tau_0)} : [0, \tau_0] \to S$ given by

$$\beta^{(\tau_0)}(\tau) = \left(\varphi^{(\tau)}\right)^{-1}(\gamma(1)), \tag{4–4}$$

so that $s_{X^{(\tau_0)}}(y^{(\tau_0)}) = s(y^{(\tau_0)}) = \gamma(1)$ and $t_{X^{(\tau_0)}}(y^{(\tau_0)}) = \varphi^{(\tau_0)}(t(y^{(\tau_0)})) = \gamma(1)$. Thus, $y^{(\tau_0)} \in X^{(\tau_0)}_{\gamma(1)}$. For each $\tau_0 \in [0, 1]$, let $\alpha^{(\tau_0)} : X^{(0)} \to X^{(\tau_0)}$ be the map $x \mapsto y^{(\tau_0)} \circ x$. Then

$$s_{X^{(\tau_0)}}(\alpha^{(\tau_0)}(x)) = s(x), \quad t_{X^{(\tau_0)}}(\alpha^{(\tau_0)}(x)) = t(x).$$

This implies that $\alpha^{(\tau_0)}$ sends the identity bisection of the symplectic groupoid $X^{(0)}$ to the identity bisection of the bimodule $X^{(\tau_0)}$, and thus defines a bimodule isomorphism $X^{(0)} \cong X^{(\tau_0)}$. Since $\varphi^{(\tau)}|_{C_i} = \mathrm{id}|_{C_i}$, it follows that this isomorphism preserves the chosen trivialization, $\alpha^{(\tau_0)}(x_i) = x_i$ for all $i = 1, \ldots, k$. Thus the map θ is indeed well-defined. □

Let $q : \mathcal{P}(S) \to \mathrm{Pic}(S, \partial S)$ be the projection map taking the pair $(X, (x_1, \ldots, x_k))$ to X, and let $\mathrm{pr} : \mathcal{M}(S) \to \mathrm{PMod}(S)$ be the natural map that takes the class of a diffeomorphism φ in $\mathcal{M}(S)$ to its class in $\mathrm{PMod}(S)$. Note that both maps are surjective group homomorphisms. We then have the following commutative diagram:

$$\begin{tikzcd} \mathcal{M}(S) \arrow[r, "\theta"] \arrow[d, "\mathrm{pr}"'] & \mathcal{P}(S) \arrow[d, "q"'] \\ \mathrm{PMod}(S) \arrow[r, "w^{-1}"] & \mathrm{Pic}(S, \partial S). \end{tikzcd}$$

The bottom arrow in the diagram is described in Proposition 15.

Claim 19. *θ is onto.*

Proof. Since pr and q are onto and w is an isomorphism, it is sufficient to prove that $\ker(q) \subseteq \mathrm{Im}(\theta)$.

Let $(X, (x_1, \ldots, x_k)) \in \ker q$. Thus we may assume that X is the identity bimodule $\Gamma(S)$ and that $x_1, \ldots, x_k \in \Gamma(S)$ are points satisfying $s(x_j) = t(x_j) = p_j$, $j = 1, \ldots, k$. Let γ_j be a curve based at p_j whose homotopy class is x_j. The condition that

$$\mathrm{Hol}_{x_j}([T_j]) = ([T_j])$$

means by definition that if λ_j is a curve based at p_j and parallel to T_j, then $\gamma_j \lambda_j \gamma_j^{-1}$ has the same class in $\pi_1(S)$ as λ_j. By Lemma 8, the only elements of the fundamental group of S that commute with the class of a boundary curve $[\lambda_j] = [T_j]$ are the powers of the class of this boundary curve. It follows that x_j must be represented by a power of λ_j, i.e., $x_j = \left[\lambda_j\right]^{n_j}$ for some $n_j \in \mathbb{Z}$. Thus $(X, (x_1, \dots, x_n)) \cong (\Gamma(S), ([T_1]^{n_1}, \dots, [T_k]^{n_k}))$ for some integers $n_1, \dots, n_k$.

We will now construct a class in $\mathcal{M}(S)$ which is mapped to this bimodule by θ. Let D_i be a cylindrical neighborhood parallel to C_i and adjacent to C_i, and set $S' = S \setminus \left(\bigcup_i C_i \cup \bigcup_i D_i\right)$. Let $\varphi : S \to S$ be a symplectomorphism such that

- $\varphi|_{D_i}$ is (homotopic to) the n_j-th power of the Dehn twist on D_i for all $i = 1, \dots, k$, and
- φ is trivial outside of $\bigcup D_i$, i.e., $\varphi|_{S \setminus \bigcup_i D_i} = \mathrm{id}$.

We claim that $\theta([\varphi]_{\mathcal{M}(S)}) = (X, (x_1, \dots, x_k))$. Indeed, let $\varphi^{(\tau)}$, $\tau \in [0, 1]$ be an isotopy between $\mathrm{id} = \varphi^{(0)}$ and $\varphi = \varphi^{(1)}$. Of course $\varphi^{(\tau)}$ no longer preserve pointwise the boundary of S for $0 < \tau < 1$; in fact, for $\tau \in [0, 1]$, the map $\tau \mapsto \varphi^{(\tau)}(p_i)$ traces out a curve which is homotopic to $T_i^{n_i}$.

For each $\tau_0 \in [0, 1]$, let $X^{(\tau_0)} = \Gamma_{\varphi^{(\tau_0)}}(S)$ and let $\alpha^{(\tau_0)} : X^{(0)} \to X^{(\tau_0)}$ be the bimodule isomorphism described in the proof of Claim 1. Identifying $\Gamma(S)$ with the space of homotopy classes of paths in X, we view $\alpha^{(\tau_0)}(x_j)$ as the class of the path $\tau \mapsto \varphi^{(\tau)}(p_j)$, $0 \le \tau \le \tau_0$. It follows that $\alpha^{(1)}(x_j)$ is homotopic to ${T_j}^{n_j}$. Thus, $\alpha^{(1)}$ is an isomorphism between $(X, (x_1, \dots, x_n))$ and $\theta([\varphi]) = (\Gamma(S), \varepsilon(p_1), \dots, \varepsilon(p_k))$. Hence θ is onto. □

Claim 20. *θ is injective.*

Proof. Let $\varphi \in \mathrm{Symp}(S, \partial S)$ be such that $[\varphi]_{\mathcal{M}(S)} \in \ker\theta$. Since w is an isomorphism, it follows that $[\varphi]_{\mathcal{M}(S)} \in \ker(\mathrm{pr})$. We may therefore assume (Lemma 10) that there are cylindrical neighborhoods $D_1, \dots, D_k$ adjacent to $C_1, \dots, C_k$ and parallel to them, so that φ is identity outside of $D_1 \cup \dots \cup D_k$ and the restriction of φ to each D_j is the n_j-th power of the Dehn twist on D_j.

We saw earlier that $\theta([\varphi_{\mathcal{M}(S)}])$ is isomorphic to $(\Gamma(S), ([T_1]^{n_1}, \dots, [T_k]^{n_k}))$. Since $[\varphi] \in \ker\theta$, it follows that there is a bimodule isomorphism $\Phi : \Gamma(S) \to \Gamma(S)$ such that $\Phi([T_i]^{n_i}) = \varepsilon(p_i)$ for all $i = 1, \dots, k$, where ε denotes the identity bisection. Since $\varepsilon : S \to \Gamma(S)$ is an identity bisection of $\Gamma(S)$, $\sigma = \Phi^{-1} \circ \varepsilon : S \to \Gamma(S)$ is also an identity bisection. Thus there is an identity bisection $\sigma : S \to \Gamma(S)$ passing through $x_j = [T_j]^{n_j}$ over p_j for all $j = 1, \dots, k$. By property 1 of an identity bisection (Definition 1), for any $p \in S$ the element $\sigma(p)$ lies in the discrete set $s^{-1}(p) \cap t^{-1}(p) = \pi_1(S, p)$, and, therefore, $\sigma(p) = \gamma(p) \cdot \varepsilon(p)$ for some (locally constant) map $\gamma : S \to \pi_1(S)$. By property 2 in the same definition, the left and right groupoid actions by an element $\beta \in \Gamma_p(S) = \pi_1(S, p)$ on $\sigma(p) = \gamma(p) \cdot \varepsilon(p)$ commute, which implies that $\gamma(p)$ lies in the center of $\pi_1(S, p)$ for all $p \in S$. Moreover, the map $p \mapsto \gamma(p)$ is globally constant.

Since $\partial S \neq \varnothing$, the fundamental group of S can be identified with a free group. Hence S is either a cylinder, or its fundamental group is a free group on two or more generators and thus has a trivial center. In the latter case, we get that $\sigma = \varepsilon$ so that $x_j = \varepsilon(p_j)$, $j = 1, \ldots, k$, and thus $n_j = 0$ for all j. This implies $[\varphi] = \mathrm{id} \in \mathcal{M}(S)$.

If S is a cylinder, $\partial S = T_1 \cup T_2$. Let T be a separating circle on the cylinder, so that $[T]$ is a generator of the fundamental group of S. The identity bisections $\varepsilon_k : S \to \Gamma(S)$ of the symplectic groupoid of S have the form $\varepsilon_k(p) = [T(p)]^k$, where $k \in \mathbb{Z}$, and $T(p)$ is a closed curve based at p and homotopic to T. Thus if there is an identity bisection of X through x_1 and x_2, it must be that x_1 and x_2 are the homotopy classes of T^k for the same k. In this case the two Dehn twists making up φ cancel and $[\varphi] = \mathrm{id} \in \mathcal{M}(S)$.

This concludes the proof that θ is injective, and so proves the proposition. □

4.5. ***A remark on bisections of bimodules.*** Let X be an (S, S)-bimodule.

Definition 21. A map $\sigma : S \to X$ is called a *bisection* of X, if

(1) $s_X \circ \sigma = \mathrm{id}$, and $t_X \circ \sigma = \varphi$ is a symplectomorphism of S.

(2) $\sigma(S)$ is a lagrangian submanifold of X.

The first condition means that σ is a section of the source map, while $\sigma \circ \varphi^{-1}$ is a section of the target map.

Let $\mathrm{Bis}(X)$ be the set of all bisections of X, and let

$$\mathcal{B}(S) = \{(X, \sigma) : X \in \mathrm{Pic}(S),\ \sigma \in \mathrm{Bis}(X)\}$$

be the set of self-equivalence bimodules with chosen bisections. Then $\mathcal{B}(S)$ is a group under the relative tensor product operation. The inverse of a pair (X, σ) is given by $(X, \sigma) = (X^{-1}, \sigma \circ \varphi^{-1})$, where $X^{-1} = X^{\mathrm{op}}$ is the opposite bimodule, and φ is the symplectomorphism given by $t \circ \sigma$, so that $s_{X^{-1}} \circ \sigma = \mathrm{id}$ and $t_{X^{-1}} \circ \sigma = \varphi^{-1}$.

The map $\Phi : \mathrm{Symp}(S) \to \mathcal{B}(S)$ given by $\varphi \mapsto (\Gamma_\varphi(S), \varepsilon)$ is a group isomorphism.

Endow the space of all symplectomorphisms $\mathrm{Symp}(S)$ of S with the C^∞ topology. The space $\mathcal{B}(S)$ can be endowed with a topology, making this isomorphism into a homeomorphism. We will say that (X, σ) and (X', σ') are *isotopic* if there is a continuous path $(X^{(t)}, \sigma^{(t)})$ in $\mathcal{B}(S)$, joining (X, σ) and (X', σ').

From the definition of $\mathcal{M}(S)$ and the isomorphism between $\mathcal{B}(S)$ and the group of all symplectomorphisms of S we deduce that

$$\mathcal{B}(S)/\mathrm{isotopy} \cong \mathcal{M}(S).$$

Thus our result that $\mathcal{M}(S) \cong \mathcal{P}(S)$ implies that

$$\mathcal{P}(S) \cong \mathcal{B}(S)/\mathrm{isotopy} = \pi_0(\mathrm{Symp}(S)).$$

5. The Picard group of a TSS

5.1. ***Topologically stable structures on compact oriented surfaces.*** From now on, let P be a compact connected oriented surface and let π be a Poisson structure on P with at most linear degeneracies. The zero set $Z \subset P$ of such a structure consists of a finite number of simple closed curves, $Z = \cup_{i=1}^{n} T_i$. Such structures are called *topologically stable structures* (or TSS), since the topology of their zero set does not change under small perturbation of the Poisson tensor.

By an easy application of the integrability criteria of [Crainic and Fernandes 2003], we know that TSS's are integrable. (Alternatively, integrability follows from a result of Debord [2000], since the anchor of the Lie algebroid associated to a TSS is injective on a dense open set $P \setminus Z$.)

By [Radko 2002], TSS's are completely classified up to orientation-preserving Poisson diffeomorphisms by three invariants: the topological class of the oriented zero set; the modular periods around the zero curves; and a generalized Liouville volume. Moreover, according to [Bursztyn and Weinstein 2004] (see also [Bursztyn and Radko 2003]), the topology of the oriented zero set together with the modular periods around the zero curves classify TSS's up to Morita equivalence.

5.2. ***The Picard group of*** $(\mathbb{R}^2, \pi = x\partial_x \wedge \partial_y)$**.** Since a TSS vanishes linearly on a zero curve T, in a neighborhood of a point $p \in T$ it is isomorphic to $\mathbb{R}^2$ with the Poisson structure $\pi = x\partial_x \wedge \partial_y$. Let us start with the description of the identity bimodule $\Gamma(\mathbb{R}^2)$ (i.e., the symplectic groupoid) of this structure. Since $\pi = x\partial_x \wedge \partial_y$ is the Lie–Poisson structure on the dual of the Lie algebra of the group G of affine transformation of the line, it follows that $\Gamma(\mathbb{R}^2) \cong T^*G$. As a manifold, $\Gamma(\mathbb{R}^2)$ is diffeomorphic to $\mathbb{R}^4$ with coordinates (x, y, p, q), in which the Lie groupoid structure is given by

$$s((x,y,p,q)) = (x,y), \qquad t((x,y,p,q)) = (xe^p, y+xq),$$
$$(x,y,p,q)\cdot(x',y',p',q') = (x,y,p+p',q+e^p q'), \quad \text{where } x' = xe^p,\ y' = y+xq,$$
$$(x,y,p,q)^{-1} = (xe^p, y+xq, -p, -qe^{-p}).$$

The symplectic form on $\Gamma(\mathbb{R}^2)$ is given by

$$\begin{aligned}\Omega &= t^*(d(\ln x)\wedge dy) - s^*(d(\ln x)\wedge dy)) \\ &= -q\,dx\wedge dp + dx\wedge dq - dy\wedge dp + x\,dp\wedge dq,\end{aligned} \tag{5–1}$$

and the corresponding Poisson tensor is

$$\Pi = -x\partial_x\wedge\partial_y + \partial_x\wedge\partial_q - \partial_y\wedge\partial_p - q\partial_y\wedge\partial_q. \tag{5–2}$$

Using this description of the symplectic groupoid, we can compute the Picard group:

Proposition 22. *Let $\pi = x\partial_x \wedge \partial_y$ be a Poisson structure on $\mathbb{R}^2$.*

(1) *The static Picard group* $\mathrm{StatPic}(\mathbb{R}^2, \pi)$ *is trivial.*

(2) *The full Picard group is given by*

$$\mathrm{Pic}(\mathbb{R}^2, \pi) \cong \mathbb{Z}_2 \times \mathbb{R} \cong \mathrm{Out}(\mathbb{R}^2, \pi),$$

where the generator of $\mathbb{Z}_2$ corresponds to the flip $(x, y) \mapsto (-x, y)$, and $t \in \mathbb{R}$ corresponds to a shift $(x, y) \mapsto (x, y + t)$.

Proof. Let $(\mathbb{R}^2, \pi) \xleftarrow{s_X} (X, \Omega_X) \xrightarrow{t_X} (\mathbb{R}^2, \pi)$ be a Morita self-equivalence bimodule of $(\mathbb{R}^2, \pi)$, inducing the identity map on the leaf space. Define on X the functions $x_1(s) = x(s_X(s))$, $y_1(s) = y(s_X(s))$, $x_2(s) = x(t_X(s))$ and $y_2(s) = y(t_X(s))$. Let $T = \{(0, y) | \, y \in \mathbb{R}\}$ be the zero set of the Poisson structure. Define

$$p(s) = \ln\frac{x_2}{x_1}, \qquad q(s) = \frac{y_2 - y_1}{x_1}.$$

The functions p and q are well-defined on $X \setminus {s_X}^{-1}(T)$, where both x_1 and x_2 are nonzero. We claim that p and q extend smoothly to all of X.

To see this for p, note that $p = \ln x_2 - \ln x_1$ is a function whose Hamiltonian vector field $H_p = \tilde{\pi}(dp)$ projects by the moment maps to the modular vector field of the Poisson structure with respect to the standard area form $dx \wedge dy$ on $\mathbb{R}^2$, that is, $(s_X)_*(H_p) = (t_X)_*(H_p) = \partial_y$. By [Ginzburg and Lu 1992], there exists a *smooth* function $h \in C^\infty(X)$ which has the same property, $(s_X)_*(H_h) = (t_X)_*(H_h) = \partial_y$. Since on the dense subset $X \setminus {s_X}^{-1}(T) \subset X$ the map $x \mapsto (s_X(x) \; , t_X(x))$ is one-to-one and onto, it follows that such a vector field is unique, i.e., $H_p = H_h$ on $X \setminus {s_X}^{-1}(T)$. Thus, $p - h$ is a locally constant function on $X \setminus s_X^{-1}(T)$. Since $x_2 = x_1 e^p$ is smooth on X, it follows that $p = h + \text{const}$, and is therefore, smooth.

To prove that q extends to all of X, it is enough to recall that $y_1 = y_2$ on $s_X^{-1}(T)$, which follows from the assumption that the map induced by X on the leaf space is the identity.

Note that because $X \setminus s_X^{-1}(T)$ is symplectomorphic to $(\mathbb{R}^2 \setminus T) \times (\mathbb{R}^2 \setminus T)^{\mathrm{op}}$, it follows that the symplectic form on $X \setminus s_X^{-1}(T)$ is given by

$$\Omega = -q\,dx_1 \wedge dp + dx_1 \wedge dq - dy_1 \wedge dp + x_1 dp \wedge dq, \tag{5–3}$$

and thus on all of X by continuity.

We claim that (x_1, y_1, p, q) is a coordinate system on X. First, since the map $\psi : X \to \Gamma(\mathbb{R}^2)$ given by $s \mapsto (x_1(s), y_1(s), p(s), q(s))$ preserves the symplectic form, and, hence, the volume form, it is a local diffeomorphism. Since X is a Morita self-equivalence bimodule, there is a diffeomorphism $\Phi : X \otimes_{\mathbb{R}^2} X^{-1} \to \Gamma(\mathbb{R}^2)$. This implies that the map ψ is one-to-one, and, therefore (x_1, y_1, p, q) is a coordinate system on X, which establishes the diffeomorphism of X and $\Gamma(\mathbb{R}^2)$.

Thus, every invertible bimodule preserving the leaf space pointwise is isomorphic to the identity bimodule, and, therefore, the static Picard group is trivial.

To compute the full Picard group, we apply the exact sequence (2–1). Any automorphism of the leaf space $L(\mathbb{R}^2)$ that comes from a Morita self-equivalence bimodule must preserve the modular vector field (see [Ginzburg and Lu 1992]). Thus, the restriction of an automorphism of $L(\mathbb{R}^2)$ to the zero set must be a translation by the flow of the restriction of a modular vector field. Thus the image of the map $h : \mathrm{Pic}(\mathbb{R}^2, \pi) \to \mathrm{Aut}(L(\mathbb{R}^2)) \cong \mathbb{Z}_2 \times \mathrm{Diff}(\mathbb{R})$ is contained in $\mathbb{Z}_2 \times \mathbb{R}$, where $\mathbb{Z}_2$ is generated by the interchange of the two-dimensional leaves, and $\mathbb{R}$ is generated by a shift along the line of zero-dimensional leaves. Since any automorphism $(\sigma, t) \in \mathbb{Z}_2 \times \mathbb{R}$ can be realized by the Poisson automorphism $\theta_{(\sigma,t)}(x, y) = ((-1)^\sigma \cdot x,\; y + t)$, which gives rise to a self-equivalence bimodule $\Gamma_{\theta_{(\sigma,t)}}(\mathbb{R}^2)$, it follows that $\mathrm{Im}(h) \simeq \mathbb{Z}_2 \times \mathbb{R}^2$. Since an automorphism $\theta_{(\sigma,t)}$ is inner if and only if $(\sigma, t) = (1, 0) = e_{\mathbb{Z}_2 \times \mathbb{R}}$, we conclude that

$$\mathrm{Pic}(\mathbb{R}^2, \pi) \simeq \mathrm{Out}(\mathbb{R}^2, \pi) \simeq \mathbb{Z}_2 \times \mathbb{R}. \qquad \square$$

5.3. ***The static Picard group of a TSS.*** Let $X \in \mathrm{StatPic}(P)$. Let $S \subset P$ be a 2-dimensional symplectic leaf. Then $X_{|S} \in \mathrm{Pic}(S, \partial S)$, and for $p \in S$ the isotropy $X_p = s_X^{-1}(p) \cap t_X^{-1}(p)$ is a discrete set, isomorphic to the fundamental group $\pi_1(S, p)$.

The following lemma shows that locally, in a neighborhood of a zero curve T of a TSS, there is at most one lift of a curve which crosses T to the "isotropy subbundle" $\cup_{p \in P} X_p$ of the bimodule.

Lemma 23. *Let S be a 2-dimensional symplectic leaf of P and $T \in \partial S$ be a zero curve. Let $\gamma : [0, 1] \to P$ be a curve in P such that $\gamma(1) \in T$ and $\gamma(t) \in S$ for $t \in [0, 1)$.*

For $\tau \in [0, 1)$, let $\gamma_\tau : [0, \tau] \to S$ be the curve $\gamma_\tau(t) = \gamma(t)$.

Let $p = \gamma(0)$ and $x_1, x_2 \in X_p$. For $\tau \in [0, 1)$, define

$$x_j(\tau) = [\gamma_\tau] \cdot x_j \cdot [\gamma_\tau]^{-1}, \quad j = 1, 2,$$

where $[\gamma_\tau] \in \Gamma(P)$ is the class of the cotangent path $\tilde{\pi}^{-1}(\gamma_\tau)$, and $\cdot$ denotes the left and right actions of $\Gamma(P)$ on X.

If both limits $\lim_{\tau\to 1} x_1(\tau)$ and $\lim_{\tau\to 1} x_2(\tau)$ exist, then $x_1 = x_2$.

Proof. Assume that both limits, $\lim_{\tau\to 1} x_1(\tau)$ and $\lim_{\tau\to 1} x_2(\tau)$, exist.

Since X is invertible, the relative tensor product $X \otimes_P X^{\mathrm{op}}$ is isomorphic to the identity bimodule $\Gamma(P)$. Let

$$y_1(\tau) = (x_1(\tau), x_1(\tau)) \in X \otimes_P X^{\mathrm{op}} \cong \Gamma(P),$$
$$y_2(\tau) = (x_2(\tau), x_1(\tau)) \in X \otimes_P X^{\mathrm{op}} \cong \Gamma(P).$$

Then $x_1(0) = x_2(0)$ if and only if $y_1(0) = y_2(0)$. Moreover, $y_j(\tau)$ satisfy

$$s(y_j(\tau)) = t(y_j(\tau)) = \gamma(\tau), \quad j = 1, 2. \tag{5–4}$$

Let $z(\tau) = y_1(\tau) \cdot y_2(\tau)^{-1} \in \Gamma(P)$. As usual, we will view the elements of $\Gamma(P)$ as classes of cotangent paths up to cotangent homotopy. For each $\tau \in [0, 1)$, let $(a^{(\tau)}, \eta^{(\tau)})$ be a cotangent path representing $z(\tau)$. Here $a^{(\tau)} : [0, 1] \to T^*P$ is a path in the cotangent bundle, $\eta^{(\tau)} : [0, 1] \to P$ is the base path, and the compatibility condition states that

$$\tilde{\pi}(a^{(\tau)}(t)) = \frac{d\eta^{(\tau)}}{dt}(t). \tag{5–5}$$

In particular, since $y_j(\tau)$ is in the isotropy group of $\Gamma(P)$ at $\gamma(\tau)$, the homotopy class of $\eta^{(\tau)}$ is trivial if and only if $z(\tau)$ is the identity element, i.e., if and only if $x_1(\tau) = x_2(\tau)$.

Endow P with a fixed metric. If the homotopy class of $\eta^{(\tau)}$ is nontrivial, we have

$$\inf_{\tau} \inf_{\substack{(a^{(\tau)},\, \eta^{(\tau)}) \\ \text{representing } z(\tau)}} \sup_{t} \left\| \frac{d\eta^{(\tau)}}{dt}(t) \right\| = C > 0$$

for the norm $\|\cdot\|$ on TP coming from our choice of a metric on P. But since $\gamma(1) \in T$, where the Poisson structure vanishes, condition (5–5) implies that

$$\lim_{\tau\to 1} \inf_{\substack{(a^{(\tau)},\eta^{(\tau)}) \\ \text{representing } z(\tau)}} \sup_{t} \|a^{(\tau)}(t)\| \to \infty.$$

This contradicts the existence of the limit

$$\lim_{\tau\to 1} z(\tau) = \lim_{\tau\to 1} y_1(\tau) \cdot \lim_{\tau\to 1} y_2(\tau)^{-1},$$

which follows from the assumption that $\lim_{\tau\to 1} x_i(\tau)$ exists for $i = 1, 2$. Thus we must have that $x_1 = x_2$. □

The following corollary is immediate:

Corollary 24. *Let S be a 2-dimensional symplectic leaf of P and $T \in \partial S$ be a zero curve bounding S. Let $\gamma : [0, 1] \to P$ be a curve in P such that $\gamma(1) \in T$ and $\gamma(t) \in S$ for $t \in [0, 1)$.*

If $\sigma_1, \sigma_2 : [0, 1] \to X$ are curves in the isotropy subbundle lying over γ, i.e.,

$$s_X(\sigma_i(t)) = t_X(\sigma_i(t)) = \gamma(t), \quad i = 1, 2,$$

then $\sigma_1 = \sigma_2$.

Next, using this corollary, we will show that the restriction of a static Morita self-equivalence bimodule $X \in \mathrm{StatPic}(P)$ to a neighborhood of each zero curve has a unique identity bisection. Thus, we will obtain the local identity bisection of X on a neighborhood of the zero set.

Lemma 25. *Let $X \in \mathrm{StatPic}(P)$ and $T \subset P$ be a zero curve. Then there is a neighborhood N of T and a map $\sigma : N \to X$ so that $t_X(\sigma(x)) = s_X(\sigma(x))$ for all $x \in N$. Moreover, σ is uniquely determined by this property.*

Proof. First, we will show that there is a unique identity bisection on a neighborhood of a point on the zero curve.

Let $p_0 \in T$ be a point. Since s_X is a submersion, we can find a cross-section defined on a neighborhood N_0' of p_0, i.e., there is a map $\sigma : N_0' \to X$, so that $s_X \circ \sigma = \mathrm{id}$. Since the source and target maps coincide on $X_{|T}$, it follows that on a neighborhood $N_0 \subset N_0'$, the composition $t_X \circ \sigma$ is a diffeomorphism. We can also assume that the Poisson manifold $(N_0, \pi|_{N_0})$ is isomorphic to $(\mathbb{R}^2,\ x\partial_x \wedge \partial_y)$.

To construct an identity bisection of $X_{|N_0}$, we would like to apply the result of Proposition 22, which states that every bimodule over $(\mathbb{R}^2,\ x\partial_x \wedge \partial_y)$ is trivial, and therefore, by Lemma 3, has a unique identity bisection. The problem is that $X_{|N_0}$ has disconnected fibers and thus is not a bimodule in our sense. To find a bimodule $X_0 \subset X_{|N_0}$, identify the symplectic groupoid $\Gamma(N_0)$ with the connected component of the identity bisection in $\Gamma(P)_{|N_0}$. Denote by $\cdot$ the left and right actions of $\Gamma(N_0)$ on X obtained by restricting the actions of $\Gamma(P)$. Let

$$X_0 = \Gamma(N_0) \cdot \sigma(N_0) \cdot \Gamma(N_0),$$

where $\sigma : N_0 \to X$ is as above. Then X_0 is the connected component of $\sigma(N_0)$ inside of $X_{|N_0}$, and thus a symplectic manifold of the same dimension. Moreover, X_0 is clearly a $(\Gamma(N_0),\Gamma(N_0))$-bimodule. Furthermore, since the isotropy groups of $\Gamma(N_0)$ are trivial (they are isomorphic to the trivial fundamental group $\pi_1(N_0)$), it follows that the action of $\Gamma(N_0)$ on the fibers of X_0 is free; it is by definition transitive. Hence $X_0 \in \mathrm{StatPic}(N_0)$. By Proposition 22, $\mathrm{StatPic}(N_0) = \mathrm{StatPic}(\mathbb{R}^2)$ is trivial, and thus X_0 is isomorphic to $\Gamma(\mathbb{R}^2)$. Identify X_0 from now on with $\Gamma(\mathbb{R}^2)$. Let $\varepsilon_0 : N_0 \to \Gamma(N_0) = X_0 \subset X$ be the identity bisection, which exists by Lemma 3.

We will now extend this identity bisection to a cylindrical neighborhood of T. Let ξ be a modular vector field of π with the property that its orbits are periodic with the same period in a neighborhood of T. (See [Radko 2002] for existence of such a modular vector field). Let now N be the annular neighborhood of T obtained by translating N_0 along the flow of ξ. We will extend the identity bisection $\varepsilon_0 : N_0 \to X_0 \cong \Gamma(N_0) \subset X$ to an identity bisection $\varepsilon : N \to X$. Let ζ be the unique lift of ξ to X satisfying

$$(s_X)_*\zeta = (t_X)_*\zeta = \xi.$$

(See [Ginzburg and Lu 1992] for the existence of such a lift). Let Φ_t be the flow of ξ, and Ψ_t be the flow of ζ at time t. Note also that on N_0 the lift of the modular vector field satisfies

$$\varepsilon_0(\Phi_t(p)) = \Psi_t(\varepsilon_0(p)).$$

Now define $\varepsilon : N \to X$ by

(1) $\varepsilon|_{N_0} = \varepsilon_0$,

(2) $\varepsilon(p) = \Psi_t(\varepsilon(p_0))$ for $p = \Phi_t(p_0) \in N$.

To prove that ε is defined unambiguously , we need to check that if T_ξ is the period of ξ near the zero curve, then

$$\Psi_{T_\xi}(\varepsilon(p)) = \varepsilon(p) \quad \text{for all } p \in N.$$

Let $\gamma : [0, 1] \to P$ be a path such that $\gamma(0) = p$ and $\gamma(1) \in T$. Then the limits

$$\lim_{t\to 1} \varepsilon(\gamma(t)), \qquad \lim_{t\to 1} \Psi_{T_\xi}(\varepsilon(\gamma(t))) = \Psi_{T_\xi}(\lim_{t\to 1} \varepsilon(\gamma(t)))$$

both exist. By Lemma 23, this implies that

$$\varepsilon(p) = \Psi_{T_\xi}(\varepsilon(p)).$$

Thus ε is a well-defined identity bisection.

To prove uniqueness, assume that $\varepsilon, \varepsilon' : N \to X$ both satisfy

$$s_X \circ \varepsilon = t_X \circ \varepsilon = \mathrm{id} = t_X \circ \varepsilon' = s_X \circ \varepsilon'.$$

Then for any $p \in N \setminus T$, consider a path $\gamma : [0, 1] \to N$ such that $\gamma(t) \in N \setminus T$ for $t \in [0, 1)$ and $\gamma(1) \in T$. Then the existence of the limits

$$\lim_{t\to 1} \varepsilon(\gamma(t)), \qquad \lim_{t\to 1} \varepsilon'(\gamma(t))$$

implies that $\varepsilon(p) = \varepsilon'(p)$ by Lemma 23. The last corollary implies that ε is uniquely defined. □

Theorem 26. *Let π be a TSS on a surface P and $Z \subset P$ be the zero set of the Poisson structure. Then*

$$\mathrm{StatPic}(P) \cong \mathcal{M}(P \setminus Z).$$

Proof. Let $X \in \mathrm{StatPic}(P)$. Let $L \subset P$ be a symplectic leaf with the boundary $\partial L = \bigcup_{i=1}^k T_i$. By Lemma 25, for each zero curve $T_i \subset \partial L$ there exists a canonical local identity bisection $\varepsilon_i : N_i \to X$ in a neighborhood N_i of T_i. Fix points $p_1, \dots, p_k$ in the collars $C_1 = N_1 \cap L, \dots, C_k = N_k \cap L$ of the boundary curves. As in section 4, let $\mathcal{P}(L)$ be the set of pairs $(X, (x_1, \dots, x_k))$, where $x_i \in X_{p_i}$ are such that $\mathrm{Hol}_{x_i}([T_i]) = [T_i]$. Define a map $\psi_L : \mathrm{StatPic}(P) \to \mathcal{P}(L)$ by

$$\psi_L(X) = (X|_L,\ (\varepsilon_1(p_1), \dots, \varepsilon_k(p_k)) \in \mathcal{P}(L).$$

We claim that the map

$$\psi = \Pi_L \psi_L : \mathrm{StatPic}(P) \to \Pi_L \mathcal{P}(L) \cong \Pi_L \mathcal{M}(L) = \mathcal{M}(P \setminus Z)$$

is an isomorphism.

To prove the injectivity, observe that if $X \in \ker \psi_L$ for all L, then the local identity bisections can be extended to all of P. Thus X has a global identity bisection and so is trivial by Lemma 3.

We next claim that this map is surjective. Note that any element in $\Pi_L \mathcal{P}(L) \cong \Pi_L \mathcal{M}(L)$ can be represented by a symplectomorphism of $P \setminus Z$, which is identity near Z. Such a symplectomorphism extends to a Poisson isomorphism φ of P. If we set X to be $\Gamma_\varphi(P)$, it is easily seen that $\psi_L(X)$ is exactly the element of $\mathcal{P}(L)$ corresponding via the isomorphism with $\mathcal{M}(L)$ to φ. Thus $\psi(X) = \varphi$ and hence our map is onto. □

Corollary 27. *Any bimodule in the static Picard group can be represented by a Poisson diffeomorphism, i.e., for all $X \in \mathrm{StatPic}(P)$ there exists $\varphi \in \mathrm{Poiss}(P)$ such that $X \cong \Gamma_\varphi(P)$.*

Proof. For $X \in \mathrm{StatPic}(P)$ choose a symplectomorphism $\varphi \in \mathrm{Symp}(P \setminus Z)$ representing the corresponding class $\psi(X) \in \mathcal{M}(P \setminus Z)$. Since X preserves the leaf space pointwise, one can choose φ to be trivial near Z. Since φ preserves the restrictions of a modular vector field to the zero curves, it can be extended to a Poisson diffeomorphism $\varphi \in \mathrm{Poiss}(P)$ of the surface. □

Remark 28. In the case of a compact surface we have

$$\mathcal{M}(P \setminus Z) = \pi_0(\mathrm{Diff}\, P \text{ fix } Z) = \mathcal{M}(P \text{ fix } Z),$$

the group of isotopy classes of diffeomorphisms of P fixing a neighborhood of Z pointwise. If π is a TSS on a open surface P such that the zero set does not intersect the boundary ($Z \cap \partial P = \varnothing$), the answer for the Picard group is the same

$$\mathrm{StatPic}(P) \cong \mathcal{M}(P \text{ fix } Z). \tag{5–6}$$

Example 29. Let $C \simeq I \times S^1$ be a cylinder with coordinates (r, θ), where $r \in (-1, 1)$ and the Poisson structure

$$\pi = r\partial_r \wedge \partial_\theta.$$

The symplectic groupoid of C is given by $\Gamma(C) \simeq C \times \mathbb{R}^2$ with coordinates (r, θ, p, q), structure maps

$$s((r, \theta, p, q)) = (r, \theta),$$
$$t((r, \theta, p, q)) = (re^p,\ (\theta + q \cdot r) \mod 2\pi),$$

and a symplectic structure Ω. The source and target fibers at all points are isomorphic to $\mathbb{R}^2$, and the isotropy groups are given by

$$\Gamma_{(0,\theta_0)}(C) = \{(0, \theta_0, p, q) \mid (p, q) \in \mathbb{R}^2\} \simeq \mathbb{R}^2,$$
$$\Gamma_{(r_0,\theta_0)}(C) = \{(r_0, \theta_0, 0, 2\pi k/r_0) \mid k \in \mathbb{Z}\} \simeq \mathbb{Z}, \quad r_0 \neq 0.$$

Away from the zero curve, there are $\mathbb{Z}$ choices of an identity bisection, given by

$$\sigma_k = \{(r, \theta, 0, 2\pi k/r) \mid k \in \mathbb{Z}\}, \quad r \neq 0.$$

However, only one of these "almost identity bisections", σ_0, extends to the identity bisection $\varepsilon = \sigma_0 : C \to \Gamma(C)$ on the whole cylinder. This leads to the conclusion that the static Picard group of the cylinder is trivial, $\mathrm{StatPic}(C) = \{e\}$. This corresponds to the general answer (5–6) as follows: there are two 2-dimensional symplectic leaves, each diffeomorphic to a cylinder, with one boundary curve being T and the other a boundary curve of C. Since $\mathcal{M}(C \text{ fix } Z)$ is trivial, it follows that $\mathrm{StatPic}(C)$ is trivial. (By comparison, the (static) Picard group of a *symplectic* cylinder is $\mathrm{Pic}(S) \cong \mathrm{Out}(\mathbb{Z}) = \mathbb{Z}_2$).

Example 30. Let $C \simeq \mathbb{R} \times S^1$ be a cylinder with coordinates (r, θ), $r \in (-2, 2)$, and the Poisson structure

$$\pi = (r^2 - 1)\partial_r \wedge \partial_\theta,$$

vanishing linearly on two parallel circles, $r = \pm 1$. The symplectic groupoid $\Gamma(C)$ of this structure is given by $\Gamma(C) = C \times \mathbb{R}^2$ with coordinates (r, θ, p, q). The source and target maps are

$$s((r, \theta, p, q)) = (r, \theta),$$
$$t((r, \theta, p, q)) = (\alpha(r, p),\ (\theta + q \cdot (r^2 - 1)) \mod 2\pi),$$

where $\alpha(r, p) = \dfrac{(r+1) + (r-1)e^{2p}}{(r+1) - (r-1)e^{2p}}$ and the symplectic structure is

$$\Omega = -2qr\,dr \wedge dp + dr \wedge dq - d\theta \wedge dp + (r^2 - 1)dp \wedge dq.$$

Similarly to the previous example, the isotropy at each point away from the zero curves is isomorphic to $\mathbb{Z}$, while at a point on a zero curve the isotropy is $\mathbb{R}^2$. Let $C_{-1} = \{(r, \theta) \in C \mid r < 0\}$ and $C_1 = \{(r, \theta) \in C \mid r > 0\}$ be disjoint cylindrical neighborhoods of the zero circles $r = -1$ and $r = 1$ respectively, so that $C = C_{-1} \cup C_1 \cup \{r = 0\}$, and $\{r = 0\} \in \partial C_1, \partial C_{-1}$ is the common bounding circle. A bimodule $X \in \mathrm{Pic}(C)$ is obtained by "gluing" two bimodules $X_{-1} \in \mathrm{Pic}(C_{-1})$ and $X_1 \in \mathrm{Pic}(C_1)$. Since by the previous example the static Picard groups of C_1 and C_{-1} are trivial, $X_{-1} \cong \Gamma(C_{-1})$ and $X_1 \cong \Gamma(C_1)$. Each of the groupoids $\Gamma(C_{-1})$ and $\Gamma(C_1)$ has $\mathbb{Z}$ "almost identity bisections", called $\sigma_k^{(-1)}$ and $\sigma_k^{(1)}$ respectively (see previous example). The nonisomorphic bimodules $X \in \mathrm{Pic}(C)$ arise from various mismatches of these almost identity bisections (i.e., from gluing the unique identity bisection $\varepsilon^{(-1)} = \sigma_0^{(-1)}$ to all possible $\sigma_k^{(1)}$, $k \in \mathbb{Z}$). Thus, $\mathrm{StatPic}(C) \cong \mathbb{Z}$. According to our general formula (5–6),

$$\mathrm{StatPic}(C) = \mathcal{M}(C \text{ fix } Z) \cong \mathcal{M}(M_{-1} \text{ fix } \{r=-1\}) \times \mathcal{M}(L) \times \mathcal{M}(M_1 \text{ fix } \{r=1\}),$$

where $M_{-1} = \{r < -1\}$, $L = \{-1 < r < 1\}$ and $M_1 = \{r > 1\}$ are the two-dimensional symplectic leaves. Note that $\mathcal{M}(M_i \text{ fix } \{r=i\})$ is trivial for both $i = \pm 1$, and $\mathcal{M}(L) \cong \mathbb{Z}$ is generated by the Dehn twists.

If we "close up" the cylinder in this example to obtain a Poisson structure vanishing linearly on two parallel nonseparating circles on the torus, the static Picard group of the resulting structure will be $\mathbb{Z} \times \mathbb{Z}$. In our picture with almost-identity bisections each of the copies of $\mathbb{Z}$ corresponds to a possible "mismatch" of identity bisections in each of the symplectic cylinders between the zero curves. In our general description in terms of mapping class groups, this corresponds to Dehn twists in each of the cylinders between the zero curves.

More generally, a TSS $\pi = (r-1)(r-2)\dots(r-n)\partial_r \wedge \partial_\theta$ on a cylinder $C \simeq \mathbb{R} \times S^1 = \{(r,\theta), r \in (0, n+1)\}$ with n parallel zero curves, which are separating, has the static Picard group isomorphic to $\mathbb{Z}^{\times(n-1)}$. For $n = 2k$, the static Picard group of the corresponding structure on the torus obtained by "closing up" the cylinder, is isomorphic to $\mathbb{Z}^n$.

5.4. ***The Picard group of a TSS.*** To a TSS (P, π) we associate a graph $\mathcal{G}(P)$ in the following way:

(1) A vertex of the graph represents a 2-dimensional symplectic leaf.

(2) An edge represents a common bounding zero curve of two symplectic leaves.

(3) Each edge is oriented so that it points toward the vertex for which the Poisson structure is positive with respect to the orientation of the surface.

We label the graph as follows:

(1) A vertex is labeled by the genus of the corresponding leaf.

(2) An edge is labeled by the modular period of π around the corresponding zero curve.

Let $\mathrm{Aut}(\mathcal{G}(P))$ be the group of all automorphism of the graph $\mathcal{G}(P)$ and let $G \subset \mathrm{Aut}(\mathcal{G}(P))$ be its subgroup consisting of automorphisms preserving the labeling. Since any homeomorphism of the leaf space $\mathcal{L}(P)$ gives rise to an automorphism of the graph, we have a natural group homomorphism $\rho : \mathrm{Aut}(\mathcal{L}(P)) \to \mathrm{Aut}(\mathcal{G}(P))$.

For a bimodule $X \in \mathrm{Pic}(P)$, let $h_X \in \mathrm{Aut}(\mathcal{L}(P))$ be the associated homeomorphism of the leaf space, $h_X = t_X \circ s_X^{-1}$. Let $j : \mathrm{Poiss}(P) \to \mathrm{Pic}(P)$ be the map

$\varphi \mapsto \Gamma_\varphi(P)$. Thus we have the following group homomorphisms:

$$\mathrm{Poiss}(P) \xrightarrow{j} \mathrm{Pic}(P) \xrightarrow{h} \mathrm{Aut}(\mathcal{L}(P)) \xrightarrow{\rho} \mathrm{Aut}(\mathcal{G}(P)). \tag{5–7}$$

Lemma 31. *The graph automorphism $\rho(h_X)$ induced by an invertible bimodule $X \in \mathrm{Pic}(P)$ automatically preserves the labeling. In symbols, $Im(\rho \circ h) \subset G$.*

Any graph automorphism that preserves the labeling comes from a Poisson automorphism of the structure, i.e., for all $\theta \in G \subset \mathrm{Aut}(\mathcal{G}(P))$ there exists $\varphi \in \mathrm{Poiss}(P)$ such that $\rho(h(j(\varphi))) = \theta$. In other words, the composition $\rho \circ h \circ j$ maps $\mathrm{Poiss}(P, \pi)$ onto G.

Proof. Let $X \in \mathrm{Pic}(P)$ be a bimodule. Since by [Bursztyn and Radko 2003] the modular periods are invariant under Morita equivalence, $\rho(h_X) \in \mathrm{Aut}(\mathcal{G}(P))$ preserves the labeling of the edges. Since for any 2-dimensional leaf L the restriction $X|_L$ is a Morita equivalence between L and $h_X(L)$, by a result of Ping Xu [1991a], we have $\pi_1(L) = \pi_1(h_X(L))$. Since h_X is a homeomorphism, L and $h_X(L)$ have the same number of boundary components. Thus, L and $h_X(L)$ have the same genus. Therefore, the labeling of the vertices is also preserved.

Let $\theta \in G$ be a graph automorphism preserving the labeling. Since the graph $\mathcal{G}(P)$ completely encodes the topology of the decomposition of the surface into the 2-dimensional symplectic leaves, there exists a diffeomorphism $\alpha \in \mathrm{Diff}(P)$, sending leaves to leaves, and inducing $\theta \in G \subset \mathrm{Aut}(\mathcal{G}(P))$. We may furthermore assume that α preserves the restrictions of a modular vector field to the zero curves.

Consider the original Poisson structure π and the Poisson structure $\pi' = \alpha_*\pi$ induced by the diffeomorphism α. The zero sets of these two structures are the same, and for any zero curve $T \in Z(\pi) = Z(\pi')$ the restrictions of the modular vector fields to T are equal. Thus, there exists a smooth function $f \in C^\infty(P)$ such that $\alpha_*\pi = f \cdot \pi$, with $f \neq 0$.

We claim that $f > 0$. Since f is continuous and nonzero, it is sufficient to prove that $f > 0$ at a point. Let I be a segment of a common zero curve of π and π' and (x, y) and (x', y') be the coordinates in a neighborhood N of I such that $\pi = x\partial_x \wedge \partial_y$ and $\pi' = x'\partial_{x'} \wedge \partial_{y'}$. Note that $x = x' = 0$ on I. Since the restriction of the modular vector field to I is by assumption preserved by α, it follows that $\partial_y = \partial_{y'}$ on I.

Let $p \in I$ be a point corresponding to $x = y = 0$. It follows that the Jacobian J_α of α at p has the form

$$J_\alpha(p) = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

We have

$$f(x, y) = \frac{x'}{x} \det J_\alpha(x, y).$$

Since $x' = a(y) \cdot x + O(x^2)$, where $a(y)$ is such that $a(0) = a$, in a neighborhood of p, we get finally that

$$f((0,0)) = \lim_{x \to 0} \frac{x'}{x} \det J_\alpha(x, y) = a^2 > 0.$$

Thus $f > 0$. In particular, the symplectic forms corresponding to π and π' on the same two-dimensional symplectic leaf have the same sign (with respect to a chosen symplectic form on the surface).

By applying Moser's argument to each two-dimensional symplectic leaf, we conclude that α is isotopic to a Poisson diffeomorphism of π, i.e., there is a family α_t, $t \in [0, 1]$, of diffeomorphisms such that

- $\alpha_0 = \alpha$;
- $\alpha_1 \in \mathrm{Poiss}(P)$;
- α_t maps leaves to leaves;
- the restriction of α_t to the zero set is equal to a translation by the flow of a modular vector field;

It follows that each α_t induces the same automorphism $\theta \in G$, and so

$$\theta = \rho(h(j(\alpha_1))). \qquad \square$$

Lemma 32. *For a TSS, the map $j : \mathrm{Poiss}(P) \to \mathrm{Pic}(P)$ is surjective.*

Proof. Let $X \in \mathrm{Pic}(P)$. By the previous Lemma, the map $\rho \circ h \circ j : \mathrm{Poiss}(P) \to G$ is onto. Thus, there exists a Poisson diffeomorphism $\varphi \in (\rho \circ h \circ j)^{-1}((\rho \circ h)(X)) \subset \mathrm{Poiss}(P)$ inducing the same graph automorphism as X. By composing X with $\Gamma_{\varphi^{-1}}(P)$ we may assume that the automorphism $\rho(h_X)$ of the labeled graph induced by X is the identity map. Next, by composing X with $\Gamma_\psi(P)$ (where the ψ is determined by the flows of a modular vector field around the zero curves), we may assume that the leaf space automorphism h_X fixes pointwise the set of zero-dimensional leaves. Thus, $X \in \mathrm{StatPic}(P)$. By Corollary 27, X comes from a Poisson automorphism. $\square$

Thus, for a TSS, the map $j : \mathrm{Poiss}(P) \to \mathrm{Pic}(P)$ is onto. Its kernel consists of the inner Poisson isomorphisms (i.e., the Poisson isomorphisms implemented by lagrangian bisections, see [Bursztyn and Weinstein 2004]).

Corollary 33. *For a TSS,*

$$\mathrm{Pic}(P, \pi) \cong (\mathbb{T}^n \times \prod_{j \in Jd} \mathcal{M}(L_j)) \rtimes G.$$

Here for each zero curve, $\mathbb{T}$ is the 1-torus of translations by the flow of the restriction of a modular vector field to this curve. $\mathcal{M}(L_j)$ denotes the mapping class *group $\mathcal{M}$ for each of the 2-dimensional symplectic leaves L_j, and $G \subset \mathrm{Aut}(\mathcal{G}(P))$ is the group of the graph automorphisms preserving the labeling.*

Corollary 34. *For a TSS*, $\mathrm{Pic}(P) \cong \mathrm{Out}\,\mathrm{Poiss}(P)$.

Definition 35. Let $\mathcal{M}(P, \pi)$ be the group of classes of diffeomorphisms $\varphi : P \to P$ that map zero curves to zero curves and preserve the restrictions of the modular vector field to the zero curves, up to isotopies by diffeomorphisms inducing the identity map on the leaf space.

Note that $\mathcal{M}(P \setminus Z)$ can be considered as a subgroup of $\mathcal{M}(P, \pi)$ consisting of classes of diffeomorphisms which preserve the leaf space.

Theorem 36. *For a TSS*, $\mathrm{Pic}(P) \cong \mathcal{M}(P, \pi)$.

Remark 37. By an argument similar to the one in Lemma 12, we see that $\mathcal{M}(P, \pi)$ is contractible (by isotopies that fix neighborhoods of zero curves) to the set of Poisson diffeomorphisms of P.

Proof. Using the Remark, represent an element of $\mathcal{M}(P, \pi)$ by a Poisson automorphism φ. Define the map $\eta : \mathcal{M}(P, \pi) \to \mathrm{Pic}(P)$ by sending φ to the associated bimodule $\Gamma_\varphi(P)$. The restriction of this map to $\mathcal{M}(P \setminus Z) \subset \mathcal{M}(P, \pi)$ is the isomorphism between $\mathcal{M}(P \setminus Z)$ and $\mathrm{StatPic}(P)$. The map η is clearly multiplicative. To check that this map is well-defined, it is enough to verify this on the class of the identity, which reduces to the fact that the map from $\mathcal{M}(P \setminus Z) \subset \mathcal{M}(P, \pi)$ to $\mathrm{StatPic}(P)$ is well-defined. By Theorem 26, this map is surjective. The kernel of η clearly lies in $\mathrm{StatPic}(P)$, and hence is trivial, since the restriction of η to $\mathcal{M}(P \setminus Z) \subset \mathcal{M}(P, \pi)$ is an isomorphism. □

Remark 38. The answer remains the same for a TSS π on an open surface P satisfying $Z \cap \partial P = \varnothing$.

Example 39. For a TSS on the cylinder with one separating zero curve (see Example 29), $\mathrm{Pic}(C) \cong \mathbb{Z}_2 \times \mathbb{T}^1 \cong \mathrm{Out}\,\mathrm{Poiss}(C)$, where $\mathbb{Z}_2$ (isomorphic to the group of automorphisms of the corresponding graph, which preserve the labeling) is generated by the (orientation-reversing) Poisson diffeomorphism $\Phi(r, \theta) = (-r, \theta)$, and the 1-torus $\mathbb{T}^1$ is generated by the flow of a modular vector field around the zero curve.

Example 40. For a TSS on the cylinder with two separating zero curves (see Example 30), we have $\mathrm{Pic}(C) \cong \mathbb{Z} \times \mathbb{T} \times \mathbb{Z}_2$, where $\mathbb{Z}$ is the static Picard group (generated by the Dehn twist of the middle symplectic cylinder), $\mathbb{T}^2$ is the torus generated by rotations of the zero curves, and $\mathbb{Z}_2$ (isomorphic to the group of graph automorphisms which preserve the labeling) is generated by the flip diffeomorphism $\Phi(r, \theta) = (-r, \theta)$. Denote by $X_{(k,\phi_1,\phi_2,\delta)} \in \mathrm{Pic}(C)$ the bimodule corresponding to $(k, \phi_1, \phi_2, \delta) \in \mathbb{Z} \times \mathbb{T}^2 \times \mathbb{Z}_2$. Then $X_{(k,0,0,1)} \in \mathrm{Pic}(C) \cong \mathbb{Z}$ are the bimodules in the static Picard group. Notice that $X_{(k,0,0,1)}$ can be "connected" to the next static bimodule $X_{((k+1),0,0,1)}$ by a path of bimodules $X(t) = X_{(k,t,0,1)}$, where $t \in [0, 2\pi]$, in the full Picard group. Indeed, we have $X(0) = X_{(k,0,0,1)}$ and $X(2\pi) = X_{(k,2\pi,0,1)} \cong X_{(k+1,0,0,1)}$.

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