Pacific Journal of Mathematics

MINIMAL CAPILLARY GRAPHS OVER REGULAR $2n$-GONS

Robert Huff

Volume 224 No. 2 April 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 224, No. 2, 2006

MINIMAL CAPILLARY GRAPHS OVER REGULAR 2*n*-GONS

ROBERT HUFF

This paper follows previous work by Huff and McCuan, who provided for $0 < \gamma < \pi$ a geometric construction of minimal capillary graphs over a square with constant contact angles on the edges alternating between γ and $\pi - \gamma$. Here the result is extended to regular $2n$-gons. Regularity results are obtained for these graphs, and explicit, conformal parametrizations are given for the Jenkins–Serrin graphs corresponding to $\gamma \in \{0, \pi\}$.

Introduction

In this paper, we prove the following theorem, where the uniqueness statement follows from [Finn and Lu 1998, Theorem 3.1].

Theorem 1. *Let Q_n be a regular $2n$-gon and $0 \le \gamma \le \pi$. There is a unique* (*up to vertical translation*) *minimal graph over Q_n with constant contact angles on the edges alternating between γ and $\pi - \gamma$. If*

$$0 < \gamma < \frac{(n-1)\pi}{2n} \quad or \quad \frac{(n+1)\pi}{2n} < \gamma < \pi,$$

then there is a finite jump discontinuity over each vertex. If $\gamma \in \{0, \pi\}$, then the corresponding graph is a Jenkins–Serrin graph.

The case $n = 2$ and $0 < \gamma < \pi$ has previously been studied in [Huff and McCuan 2006], and by Concus, Finn, and McCuan in [Concus et al. 2001]. Existence was proved in the latter paper, while regularity and existence of the jump discontinuity was shown in the former. To prove existence here, we assume symmetries and then determine the image under the Gauss map, which is conformal on a minimal surface, of our fundamental piece. Next, we determine the image of the conformal map developing the (square root of) the complexified second fundamental form on the graph. As a result, we obtain conformal parametrizations of the graphs, and those corresponding to $\gamma \in \{0, \pi\}$ (Jenkins–Serrin graphs [1966]) and $(n-1)\pi/(2n) \le \gamma \le (n+1)\pi/(2n)$ can be made explicit. Another consequence of the construction is that Sobolev embedding theorems can be used to compute appropriate regularity properties of the graphs.

MSC2000: primary 76B45; secondary 53A10.

Keywords: capillarity, contact angle, minimal surface, Weierstrass representation.

1. Background

The Weierstrass representation. Given a domain $\Omega \subset \mathbb{C}$, the Weierstrass representation theorem says that any orientation-preserving conformal minimal immersion

$$X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$$

can be expressed, up to translation, in terms of a meromorphic function g and a holomorphic one-form dh by the formula

$$X(z) = \mathrm{Re} \int_{\cdot}^{z} \left(\frac{1}{2}(g^{-1} - g)\, dh,\ \frac{i}{2}(g^{-1} + g)\, dh,\ dh \right), \tag{1}$$

where g is the stereographic projection of the Gauss map and

$$dh = \left(\frac{\partial X_3}{\partial x} - i \frac{\partial X_3}{\partial y} \right) dz$$

is called the complex height differential (note that $\mathrm{Re}\, dh = dX_3$). Conversely, the theorem states that if g is a meromorphic function and dh a holomorphic one-form on Ω such that dh has a zero of order n at z if and only if g has a zero or pole of order n at z, then (1) gives an orientation-preserving conformal minimal immersion on Ω that is well-defined, provided that

$$\mathrm{Re} \int_{c} \left(\frac{1}{2}(g^{-1} - g)\, dh,\ \frac{i}{2}(g^{-1} + g)\, dh,\ dh \right) = 0$$

for every simple closed curve $c \subset \Omega$; this condition is satisfied automatically if Ω is simply connected.

Determining dh via the second fundamental form. For a minimal surface given by Weierstrass data g and dh, we have, for tangent vectors v and w,

$$\frac{dg(v)\, dh(w)}{g} = II(v, w) - i\, II(v, iw),$$

where II is the second fundamental form on the surface (for details, see [Hoffman and Karcher 1997]). It follows that

$$c \text{ is a principal curve} \iff \frac{dg(\dot{c})\, dh(\dot{c})}{g} \in \mathbb{R} \tag{2}$$

and

$$c \text{ is an asymptotic curve} \iff \frac{dg(\dot{c})\, dh(\dot{c})}{g} \in i\mathbb{R}. \tag{3}$$

We see from these two equivalences that the function ζ given by

$$\zeta(z) = \int_{\cdot}^{z} \sqrt{\frac{dg\,dh}{g}} \tag{4}$$

maps principal curves into vertical or horizontal lines and asymptotic curves into lines in one of the directions $e^{\pm i\pi/4}$.

The map ζ is called the *developing map* of the one-form $\sqrt{dg\,dh/g}$. It is a local isometry between the minimal surface equipped with the conformal cone metric $|dg\,dh/g|$ and $\mathbb{R}^2$ equipped with the Euclidean metric.

Each surface considered in this paper will have boundary consisting of principal and asymptotic curves, which will allow us to determine the function ζ. Once this is done, we can use (4) to conclude that

$$dh = \frac{g(d\zeta)^2}{dg}. \tag{5}$$

Extremal length. To prove the existence of an appropriate ζ, we will need to show the existence of a biholomorphic, edge-preserving map between two curvilinear polygons (polygons whose edges are arcs of circles or Euclidean line segments). To do this, we will need some properties of the conformal invariant extremal length. We will restrict our attention to curvilinear polygons, although in general extremal length is defined on arbitrary domains.

Given a curvilinear polygon Δ, a Borel measurable function $\rho > 0$ on Δ defines a conformal metric $\rho(dx^2 + dy^2)$. The length of a curve $\gamma \subset \Delta$ with respect to ρ is denoted $\ell_\rho(\gamma)$ (with $|\gamma|$ denoting Euclidean length), and the ρ-area of Δ is denoted by A_ρ. With this notation, we define the extremal length between edges A and B by

$$\operatorname{Ext}_\Delta(A, B) = \sup_\rho \frac{\inf_\gamma \ell_\rho^2(\gamma)}{A_\rho},$$

where the infimum is taken over all curves $\gamma : [0, 1] \to \Delta$ such that $\gamma(0) \in A$, $\gamma(1) \in B$, and $\gamma(t) \subset \mathring{\Delta}$ for $t \in (0, 1)$. Extremal length is invariant under biholomorphisms and has the following properties, which we record here (see [Ahlfors 1973] for details).

Proposition. (i) *If A and B are adjacent, then* $\operatorname{Ext}_\Delta(A, B) = 0$.

(ii) *If B is degenerate* (*a point*) *and* $\operatorname{dist}(A, B) > 0$, *then* $\operatorname{Ext}_\Delta(A, B) = \infty$.

(iii) *If $\Delta_1 \subset \Delta_2$ are such that edges $A_k, B_k \subset \Delta_k$, $k = 1, 2$, satisfy $A_1 \subset A_2$ and $B_1 \subset B_2$, then*

$$\operatorname{Ext}_{\Delta_2}(A_2, B_2) \le \operatorname{Ext}_{\Delta_1}(A_1, B_1),$$

where the inequality is strict if $A_1 \neq A_2$ or $B_1 \neq B_2$.

2. Construction

Determining the image of the Gauss map. Given $0 \le \gamma \le \pi$ and a regular $2n$-gon Q_n centered at the origin, let's assume the existence of a minimal graph $M_{\gamma,n}$ over Q_n with constant contact angles on the edges alternating between γ and $\pi - \gamma$. Such a graph, should it exist, is unique up to vertical translation, and so we normalize so that $0 \in M_{\gamma,n}$.

By the symmetry of the contact angle condition, it is sufficient to consider only $0 \le \gamma < \pi/2$ (Note that $M_{\pi/2,n} = Q_n$), and we can simplify the problem further if we assume the following additional symmetries:

(S1) $M_{\gamma,n}$ is symmetric with respect to reflection through any vertical plane containing a bisector of two opposite edges of Q_n.

(S2) $M_{\gamma,n}$ is symmetric with respect to 180 degree rotation around any line connecting two opposite vertices of Q_n.

If we take the quotient by the symmetries (S1) and (S2), we are left with a fundamental piece $\hat{M}_{\gamma,n}$ that is a graph over a triangle T_n (see Figure 1) which is the quotient of Q_n by its symmetry group. For computational purposes, we rotate Q_n if necessary so that the edge s_1 of T_n connecting the center of Q_n to the midpoint of one of its edges lies on the positive x_1-axis (again, see Figure 1).

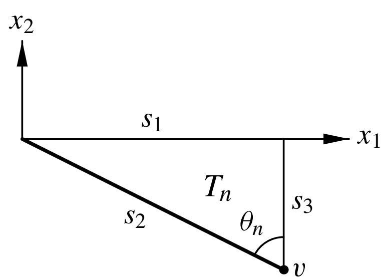


Figure 1. The fundamental triangle T_n.

We now wish to determine the image of the (downward pointing) Gauss map N on $\partial\hat{M}_{\gamma,n}$ under the stereographic projection σ that takes the south pole $(0, 0, -1)$ of S^2 to $0 \in \mathbb{C}$, the north pole to ∞, and the equator to the unit circle. Beginning with s_1, we assume the corresponding curve of $\partial\hat{M}_{\gamma,n}$ given by $f(x_1)$ is such that $f'' > 0$. Then it follows from the symmetries (S1) that the image of $\sigma \circ N$ along this curve is contained in the positive x-axis. Continuing, from the symmetries (S2) we have

$$s_2 \subset \partial\hat{M}_{\gamma,n},$$

and hence it follows that $\sigma \circ N(s_2)$ is contained in the line

$$L_n = \mathbb{R}e^{i\theta_n},$$

where

$$\theta_n = \frac{(n-1)\pi}{2n}.$$

For s_3, we have from the contact angle condition that $\sigma \circ N$ is contained in the circle

$$C_\gamma = \partial B(\sec\gamma, \tan\gamma),$$

where $B(\sec\gamma, \tan\gamma)$ is the disk centered at $\sec\gamma$ with radius $\tan\gamma$. Note that if $\gamma = 0$, then C_γ is just a point. In this case, as we will see below, the (Jenkins–Serrin) graph $M_{0,n}$ is infinite over the edges of Q_n.

To conclude our analysis of N on $\partial\hat{M}_{\gamma,n}$, we consider the behavior of the graph at the vertex v labeled in Figure 1. This behavior, which depends on the relation of the contact angle γ to the wedge angle $2\theta_n$, falls into one of the three cases below, as illustrated by Figure 2. (In the first two cases, we denote both the vertex and the jump discontinuity over the vertex by v.)

(C1) $\gamma = 0$: We assume there is an infinite jump discontinuity at v. That is, the vertical line in $\mathbb{R}^3$ passing through v is contained in $\partial M_{0,n}$. Since $\sigma \circ N$ along a vertical line is contained in the unit circle S^1, we conclude $\Omega_{0,n}$ is the curvilinear triangle shown in Figure 2 bounded by a segment of the positive real axis, a segment of L_n, and an arc of S^1.

(C2) $0 < \gamma < \theta_n$: We assume there is a finite jump discontinuity at v. That is, a vertical line segment passing through v is contained in $\partial M_{\gamma,n}$. Here $\Omega_{\gamma,n}$ is a curvilinear quadrilateral, as shown in Figure 2, bounded by a segment of the positive real axis, a segment of L_n, an arc of C_γ, and an arc of S^1.

(C3) $\theta_n \le \gamma < \pi/2$: In this case, Concus and Finn [Concus and Finn 1996] have shown $u_{\gamma,n}$ must be continuous at v if $\gamma \ne \theta_n$, where $\mathrm{Graph}(u_{\gamma,n}) = M_{\gamma,n}$, and we assume continuity for the case $\gamma = \theta_n$. Thus, we conclude $\Omega_{\gamma,n}$ is a curvilinear triangle as shown in Figure 2 bounded by a segment of the positive real axis, a segment of L_n, and an arc of C_γ.

Determining the developed image of $\sqrt{dg\,dh/g}$. We wish to parametrize $\hat{M}_{\gamma,n}$ on $\Omega_{\gamma,n}$ by finding the appropriate Weierstrass data g and dh. Since $\Omega_{\gamma,n}$ is the image of $\hat{M}_{\gamma,n}$ under stereographic projection of the Gauss map, we take

$$g(z) = z$$

for our first piece of data. For the second piece of data, we determine the conformal map $\zeta = \zeta_{\gamma,n}$ on $\Omega_{\gamma,n}$ given by (4). Then we solve for dh in terms of $\zeta_{\gamma,n}$ and obtain equation (5).

To determine $\zeta_{\gamma,n}$, we first note that each curve in $\partial\hat{M}_{\gamma,n}$ is either an asymptotic curve or a principal curve. Indeed, since s_2 and v are straight lines or straight

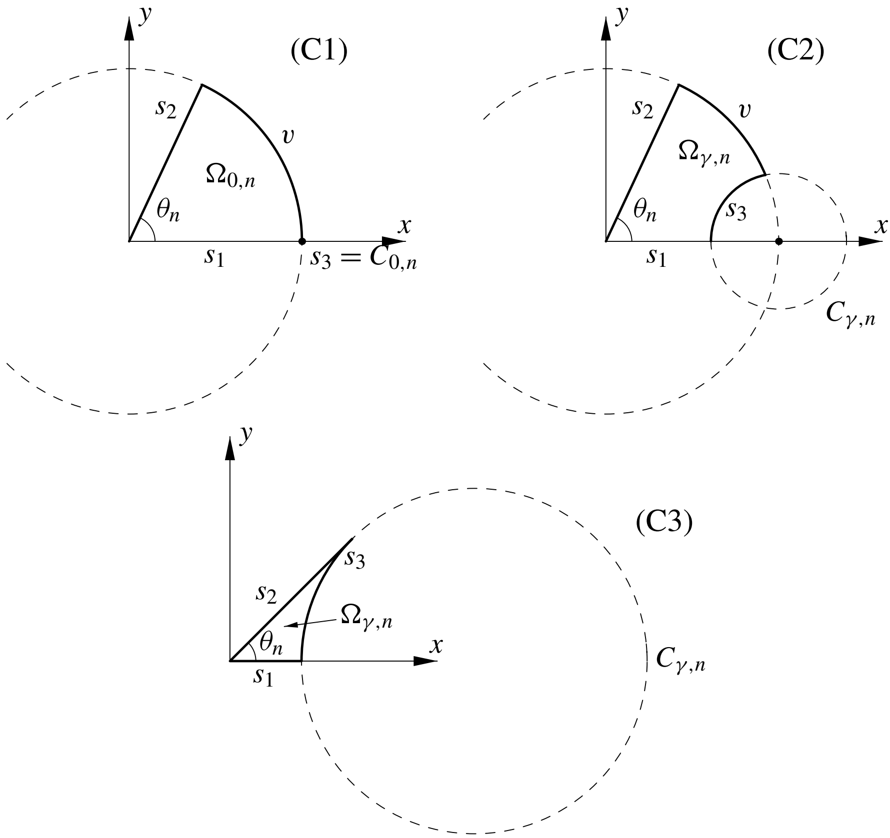
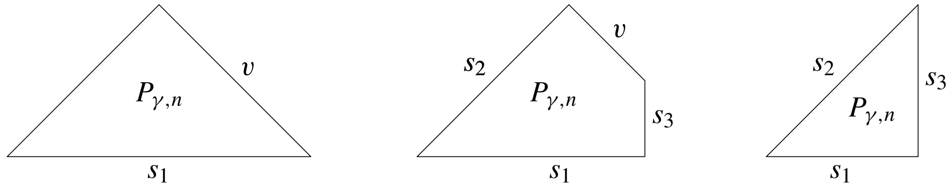


Figure 2. The image $\Omega_{\gamma,n}$ corresponding to cases (C1)–(C3).

segments, it follows immediately that they are asymptotic. For s_1 and s_3, we have that each is a planar curve along which the surface meets the plane of the curve at a constant angle. By Joachimstahl's theorem, such curves are principal. Thus, by (3), the curves s_2 and v are mapped by ζ into lines in one of the directions $e^{\pm i\pi/4}$, while s_1 and s_3 are mapped into horizontal or vertical lines. Based on this information, we conclude the image of ζ is a Euclidean polygon $P_{\gamma,n}$ with edges oriented and labeled as in Figure 3, where the number of edges and the labeling of the edges depend on the cases (C1), (C2), and (C3). Now, scaling $P_{\gamma,n}$ by a real number $\lambda > 0$ results in scaling dh, and thus $\hat{M}_{\gamma,n}$, by λ^2. Therefore, we can select one graph from each homothety class by normalizing $P_{\gamma,n}$ so that $|s_1| = 1$. Note that with this normalization, there is only one $P_{\gamma,n}$ corresponding to case (C1) and only one $P_{\gamma,n}$ corresponding to case (C3). In case (C2), the space $\{P_{\gamma,n}\}$ is one-dimensional. This space can be parametrized by the length of the edge s_3, where $0 < |s_3| < 1$.



(C1) (C2) (C3)

Figure 3. The developed image of ζ corresponding to cases (C1)–(C3).

Therefore, the map $\zeta_{\gamma,n}$ is an edge-preserving biholomorphism between $\Omega_{\gamma,n}$ and some $P_{\gamma,n}$. Since each of the two domains is simply connected and bounded by a simple closed curve, it follows that there exists a biholomorphism between them. Furthermore, we are allowed to specify the images of three points on the boundary. Thus, if in cases (C1) and (C3) we specify that the vertices of $\Omega_{\gamma,n}$ are mapped to the corresponding vertices of $P_{\gamma,n}$, then the edge-preserving property follows immediately.

For case (C2), the two domains are quadrilaterals, and so the result is not immediate. Here we normalize by specifying the images of three vertices of $\Omega_{\gamma,n}$ so that the edges s_1 and s_2 are preserved. What remains is a one-parameter family of biholomorphisms, and we aim to show there exists a map within this family that preserves all four edges. To prove this, we consider the quantity

$$\mathrm{Ext}_{P_{\gamma,n}}(s_2, s_3).$$

First of all, it follows from part (i) of the Proposition (page 265) that

$$\mathrm{Ext}_{P_{\gamma,n}}(s_2, s_3) \to 0 \quad \text{as} \quad |s_3| \to 1.$$

Then, by part (ii), it follows that

$$\mathrm{Ext}_{P_{\gamma,n}}(s_2, s_3) \to \infty \quad \text{as} \quad |s_3| \to 0.$$

Hence, it follows by continuity that there is some intermediate $|\hat{s}_3|$ and corresponding $\hat{P}_{\gamma,n}$ such that

$$\mathrm{Ext}_{\Omega_{\gamma,n}}(s_2, s_3) = \mathrm{Ext}_{\hat{P}_{\gamma,n}}(s_2, s_3).$$

Using part (iii) of the Proposition and the conformal invariance of extremal length, we see that the fourth vertex $v \cap s_3$ must also be preserved. Thus, the normalized conformal biholomorphism

$$\zeta_{\gamma,n} : \Omega_{\gamma,n} \to \hat{P}_{\gamma,n}$$

is the desired edge-preserving map.

3. Verification of parametrizations

Let $X_{\gamma,n}$ on $\Omega_{\gamma,n}$ have the form (1), where

$$g(z)=z \quad \text{and} \quad dh=\frac{g(d\zeta_{\gamma,n})^2}{dz}.$$

Here we choose the base point of integration to be $0=s_1\cap s_2$ so that

$$X_{\gamma,n}(0)=0.$$

We seek to verify that the image of $X_{\gamma,n}$ gives a surface in $\mathbb{R}^3$ that can be extended by symmetry to the desired capillary graph over a regular $2n$-gon Q_n. By construction, we know $X_{\gamma,n}$ is a minimal immersion. What remains is to verify its image is also a graph over T_n that has the desired properties. To accomplish this, we investigate $X_{\gamma,n}$ along $\partial\Omega_{\gamma,n}$, and we separate this investigation into the three cases (C1)–(C3).

Case (C3). The first observation is that

(6) $$X_{\gamma,n} \textit{ is continuous on } \overline{\Omega}_{\gamma,n}.$$

To see this, let ϕ_j denote the angle between any two adjacent edges e_1 and e_2 on $\Omega_{\gamma,n}$, and let ψ_j denote the angle between the corresponding edges on $P_{\gamma,n}$. Then we have

$$\zeta_{\gamma,n}(z)=\zeta_{\gamma,n}(e_1\cap e_2)+(z-e_1\cap e_2)^{\psi_j/\phi_j}\zeta_0(z)$$

in an $\Omega_{\gamma,n}$-neighborhood of $e_1\cap e_2$, where ζ_0 is holomorphic and nonzero at $e_1\cap e_2$. Hence, it follows that

$$\zeta'_{\gamma,n}(z)^2=(z-e_1\cap e_2)^{2(\psi_j/\phi_j-1)}\tilde{\zeta}_0(z),$$

where $\tilde{\zeta}_0$ is holomorphic and nonzero at $e_1\cap e_2$. Clearly, from the geometry of $\Omega_{\gamma,n}$ and $\hat{P}_{\gamma,n}$ we have $\psi_j/\phi_j>\frac{1}{2}$, so

$$2\Big(\frac{\psi_j}{\phi_j}-1\Big)>-1.$$

Thus, it follows that

$$dh=\frac{g(d\zeta_{\gamma,n})^2}{dg}=z\zeta'(z)^2\,dz$$

is integrable on $\overline{\Omega}_{\gamma,n}$, proving (6).

Beginning our analysis on $\partial\Omega_{\gamma,n}$, we parametrize s_1 from 0 to $\sec\gamma-\tan\gamma$ by

$$z_1(t)=t, \qquad 0<t<\sec\gamma-\tan\gamma.$$

Then

$$dz(\dot{z}_1)=1,$$

and from the geometry of $\zeta_{\gamma,n}$ it follows that

$$d\zeta_{\gamma,n}(\dot{z}_1)^2 > 0.$$

Using this information, we compute

$$\begin{aligned} d\left(X_{\gamma,n}\right)_1(\dot{z}_1) &= \mathrm{Re}\big(\tfrac{1}{2}(1-t^2)\, d\zeta_{\gamma,n}(\dot{z}_1)^2\big) > 0, \\ d\left(X_{\gamma,n}\right)_2(\dot{z}_1) &= \mathrm{Re}\big(\tfrac{i}{2}(1+t^2)\, d\zeta_{\gamma,n}(\dot{z}_1)^2\big) = 0, \\ d\left(X_{\gamma,n}\right)_3(\dot{z}_1) &= \mathrm{Re}(t d\zeta_{\gamma,n}(\dot{z}_1)^2) > 0. \end{aligned} \tag{7}$$

Thus, the computations above show that

$$X_{\gamma,n}(s_1) \subset \{x_2 = 0\} \textit{ is a curve of mirror symmetry}, \tag{8}$$

where the statement about mirror symmetry follows from the fact that $g(z) = z$. Moreover, the equations (7) yield

$$\left(X_{\gamma,n}\right)_1 \textit{ and } \left(X_{\gamma,n}\right)_3 \textit{ increase as } t \textit{ increases},$$

so that

$$X_{\gamma,n}(s_1) \textit{ is a graph over it projection into the } x_1x_2\textit{-plane}. \tag{9}$$

Continuing, we parametrize s_2 from 0 to $e^{i\theta_n}$ by

$$z_2(t) = te^{i\theta_n}, \quad 0 < t < \rho, \qquad \text{where } \rho < 1.$$

Hence, it follows that

$$dz(\dot{z}_2) = e^{i\theta_n},$$

and since

$$\frac{1}{i} d\zeta_{\gamma,n}(\dot{z}_2)^2 > 0,$$

we have

$$\begin{aligned} d\left(X_{\gamma,n}\right)_1(\dot{z}_2) &= \mathrm{Re}\left(\frac{1}{2}(1-t^2 e^{i2\theta_n})\frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{e^{i\theta_n}}\right) \\ &= \frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}\mathrm{Re}(i(e^{-i\theta_n} - t^2 e^{i\theta_n})) = \frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}(1+t^2)\sin\theta_n > 0, \\ d\left(X_{\gamma,n}\right)_2(\dot{z}_2) &= \mathrm{Re}\left(\frac{i}{2}(1+t^2 e^{i2\theta_n})\frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{e^{i\theta_n}}\right) \\ &= \frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}\mathrm{Re}(-(e^{-i\theta_n} + t^2 e^{i\theta_n})) = -\frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}(1+t^2)\cos\theta_n < 0, \\ d\left(X_{\gamma,n}\right)_3(\dot{z}_2) &= \mathrm{Re}(t d\zeta_{\gamma,n}(\dot{z}_2)^2) = 0. \end{aligned}$$

It follows that

$$\frac{d(X_{\gamma,n})_2(\dot{z}_2)}{d(X_{\gamma,n})_1(\dot{z}_2)} = -\cot\theta_n,$$

and since $d\left(X_{\gamma,n}\right)_3(\dot{z}_2) = 0$,

(10) *$X_{\gamma,n}$ maps s_2 monotonically onto a straight segment contained in the ray* $R_{\theta_n} = \{(x_1, x_2, 0) \mid x_1 > 0 \text{ and } x_2 = -(\cot\theta_n)x_1\}$.

Finally, we parametrize the contact curve s_3 from $s_2 \cap s_3$ to $s_1 \cap s_3$ by

$$z_3(t) = \sec\gamma + \tan\gamma\, e^{it}, \quad T_{\gamma,n} < t < \pi.$$

Now, the value $T_{\gamma,n}$ is greatest in the borderline case $\gamma = \theta_n$. Here the circle C_γ intersects the line L_n tangentially at $z = e^{i\theta_n}$, and a simple calculation yields

$$T_{\theta_n,n} = \frac{\pi}{2} - \gamma.$$

Thus, we have

$$T_{\gamma,n} \le \frac{\pi}{2} - \gamma, \quad \theta_n \le \gamma < \frac{\pi}{2},$$

so that

$$\cos T_{\gamma,n} \le -\sin\gamma, \quad \theta_n \le \gamma < \frac{\pi}{2}. \tag{11}$$

Continuing, we have $dz(\dot{z}_3) = i\tan\gamma\, e^{it}$ and $d\zeta_{\gamma,n}(\dot{z}_3)^2 < 0$, so that

$$\begin{aligned} d\left(X_{\gamma,n}\right)_1(\dot{z}_3) &= \operatorname{Re}\left(\frac{1}{2}(1-\sec^2\gamma - 2\sec\gamma\tan\gamma\, e^{it} - \tan^2\gamma\, e^{i2t})\frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{i\tan\gamma\, e^{it}}\right) \\ &= \tfrac{1}{2} d\zeta_{\gamma,n}(\dot{z}_3)^2 \operatorname{Re}(ie^{-it}(\tan\gamma + 2\sec\gamma\, e^{it} + \tan\gamma\, e^{i2t})) = 0, \end{aligned}$$

$$\begin{aligned} d\left(X_{\gamma,n}\right)_2(\dot{z}_3) &= \operatorname{Re}\left(\frac{i}{2}(1+\sec^2\gamma + 2\sec\gamma\tan\gamma\, e^{it} + \tan^2\gamma\, e^{i2t})\frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{i\tan\gamma\, e^{it}}\right) \\ &= \frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{2\tan\gamma}\operatorname{Re}\left(e^{-it}(1+\sec^2\gamma + 2\sec\gamma\tan\gamma\, e^{it} + \tan^2\gamma\, e^{i2t})\right) \\ &= \frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{\sin\gamma\cos\gamma}(\cos t + \sin\gamma) > 0 \end{aligned} \tag{12}$$

— the inequality being due to (11) — and

$$\begin{aligned} d\left(X_{\gamma,n}\right)_3(\dot{z}_3) &= \operatorname{Re}\left((\sec\gamma + \tan\gamma\, e^{it})\frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{i\tan\gamma\, e^{it}}\right) \\ &= -\frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{\tan\gamma}\operatorname{Re}(ie^{-it}(\sec\gamma + \tan\gamma\, e^{it})) = -\frac{d\zeta_{\gamma,n}(\dot{z}_3)^2}{\sin\gamma}\sin t > 0. \end{aligned}$$

Therefore, from our computations on s_3 and the fact that $g(z) = z$, we conclude that

(13) $\quad X_{\gamma,n}$ *maps* s_3 *onto a contact curve of angle* γ *contained in a plane parallel to the* x_2x_3*-plane.*

Moreover, from (12) we get that

(14) $\quad X_{\gamma,n}(s_3)$ *is a graph over its projection into the* x_1x_2*-plane.*

So, from (6), (8), (9), (10), (13) and (14) it follows that $X_{\gamma,n}(\overline{\Omega}_{\gamma,n})$ is compact and $X_{\gamma,n}(\partial\Omega_{\gamma,n})$ projects into the x_1x_2-plane in a one-to-one fashion onto the boundary of T_n. Using a theorem of Radó, it then follows that $X(\Omega_{\gamma,n})$ is a projection over T_n.

Case (C2). This case differs from case (C3) by the addition of the edge v into $\Omega_{\gamma,n}$ and $\hat{P}_{\gamma,n}$. Clearly, statements (6), (8), (9), (10) and (13) still hold. To show (14) is also true, we write

$$T_{\gamma,n} = \frac{\pi}{2} - \gamma, \quad 0 < \gamma < \theta_n.$$

Thus, inequality (11) holds, and this implies (14). So, it remains to check $X_{\gamma,n}$ along v.

Parameterizing v from $s_3 \cap v$ to $s_2 \cap v$ by

$$z_v = e^{it}, \quad \gamma < t < \theta_n,$$

we have

$$dz(\dot{z}_v) = ie^{it}$$

and

$$\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{i} < 0.$$

Computing, we obtain

$$d\left(X_{\gamma,n}\right)_1(\dot{z}_v) = \mathrm{Re}\left(\frac{1}{2}(1-e^{i2t})\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{ie^{it}}\right) = \frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{2i}\mathrm{Re}(e^{-it}-e^{it}) = 0,$$

$$d\left(X_{\gamma,n}\right)_2(\dot{z}_v) = \mathrm{Re}\left(\frac{i}{2}(1+e^{i2t})\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{ie^{it}}\right) = \frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{2i}\mathrm{Re}(i(e^{-it}+e^{it})) = 0,$$

$$d\left(X_{\gamma,n}\right)_3(\dot{z}_v) = \mathrm{Re}\left(e^{it}\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{ie^{it}}\right) < 0.$$

Thus, it follows that $X_{\gamma,n}$ *maps* v *monotonically onto a vertical line segment*. Fortunately, the theorem of Radó used in case (C3) can be generalized to allow for vertical line segments in the boundary. Hence, it follows that in case (C2) $X_{\gamma,n}(\Omega_{\gamma,n})$ is also a graph over T_n.

Case (C1). If $\gamma = 0$, we cannot use Radó's theorem to show $X_{0,n}(\Omega_{0,n})$ is a graph over some T_n because $X_{0,n}(\overline{\Omega}_{0,n})$ may no longer be compact. In particular, we can argue as above to show that $X_{0,n}$ *is continuous on* $\overline{\Omega}_{0,n} \backslash \{s_1 \cap v\}$, so that only neighborhoods of $s_1 \cap v$ may fail to be compact.

To show the surface $X_{0,n}(\Omega_{0,n})$ is a graph over some T_n, we consider it as a limit of graphs $X_{\gamma,n}(\Omega_{\gamma,n})$ corresponding to (C2). Indeed, in $\overline{\Omega}_{\gamma,n}$ it follows immediately that

$$s_1 \cap s_3,\ \ s_3 \cap v \to s_1 \cap v \text{ as } \gamma \to 0. \tag{15}$$

Furthermore, we have

$$\mathrm{Ext}_{\Omega_{\gamma,n}}(s_2, s_3) \to \infty \text{ as } \gamma \to 0,$$

which implies

$$|\hat{s}_3| \to 0 \text{ as } \gamma \to 0. \tag{16}$$

At this point, we consider the map $2X_{\gamma,n}$ obtained by extending the parametrization through reflection across s_1. From (15) and (16) it follows that the domains $2\Omega_{\gamma,n}$ and $2\hat{P}_{\gamma,n}$ converge to $2\Omega_{0,n}$ and $2P_{0,n}$, respectively, as γ approaches 0. Thus, we can use results from [Pommerenke 1992] to conclude that

$$(2\zeta_{\gamma,n}) \circ (2f_{\gamma,n}) \to 2\zeta_{0,n} \text{ as } \gamma \to 0,$$

where $f_{\gamma,n}$ maps $\Omega_{0,n}$ conformally onto $\Omega_{\gamma,n}$ in such a way that

$$f_{\gamma,n}(0) = 0, \quad f_{\gamma,n}(s_2 \cap v) = s_2 \cap v \ \text{ and } \ f_{\gamma,n}(s_1 \cap v) = s_1 \cap s_3.$$

Moreover, the convergence is uniform on compact subsets of

$$\overline{2\Omega}_{0,n} \backslash \{s_1 \cap v\},$$

and so we have that on this set $(2X_{\gamma,n}) \circ (2f_{\gamma,n})$ converges to $2X_{0,n}$.

Because of the convergence we can compute

$$\frac{X_{0,n}(s_2 \cap v) - X_{0,n}(\tilde{s}_2 \cap \tilde{v})}{|X_{0,n}(s_2 \cap v) - X_{0,n}(\tilde{s}_2 \cap \tilde{v})|} = \lim_{\gamma \to 0} \frac{X_{\gamma,n}(s_2 \cap v) - X_{\gamma,n}(\tilde{s}_2 \cap \tilde{v})}{|X_{\gamma,n}(s_2 \cap v) - X_{\gamma,n}(\tilde{s}_2 \cap \tilde{v})|} = (0, 1, 0),$$

where, for example, the notation $\tilde{s}_2$ refers to the image of s_2 under reflection across s_1. Therefore, we know the projection of $2X_{0,n}(2\Omega_{0,n})$ into the x_1x_2-plane is contained in some triangle $2T_n$. To show that this surface is actually a graph over $2T_n$, assume the contrary. That is, suppose there is a vertical line L_x over some point $x \in 2T_n$ such that L_x intersects $2X_{0,n}(2\Omega_{0,n})$ more than once or not at all. Then there must be some point $y \in 2T_n$ such that L_y is tangent to the surface. At such a point, the Gauss map must be horizontal, and this is a contradiction since no interior points of $2\Omega_{0,n}$ lie in the unit circle S^1.

4. Explicit parametrizations and regularity

The parametrizations of the Jenkins–Serrin graphs of case (C1) can be made explicit. To see this, we first conformally change coordinates to the upper half-plane $\mathbb{H}$ via the conformal map

$$\Phi = \Phi_n : \mathbb{H} \to \Omega_{\gamma,n},$$

normalized so that

$$\Phi(-1) = 0, \quad \Phi(0) = 1, \quad \Phi(\infty) = e^{i\theta_n}.$$

Then the Weierstrass data on $\mathbb{H}$ is given by

$$g = \Phi \quad dh = \frac{\Phi(d\Psi)^2}{d\Phi}, \tag{17}$$

where

$$\Psi : \mathbb{H} \to \hat{P}_{\gamma,n}$$

is the conformal map normalized so that

$$\Psi(-1) = 0, \quad \Psi(0) = 1, \quad \Psi(\infty) = \frac{1}{\sqrt{2}} e^{i\pi/4}.$$

To determine Φ explicitly, we map $\mathbb{H}$ to the first quadrant by the map $\sqrt{z}$, where we assume here and in what follows that any map z^q for $q \in \mathbb{R}$ is defined for $0 \le \theta < 2\pi$ by

$$re^{i\theta} \mapsto r^q e^{iq\theta}.$$

Then we compose with the Möbius transformation

$$z \to \frac{-z+i}{z+i},$$

taking the first quadrant onto the upper hemisphere of the unit disk. Finally, we map this upper hemisphere onto $\Omega_{\gamma,n}$ via the map $z^{\theta_n/\pi} = z^{(n-1)/(2n)}$. Thus

$$g(z) = \Phi(z) = \left(\frac{-\sqrt{z}+i}{\sqrt{z}+i} \right)^{(n-1)/(2n)}. \tag{18}$$

For Ψ, we can use the Schwarz–Christoffel formula to conclude that

$$\Psi(z) = \mathscr{C} \int_{-1}^{z} (w+1)^{-3/4} w^{-3/4} dz,$$

where $\mathscr{C}$ is determined by the fact that $\Psi(0) = 1$. In particular, we have

$$\mathscr{C} = \frac{1}{\int_{-1}^{0} (w+1)^{-3/4} w^{-3/4} dz},$$

and if we parametrize the interval $(-1,0)$ by $w=t-1$, this expression takes the form

$$\mathscr{C}=\frac{e^{i(3\pi/4)}}{\Lambda},$$

where

$$\Lambda=\int_0^1\frac{1}{(t-t^2)^{3/4}}dt.$$

Thus

$$(d\Psi)^2=\mathscr{C}^2(z+1)^{-3/2}z^{-3/2}dz=-\frac{i}{\Lambda^2}(z+1)^{-3/2}z^{-3/2},$$

and from (18) we can compute

$$\frac{\Phi}{d\Phi}=-\frac{i2n}{n-1}(z+1)\sqrt{z}.$$

Therefore, it follows from (17) that

$$dh=-\left(\frac{2n}{\Lambda^2(n-1)}\right)\frac{1}{z\sqrt{z+1}}.$$

Similarly, one can find explicit parametrizations for the graphs of case (C3). Here the map Ψ is a Schwarz–Christoffel map to the triangle $P_{\gamma,n}$ corresponding to (C3), and the Gauss map $\Phi=\Phi_{\gamma,n}$ is given in terms of hypergeometric functions. The procedure for finding the parametrization is similar for each n, and the interested reader is referred to [Huff and McCuan 2006] to see the result when $n=2$. Also, the reference [Carathéodory 1954] will prove useful in calculating the constants appearing in the hypergeometric functions. For the case (C2), the map $\Psi=\Psi_{\gamma,n}$ is again a Schwarz–Christoffel map onto the quadrilateral $\hat{P}_{\gamma,n}$, and the Gauss map $\Phi_{\gamma,n}$ is given in terms of hypergeometric functions. However, we can not determine Ψ explicitly as the exact locations of the vertices $s_3\cap v$ and $s_2\cap v$ are unknown. Additionally, the fact that $\Omega_{\gamma,n}$ is four-sided makes it difficult to determine the coefficients of Φ explicitly.

We can investigate the regularity of the graphs in the cases (C2) and (C3). The proofs are similar for each n; for the case $n=2$, see [Huff and McCuan 2006] (where the notation for our Q_2 is Ω). To begin with, we have a statement about the subcase of (C3) defined by $\theta_n<\gamma<\pi/2$. The dependency of the Hölder exponent on γ and n comes from the changing value of the angle between s_2 and s_3 of $\Omega_{\gamma,n}$ and the fact that $\zeta_{\gamma,n}$ always maps this angle to an angle of $\pi/4$ on $\hat{P}_{\gamma,n}$.

Theorem 2. *For $\theta_n<\gamma<\pi/2$, the graphing function $u_{\gamma,n}$ satisfies*

$$u_{\gamma,n}\in C^{1,\beta-\epsilon}(\overline{Q}_n)\backslash C^{1,\beta+\epsilon}(Q_n)$$

for small ϵ, where $0<\beta<1$ depends on γ and n.

In the boundary case $\gamma = \theta_n$, the unit normal is horizontal at v, and so $u_{\gamma,n}$ cannot be C^1. However, we can measure the continuity of $u_{\gamma,n}$ as recorded in the following theorem. In the conformal category, this case is distinguished by the fact that $\Omega_{\gamma,n}$ has an outward pointing cusp at $s_2 \cap s_3$. This means $\zeta_{\gamma,n}$ vanishes to all orders at this vertex, and it is this property that determines the range of the Hölder exponent.

Theorem 3. *If $\gamma = \theta_n$, then $u_{\gamma,n} \in C^{0,\beta}(\overline{Q}_n)$ for any $0 \le \beta < 1$.*

Functions $u_{\gamma,n}$ corresponding to (C2) are discontinuous at the vertices of Q_n, but we can investigate the regularity of the trace of $u_{\gamma,n}$ over an edge of Q_n. The crucial property from which the theorem below follows is that the function $\zeta_{\gamma,n}$ can be expressed in a neighborhood $\mathcal{U}$ of the vertex $v_2 = v \cap s_3$ of $\Omega_{\gamma,n}$ by the formula

$$\zeta_{\gamma,n}(z) = \zeta_{\gamma,n}(v_2) + (z - v_2)^{3/2}\zeta_0(z),$$

where ζ_0 is holomorphic and nonzero on $\mathcal{U}$.

Theorem 4. *If $0 < \gamma < \theta_n$, and $f_{\gamma,n}$ is the restriction of $u_{\gamma,n}$ to the interior of an edge S of Q_n, then*

$$f_{\gamma,n} \in C^{2/3}(\bar{S}) \setminus C^{2/3+\epsilon}(\mathring{S}).$$

References

[Ahlfors 1973] L. V. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill, New York, 1973. MR 50 #10211 Zbl 0272.30012

[Carathéodory 1954] C. Carathéodory, *Theory of functions of a complex variable*, vol. 2, Chelsea, New York, 1954. MR 16,346c Zbl 0056.06703

[Concus and Finn 1996] P. Concus and R. Finn, "Capillary wedges revisited", *SIAM J. Math. Anal.* **27**:1 (1996), 56–69. MR 96m:76006 Zbl 0843.76012

[Concus et al. 2001] P. Concus, R. Finn, and J. McCuan, "Liquid bridges, edge blobs, and Scherk-type capillary surfaces", *Indiana Univ. Math. J.* **50**:1 (2001), 411–441. MR 2002g:76023 Zbl 0996.76014

[Finn and Lu 1998] R. Finn and J. Lu, "Regularity properties of H-graphs", *Comment. Math. Helv.* **73**:3 (1998), 379–399. MR 99e:53007 Zbl 0912.53006

[Hoffman and Karcher 1997] D. Hoffman and H. Karcher, "Complete embedded minimal surfaces of finite total curvature", pp. 5–93 in *Geometry, V*, Encyclopaedia Math. Sci. **90**, Springer, Berlin, 1997. MR 98m:53012 Zbl 0890.53001

[Huff and McCuan 2006] R. Huff and J. McCuan, "Scherk-type capillary graphs", *J. Math. Fluid Mech.* (2006). To appear.

[Jenkins and Serrin 1966] H. Jenkins and J. Serrin, "The Dirichlet problem for the minimal surface equation, with infinite data", *Bull. Amer. Math. Soc.* **72** (1966), 102–106. MR 32 #2988 Zbl 0134.08502

[Pommerenke 1992] C. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Math. Wissenschaften **299**, Springer, Berlin, 1992. MR 95b:30008 Zbl 0762.30001

Received August 20, 2004.

ROBERT HUFF
DEPARTMENT OF MATHEMATICS
INDIANA UNIVERSITY
BLOOMINGTON, IN 47401
UNITED STATES
rohuff@indiana.edu