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THE ASCENT OF A LIQUID ON A CIRCULAR NEEDLE

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Dedicated to the memory of Herbert Beckert

It is shown that there exists an asymptotic expansion of the ascent of a liquid on a circular needle if the radius of the cross section tends to zero. In particular, a formula derived formally by Derjaguin in 1945 is confirmed.

1. Introduction

We consider the following nonparametric capillary problem in the presence of gravity (see [Finn 1986, Chapter 1]). We seek a function $U = U(x)$, $x = (x_1, x_2)$, defined over the base domain $\Omega := \mathbb{R}^2 \setminus \overline{B_a(0)}$, where $B_a(0)$ is a disk with (small) radius a and center at $x = 0$, and satisfying the nonlinear elliptic boundary value problem

$$\operatorname{div} TU = \kappa U \quad \text{in } \Omega, \tag{1}$$

$$\nu \cdot TU = \cos\theta \quad \text{on } \partial\Omega, \tag{2}$$

where

$$TU = \frac{\nabla U}{\sqrt{1 + |\nabla U|^2}},$$

κ and θ are constants with $0 \le \theta \le \pi$, and ν is the exterior unit normal on $\partial\Omega$ (equivalently, the interior normal on $\partial B_a(0)$). The graph of U describes the capillarity-driven equilibrium interface in the exterior of a vertical cylinder (the needle) with cross section $B_a(0)$, in the presence of a constant gravity field directed downward; θ is the constant contact angle between the capillary surface and the tube and κ is the (positive) capillary constant, given by $\kappa = \rho g/\sigma$, where ρ is the density change across the interface, g is the acceleration of gravity, and σ is the surface tension.

No explicit solution of (1)–(2) is known. It was shown by Johnson and Perko [1968] that there exists a radially symmetric solution. From a maximum principle of Finn and Hwang [1989] for unbounded domains it follows that this symmetric solution is the only one.

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Set

$$u(r) = U(x), \quad r = \sqrt{x_1^2 + x_2^2}. \tag{3}$$

We will prove that there is an asymptotic expansion for the ascent $u(a)$ of the liquid in this problem. More precisely:

Theorem 1.1. *Set $B = \kappa a^2$ and let $\gamma = 0.5772\ldots$ be Euler's constant. Then the ascent $u(a)$ of a liquid on a circular needle with radius a satisfies*

$$\frac{u(a)}{a} = -\cos\theta\left(\tfrac{1}{2}\ln B + \gamma - 2\ln 2 + \ln(1+\sin\theta) + O(B^{1/5}\ln^2 B)\right)$$

as $B \to 0$, uniformly in $\theta \in [0, \pi]$.

Uniformly means that the remainder satisfies $|O(B^{1/5}\ln^2 B)| \le cB^{1/5}|\ln^2 B|$ for all $0 < B \le B_0$, if B_0 is sufficiently small, where the constant c depends only on B_0 and not on the contact angle θ.

It is noteworthy that the special nonlinearity of the problem implies that the expansion is uniform with respect to $\theta \in [0, \pi]$ although $|Du|$ tends to infinity as $\theta \to 0$ or $\theta \to \pi$ and therefore the differential equation (1) will be singular on $\partial\Omega$. Moreover, as a further consequence of the strong nonlinearity of the problem, we do not need any growth assumption at infinity.

In the case of complete wetting, that is, if $\theta = 0$, the formula

$$u(a) \sim -a\left(\tfrac{1}{2}\ln B - 0.809\ldots\right)$$

as $a \to 0$ was derived formally by Derjaguin [1946] by expansion matching. We recall that $B = \kappa a^2$. Higher-order approximations where obtained formally by James [1974] and Lo [1983], also by matching arguments.

(Matching means that some free constants which occur in two asymptotic expansions with an overlapping domain of their definition will be determined in an appropriate way; see [Van Dyke 1964; Fraenkel 1969], for example.)

Turkington [1980] proved that $u(a) \sim -\frac{1}{2}\cos\theta\, a\ln B$ as $a \to 0$ under an additional growth assumption at infinity. This assumption is superfluous because of the comparison principle of Finn and Hwang [1989].

The proof of the existence of the asymptotic expansion is based on a construction of an upper and a lower C^1-solution of (1)–(2) and on the maximum principle of Finn and Hwang for unbounded domains. We obtain the lower and the upper solution by gluing together a boundary layer expansion near the needle with a second expansion far from the needle such that the resulting function is in C^1. This method of composing of functions on different annular domains was used in [Miersemann 1996], where a numerical method for the circular tube was proposed.

Theorem 1.1 and the calculations of the appendix, together with those of [Lo 1983], suggest:

Conjecture. *For given* $N \in \mathbb{N} \cup \{0\}$ *the ascent* $u(a)$ *satisfies*

$$\frac{u(a)}{a} = -\cos\theta \left(\sum_{k=0}^{N} \sum_{l=0}^{M(k)} c_{kl}(\theta) B^k (\ln B)^l + o(B^N) \right)$$

as $B \to 0$, *uniformly in* $\theta \in [0, \pi]$.

2. Expansion near the needle

Since $U(x)$ is rotationally symmetric, the boundary value problem (1)–(2) reads, with the notation (3),

$$\frac{1}{r}\left(\frac{ru'(r)}{\sqrt{1+(u'(r))^2}}\right)' = \kappa u(r) \quad \text{in} \quad a < r < \infty,$$

$$\lim_{r \to a+0} \frac{u'(r)}{\sqrt{1+(u'(r))^2}} = -\cos\theta.$$

Set

$$r = as, \quad v(s) = \frac{1}{a} u(as), \quad B = \kappa a^2.$$

Then the problem becomes

$$\frac{1}{s}\left(\frac{sv'(s)}{\sqrt{1+(v'(s))^2}}\right)' = Bv(s) \quad \text{in} \quad 1 < s < \infty, \tag{4}$$

$$\lim_{s \to 1+0} \frac{v'(s)}{\sqrt{1+(v'(s))^2}} = -\cos\theta. \tag{5}$$

For a fixed q, $1 < q < \infty$, $b_0 := -\cos\theta$, $\theta \in [0, \pi]$ and $b_1 \in [-1, 1]$ let

$$v_1(s) \equiv v_1(B, q, b_0, b_1; s)$$

be the solution of

$$\frac{1}{s}\left(\frac{sv'(s)}{\sqrt{1+(v'(s))^2}}\right)' = Bv(s) \quad \text{for } 1 < s < q, \tag{6}$$

$$\lim_{s \to 1+0} \frac{v'(s)}{\sqrt{1+(v'(s))^2}} = b_0, \qquad \lim_{s \to q-0} \frac{v'(s)}{\sqrt{1+(v'(s))^2}} = b_1. \tag{7}$$

Set

$$\text{div } Tv = \frac{1}{r}\left(\frac{rv'}{\sqrt{1+(v')^2}}\right)'.$$

It was shown in [Miersemann 1993; 1994] that for fixed q there exists a complete asymptotic expansion of v_1 as $B \to 0$, uniformly in $b_0,\ b_1 \in [-1, 1]$:

$$v_1 = \frac{C}{B} + \sum_{k=0}^{m} \varphi_k(s) B^k + O(B^{m+1}),$$

here $\varphi_k(s) \equiv \varphi_k(q, b_0, b_1; s)$ and

$$C \equiv C(q, b_0, b_1) = \frac{2(qb_1 - b_0)}{q^2 - 1}.$$

The function φ_0 is a solution of a boundary value problem for a nonlinear second order ordinary differential equation and the φ_k, for $k \geq 1$, are solutions of linear boundary value problems.

It turns out that we have to change q if $B \to 0$. More precisely, $q = B^{-\tau}$, for $\tau > 0$ small, will be an appropriate choice. Therefore, we need some information about how the functions, for example φ_k, depend on q.

Set

$$b_1 := \frac{b_0}{q}(1 + \epsilon), \quad 0 \leq |\epsilon| < \epsilon_0 < 1,$$

$$\phi_k(s) \equiv \phi_k(q, b_0, \epsilon; s) := \varphi_k\Big(q, b_0, \frac{a_0}{q}(1 + \epsilon); s\Big)$$

and for $m \geq 0$

$$v_{1,m}(s) \equiv v_{1,m}(B, q, b_0, \epsilon; s) := \frac{2\epsilon b_0}{B(q^2 - 1)} + \sum_{k=0}^{m} \phi_k(s) B^k. \tag{8}$$

Assume that

$$\lambda := Bq^2 \ln q \leq \lambda_0$$

for a sufficiently small positive λ_0, independent of B and q. We will choose $q = B^{-\tau}$ for $\tau \in \big(0, \frac{1}{2}\big)$.

Proposition 2.1. *Suppose $q \geq 3$. For a given $m \in \mathbb{N} \cup \{0\}$ there exist functions $\varphi_k(s) \equiv \varphi_k(q, b_0, b_1; s)$ for $k = 0, 1, \ldots, m$, analytic in $1 < s < q$ and continuous in $1 \leq s \leq q$, as well as functions $\phi_k(s) \equiv \phi_k(q, b_0, \epsilon; s)$, continuous in $|\epsilon| < \frac{1}{4}$, such that for $|\epsilon| \leq \frac{1}{4}$ and $s \in (1, q)$ we have*

$$\phi_k(s) = \sum_{l=0}^{N} \phi_{k,l}(q, b_0; s)\epsilon^l + R_{N+1}\epsilon^{N+1},$$

where

$$|\phi_{k,l}(q, b_0; s)| \leq c\,|b_0|\,(\ln q)^{k+1} q^{2k}, \quad |R_{N+1}| \leq c\,|b_0|\,(\ln q)^{k+1} q^{2k}$$

and

$$|\text{div}\, Tv_{1,m} - Bv_{1,m}| \le c\,|b_0|\,(\ln q)^{m+1} q^{2m} B^{m+1}; \tag{9}$$

here $v_{1,m}$ *is the sum* (8). *The constants* c *depend only on* λ_0 *and on* k, N, m, *and not on* $b_0 \in [-1, 1]$.

In particular,

$$\phi_{0,0}(q, b_0; 1) = -b_0\Big(\ln q + \ln 2 - \tfrac{1}{2} - \ln\big(1 + \sqrt{1 - b_0^2}\big) + O(q^{-2} \ln q)\Big)$$

as $q \to \infty$.

The proof is given in Section A.1 of the Appendix.

3. Expansion far from the needle

Let $v_2(s) \equiv v_2(B, q, b_1; s)$ be the solution of

$$\frac{1}{s}\left(\frac{s v'(s)}{\sqrt{1 + (v'(s))^2}}\right)' = Bv(s) \quad \text{in} \quad q < s < \infty, \tag{10}$$

$$\lim_{s \to q+0} \frac{v'(s)}{\sqrt{1 + v'(s)^2}} = b_1. \tag{11}$$

In contrast to the earlier expansion with respect to B near the needle, we expand v_2 with respect to b_1 for fixed Bond number $0 < B < 1$.

For small $|b_1|$ we have

$$v'(q) = \frac{b_1}{\sqrt{1 - b_1^2}} = b_1 \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(-b_1^2\right)^k.$$

We make the following ansatz for a solution of the differential equation (10), where $n \in \mathbb{N} \cup \{0\}$, $\rho \in \mathbb{R}$, $|\rho|$ small:

$$v_{2,n}(s) \equiv v_{2,n}(B, q, \rho; s) := \sum_{k=0}^{n} \psi_k(B, q; s)\, \rho^{2k+1} \tag{12}$$

with unknown functions $\psi_k(s) :\equiv \psi_k(B, q; s)$ such that

$$\psi_k'(q) = (-1)^k \binom{-\frac{1}{2}}{k}. \tag{13}$$

Since

$$v_{2,n}'(q) = \rho \sum_{k}^{n} \binom{-\frac{1}{2}}{k} \left(-\rho^2\right)^k,$$

it follows that $v_{2,n}$ satisfies the boundary condition (11) at $s=q$ if

$$\sum_{k=0}^{n}(-1)^k\binom{-\frac{1}{2}}{k}\rho^{2k+1}=\frac{b_1}{\sqrt{1-b_1^2}}. \tag{14}$$

Thus, since $b_1=b_0(1+\epsilon)/q$,

$$\rho=b_1+O(b_1^{2n+3})=\frac{b_0}{q}+\epsilon\frac{b_0}{q}+O\left(\left(\frac{b_0}{q}\right)^{2n+3}\right)$$

as $b_0/q\to 0$.

Definition 3.1. We write $w(\delta)=P(\delta,\ln\delta)$, where $0<\delta<\delta_0$, if for given $N\in\mathbb{N}$ we have

$$w(\delta)=\sum_{\alpha=1}^{N}\sum_{\beta=0}^{M(\alpha)}c_{\alpha\beta}\delta^\alpha(\ln\delta)^\beta+R_N(\delta),$$

where $c_{\alpha\beta}\in\mathbb{R}$, $R_N(\delta)$ is continuous in $0\le\delta<\delta_0$, $\lim_{N\to\infty}R_N(\delta)=0$ for fixed δ and $R_N(\delta)=o(\delta^N)$ as $\delta\to 0$.

Proposition 3.2. *Assume that* $0<B<1$, $q=B^{-\tau}$, $\tau\in(0,\tau_1]$, $0<\tau_1<\frac{1}{2}$ *and* $|\rho|<\rho_0$, *for* ρ_0 *sufficiently small. For a given* $n\in\mathbb{N}\cup\{0\}$ *there exist functions* $\psi_k(s)\equiv\psi_k(B,q;s)$, $k=0,\dots,n$, *analytic on* $q\le s<\infty$, *such that the sum* $v_{2,n}$ *of* (12) *satisfies*

$$|\operatorname{div} Tv_{2,n}-Bv_{2,n}|\le c\,|\rho|^{2n+3} \tag{15}$$

on $s\in[q,\infty)$, *where the constant* c *depends only on* τ_1, ρ_0 *and* n. *Further, for* $\delta:=\sqrt{B}q$ *there are functions* $w_k(\delta)=P(\delta,\ln\delta)$ *such that*

$$\psi_k(B,q;q)=\frac{1}{\sqrt{B}}w_k(\delta). \tag{16}$$

In particular,

$$\psi_0(B,q;q)=\frac{1}{\sqrt{B}}\frac{K_0(\delta)}{K_0'(\delta)},$$

where $K_0(\delta)$ *is a modified Bessel function of second kind and of order zero.*

The proof is given in Section A.2 of the Appendix.

Siegel [1980] observed that the function $\psi_0:=cK_0(\sqrt{B}s)$, where c is a positive constant, defines for a fixed $q>1$ a supersolution of the differential equation (10) on (q,∞). We will show that there is a positive constant A such that $v_{2,n}\pm A$ defines a supersolution and a subsolution, respectively, on (q,∞) if $q:=B^{-\tau}$ for appropriate τ satisfying $0<\tau\le\tau_1<\frac{1}{2}$ and if ρ is defined by (14). In particular,

$$v_{2,0}=\frac{K_0(\sqrt{B}s)}{\sqrt{B}K_0'(\sqrt{B}q)}\rho.$$

4. Composing of the inner and outer solutions

By the inner solution we mean the expansion $v_{1,m}$ near the needle and the outer solution is $v_{2,n}$, the expansion far from the needle.

We glue together these two expansions at $s = q$ in such a way that the composite function is in $C^1(1, \infty)$.

Set

$$v_{c,m,n}(s) := \begin{cases} v_{1,m}(B, q, b_0, \epsilon; s) & \text{for } 1 \le s \le q, \\ v_{2,n}(B, q, \rho; s) & \text{for } q < s < \infty. \end{cases}$$

This composite function is in $C^1(1, \infty)$ if and only if ρ satisfies (14) and $v_{1,m}$, $v_{2,n}$ coincide at $s = q$, that is, if

$$v_{1,m}(B, q, b_0, \epsilon; q) = v_{2,n}(B, q, \rho; q), \tag{17}$$

where $\rho = \rho(b_0, q, \epsilon)$ is defined by (14). Now set

$$\delta := \sqrt{B}q.$$

We choose $q = B^{-\tau}$ for a fixed $\tau \in \left(0, \frac{1}{2}\right)$; then $\delta \to 0$ if $B \to 0$.

Proposition 4.1. *Assume that $q = B^{-\tau}$ for a fixed $\tau \in \left(0, \frac{1}{2}\right)$. Then there is a solution ϵ of equation* (17). *In particular, we have*

$$\epsilon = \tfrac{1}{2}\delta^2 \ln \delta + \tfrac{1}{2}\left(\gamma - \ln 2 - \tfrac{1}{2}\right)\delta^2 + R(b_0, B, B^{-\tau})\delta^2$$

with

$$R(b_0, B, B^{-\tau}) = O(B^{2\tau}(\ln B)^{l+1}) + O(B^{1-2\tau}\ln^2 B)$$

uniformly in $b_0 \in [-1, 1]$ as $B \to 0$, where $l \in \mathbb{N} \cup \{0\}$ and

$$\gamma := \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{1}{k} - \ln m\right) = 0.5772\ldots$$

is Euler's constant.

The proof is given in Section A.3 of the Appendix.

Assume that $q := B^{-\tau}$ for $0 < \tau \le \tau_1 < \frac{1}{2}$. Then, since

$$b_1 = \frac{b_0}{q}\left(1 + O(B^{1-2\tau}\ln B)\right),$$

it follows from the three propositions above that the $C^1(1, \infty)$ function $v_{c,m,n}$ satisfies, for $0 < B \le B_0 < 1$ with B_0 sufficiently small,

$$|\text{div}\, Tv_{c,m,n} - Bv_{c,m,n}| \le \begin{cases} c\,|b_0|\,(-\ln B)^{m+1}B^{(1-2\tau)m+1} & \text{for } 1 \le s \le q, \\ c\,|b_0|B^{(2n+3)\tau} & \text{for } q < s < \infty. \end{cases}$$

The constant c depends only on m, n, B_0 and τ_1.

5. Asymptotic expansion

Let A be a positive constant. Set

$$v^+_{c,m,n} := v_{c,m,n} + A.$$

This function $v^+_{c,m,n}$ is in $C^1(1,\infty)$ and satisfies the boundary condition (5) at $s=1$. From the above estimate it follows

$$\begin{aligned}\mathrm{div} T v^+_{c,m,n} - B v^+_{c,m,n} &= \mathrm{div}\ T v_{c,m,n} - B v_{c,m,n} - AB \\ &\leq B \begin{cases} c\,|b_0|\,(-\ln B)^{m+1} B^{(1-2\tau)m} - A & \text{for } 1 \leq s \leq q, \\ c\,|b_0| B^{(2n+3)\tau-1} - A & \text{for } q < s < \infty. \end{cases}\end{aligned}$$

The constant c depends only on m, n, B_0 and τ_1.

For $\tau \in \left(0, \frac{1}{2}\right)$ and $m, n \in \mathbb{N} \cup \{0\}$, set

$$p(m,n;\tau) := min\{(1-2\tau)m,\ (2n+3)\tau - 1\}$$

and let $\tau_0 \equiv \tau_0(m,n)$ be the solution of $(1-2\tau)m = (2n+3)\tau - 1$, that is,

$$\tau_0 = \frac{m+1}{2(m+1)+2n+1}.$$

Thus τ_0 is the solution of

$$\max_{0<\tau<1/2} p(m,n;\tau).$$

Set $p_0 \equiv p_0(m,n) := p(m,n;\tau_0)$; that is,

$$p_0 = \frac{2mn+m}{2m+2n+3}.$$

Choose

$$A := c\,|b_0|\,(-\ln B)^{m+1} B^{p_0}; \tag{18}$$

then the preceding inequality implies

$$\mathrm{div}\ T v^+_{c,m,n} - B v^+_{c,m,n} \leq 0$$

for all B such that $0 < B \leq B_0$ and for all s in $(1,q] \cup (q,\infty)$. The maximum principle of Finn and Hwang [1989] yields

$$v(s) \leq v^+_{c,m,n}(s)$$

on $(1,\infty)$. By the same reasoning it follows that

$$v^-_{c,m,n} := v_{c,m,n} - A,$$

satisfies $v(s) \geq v^-_{c,m,n}(s)$ on $(1,\infty)$, where A is given by (18).

Summarizing, we have shown that $|v(s) - v_{c,m,n}(s)| \leq c\,|b_0|\,(-\ln B)^{m+1} B^{p_0}$. We can choose p_0 arbitrarily large provided m and n are large enough; see the definition of p_0 above.

In particular, the height rise at $s = 1$ satisfies

$$|v(1) - v_{1,m}(1)| \leq c\,|b_0|\,(-\ln B)^{m+1} B^{p_0}.$$

Thus

$$v(1) = \frac{C(q, b_0, b_1)}{B} + \sum_{k=0}^{m} \varphi_k(q, b_0, b_1; 1) B^k + O(b_0 B^{p_0} \ln^{m+1} B),$$

where $b_1 = b_0(1+\epsilon)/q$, $q = B^{-\tau_0}$ and ϵ is the solution of (17); see Proposition 4.1.

Thus, we consider

$$v_{1,m}(1) := \frac{C(q, b_0, b_1)}{B} + \sum_{k=0}^{m} \varphi_k(q, b_0, b_1; 1) B^k$$

as an *approximation of order* p_0 of the value $v(1)$.

Then, since $B = \kappa a^2$ and $u(a) = a v(1)$, we have

$$\frac{u(a)}{a} = v_{1,m}(1) + O(b_0 B^{p_0} \ln^{m+1} B) \tag{19}$$

as $B \equiv \kappa a^2 \to 0$.

Proof of Theorem 1.1. Set $m = 1$ and $n = 0$. Then $\tau_0 = \frac{2}{5}$, $p_0 = \frac{1}{5}$, $q \equiv B^{-\tau_0} = B^{-2/5}$ and $\delta \equiv \sqrt{B} q = B^{1/10}$. We obtain from Proposition 4.1

$$\epsilon = \tfrac{1}{2}\delta^2 \ln \delta + \tfrac{1}{2}(\gamma - \ln 2 - \tfrac{1}{2})\delta^2 + O(\delta^2 B^{1/5} \ln^2 B)$$

and Proposition 2.1 yields

$$\phi_0(1) = -b_0\Big(\ln q + \ln 2 - \tfrac{1}{2} - \ln\big(1 + \sqrt{1 - b_0^2}\big)\Big) + O(b_0 B^{1/5} \ln^2 B)$$

and $\phi_1(1) B = O(b_0 B^{1/5} \ln^2 B)$.

Thus

$$\begin{aligned} v_{1,1}(1) &= \frac{2\epsilon b_0}{B(q^2-1)} + \phi_0(1) + \phi_1(1) B + O(b_0 B^{1/5} \ln^2 B) \\ &= b_0 \left(\ln \delta - \ln 2 - \tfrac{1}{2} + \gamma + O(B^{1/5} \ln^2 B)\right) \left(1 - \frac{1}{q^2}\right)^{-1} \\ &\qquad - b_0\big(\ln q + \ln 2 - \tfrac{1}{2} - \ln(1 + \sqrt{1 - b_0^2})\big) + O(b_0 B^{1/5} \ln^2 B) \\ &= b_0\big(\tfrac{1}{2} \ln B - 2\ln 2 + \gamma + \ln(1 + \sqrt{1 - b_0^2})\big) + O(b_0 B^{1/5} \ln^2 B). \end{aligned}$$

The theorem follows from formula (19) for $u(a)/a$. □

Appendix: Proof of the propositions

Here we prove the propositions of the previous sections. The argument concerns mainly expansions of nonlinear expressions with respect to appropriate parameters. In the expansion near the needle the special nonlinearity of the problem is exploited. The expansion far from the needle ensues by linearization of the problem with respect to the zero solution.

A.1. ***Expansion near the needle.*** Set for $0 < B < B_0$

$$v_m = \frac{C}{B} + \sum_{k=0}^{m} \varphi_k(s) B^k,$$

where C is a constant and φ_k are functions in $C^2(1,q)$, $1 < q < \infty$.

The sum v_m is said to be an *approximate solution* of (6)–(7) if v_m satisfies the boundary conditions (7) and if

$$|\text{div } Tv_m - Bv_m| \le cB^{m+1}$$

on $(1,q)$, where $c = c(m,q)$ and c is independent on $b_0,\ b_1 \in [-1,1]$.

In the following we will define C and φ_k so that v_m is an approximate solution. It turns out that C is given explicitely, φ_0 is the solution of a nonlinear boundary value problem for a second order differential equation and φ_k, for $k \ge 1$, are solutions of linear boundary value problems of second order, defined iteratively. The main idea here is to preserve the properties of the special nonlinearity also in the expansions.

In

$$\text{div } Tv_m \equiv \frac{1}{s}\left(\frac{sv_m'}{\sqrt{1+{v_m'}^2}}\right)'$$

there appears the quotient $v_m'/\sqrt{1+{v_m'}^2}$. We now derive some expansions in B related to this quotient.

Definition of C and φ_k. Since

$$\begin{aligned}1+{v_m'}^2 &= 1+\left(\sum_{l=0}^{m}\varphi_l' B^l\right)^2\\ &= (1+{\varphi_0'}^2)\left(1+2\frac{\varphi_0'}{\sqrt{1+{\varphi_0'}^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+{\varphi_0'}^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+{\varphi_0'}^2}}B^l\right)^2\right),\end{aligned}$$

it follows that

$$\frac{v_m'}{\sqrt{1+v_m'^2}} = \frac{v_m'}{\sqrt{1+\varphi_0'^2}}\left(1+2\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l\right)^2\right)^{-1/2}.$$

Set, for $l=1,\dots,m$,

$$d_l := \frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}$$

and assume that

$$\text{(A–1)}\qquad \sup_{s\in(1,q)} |d_l| \le c_l^{(1)}(q) < \infty.$$

Then for $M\in\mathbb{N}$, provided $0<B\le B_0(q)$ with B_0 sufficiently small, we have

$$\text{(A–2)}\qquad \frac{v_m'}{\sqrt{1+v_m'^2}} = \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} + \sum_{k=1}^{M} f_{m,k}(\varphi_0',\dots,\varphi_m')B^k + \tilde f_{m,M+1}B^{M+1},$$

where $f_{m,k}$ and $\tilde f_{m,M+1}$ are defined as follows. Set $g_m(B) := v_m'/\sqrt{1+v_m'^2}$, then

$$f_{m,k} = g_m^{(k)}(0)/k! \quad\text{and}\quad \tilde f_{m,k} = g_m^{(k)}(tB)/k! \quad \text{for } 0<t<1.$$

From assumption (A–1) on φ_k' we obtain

$$|f_{m,k}| \le c_{m,k}(q) < \infty \quad\text{and}\quad |\tilde f_{m,M+1}| \le \tilde c_{m,M+1}(q) < \infty.$$

We have, from (A–2), $f_{0,k}\equiv 0$ and $\tilde f_{0,k}\equiv 0$ for all $k\in\mathbb{N}$.

This argument exploits the special nonlinearity of the problem. More precisely, we have used that

$$\frac{|\varphi_0'|}{\sqrt{1+\varphi_0'^2}}$$

remains bounded even if $|\varphi_0'(s)|\to\infty$ if $s\to 1$ or $s\to q$.

We obtain from (A–2) the expansion

$$\text{(A–3)}\qquad \operatorname{div} Tv_m = \frac{1}{s}\left(\frac{s\varphi_0'}{\sqrt{1+\varphi_0'^2}}\right)' + \sum_{k=1}^{M}\frac{1}{s}\big(sf_{m,k}\big)'B^k + \frac{1}{s}\big(s\tilde f_{m,M+1}\big)'B^{M+1}.$$

We next need some information on how the derivatives $(f_{m,k})'$ and $(\tilde f_{m,l})'$ depend on b_0, b_1 and q.

Since $v_m' = \sum_{l=0}^{m}\varphi_l' B^l$ and

$$\text{(A–4)}\qquad \operatorname{div} Tv \equiv \frac{1}{s}v'(1+v'^2)^{-1/2} + v''(1+v'^2)^{-3/2}$$

it follows under assumption (A–1) that for $0 < B \le B_0 \equiv B_0(q)$, with B_0 sufficiently small,

$$\begin{aligned}\operatorname{div} Tv_m &= \frac{1}{s}\frac{v_m'}{\sqrt{1+\varphi_0'^2}}\left(1+2\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l\right)^2\right)^{-1/2}\\ &\quad+\frac{v_m''}{(1+\varphi_0'^2)^{3/2}}\left(1+2\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l\right)^2\right)^{-3/2}.\end{aligned}$$

Thus

$$\operatorname{div} Tv_m = \frac{1}{s}\left(\frac{s\varphi_0'}{\sqrt{1+\varphi_0'^2}}\right)' + \sum_{k=1}^{M} F_{m,k}B^k + \tilde{F}_{m,M+1}B^{M+1}, \tag{A–5}$$

where $F_{m,k}$ and $\tilde{F}_{m,M+1}$ are defined as follows. Set

$$h_m(B) := \frac{1}{s}\frac{v_m'}{\sqrt{1+v_m'^2}} + v_m''\left(1+v_m'^2\right)^{-3/2}.$$

Then $F_{m,k} = h_m^{(k)}(0)/k!$ and $\tilde{F}_{m,k} = h_m^{(k)}(tB)/k!$ for $0 < t < 1$. We have $F_{0,k} \equiv 0$ and $\tilde{F}_{0,k} \equiv 0$ for all $k \in \mathbb{N}$.

Set for $l = 1, \dots, m$

$$e_l := \frac{\varphi_l''}{(1+\varphi_0'^2)^{3/2}}$$

and assume

$$\sup_{s\in(1,q)} |e_l| \le c^{(2)}(q) < \infty. \tag{A–6}$$

Then the functions $F_{m,k}$ and $\tilde{F}_{m,M+1}$ are bounded.

Since

$$\frac{1}{s}(sf_{m,k})' \equiv F_{m,k}, \quad \frac{1}{s}(s\tilde{f}_{m,k})' \equiv \tilde{F}_{m,k},$$

it follows, under assumptions (A–1) and (A–6), that the derivatives $(f_{m,k})'$, $(\tilde{f}_{m,k})'$ are bounded.

In the following considerations we derive boundary value problems which define the functions $\varphi_0, \varphi_1, \dots, \varphi_m$. Then we prove that these functions φ_l satisfy inequalities (A–1) and (A–6) *uniformly* in $q \ge 3$ and in $b_0 \in [-1, 1]$, where $b_1 = b_0(1+\epsilon)/q$, with $|\epsilon| \le \frac{1}{4}$.

The following lemma is useful in order to iteratively find the appropriate boundary value problem which defines φ_{m+1} for given $\varphi_0, \dots, \varphi_m$.

Lemma A.1.1. *Let assumption* (A–1) *on* φ_l, *for* $l = 1, \dots, m+1$, *be satisfied. Then*

$$\frac{v'_{m+1}}{\sqrt{1+{v'_{m+1}}^2}} = \frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}} B^{m+1} + R,$$

where $|R| \le c(q)B^{m+2}$, $0 < B \le B_0(q)$, B_0 *sufficiently small.*

Proof.

$$\begin{aligned}
\frac{v'_{m+1}}{\sqrt{1+{v'_{m+1}}^2}} &= \big(v'_m + \varphi'_{m+1}B^{m+1}\big)\big(1+{v'_m}^2 + 2v'_m\varphi'_{m+1}B^{m+1} + {\varphi'_{m+1}}^2B^{2m+2}\big)^{-1/2}\\
&= \big(v'_m + \varphi'_{m+1}B^{m+1}\big)\,\big(1+{v'_m}^2\big)^{-1/2}\\
&\qquad\cdot\left(1+2\frac{v'_m}{\sqrt{1+{v'_m}^2}}\frac{\varphi'_{m+1}}{\sqrt{1+{v'_m}^2}}B^{m+1} + \frac{(\varphi'_{m+1})^2}{1+{v'_m}^2}B^{2m+2}\right)^{-1/2}\\
&= \left(\frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{\varphi'_{m+1}}{\sqrt{1+{v'_m}^2}}B^{m+1}\right)\left(1-\frac{v'_m\varphi'_{m+1}}{1+{v'_m}^2}B^{m+1} + R_1\right)\\
&= \frac{v'_m}{\sqrt{1+{v'_m}^2}} + \left(-\frac{{v'_m}^2\varphi'_{m+1}}{(1+{v'_m}^2)^{3/2}} + \frac{\varphi'_{m+1}}{\sqrt{1+{v'_m}^2}}B^{m+1}\right) + R_2\\
&= \frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{\varphi'_{m+1}}{(1+{v'_m}^2)^{3/2}}B^{m+1} + R_2.
\end{aligned}$$

The remainders above satisfy $|R_1|,\ |R_2| \le c(q)B^{2m+2}$. Since

$$\begin{aligned}
&\frac{\varphi'_{m+1}}{(1+{v'_m}^2)^{3/2}}\\
&\quad= \frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\left(1+2\frac{\varphi'_0}{\sqrt{1+{\varphi'_0}^2}}\sum_{l=1}^{m}\frac{\varphi'_l}{\sqrt{1+{\varphi'_0}^2}}B^l + \left(\sum_{l=1}^{m}\frac{\varphi'_l}{\sqrt{1+{\varphi'_0}^2}}B^l\right)^2\right)^{-3/2}\\
&\quad= \frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}} + R_3,
\end{aligned}$$

where $|R_3| \le c(q)B$, the expansion of the lemma is shown. □

Lemma A.1.2. *Suppose assumptions* (A–1) *and* (A–6) *are satisfied. Then*

$$\operatorname{div}\, Tv_{m+1} = \operatorname{div}\, Tv_m + \frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)' B^{m+1} + O(B^{m+2})$$

as $B \to 0$, *uniformly in* $s \in (1, q)$.

Proof. We conclude from (A–4) and Lemma A.1.1 that

$$\begin{aligned}\operatorname{div} Tv_{m+1} &\equiv \frac{1}{s}\frac{v'_{m+1}}{\sqrt{1+{v'_m}^2}} + \frac{v''_{m+1}}{(1+{v'_m}^2)^{3/2}} \\ &= \frac{1}{s}\frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{1}{s}\frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}B^{m+1} + \frac{v''_{m+1}}{(1+{v'_{m+1}}^2)^{3/2}} + O(B^{m+2}).\end{aligned}$$

Since

$$\frac{v''_{m+1}}{(1+{v'_{m+1}}^2)^{3/2}} = \frac{v''_m}{(1+{v'_m}^2)^{3/2}} + \left(\frac{\varphi''_{m+1}}{(1+{\varphi'_0}^2)^{3/2}} - \frac{3\varphi'_0\varphi''_0\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{5/2}}\right)B^{m+1} + O(B^{m+2}),$$

which follows by similar calculations as in the proof of Lemma A.1.1, we obtain

$$\operatorname{div} Tv_{m+1} = \frac{1}{s}\frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{v''_m}{(1+{v'_m}^2)^{3/2}} + \frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)' B^{m+1} + O(B^{m+2}).$$

□

Lemma A.1.2 implies

$$\begin{aligned}&\operatorname{div} Tv_{m+1} - Bv_{m+1} \\ &\quad = \operatorname{div} Tv_m + \frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)' B^{m+1} - (C + B\varphi_0 + \cdots + B^{m+1}\varphi_m) + O(B^{m+2}).\end{aligned}$$

Then from expansion (A–3) for $\operatorname{div} Tv_m$, with $M := m+1$, and from the condition

$$\operatorname{div} Tv_{m+1} - Bv_{m+1} = O(B^{m+2}) \quad \text{as } B \to 0,$$

there follows for $m \geq 0$ the differential equation

$$\frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)' + \frac{1}{s}(sf_{m,m+1})' = \varphi_m \tag{A–7}$$

on $1 < s < q$. We recall that $f_{m,m+1} = g_m^{(m+1)}(0)/(m+1)!$, where $g_m(B) = v'_m/\sqrt{1+{v'_m}^2}$.

We conclude from $\operatorname{div} Tv_0 - Bv_0 = O(B)$ that

$$\operatorname{div} T\varphi_0 \equiv \frac{1}{s}\left(\frac{s\varphi'_0}{\sqrt{1+{\varphi'_0}^2}}\right)' = C \tag{A–8}$$

on $1 < s < q$.

From the assumptions

$$\lim_{s\to 1+0}\frac{v'_m}{\sqrt{1+{v'_m}^2}} = b_0, \qquad \lim_{s\to q-0}\frac{v'_m}{\sqrt{1+{v'_m}^2}} = b_1$$

for fixed q and $0 < B \le B_0(q)$, and from the expansion (A–2), we get

$$\text{(A–9)} \qquad \lim_{s\to 1+0} \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} = b_0, \quad \lim_{s\to q-0} \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} = b_1.$$

Further, we obtain from Lemma A.1.1 that for $m \ge 1$

$$\text{(A–10)} \qquad \lim_{s\to 1+0} \frac{\varphi_{m+1}'}{(1+\varphi_0'^2)^{3/2}} = 0, \quad \lim_{s\to q-0} \frac{\varphi_{m+1}'}{(1+\varphi_0'^2)^{3/2}} = 0,$$

and (A–2) implies the boundary conditions

$$\text{(A–11)} \qquad \lim_{s\to 1+0} f_{m,k}(\varphi_0', \dots, \varphi_m') = 0, \quad \lim_{s\to q-0} f_{m,k}(\varphi_0', \dots, \varphi_m') = 0$$

for $k \ge 1$ and $m \ge 0$.

After integration of the differential equation from 1 to q it follows from the boundary conditions (A–11) and (A–12) that, for $m \ge 0$,

$$\text{(A–12)} \qquad \int_1^q s\varphi_m(s)\, ds = 0.$$

Applying the differential equation (A–8) for φ_0 and the boundary conditions (A–9), we find

$$\text{(A–13)} \qquad C = \frac{2(qb_1 - b_0)}{q^2 - 1}.$$

Set

$$\text{(A–14)} \qquad f(s) \equiv f(q, b_0, b_1; s) := b_0 f_0 + b_1 f_1,$$

where

$$f_0 := \frac{q^2 - 1 - (s^2 - 1)}{s(q^2 - 1)}, \quad f_1 := \frac{q(s^2 - 1)}{s(q^2 - 1)}.$$

Then it follows from (A–8) and the formula (A–13) for C that

$$\text{(A–15)} \qquad \frac{\varphi_0'(s)}{\sqrt{1 + (\varphi_0'(s))^2}} = f(s)$$

or, equivalently,

$$\text{(A–16)} \qquad \varphi_0'(s) = \frac{f(s)}{\sqrt{1 - f^2(s)}}.$$

Set for $1 \le s \le q$

$$\text{(A–17)} \qquad \tilde{\varphi}_0(s) := \int_1^s \frac{f(\tau)}{\sqrt{1 - f^2(\tau)}}\, d\tau,$$

then $\varphi_0(s) = \tilde{\varphi}_0(s) + K$, where the constant K will be determined by the side condition (A–12). That is, $\varphi_0(s) \equiv \varphi_0(q, b_0, b_1; s)$ is given by

$$\varphi_0(s) = \tilde{\varphi}_0(s) - \frac{2}{q^2-1}\int_1^q \tau\tilde{\varphi}_0(\tau)\,d\tau. \tag{A–18}$$

Then we obtain $\varphi_l(s) \equiv \varphi_l(q, b_0, b_1; s)$ for $l \geq 1$, by the iterative application of (A–7), (A–9), (A–10) and (A–11). That is,

$$\varphi_{l+1}(s) = \tilde{\varphi}_{l+1}(s) - \frac{2}{q^2-1}\int_1^q \tau\tilde{\varphi}_{l+1}(\tau)\,d\tau, \tag{A–19}$$

where

$$\tilde{\varphi}_{l+1}(s) := \int_1^s \varphi'_{l+1}(\tau)\,d\tau \tag{A–20}$$

and

$$\varphi'_{l+1}(s) := (1+\varphi_0'^2)^{3/2}\left(-f_{l,l+1} + \frac{1}{s}\int_1^s \tau\varphi_l(\tau)\,d\tau\right). \tag{A–21}$$

Set for the unknown b_1

$$b_1 := \frac{b_0}{q}(1+\epsilon), \tag{A–22}$$

where

$$|\epsilon| \leq \tfrac{1}{4} \ \text{ and } \ q \geq 3. \tag{A–23}$$

We will determine ϵ in Section A.3 by gluing together two expansions at $s = q$, where $q = B^{-\tau}$ for $\tau > 0$ small.

Expansions with respect to ϵ. In this section we expand related functions with respect to ϵ.

Definition. Let $h \equiv h(q, b_0, \epsilon; s)$, where $1 \leq s \leq q$, $q \geq 3$, $|\epsilon| \leq \frac{1}{4}$ and $b_0 \in [-1, 1]$. We will write $h = \mathbb{O}(\epsilon; K)$ if for any fixed $M \in \mathbb{N} \cup \{0\}$

$$h = \sum_{l=0}^{M} h_l\epsilon^l + \tilde{h}_{M+1}\epsilon^{M+1},$$

where $h_l \equiv h_l(q, b_0; s)$, $\tilde{h}_{M+1} \equiv \tilde{h}_{M+1}(q, b_0, \epsilon; s)$, and $|h_l|, |\tilde{h}_{M+1}| \leq c_M|K|$. The constant c_M is independent on q, b_0, s, ϵ and K, it can depend on q, b_0 and s but not on ϵ.

From formula (A–14) for f and from (A–22) it follows that on $1 < s \leq q$

$$f = \frac{b_0}{s}\left(1 + \epsilon\frac{s^2-1}{q^2-1}\right). \tag{A–24}$$

Then

$$1-f^2=\Big(1-\Big(\frac{b_0}{s}\Big)^2\Big)(1+C_1\epsilon+C_2\epsilon^2), \tag{A–25}$$

where

$$C_1\equiv C_1(q,b_0;s)=-2b_0^2\frac{1}{q^2-1}\frac{s^2-1}{s^2-b_0^2},$$
$$C_2\equiv C_2(q,b_0;s)=-b_0^2\frac{1}{q^2-1}\frac{(s^2-1)^2}{s^2-b_0^2}.$$

Using (A–23), it follows that $|C_1\epsilon+C_2\epsilon^2|\le\frac{1}{2}$.

Set

$$\phi_k(s)\equiv\phi_k(q,b_0,\epsilon;s):=\varphi_k\Big(q,b_0,\frac{b_0}{q}(1+\epsilon);s\Big).$$

Then we obtain from formula (A–16) for φ_0'

$$\begin{aligned}\phi_0'&=\frac{b_0}{s}\Big(1+\epsilon\frac{s^2-1}{q^2-1}\Big)\Big(1-\Big(\frac{b_0}{s}\Big)^2\Big)^{-1/2}(1+C_1\epsilon+C_2\epsilon^2)^{-1/2}\\&=\frac{b_0}{\sqrt{s^2-b_0^2}}\big(1+\epsilon\mathbb{O}(\epsilon;1)\big).\end{aligned} \tag{A–26}$$

Formula (A–17) implies

$$\tilde\phi_0(s)=\tilde\phi_{0,0}(s)+\epsilon\mathbb{O}(\epsilon;b_0\ln s),$$

where

$$\tilde\phi_{0,0}(s)=b_0\Big(\ln\big(s+\sqrt{s^2-b_0^2}\big)-\ln\big(1+\sqrt{1-b_0^2}\big)\Big).$$

Finally, it follows from (A–18) that

$$\phi_0(s)=\phi_{0,0}(s)+\epsilon\mathbb{O}(\epsilon;b_0\ln q),$$

where

$$\begin{aligned}\phi_{0,0}(s)&=b_0\Big(\ln\big(s+\sqrt{s^2-b_0^2}\big)-\ln\big(1+\sqrt{1-b_0^2}\big)\Big)\\&+\frac{b_0}{q^2-1}\Big(\frac{q}{2}\sqrt{q^2-b_0^2}+\frac{b_0^2}{2}\ln\big(q+\sqrt{q^2-b_0^2}\big)-\frac{1}{2}\sqrt{1-b_0^2}-\frac{b_0^2}{2}\ln\big(1+\sqrt{1-b_0^2}\big)\Big).\end{aligned}$$

Using (A–24), (A–25) and (A–26), we immediately obtain

$$1+\phi_0'^2 \equiv (1-f^2)^{-1} = \frac{s^2}{s^2-b^2}\left(1+\epsilon\mathbb{O}(\epsilon;1)\right), \tag{A–27}$$

$$\frac{\phi_0'}{\sqrt{1+\phi_0'^2}} \equiv f = \frac{b_0}{s}\left(1+\epsilon\frac{s^2-1}{q^2-1}\right), \tag{A–28}$$

$$\frac{\phi_0''}{(1+\phi_0'^2)^{3/2}} \equiv f' = -\frac{b_0}{s^2}\left(1-\epsilon\frac{s^2+1}{q^2-1}\right). \tag{A–29}$$

Lemma A.1.3. *The functions ϕ_l, $l \geq 1$ are continuous in ϵ, $|\epsilon| \leq \frac{1}{4}$, and satisfy*

$$\phi_l(s) = \mathbb{O}\left(\epsilon; b_0(\ln q)^l q^{2l}\right), \tag{A–30}$$

$$d_l \equiv \frac{\phi_l'}{\sqrt{1+\phi_0'^2}} = \mathbb{O}\left(\epsilon; b_0(\ln q)^l q^{2l-1}\right), \tag{A–31}$$

$$e_l \equiv \frac{\phi_l''}{(1+\phi_0'^2)^{3/2}} = \mathbb{O}\left(\epsilon; b_0(\ln q)^l q^{2l-2}\right). \tag{A–32}$$

We will prove this lemma by induction based on formulas (A–15)–(A–17) and on the next lemma.

Lemma A.1.4. *Assume that equations* (A–30)–(A–32) *hold for* $1 \leq l \leq m$. *Then*

$$F_{m,m+1} = \mathbb{O}\left(\epsilon; b_0(\ln q)^{m+1} q^{2m}\right)$$

and, if $\lambda := Bq^2 \ln q \leq \lambda_0$, *for* $\lambda_0 > 0$ *sufficiently small, then*

$$|\tilde{F}_{m,m+1}| \leq c_m |b_0| (\ln q)^{m+1} q^{2m},$$

where $c_m = c_m(\lambda_0)$ *is independent on* b_0 *and* q.

Proof. Set

$$h_m(B) = \frac{1}{s}(d_0+P)F(d_0,P) + (e_0+Q)G(c_0,P),$$

where $F = (1+2d_0P+P^2)^{-1/2}$, $G = (1+2d_0P+P^2)^{-3/2}$, $P = \sum_{l=1}^m d_l B^l$, $Q = \sum_{l=1}^m e_l B^l$.

From assumption (A–1) on d_l it follows $|2d_0P+P^2| \leq \frac{1}{2}$, provided λ_0 is sufficiently small. Since

$$F_{m,m+1} = \frac{h_m^{(m+1)}(0)}{(m+1)!} \quad \text{and} \quad \tilde{F}_{m,m+1} = \frac{h_m^{(m+1)}(tB)}{(m+1)!}, \quad \text{for } 0<t<1,$$

the lemma is a consequence of the Leibniz rule and the chain rule. We find from these rules for $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_l \in \mathbb{N}$ and $t = (t_1, \ldots, t_m)$, $t_l \in \mathbb{N} \cup \{0\}$ and

$0 \le k \le m$ that

$$\text{(A–33)}\quad h_m^{(m+1)}(B) = \sum_{\sum_{l=1}^m \alpha_l t_l = m+1} \frac{1}{s} C_{m,\alpha,t} (P^{(\alpha_1)})^{t_1} \dots (P^{(\alpha_m)})^{t_m} + \sum_{k+\sum_{l=1}^m \alpha_l t_l = m+1} D_{m,k,\alpha,t}\, Q^{(k)} (P^{(\alpha_1)})^{t_1} \dots (P^{(\alpha_m)})^{t_m},$$

where

$$C_{m,\alpha,t} = C_{m,\alpha,t}(d_0, e_0, P), \quad D_{m,k,\alpha,t} = D_{k,\alpha,t}(d_0, P)$$

and

$$\hat{C}_{m,\alpha,t} := C_{m,\alpha,t}(s, d_0, e_0, 0) = \mathbb{O}(\epsilon; 1), \quad \hat{D}_{m,\alpha,t} := D_{m,k,\alpha,t}(d_0, 0) = \mathbb{O}(\epsilon; 1).$$

We recall that $d_0 = \mathbb{O}(\epsilon; b_0/s)$ and $e_0 = \mathbb{O}(\epsilon; b_0/s^2)$. From (A–33) it follows that

$$h_m^{(m+1)}(0) = \sum_{\sum_{l=1}^m \alpha_l t_l = m+1} \frac{1}{s} \hat{C}_{m,\alpha,t} (d_{\alpha_1})^{t_1} \dots (d_{\alpha_m})^{t_m} + \sum_{k+\sum_{l=1}^m \alpha_l t_l = m+1} \hat{D}_{m,k,\alpha,t}\, e_k (d_{\alpha_1})^{t_1} \dots (d_{\alpha_m})^{t_m}.$$

Using the assumptions on d_l and e_l (Lemma A.1.3), we have

$$h_m^{(m+1)}(0) = \mathbb{O}\left(\epsilon; b_0 (\ln q)^{\sum_l^m \alpha_i t_i} q^{\sum_l^m (2\alpha_l t_l - 1)}\right) + \mathbb{O}\left(\epsilon; b_0 (\ln q)^{k+\sum_l^m \alpha_i t_i} q^{2k-2\sum_l^m (2\alpha_l t_l - 1)}\right),$$

where in the first term on the right we have $\sum_{l=1}^m \alpha_l t_l = m+1$, and $k + \sum_{l=1}^m \alpha_l t_l = m+1$ in the second term. Hence, since in the first term $\sum_{l=1}^m t_l \ge 2$ holds because of $\sum_{l=1}^m \alpha_l t_l \ge 2$, $\alpha_l \ge 1$ and $t_l \ge 0$, it follows that

$$h_m^{(m+1)}(0) = \mathbb{O}(\epsilon; b_0 (\ln q)^{m+1} q^{2m}).$$

The estimate of $h_m^{(m+1)}(tB)$, $0 < t < 1$, is a consequence of (A–33) since

$$|P^{(l)}| \le c_l \left(|d_l| + |d_{l+1}| B + \dots + |d_{m-l}| B^{m-l}\right). \qquad \square$$

We recall that $\lambda := Bq^2 \ln q \le \lambda_0$.

Corollary A.1.5. $f_{m,m+1} = \mathbb{O}(\epsilon; b_0 (\ln q)^{m+1} q^{(2m)} (s-1))$.

Proof. Since $F_{l,k} \equiv (1/s)(s f_{l,k})'$, it follows from the boundary condition $f_{l,k}(1) = 0$ (see (A–11)) that

$$\text{(A–34)}\qquad f_{m,m+1} = \frac{1}{s} \int_1^s \tau F_{m,m+1}(\tau)\, d\tau. \qquad \square$$

Proof. Proof of Lemma A.1.3 Assume that the lemma holds for $1 \le l \le m$. Then

$$\frac{1}{s}\int_1^s \tau\phi_m(\tau)\,d\tau = \mathbb{O}\bigl(\epsilon;\,(\ln q)^m q^{2m}(s-1)\bigr). \tag{A–35}$$

Using formula (A–21) for φ'_{m+1}, Corollary A.1.5, (A–35) and the formula (A–27) for $1+\phi_0'^2$ we conclude that

$$\frac{\phi'_{m+1}}{\sqrt{1+\phi_0'^2}} = \mathbb{O}(\epsilon;\,b_0(\ln q)^{m+1}q^{2m+1})$$

and

$$\phi'_{m+1} = \mathbb{O}\left(\epsilon;\,b_0(\ln q)^{m+1}q^{2m+1}\frac{s^{3/2}}{(s-1)^{1/2}}\right).$$

Thus, it follows from (A–19) and (A–20) that

$$\phi_{m+1} = \mathbb{O}(\epsilon;\,b_0(\ln q)^{m+1}q^{2m+2}).$$

Formula (A–17) implies

$$\frac{\phi''_{m+1}}{(1+\phi_0'^2)^{3/2}} = 3\phi_0'\phi_0''\left(-f_{m,m+1}+\frac{1}{s}\int_1^s \tau\phi_m(\tau)\,d\tau\right) - (f_{m,m+1})' - \frac{1}{s^2}\int_1^s \tau\phi_m(\tau)\,d\tau + \phi_m.$$

Since, by (A–34),

$$f'_{m,m+1} = F_{m,m+1} - \frac{1}{s}f_{m,m+1},$$

it follows from formulas (A–27)–(A–29) for ϕ_0' and ϕ_0'', Lemma A.1.4, Corollary A.1.5, (A–35) and (A–30) that

$$\frac{\phi''_{m+1}}{(1+\phi_0'^2)^{3/2}} = \mathbb{O}(\epsilon;\,b_0(\ln q)^{m+1}q^{2(m+1)-2}).$$

It remains to show Lemma A.1.3 in the case $l=1$. Since $f_{0,1}\equiv 0$, we find from (A–21) that

$$\phi_1' = (1+\phi_0'^2)^{3/2}\,\frac{1}{s}\int_1^s \tau\phi_0(\tau)\,d\tau.$$

This equation implies Lemma A.1.3 in the case $l=1$ by using the properties of ϕ_0, see the formulas (A–27)–(A–29).

The continuity of ϕ_l in ϵ follows from formula (A–26) for ϕ_0' iteratively from (A–21), (A–20) and (A–19). □

Proof of Proposition 2.1. Because of Lemma A.1.3 it remains to show inequality (9) of Proposition 2.1, where $v_{1,m}\equiv v_m$. From Lemma A.1.4, (A–30) and the differential equations (A–8) for φ_0 and (A–7) for φ_l, where $m := l-1$ in (A–7), it follows that

$$\begin{aligned}\operatorname{div} Tv_m - Bv_m &= \frac{1}{s}\left(\frac{s\phi_0'}{\sqrt{1+\phi_0'^2}}\right)' + \sum_{k=1}^{m} F_{m,k}B^k + \tilde F_{m,m+1}B^{m+1} \\ &\qquad\qquad - B\left(\frac{C}{B} + \phi_0 + \cdots + \phi_m B^m\right) \\ &= (\tilde F_{m,m+1} - \phi_m)B^{m+1} \\ &= \big(O(b_0(\ln q)^{m+1}q^{2m}) + O(b_0(\ln q)^m q^{2m})\big)B^{m+1} \\ &= O(b_0(\ln q)^{m+1}q^{2m})B^{m+1}. \qquad \square\end{aligned}$$

A.2. ***Expansion far from the needle.*** Set, for $0 < B < 1$, $q \geq 3$ and $|\rho| < \rho_0$,

$$v_n = \sum_{k=0}^{n} \psi_k(s)\rho^{2k+1},$$

where the $\psi_k(s) \equiv \psi_k(B, q; s)$ are twice continuously differentiable functions in $q \leq s < \infty$. Suppose that $\psi_k'(q)$ satisfies the condition (13) and that ρ is a solution of (14) for a given b_1. We will set $b_1 = b_0(1+\epsilon)/q$, where $|\epsilon|$ is small and q is large. Thus, ρ will be small. Then v_n satisfies the boundary condition (11).

The sum v_n is said to be an *approximate solution* of (10)–(11) if v_n satisfies the boundary condition (11) and if

$$|\operatorname{div} Tv_n - Bv_n| \leq c|\rho|^{2n+3}$$

on $[q, \infty)$, where the constant $c = c(n, \rho_0)$ is independent on B, ρ and s. We will see that ψ_k satisfies a linear second order boundary value problem, provided v_n is an approximate solution. In particular, ψ_0 is a solution of the linearized equation to (10) about the zero solution.

Definition of ψ_k. Assume for $k \in \mathbb{N} \cup \{0\}$ that

$$\sup_{s\in(q,\infty)} |\psi_k'(s)| < \infty, \tag{A–36}$$

uniformly in $0 < B < 1$ and $q \geq 3$.

Then, for given $N \in \mathbb{N}$ and $|\rho| < \rho_0$ with ρ_0 sufficiently small, we have

$$\begin{aligned}\frac{v_n'}{\sqrt{1+v_n'^2}} &\equiv \left(\sum_{k=0}^{n} \psi_k'\rho^{2k+1}\right)\left(1 + \left(\sum_{k=0}^{n} \psi_k'\rho^{2k+1}\right)^2\right)^{-1/2} \\ &= \rho\psi_0' + \sum_{k=1}^{N} f_{n,k}(\psi_0', \dots, \psi_n')\rho^{2k+1} + \tilde f_{n,N+1}\rho^{2N+3}.\end{aligned}$$

Set $g_n(\rho) := v_n'/\sqrt{1+v_n'^2}$. Then

$$f_{n,k} = g_n^{(2k+1)}(0)/(2k+1)! \quad \text{and} \quad \tilde{f}_{n,k} = g_n^{(2k+1)}(t\rho)/(2k+1)! \quad \text{for } 0<t<1.$$

From assumption (A–36) on ψ_k' it follows that

$$|f_{n,k}| \le c_{n,k}(q) < \infty \quad \text{and} \quad |\tilde{f}_{n,N+1}| \le \tilde{c}_{n,N+1}(q) < \infty.$$

Above we have used that $v_n(1+(v_n')^2)^{-1/2}$ is an odd function in ρ.

Thus

$$\text{div}\, Tv_n = \frac{1}{s}(s\psi_0')'\rho + \frac{1}{s}\sum_{k=1}^{N}(sf_{n,k})'\rho^{2k+1} + \frac{1}{s}(s\tilde{f}_{n,N+1})'\rho^{2N+3}. \tag{A–37}$$

As in the previous section we need estimates on the derivatives $(f_{n,k})'$ and $(\tilde{f}_{n,N+1})'$. Assume for $k \in \mathbb{N}\cup\{0\}$ that

$$\sup_{s\in(q,\infty)} |\psi_k''(s)| < \infty, \tag{A–38}$$

uniformly in $0<B<1$ and $q \ge 3$.

Applying identity (A–4) and the assumptions (A–36) and (A–38) on ψ_k' and ψ_k'', we get

$$\text{div}\, Tv_n = \frac{1}{s}(s\psi_0')'\rho + \sum_{k=1}^{N} F_{n,k}\rho^{2k+1} + \tilde{F}_{n,N+1}\rho^{2N+3}$$

and $F_{n,k}$, $\tilde{F}_{n,N+1}$ are bounded on $[q,\infty)$. Set

$$h_n(\rho) := \frac{1}{s}\frac{v_n'}{\sqrt{1+v_n'^2}} + v_n''\big(1+v_n'^2\big)^{-3/2}.$$

Then

$$F_{n,k} = \frac{h_n^{(2k+1)}(0)}{(2k+1)!} \quad \text{and} \quad \tilde{F}_{n,N+1} = \frac{h_n^{(2N+3)}(t\rho)}{(2N+3)k!} \quad \text{for } 0<t<1.$$

Lemma A.2.6. *Assume that* ψ_l', $l=0,\dots,n+1$ *satisfies* (A–36). *Then*

$$\frac{v_{n+1}'}{\sqrt{1+v_{n+1}'^2}} = \frac{v_n'}{\sqrt{1+v_n'^2}} + \psi_{n+1}'\rho^{2(n+1)+1} + R,$$

where $|R| \le c(q)\rho^{2(n+1)+3}$ *and* $0<\rho\le\rho_0(q)$ *for* ρ_0 *sufficiently small.*

Proof.

$$
\begin{aligned}
&\frac{v'_{n+1}}{\sqrt{1+{v'_{n+1}}^2}}\\
&\quad= (v'_n + \psi'_{n+1}\rho^{2(n+1)+1})\big(1+{v'_n}^2 + 2v'_n\psi'_{n+1}\rho^{2(n+1)+2} + (\psi'_{n+1})^2\rho^{4(n+1)+2}\big)^{-1/2}\\
&\quad= \big(v'_n + \psi'_{n+1}\rho^{2(n+1)+1}\big)\big(1+{v'_n}^2\big)^{-1/2}\\
&\qquad\qquad\cdot\left(1+2\frac{v'_n}{\sqrt{1+{v'_n}^2}}\frac{\psi'_{n+1}}{\sqrt{1+{v'_n}^2}}\rho^{2(n+1)+1} + \frac{(\psi'_{n+1})^2}{1+{v'_n}^2}\rho^{4(n+1)+2}\right)^{-1/2}\\
&\quad= \frac{v'_n}{\sqrt{1+{v'_n}^2}} + (1+{v'_n}^2)^{-3/2}\Big((1+{v'_n}^2)\psi'_{n+1} - {v'_n}^2\psi'_{n+1}\Big)\rho^{2(n+1)+1}\\
&\qquad\qquad + O\big(\rho^{4(n+1)+2}\big)\\
&\quad= \frac{v'_n}{\sqrt{1+{v'_n}^2}} + \frac{\psi'_{n+1}}{(1+{v'_n}^2)^{3/2}}\rho^{2(n+1)+1} + O\big(\rho^{4(n+1)+2}\big)\\
&\quad= \frac{v'_n}{\sqrt{1+{v'_n}^2}} + \psi'_{n+1}\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big).
\end{aligned}
$$

The last line follows since $1+{v'_n}^2 = 1 + O(\rho)$. □

Lemma A.2.7. *Suppose the assumptions* (A–36) *and* (A–38) *on* ψ'_k *and* ψ''_k *are satisfied. Then*

$$\operatorname{div} Tv_{n+1} = \operatorname{div} Tv_n + \frac{1}{s}(s\psi'_{n+1})'\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big)$$

as $\rho \to 0$, *uniformly in* $s \in [q,\infty)$.

Proof. From (A–4) and Lemma A.2.6 it follows that

$$
\begin{aligned}
\operatorname{div} Tv_{n+1} &\equiv \frac{1}{s}\frac{v'_{n+1}}{\sqrt{1+{v'_n}^2}} + \frac{v''_{n+1}}{(1+{v'_n}^2)^{3/2}}\\
&= \frac{1}{s}\frac{v'_n}{\sqrt{1+{v'_n}^2}} + \frac{1}{s}\psi'_{n+1}\rho^{2(n+1)+1} + \frac{v''_{n+1}}{(1+{v'_{n+1}}^2)^{3/2}} + O\big(\rho^{2(n+1)+3}\big).
\end{aligned}
$$

Since

$$\frac{v''_{n+1}}{(1+{v'_{n+1}}^2)^{3/2}} = \frac{v''_n}{(1+{v'_n}^2)^{3/2}} + \psi''_{n+1}\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big)$$

(see the proof of Lemma A.2.6), we find that

$$\begin{aligned}\operatorname{div} Tv_{n+1} &= \frac{1}{s}\frac{v_n'}{\sqrt{1+v_n'^2}} + \frac{v_n''}{(1+v_n'^2)^{3/2}} + \frac{1}{s}(s\psi_{n+1}')'\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big)\\ &= \operatorname{div} Tv_n + \frac{1}{s}(s\psi_{n+1}')'\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big).\end{aligned}$$

Lemma A.2.7 implies

$$\begin{aligned}&\operatorname{div} Tv_{n+1} - Bv_{n+1}\\ &\quad= \operatorname{div} Tv_n + \frac{1}{s}(s\psi_{n+1}')'\rho^{2(n+1)+1} - B\big(v_n + \psi_{n+1}\rho^{2(n+1)+1}\big) + O\big(\rho^{2(n+1)+3}\big)\\ &\quad= \operatorname{div} Tv_n - Bv_n + \Big(\frac{1}{s}(s\psi_{n+1}')' - B\psi_{n+1}\Big)\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big).\end{aligned}$$

Then from the expansion (A–37) of $\operatorname{div} Tv_n$, with $N := n+1$, and the condition

$$\operatorname{div} Tv_{n+1} - Bv_{n+1} = O\left(\rho^{2(n+1)+3}\right)$$

as $\rho \to 0$, it follows on $q < s < \infty$ that

(A–39) $$\frac{1}{s}(s\psi_0')' - B\psi_0 = 0$$

and for $n \geq 0$

$$\frac{1}{s}(s\psi_{n+1}')' - B\psi_{n+1} = -\frac{1}{s}(sf_{n,n+1})'.$$

Thus (see Section 3) we define ψ_k, $k \in \mathbb{N}$, iteratively by the boundary value problem

(A–40) $$\frac{1}{s}(s\psi_k')' - B\psi_k = -\frac{1}{s}\big(sf_{k-1,k}(\psi_0', \dots, \psi_{k-1}')\big)' \quad \text{on } (q,\infty),$$

(A–41) $$\psi_k'(q) = (-1)^k \binom{-\frac{1}{2}}{k}, \quad \limsup_{s\to\infty} |\psi_k(s)| < \infty. \qquad \square$$

Boundary value problem for ψ_k. The solution of the homogeneous equation (A–39) that satisfies the boundary conditions (A–41) is given by

$$\psi_0(s) = \frac{1}{\sqrt{B}}\frac{K_0(\sqrt{B}s)}{K_0'(\sqrt{B}q)}.$$

We obtain $\psi_1, \psi_2, \dots$ iteratively from the boundary value problem (A–40)–(A–41). The estimates (A–36), (A–38) on ψ_k', ψ_k'' and formula (16) of $\psi_k(B,q;q)$, see Proposition 3.2, follow iteratively from a formula for the solution ψ_k by using the properties of $f_{k-1,k}(\psi_0', \dots, \psi_{k-1}')$. Once we have shown (A–36) and (A–38), we arrive at the estimate (15) of Proposition 3.2, since

$$\begin{aligned}\operatorname{div} Tv_{2,n} - Bv_{2,n} &= \frac{1}{s}(s\psi_0')'\rho + \frac{1}{s}\sum_{k=1}^{n}(sf_{n,k})'\rho^{2k+1} + \tilde{F}_{n,n+1}\rho^{2n+3} \\ &\qquad\qquad - B(\psi_0\rho + \cdots + \psi_n\rho^{2n+1}) \\ &= \tilde{F}_{n,n+1}\rho^{2n+3}.\end{aligned}$$

The proof of Theorem 1.1 requires Proposition 3.2 in the case $n = 0$ only. That is, we have to confirm the estimates (A–36), (A–38) for ψ_0', ψ_0'' and the property (16) of Proposition 3.2. Since

$$\psi_0(B, q; s) = \frac{1}{\sqrt{B}}\frac{K_0(\sqrt{B}s)}{K_0'(\delta)}, \quad \delta = \sqrt{B}q,$$

the expansion of $w_0(\delta)$ (see Proposition 3.2) follows from the expansions of $K_0(\delta)$ and $K_0'(\delta)$ as $\delta \to 0$. Since $\lim_{s\to\infty} \psi_0'(s) = 0$, where $B > 0$ is fixed, and since $K_0''(z) > 0$ for, $z > 0$, it follows that $|\psi_0'(s)| \le 1$ on $[q, \infty)$. From the differential equation (A–39) we conclude that

$$|\psi_0''(s)| \le \frac{1}{q} + \sqrt{B} \sup_{s\in(q,\infty)} \frac{K_0(\sqrt{B})}{|K_0'(\delta)|} \le \frac{1}{q} + \sqrt{B}\frac{K_0(\delta)}{|K_0'(\delta)|},$$

where we have used that $K_0'(z) < 0$, where $z > 0$. Thus

$$\sup_{s\in(q,\infty)} |\psi_0''(s)| \le \frac{1}{q} + \sqrt{B}\ O(\delta \ln \delta) \quad \text{as } \delta \to 0.$$

We will now prove iteratively the existence of ψ_k, the estimates (A–36) and (A–38), and the formula (16) for ψ_k if $k \ge 1$.

Let $K_0(z)$ and $I_0(z)$ be the modified Bessel functions of second kind of order zero. Concerning properties of the Bessel functions $K_0(z)$ and $I_0(z)$, see [Abramowitz and Stegun 1964] and the considerations in [Siegel 1980].

For $k \in \mathbb{N}$, set

$$f := f_{k-1,k}(\psi_0', \ldots, \psi_{k-1}'), \quad F := -\frac{1}{s}(sf)', \quad \eta := (-1)^k\binom{-\frac{1}{2}}{k}.$$

Any solution of the differential equation (A–40) can be written as

$$\text{(A–42)} \quad \begin{aligned}\psi(s) = \left(c_1 - \int_q^s tI_0(\sqrt{B}t)F(t)\,dt\right) K_0(\sqrt{B}s) \\ + \left(c_2 + \int_q^s tK_0(\sqrt{B}t)F(t)\,dt\right) I_0(\sqrt{B}s),\end{aligned}$$

where $c_1,\ c_2 \in \mathbb{R}$. From the boundary conditions (A–41) it follows that

$$c_2 = -\int_q^\infty t K_0(\sqrt{B}t) F(t)\, dt, \tag{A–43}$$

$$c_1 = \frac{1}{\sqrt{B} K_0'(\delta)} \left(\eta + \sqrt{B} I_0'(\delta) \int_q^\infty t K_0(\sqrt{B}t) F(t)\, dt \right). \tag{A–44}$$

Since

$$f_{0,1}(\psi_0') = \tfrac{1}{2}\big(\psi_0'(t)\big)^3 = \frac{1}{2}\big(K_0'(\delta)\big)^{-3}\big(K_0'(\sqrt{B}t)\big)^3,$$

we expect that $f_{k-1,k}$ is a sum of such products too.

Definition. A function $f(t)$ is said to be of type (SP) if

(i) there exists an $M \in \mathbb{N}$ such that f can be written as

$$f(t) = \sum_{l=1}^{M} A_l(\delta) B_l(\sqrt{B}t),$$

where $A_l,\ B_l \in C^\infty(0, \infty)$,

(ii) there is a $k_l \in \mathbb{N} \cup \{0\}$ such that $A_l(\delta) = \delta^{k_l} P(\delta, \ln\delta),\ \ B_l(\delta) = \delta^{-k_l} P(\delta, \ln\delta)$ as $\delta \to 0$, where the expression $P(\delta, \ln\delta)$ is explained in Definition 3.1, and

(iii) $B_l(u) = O(\mathrm{e}^{-2u})$ as $u \to \infty$.

Suppose f is of type (SP). Applying (A–42)–(A–44), we find

$$\psi(s) = \frac{1}{\sqrt{B}}\big(F_1(\delta, \sqrt{B}s) K_0(\sqrt{B}s) + F_2(\delta, \sqrt{B}s) I_0(\sqrt{B}s)\big), \tag{A–45}$$

where

$$\begin{aligned} F_1 := \frac{\eta}{K_0'(\delta)} + \frac{I_0'(\delta)}{K_0'(\delta)} \left(\sum_l A_l(\delta) \int_\delta^\infty u K_0'(u) B_l(u)\, du + \delta K_0(\delta) \sum_l A_l(\delta) B_l(\delta) \right) \\ - \sum_l A_l(\delta) \int_\delta^{\sqrt{B}s} u I_0'(u) B_l(u)\, du + \sqrt{B}s\ I_0(\sqrt{B}s) \sum_l A_l(\delta) B_l(\sqrt{B}s) \\ - \delta I_0(\delta) \sum_l A_l(\delta) B_l(\delta) \end{aligned}$$

and

$$F_2 := -\sum_l A_l(\delta) \int_{\sqrt{B}s}^\infty u K_0'(u) B_l(u)\, du - \sqrt{B}s\ K_0(\sqrt{B}s) \sum_l A_l(\delta) B_l(\sqrt{B}s).$$

The derivative ψ' is given by

$$\psi'(s) = F_1(\delta, \sqrt{B}s) K_0'(\sqrt{B}s) + F_2(\delta, \sqrt{B}s) I_0'(\sqrt{B}s). \tag{A–46}$$

We conclude from (A–46) that ψ_k' is of type (SP), provided the function $f := f_{k-1,k}(\psi_0', \dots, \psi_{k-1}')$ is of type (SP). Property (i) of the definition follows immediately from formula (A–46). We omit here the considerations that (ii) and (iii) are also satisfied. Then $f_{k,k+1}(\psi_0', \dots \psi_k')$ is of type (SP) since

$$f_{k,k+1}(\psi_0', \dots, \psi_k') = \frac{1}{(2k+3)!}\frac{d^{2k+3}g_k}{d\rho^{2k+3}}(0) = \sum_{\sum_{l=0}^{k}(2\alpha_l+1)t_l=2k+3} r_{k,\alpha,t}(\psi_{\alpha_0}')^{t_0}\dots(\psi_{\alpha_k}')^{t_k},$$

where $\alpha = (\alpha_0, \dots, \alpha_k),\ t = (t_0, \dots, t_k),\ \alpha_l,\ t_l \in \mathbb{N} \cup \{0\}$ and $r_{k,\alpha,t} \in \mathbb{R}$. We recall that $g_k(\rho) = v_k'/\sqrt{1 + (v_k')^2}$ and $v_k' = \sum_{l=0}^{k} \psi_l' \rho^{2l+1}$.

Finally, we find iteratively from (A–45), (A–46) and the differential equation (A–40) that the estimates (A–36), (A–38) for ψ_k', ψ_k'' hold and that

$$\sqrt{B}\psi_k(B, q; q) = P(\delta, \ln\delta)$$

(see Proposition 3.2).

A.3. ***Composing of the inner and outer solutions.*** Set $q := B^{-\tau}$ for some $\tau \in (0, \frac{1}{2})$. Then we will show that there is a solution $\epsilon \in (-\frac{1}{4}, \frac{1}{4})$ of equation (17), that is of $G(\epsilon) = 0$, where

$$G(\epsilon) := \frac{2\epsilon b_0}{B(q^2-1)} + \sum_{k=0}^{m}\phi_k(q, b, \epsilon; q)B^k - \frac{1}{\sqrt{B}}\sum_{k=0}^{n} w_k(\delta)\rho^{2k+1}.$$

Here is $\delta = \sqrt{B}q$, $b_1 = b_0(1+\epsilon)/q$ and $\rho = \rho(b_0, q, \epsilon)$ is given by (14). In particular,

$$\rho = b_1 + O(b_1^{2n+3}) = \frac{b_0(1+\epsilon)}{q} + O\left(\frac{b_0}{q^{2n+3}}\right)$$

as $q \to \infty$. The existence of a zero of the continuous function $G(\epsilon)$ follows from the intermediate value theorem. Propositions 2.1 and 3.2 imply

$$G(\epsilon) = \frac{2\epsilon b_0}{B(q^2-1)} + \phi_0(q, b_0, \epsilon; q) + O(b_0 q^2(\ln q)^2 B) - \frac{1}{\sqrt{B}}\left(w_0(\delta)\rho + O\left(\frac{b_0}{q^3}\delta(\ln\delta)^l\right)\right)$$

for some $l \in \mathbb{N} \cup \{0\}$. Since, by Proposition 2.1 and the formula for $\phi_{0,0}$ (page 307), we have

$$\phi_0(q, b_0, \epsilon; q) = \phi_{0,0}(q, b_0; q) + O(b_0\epsilon\ln q) = \tfrac{1}{2}b_0 + O\left(b_0\frac{\ln q}{q^2}\right) + O(b_0\epsilon\ln q)$$

and

$$w_0(\delta) = \frac{K_0(\delta)}{K_0'(\delta)} = \delta\big(\ln\delta + \gamma - \ln 2 + O(\delta^2(\ln\delta)^2)\big)$$

as $\delta \to 0$, it follows that

$$G(\epsilon) = \frac{2\epsilon b_0}{\delta^2} + \frac{b_0}{2} - b_0(\ln\delta + \gamma - \ln 2) + O\Big(b_0\frac{\epsilon}{\delta}\frac{1}{q^2}\Big) + O\Big(b_0\frac{\ln q}{q^2}\Big) + O(b_0\epsilon\ln q)$$
$$+ O(b_0 q^2(\ln q)^2 B) + O(b_0\delta^2(\ln\delta)^2) + O(b_0\epsilon\ln\delta) + O\Big(b_0(\ln\delta)^l\frac{1}{q^2}\Big).$$

For R real, $|R| \le 1$, set

$$\epsilon(R) := \tfrac{1}{2}\delta^2\ln\delta + \tfrac{1}{2}(\gamma - \ln 2 - \tfrac{1}{2})\delta^2 + R\delta^2.$$

then $|\epsilon| < \frac{1}{4}$ if $\delta < \delta_0$, for δ_0 sufficiently small. We have $G(\epsilon(1)) > 0$ and $G(\epsilon(-1)) < 0$ if $0 < \delta < \delta_0$, for δ_0 sufficiently small.

Finally, we obtain from $G(\epsilon(R)) = 0$ an estimate of R. Since

$$R = O\left(\frac{1}{q^2}\ln\delta\right) + O\left(\frac{\ln q}{q^2}\right) + O(\delta^2\ln\delta\ln q) + O(\delta^2(\ln\delta)^2) + O\left(\frac{1}{q^2}(\ln\delta)^l\right),$$

we find

$$R \equiv R(b_0, B, B^{-\tau}) = O\left((\ln B)^{k+1}B^{2\tau}\right) + O\left((\ln B)^2 B^{1-2\tau}\right)$$

uniformly in $b_0 \in [-1, 1]$. Thus, Proposition 4.1 is shown.

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