Pacific Journal of Mathematics

APPROXIMATING SYMMETRIC CAPILLARY SURFACES

DAVID SIEGEL

Volume 224 No. 2 April 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 224, No. 2, 2006

APPROXIMATING SYMMETRIC CAPILLARY SURFACES

DAVID SIEGEL

An iterative method is introduced for approximating symmetric capillary surfaces which makes use of the known exact volume. For the interior and annular problems this leads to upper and lower bounds at the center or inner boundary and at the outer boundary, and to an asymptotic expansion in powers of the Bond number. For the exterior problem we determine the leading order asymptotics of the boundary height as the Bond number tends to zero, obtaining a result first proved by B. Turkington.

1. Introduction

The study of capillary surfaces goes back to Laplace [1805–1806]. The canonical modern reference is [Finn 1986]. We will consider symmetric capillary surfaces with gravity in one of three cases: interior, annular and exterior. A vertical circular cylindrical tube immersed in an infinite reservoir of fluid will create an interior and an exterior capillary surface. Two concentric circular tubes will create an annular capillary surface between them.

Let r be the radial variable and let ψ be the inclination angle of the surface $z = u(r)$. Then $\sin\psi = u_r/\sqrt{1+u_r^2}$ and $Nu = (1/r)(r\sin\psi)_r$ is twice the mean curvature of the surface. A capillary surface is determined by the capillary equation $Nu = Bu$, where B is a positive constant, the Bond number, and by specifying the contact angle $\gamma \in [0, \pi]$ on the boundary. The contact angle is the angle between the interface cross–section and vertical, measured inside the fluid. Thus, the inclination angle will be prescribed on the boundary. In order for the annular problem to be similar to the interior problem we take the contact angle to be $\pi/2$ on the inner boundary and γ on the outer boundary.

The interior and annular problems can be written

$$Nu = Bu, \quad a < r < 1, \quad \sin\psi(a) = 0, \quad \sin\psi(1) = \cos\gamma, \tag{1}$$

where $a = 0$ for the interior problem and $0 < a < 1$ for the annular problem.

MSC2000: 76B45, 34E05, 34B15, 34B40.
Keywords: capillarity, symmetric capillary surface, exterior problem, approximation, asymptotics.
This research is supported by a Natural Sciences and Engineering Research Council of Canada Discovery Grant.

The exterior problem is

$$Nu = Bu, \quad r > 1, \quad \sin\psi(1) = -\cos\gamma, \quad u \to 0 \text{ as } r \to \infty. \tag{2}$$

For all three problems we take

$$0 \le \gamma < \pi/2. \tag{3}$$

If $\gamma = \pi/2$ then $u = 0$. If $\pi/2 < \gamma \le \pi$ then $\bar{u} = -u$ satisfies $N\bar{u} = B\bar{u}$ with contact angle $\bar{\gamma} = \pi - \gamma$, so $0 \le \bar{\gamma} < \pi/2$. Scaling allows us to take one boundary at $r = 1$. It is known [Siegel 1980] that for a solution to (2), u and u_r decay exponentially fast as $r \to \infty$. Also, under (3), the solution u is positive in every case by the Comparison Principle [Finn 1986, Theorem 5.1; Siegel 1980, Theorem 1].

The volume lifted can be determined for all three problems:

$$B \int_I ru(r)\, dr = \cos\gamma,$$

where $I = [a, 1]$ for (1) and $I = [1, \infty)$ for (2).

We wish to employ approximate solutions that have the correct volume. The key observation is that if v_1 is a nonnegative function with the correct volume then we may define v_2 by $Nv_2 = Bv_1$ and v_2 will satisfy the correct boundary conditions.

Theorem 1.1. *Let v_1 be a nonnegative continuous function on I which satisfies $B \int_I rv_1(r)\, dr = \cos\gamma$ where I is $[a, 1]$ or $[1, \infty)$. Assume that v_1 is nondecreasing when I is $[a, 1]$ and $v_1(r) = O(\frac{1}{r^3})$ as $r \to \infty$ when $I = [1, \infty)$. Here $B > 0$, $0 \le \gamma < \frac{\pi}{2}$ and $0 \le a < 1$. Then there is a function v_2 defined and continuous on I, satisfying $Nv_2 = Bv_1$, given as a quadrature of v_1, which satisfies the boundary conditions of problem (1) or (2). Let ψ_2 be the inclination angle of v_2 and let*

$$h_2 = \frac{\sin\psi_2}{\sqrt{1 - \sin^2\psi_2}}.$$

For $I = [a, 1]$, let $\sin\psi_2(r) = (B/r)\int_a^r sv_1(s)\, ds$ and $v_2(r) = v_2(a) + \int_a^r h_2(s)\, ds$. Then v_2 is nondecreasing, $\sin\psi(a) = 0$ and $\sin\psi(1) = \cos\gamma$.

For $I = [1, \infty)$, let $\sin\psi_2(r) = -(B/r)\int_r^\infty sv_1(s)\, ds$ and $v_2(r) = -\int_r^\infty h_2(s)\, ds$. Then v_2 is nonincreasing, $\sin\psi_2(1) = -\cos\gamma$ and $v_2(r) = O(r^{-1})$ and $v_{2r}(r) = O(r^{-2})$ as $r \to \infty$.

For $I = [a, 1]$, by choosing

$$v_2(a) = \frac{1}{1 - a^2}\left(\frac{2\cos\gamma}{B} - \int_a^1 (1 - r^2)\frac{\sin\psi_2(r)}{\sqrt{1 - \sin^2\psi_2(r)}}\, dr\right), \tag{4}$$

v_2 will satisfy the volume condition $B \int_I rv_2\, dr = \cos\gamma$. With this choice v_2 will be nonnegative when $B \le 6$.

Proof. First consider $I = [a, 1]$. Since v_1 is nonnegative we have $\sin\psi_2 \geq 0$, which implies that v_2 is nondecreasing. Since v_1 is nondecreasing we have

$$\sin\psi_2 \leq \frac{Bv_1(r^2 - a^2)}{2r} \leq \frac{Brv_1}{2}.$$

It follows that

$$\left(\frac{\sin\psi_2}{r}\right)_r = \frac{2}{r^2}\left(\frac{Brv_1}{2} - \sin\psi_2\right) \geq 0.$$

Thus, $(\sin\psi_2)/r \leq \cos\gamma$ or $\sin\psi_2 \leq r\cos\gamma \leq r$. Since $p/\sqrt{1-p^2}$ is increasing on $[0, 1)$, we have

$$\frac{\sin\psi_2}{\sqrt{1-\sin^2\psi_2}} \leq \frac{r}{\sqrt{1-r^2}},$$

so $v_2(r) \leq v_2(a) + \sqrt{1-a^2} - \sqrt{1-r^2}$. Thus v_2 is defined and continuous on I. Requiring $B\int_I rv_2\,dr = \cos\gamma$, after changing the order of integration, results in (4). Now for $B \leq 6$, use

$$\frac{\sin\psi_2}{\sqrt{1-\sin^2\psi_2}} \leq \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}} \leq \frac{r\cos\gamma}{\sqrt{1-r^2}}$$

in (4) to see that

$$v_2(a) \geq \frac{\cos\gamma}{1-a^2}\left(\frac{2}{B} - \int_0^1 r\sqrt{1-r^2}\,dr\right) = \frac{\cos\gamma}{1-a^2}\left(\frac{2}{B} - \frac{1}{3}\right) \geq 0.$$

Thus v_2 is nonnegative.

Next consider $I = [1, \infty)$. Since v_1 is nonnegative, $\sin\psi_2 \leq 0$, which implies that v_2 is nonincreasing. From the volume condition on v_1, $\sin\psi_2(1) = -\cos\gamma$. From $(\sin\psi_2)_r = Bv_1 - (\sin\psi_2)/r \geq 0$, we get $\sin\psi_2 \geq -\cos\gamma$. Since $v_1 = O(r^{-3})$, $\sin\psi_2 = O(r^{-2})$, giving $v_{2r} = O(r^{-2})$ as $r \to \infty$. Since v_2 is nonincreasing and tends to zero, v_2 is nonnegative. From the formula for v_2, we see that $v_2 = O(r^{-1})$ as $r \to \infty$. As $(\sin\psi_2)_r(1) = Bv_1(1) + \cos\gamma > 0$, the integral for $v_2(1)$ is finite. Thus v_2 is continuous on I.

Finally, by the defining formulas, in all cases, $Nv_2 = Bv_1$ in the interior of I. □

For interior or annular capillary surfaces and $B \leq 6$, Theorem 1.1 provides a sequence of iterates $\{v_n\}$, where $Nv_{n+1} = Bv_n$ for $n \geq 0$. The simplest initial function is the constant function satisfying the volume condition

$$v_0 = \frac{2\cos\gamma}{B(1-a^2)}. \tag{5}$$

The properties of this sequence are explored in Section 2. An asymptotic expansion in powers of B is obtained. The theory is then applied to the interior problem and a formula of Rayleigh for measuring surface tension is proved.

The exterior problem is considered in Section 3. Two approximations are used to prove a result of Bruce Turkington [1980] on the asymptotic boundary height as B tends to zero.

An attractive feature of the method employed in this paper is its applicability to capillary problems with $\gamma = 0$. The general asymptotic series result in [Miersemann 1993] excludes the case $\gamma = 0$. However, for the interior problem, Miersemann [1994] has established an asymptotic expansion with $0 \le \gamma < \pi/2$.

The annular problem certainly merits further work. A start on this has been made by Alan Elcrat, Tae-Eun Kim and Ray Treinen [Elcrat et al. 2004].

2. Interior and annular capillary surfaces

The sequence of iterates $\{v_n\}$ for the interior and annular capillary problem (1) introduced after Theorem 1.1 has the properties listed in Theorem 2.3 below. The proof will make use of two lemmas whose proof is straightforward. Denote the inclination angles of two functions v and w defined on $[a, 1]$ by ψ_v and ψ_w, respectively.

Lemma 2.1. *Let $a < b < 1$. If $Nv < Nw$ for $a < r < b$ and $\psi_v(a) = \psi_w(a)$ then $\psi_v < \psi_w$, for $a < r \le b$. If $Nv < Nu$ for $b < r < 1$ and $\psi_v(1) = \psi_w(1)$, then $\psi_w < \psi_v$, for $b \le r < 1$.*

Lemma 2.2. *If $\psi_v < \psi_w$ on $(a, 1)$ and $\int_a^1 rv\,dr = \int_a^1 rw\,dr$ then there exists $b \in (a, 1)$ such that $v(b) = w(b)$ and $w(r) < v(r)$ for $r < b$ and $v(r) < w(r)$ for $r > b$.*

Theorem 2.3. *Let u be the solution to* (1) *and ψ its inclination angle. For $B \le 6$, the iterates provided by Theorem 1.1 with v_0 given by* (5) *have the following properties*:

$$\psi_0 < \psi_2 < \cdots < \psi < \cdots \psi_3 < \psi_1;$$
$$v_1(a) < v_3(a) < \cdots < u(a) < \cdots < v_2(a) < v_0 \quad \textit{for } a < r < 1;$$
$$v_0 < v_2(1) < \cdots < u(1) < \cdots < v_3(1) < v_1(1),$$
$$|u - v_n| < C(\gamma, a)\left(B\frac{\sqrt{1-a^2}}{1+a^2}\right)^n, \quad \textit{where } C(\gamma, a) = \frac{\sqrt{1-a^2\cos^2\gamma} - \sin\gamma}{\cos\gamma}.$$

Proof. From the defining equations, $\psi_1 > 0$ and $v_1 > 0$ on $(a, 1]$ and so $\sin\psi_2 > 0$ on $(a, 1]$.

Since u is positive, it follows that $\sin\psi = \frac{B}{r}\int_a^r su(s)\,ds > 0$ for $r > a$. Since v_0 is constant, $\psi_0 = 0$. Thus, $\psi_0 < \psi$.

We proceed to prove a number of statements of a recursive nature, using Lemmas 2.1 and 2.2. First we show that $\psi_{2k} < \psi$ implies that $\psi < \psi_{2k+1}$ for $k \ge 0$. By Lemma 2.2 there exists $b_{2k} \in (a, 1)$ with $v_{2k}(b_{2k}) = u(b_{2k})$, $u < v_{2k}$ for $r < b_{2k}$

and $u > v_{2k}$ for $r > b_{2k}$. Since $Nv_{2k+1} = Bv_{2k}$ and $Nu = Bu$, we conclude that $\psi < \psi_{2k+1}$ by Lemma 2.1 by arguing on the intervals $[a, b_{2k}]$ and $[b_{2k}, 1]$.

In a similar fashion, one proves that $\psi < \psi_{2k+1}$ implies that $\psi_{2k+2} < \psi$ for $k \geq 0$. Combining statements, we have $\psi_{2k} < \psi < \psi_{2k+1}$ for $k \geq 0$.

We know that $\psi_0 < \psi_2$ for $r > a$. Next we show that $\psi_{2k} < \psi_{2k+2}$ implies that $\psi_{2k+3} < \psi_{2k+1}$ for $k \geq 0$. By Lemma 2.2 there exists $c_k \in (a, 1)$ with $v_{2k}(c_k) = v_{2k+2}(c_k)$, $v_{2k+2} > v_{2k}$ for $r < c_k$ and $v_{2k+2} < v_{2k}$ for $r > c_k$. Using $Nv_{2k+3} = Bv_{2k}$ and $Nv_{2k+1} = Bv_{2k}$, we get $\psi_{2k+3} < \psi_{2k+1}$ by Lemma 2.1.

Likewise, one proves that $\psi_{2k+3} < \psi_{2k+1}$ implies that $\psi_{2k+4} > \psi_{2k+2}$ for $k \geq 0$. Combing statements gives that $\{\psi_{2k}\}$ is increasing and $\{\psi_{2k+1}\}$ is decreasing.

From $\psi_{2k} < \psi$ it follows that $u(a) < v_{2k}(a)$ and $v_{2k}(1) < u(1)$ by Lemma 2.2. From $\psi < \psi_{2k+1}$ it follows that $v_{2k+1}(a) < u(a)$ and $u(1) < v_{2k+1}(1)$ again by Lemma 2.2.

Similarly, $\psi_{2k} < \psi_{2k+2}$ implies that $v_{2k+2}(a) < v_{2k}(a)$ and $v_{2k}(1) < v_{2k+2}(1)$ for $k \geq 0$; and $\psi_{2k+3} < \psi_{2k+1}$ implies that $v_{2k+1}(a) < v_{2k+3}(a)$ and $v_{2k+3}(1) < v_{2k+1}(1)$ for $k \geq 0$. Thus $\{v_{2k+1}(a)\}$ is increasing, $\{v_{2k+1}(1)\}$ is decreasing, $\{v_{2k}(a)\}$ is decreasing and $\{v_{2k}(1)\}$ is increasing. The proof of the interleaving properties is complete.

Finally, we establish the error bound. Since $u(a) < v_0$ and $v_0 < u(1)$, and u is increasing, we have $|u - v_0| < u(1) - u(a) < v_1(1) - v_1(a)$. The latter expression can be estimated. By the defining equations we have

$$\sin\psi_1 = \frac{\cos\gamma}{1-a^2}\frac{(r^2-a^2)}{r} \quad \text{and} \quad v_1(1) - v_1(a) = \int_a^1 \frac{\sin\psi_1}{\sqrt{1-\sin^2\psi_1}}\,dr.$$

Using the inequality $\sin\psi_1 \leq r\cos\gamma$ to estimate the integral, we get

$$v_1(1) - v_1(a) \leq \int_a^1 \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}}\,dr = C(\gamma, a).$$

Thus, $|u - v_0| < C(\gamma, a)$. This is the case $n = 0$ of the bound to be established and we proceed by induction. Assume

$$|u - v_n| < \mathcal{B}_n := C(\gamma, a)\left(B\frac{\sqrt{1-a^2}}{1+a^2}\right)^n.$$

From the defining equations for $\{v_n\}$ and the equation for u we have

$$\sin\psi - \sin\psi_{n+1} = \frac{B}{r}\int_a^r s(u(s) - v_n(s))\,ds \quad \text{or} \quad -\frac{B}{r}\int_r^1 s(u(s) - v_n(s))\,ds\,.$$

This gives $|\sin\psi - \sin\psi_{n+1}| < (\mathcal{B}_n B)/(2r)\min\{r^2-a^2,\ 1-r^2\}$. Using the fact that

$$\min\{r^2 - a^2, 1 - r^2\} \leq \frac{2(r^2-a^2)(1-r^2)}{1+a^2},$$

we have

$$|\sin\psi - \sin\psi_{n+1}| < \frac{\mathcal{B}_n B}{1+a^2} r(1-r^2). \tag{6}$$

If $m := n+1$ is even, then since $\psi_m < \psi$ and $v_m(a) > u(a),\ v_m(1) < u(1)$,

$$|u - v_m| \le \max\{v_m(a) - u(a),\ u(1) - v_m(1)\} < (u(1) - v_m(1)) - (u(a) - v_m(a)).$$

Similarly, if m is odd, then $|u - v_m| < (v_m(1) - u(1)) - (v_m(a) - u(a))$. Thus $|u - v_m| < \left|\int_a^1 (u_r - v_{m\,r})\,dr\right| \le \int_a^1 |u_r - v_{m\,r}|\,dr$. We use the Mean Value Theorem to estimate the integrand, noting that

$$u_r = \frac{\sin\psi}{\sqrt{1-\sin^2\psi}} \quad \text{and} \quad v_{m\,r} = \frac{\sin\psi_m}{\sqrt{1-\sin^2\psi_m}}, \quad u_r - v_{m\,r} = \frac{\sin\psi - \sin\psi_m}{(1-\xi^2)^{3/2}},$$

where ξ is between $\sin\psi$ and $\sin\psi_m$. Using $\xi < \sin\psi_1 \le r$, we have $|u_r - v_{m\,r}| < |\sin\psi - \sin\psi_m|/(1-r^2)^{3/2}$. Combining this with previous bound (6), we have

$$|u - v_{n+1}| < \frac{\mathcal{B}_n B}{a^2+1} \int_a^1 \frac{r}{\sqrt{1-r^2}}\,dr = \mathcal{B}_n B \frac{\sqrt{1-a^2}}{1+a^2} = \mathcal{B}_{n+1}.$$

This completes the induction argument. □

The upper bound $\mathcal{B}_n = C(\gamma, a)\big(B\sqrt{1-a^2}/(1+a^2)\big)^n$ is at most B^n, so we have an upper bound independent of γ and a. For the interior problem, the result $v_1(0) < u(0)$ and $u(1) < v_1(1)$ was first proved in [Finn 1981] and the result $\psi < \psi_1$ was first proved in [Siegel 1989]. For the interior problem with $\gamma = 0$, Theorem 2.3 gives $|u - v_1| < B$, whereas [Siegel 1989] has the better estimate $|u - v_1| < B/3$.

The iterates $\{v_n\}$ can be used to establish an asymptotic expansion for u in powers of B. Denote differentiation with respect to B by D_B.

Theorem 2.4. *Let* $0 \le \gamma < \pi/2$ *and* $0 < B \le 6$. *The solution* $u(r, B)$ *to* (1) *has an asymptotic expansion in powers of* B,

$$u(r, B) = v_0 + u_0(r) + u_1(r)B + u_2(r)B^2 + \cdots,$$

where $u_n(r) = D_B^n w_k(r, 0)/n!$ *with* $w_k = v_k - v_0$ *for* $k > n \ge 0$. *There are constants* C_n *such that* $\left|u - \big(v_0 + u_0(r) + \cdots + u_n(r)B^n\big)\right| \le C_n B^{n+1}$ *for* $n \ge 0$.

Proof. The idea is to show that the w_n's have Taylor expansions in powers of B and combine that with Theorem 2.3. To do this we need to show that $D_B^\ell w_k$ exists and is continuous for $0 \le B \le 6,\ 0 \le \gamma \le \frac{\pi}{2}$ and $\ell \ge 0,\ k \ge 0$. The inclination angle for w_k is ψ_k since w_k differs by a constant from v_k. The w_k's are generated recursively by

$$\begin{cases} \sin\psi_{k+1} = \sin\psi_1 + \dfrac{B}{r}\displaystyle\int_a^r s w_k(s)\,ds, \\ w_{k+1}(r) = w_{k+1}(a) + \displaystyle\int_a^r \dfrac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds, \\ w_{k+1}(a) = -\displaystyle\int_a^1 (1-s^2)\dfrac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds, \end{cases} \tag{7}$$

for $k \geq 0$. We have $w_0 = 0$ and

$$\sin\psi_1 = \frac{\cos\gamma}{1-a^2}\frac{r^2-a^2}{r}.$$

From the volume condition for v_k it follows that

$$\int_a^1 r w_k\,dr = 0\,. \tag{8}$$

We will show by induction on k that $D_B^\ell w_k$ and $D_B^\ell \sin\psi_k$ are continuous for $\ell \geq 0$ and $D_B^\ell \sin\psi_k = O(1-r)$ for $\ell \geq 1$.

We will differentiate the recursion relation (7) repeatedly with respect to B, so we need the equality

$$D_B^\ell \frac{\sin\psi_k}{\sqrt{1-\sin^2\psi_k}} = \sum_{j=0}^{\ell} \frac{h_{\ell,j}}{(1-\sin^2\psi_k)^{(2j+1)/2}}, \tag{9}$$

where each $h_{\ell,j}$, for $\ell \geq 0$, is a homogeneous polynomial of degree $2j+1$ in $\sin\psi_k$, $D_B \sin\psi_k, \ldots, D_B^\ell \sin\psi_k$ which is of degree at least j in $D_B \sin\psi_k, \ldots, D_B^\ell \sin\psi_k$. This is seen by induction on ℓ. Statement (9) is true for $\ell = 0$. Assume it is true for ℓ; differentiation gives

$$D_B^{\ell+1}\frac{\sin\psi_k}{\sqrt{1-\sin^2\psi_k}} = \sum_{j=0}^{\ell} \frac{D_B h_{\ell,j}}{(1-\sin^2\psi_k)^{(2j+1)/2}} - \frac{(2j+1)h_{\ell,j}\sin\psi_k D_B\sin\psi_k}{(1-\sin^2\psi_k)^{(2j+3)/2}},$$

so $h_{\ell+1,\,j} = D_B h_{\ell,\,0}$,

$$h_{\ell+1,\,j} = D_B h_{\ell,\,j} + (2j-1)h_{\ell,\,j-1}\sin\psi_k D_B \sin\psi_k \quad \text{for } 1 \leq j \leq \ell,$$

and $h_{\ell+1,\,\ell+1} = (2\ell+1)h_{\ell,\,\ell}\sin\psi_k D_B\sin\psi_k$. Since $D_B h_{\ell,j}$ is homogeneous of degree $2j+1$ in $\sin\psi_k$, $D_B\sin\psi_k, \ldots, D_B^\ell \sin\psi_k$ and of degree at least j in $D_B\sin\psi_k, \ldots, D_B^\ell\sin\psi_k$, statement (9) holds with ℓ replaced by $\ell+1$.

Now, back to the induction argument on k. The case for $k=0$ is true since $w_0 = 0$, $\sin\psi_0 = 0$. Assume the statement is true for k. Taking ℓ derivatives of (7) with respect to B we obtain

$$\begin{aligned}
D_B^\ell \sin\psi_{k+1} &= \frac{B}{r}\int_a^r s D_B^\ell w_k(s)\,ds + \frac{\ell}{r}\int_a^r s D_B^{\ell-1} w_k(s)\,ds,\\
D_B^\ell w_{k+1}(r) &= D_B^\ell w_{k+1}(a) + \int_a^r D_B^\ell \frac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds,\\
D_B^\ell w_{k+1}(a) &= -\int_a^1 (1-s^2) D_B^\ell \frac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds.
\end{aligned}$$

Differentiating the volume condition (8), we have $\int_a^1 r D_B^\ell w_k\,dr = 0$ for all $\ell \geq 0$. Thus we see that $D_B^\ell \sin\psi_{k+1}$ is continuous and $D_B^\ell \sin\psi_{k+1} = O(1-r)$ for $\ell \geq 1$. It follows that the integrals defining $D_B^\ell w_{k+1}$ are convergent, so $D_B^\ell w_{k+1}$ is continuous. The induction argument is complete.

Now, for a given positive n, take $k > n$. By Taylor's Theorem, $w_k(r, B) = w_k(r, 0) + D_B w_k(r, 0)B + \cdots + D_B^n w_k(r, 0)B^n + O(B^{n+1})$, and by Theorem 2.3, $u(r, B) = v_0 + w_k(r, B) + O(B^{k+1})$. Thus $u = v_0 + w_k(r, 0) + D_B w_k(r, 0)B + \cdots + D_B^n w_k(r, B) + O(B^{n+1})$. By the uniqueness of asymptotic expansions, this may be written $u(r, B) = v_0 + u_0(r) + u_1(r)B + u_2(r)B^2 + \cdots + u_n(r)B^n + O(B^{n+1})$. $\square$

Example 2.5. As an example of Theorem 2.4, consider the interior capillary problem (1) with $a = 0$ and $\gamma = 0$. Then $v_0 = 2/B$, $\sin\psi_1 = r$, $w_1 = \frac{2}{3} - \sqrt{1-r^2}$, so $u = 2/B + \frac{2}{3} - \sqrt{1-r^2} + O(B)$. Similarly, $\sin\psi_2 = r + \frac{1}{3}(B/r)\big((1-r^2)^{3/2} + r^2 - 1\big)$ so that $w_2(r, 0) = \frac{2}{3} - \sqrt{1-r^2}$ and $D_B w_2(r, 0) = -\frac{1}{6} + \frac{1}{3}\ln\big(1 + \sqrt{1-r^2}\,\big)$, giving

$$u(r, B) = \frac{2}{B} + \frac{2}{3} - \sqrt{1-r^2} + \Big(-\frac{1}{6} + \frac{1}{3}\ln\big(1+\sqrt{1-r^2}\,\big)\Big)B + O(B^2).$$

Setting $r = 0$, we have $u(0, B) = 2/B - \frac{1}{3} + \frac{1}{3}\big(\ln 2 - \frac{1}{2}\big)B + O(B^2)$. Inverting this relationship and setting $u_0 = u(0, B)$, we obtain

$$B = \frac{2}{u_0} - \frac{2}{3u_0^2} + \frac{\frac{4}{3}\big(\ln 2 - \frac{1}{2}\big) + \frac{2}{9}}{u_0^3} + O\Big(\frac{1}{u_0^4}\Big) \quad \text{as } u_0 \to \infty.$$

This is a formula due to Rayleigh [1915]. It is the basis for the technique of measuring surface tension by means of the rise of liquid in a narrow tube.

3. Exterior capillary surface

In the exterior case, since the domain is unbounded, we must proceed differently in finding an initial approximation v_1.

Set $v_1 = AK(r)$, where $K(r) = (1/\sqrt{B})K_0(\sqrt{B}r)$ (K_0 being a modified Bessel function of the second kind) and A is a positive constant. We will make use of the fact [Siegel 1980] that v_1, which satisfies $v_{1\,rr} + v_{1\,r}/r = Bv_1$ for $r > 0$, is a supersolution: $Nv_1 < Bv_1$ for $r > 0$. The Bessel function $K_0(r)$ has the following properties [Lebedev 1965]:

$$K_0(r) > 0, \quad K_0'(r) < 0, \quad K_0(r) \sim \frac{e^{-r}}{\sqrt{2\pi r}} \text{ as } r \to \infty, \quad K_0(r) \sim -\ln r \text{ as } r \to 0.$$

We also need that $(rK_0')' = rK_0$ for $r > 0$ and $K_0'(r) \sim -r^{-1}$ as $r \to 0$. Now choose A so that $B\int_1^\infty rv_1\,dr = \cos\gamma$: namely, $A = -(\cos\gamma)/K_0'(\sqrt{B})$.

Theorem 3.1. *Let $v_1(r) = AK(r)$ be as chosen above and let v_2 be determined according to Theorem 1.1, so that $Nv_2 = Bv_1$, $v_2(r)$, $v_{2r}(r) \to 0$ as $r \to \infty$. Then $\psi_2(r) < \psi(r)$ for $r > 1$, $\psi_2(1) = \psi(1) = \gamma - \pi/2$ and $v_1(1) < u(1) < v_2(1)$. It follows that $u(1) = -\cos\gamma \ln\sqrt{B} + O(1)$ as $B \to 0$.*

Proof. By Theorem 1.1, $\psi_2(1) = \psi(1) = \gamma - \pi/2$ and $v_2(r)$, $v_{2r}(r) \to 0$ as $r \to \infty$.

If $v_1(1) \geq u(1)$, then $v_1(r) > u_1(r)$ for $r > 1$ by the comparison principle. This contradicts the volume condition. Thus $v_1(1) < u(1)$. Note that

$$v_1(1) = -\frac{K_0(\sqrt{B})\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})} = -\cos\gamma\ln\sqrt{B} + O(1) \text{ as } B \to 0. \tag{10}$$

Also, because of the volume condition, there exists a $b > 1$ so that $v_1(b) = u(b)$. Since v_1 is a supersolution, $v_1(r) > u(r)$ for $r > b$ and $v_1(r) < u(r)$ for $r < b$. This implies that $Nv_2 < Nu$ for $r < b$ and $Nv_2 > Nu$ for $r > b$. Using that $\psi_2(1) = \psi(1)$, $r\sin\psi_2(r)$, $r\sin\psi(r) \to 0$ as $r \to \infty$ and integrating on $[1, b]$ and $[b, \infty]$ gives that $\sin\psi_2(r) < \sin\psi(r)$ for $r > 1$. Thus $\psi_2(r) < \psi(r)$ for $r > 1$ or, equivalently, $v_{2r} < u_r$ for $r > 1$. Using that $u(r)$, $v_2(r) \to 0$ as $r \to \infty$, and integrating on $[1, \infty)$, gives that $u(1) < v_2(1)$.

Finally, we have

$$r\sin\psi_2(r) = -B\int_r^\infty sv_1(s)\,ds = -rv_{1r}(r),$$

so $\sin\psi_2 = AK_0'(\sqrt{B}r)$. Using that $v_{2r} = \sin\psi_2/\sqrt{1-\sin^2\psi_2}$ and integrating on $[1, \infty)$ gives

$$v_2(1) = \frac{\cos\gamma}{K_0'(\sqrt{B})}\int_1^\infty \frac{K_0'(\sqrt{B}r)}{\sqrt{1-\left(\dfrac{\cos\gamma}{K_0'(\sqrt{B})}K_0'(\sqrt{B}r)\right)^2}}\,dr. \tag{11}$$

We will show that there is an upper bound on $v_2(1)$ which is asymptotically the same as (10). Change variables in the integral with the substitution $s = \sqrt{B}r$ and write the integral as the sum of two terms, where δ is an arbitrary fixed positive number: $v_2(1) = I_1 + I_2,\ I_1 = \int_{\sqrt{B}}^{\delta} F\,ds,\ I_2 = \int_{\delta}^{\infty} F\,ds$, where

$$F = \frac{\cos\gamma}{\sqrt{\left(\dfrac{\sqrt{B}K_0'(\sqrt{B})}{K_0'(s)}\right)^2 - B\cos^2\gamma}} < F_1 = \frac{\cos\gamma}{\sqrt{s^2 - B\cos^2\gamma}}$$

and

$$F = \frac{\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})}\frac{K_0'(s)}{\sqrt{1-\left(\dfrac{\cos\gamma\,K_0'(s)}{K_0'(\sqrt{B})}\right)^2}} < F_2 = \frac{\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})}\frac{K_0'(s)}{\sqrt{1-\left(\dfrac{\cos\gamma\,K_0'(\delta)}{K_0'(\sqrt{B})}\right)^2}}.$$

The upper bound F_1 was obtained by using that $(rK_0')' = rK_0 > 0$, so that

$$\left|\sqrt{B}K_0'(\sqrt{B})\right| > |sK_0'(s)|$$

for $s > \sqrt{B}$. Using the upper bounds F_1 and F_2 for the integrals I_1 and I_2, we obtain

$$\begin{aligned} I_1 &< \cos\gamma\Big(\ln\big(\delta + \sqrt{\delta^2 - B\cos^2\gamma}\big) - \ln\big(\sqrt{B}(1+\sin\gamma)\big)\Big) \\ &= -\cos\gamma\ln\sqrt{B} + O(1), \\ I_2 &= -\frac{\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})}\frac{K_0(\delta)}{\sqrt{1-\left(\dfrac{\cos\gamma\,K_0'(\delta)}{K_0'(\sqrt{B})}\right)^2}} = O(1). \end{aligned}$$

Thus $u(1) < v_2(1) = I_1 + I_2 < -\cos\gamma\ln\sqrt{B} + O(1)$. Combining this with the lower bound (10), we have that $u(1) = -\cos\gamma\ln\sqrt{B} + O(1)$ as $B \to 0$. $\square$

Translating [Turkington 1980, Theorem 3.3] to the notation of this paper gives $u(1) \sim -\cos\gamma\ln\sqrt{B}$ as $B \to 0$. Theorem 3.1 gives a better estimate of the error.

References

[Elcrat et al. 2004] A. Elcrat, T.-E. Kim, and R. Treinen, “Annular capillary surfaces”, *Arch. Math. (Basel)* **82**:5 (2004), 449–467. MR 2006e:76055 Zbl 02106496

[Finn 1981] R. Finn, “On the Laplace formula and the meniscus height for a capillary surface”, *Z. Angew. Math. Mech.* **61**:3 (1981), 165–173. MR 83d:49080 Zbl 0485.76009

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren der Math. Wiss. **284**, Springer, New York, 1986. MR 816345 (88f:49001) Zbl 0583.35002

[de Laplace 1805–1806] P. S. de Laplace, *Traité de mécanique céleste* (suppléments au Livre X), 1805–1806. Reprinted in vol. 4 of his *Œuvres complètes*, Gaulthier-Villars, Paris, 1882. See also the annotated translation by N. Bowditch, Boston, 1839; reprinted by Chelsea, New York, 1966.

[Lebedev 1965] N. N. Lebedev, *Special functions and their applications*, Prentice-Hall, Englewood Cliffs, NJ, 1965. Reprinted Dover, New York, 1972. MR 50 #2568 Zbl 0131.07002

[Miersemann 1993] E. Miersemann, "On the rise height in a capillary tube of general cross section", *Asymptotic Anal.* **7**:4 (1993), 301–309. MR 94i:76027 Zbl 0794.35018

[Miersemann 1994] E. Miersemann, "On the Laplace formula for the capillary tube", *Asymptotic Anal.* **8**:4 (1994), 393–403. MR 95i:76027 Zbl 0809.76011

[Rayleigh 1915] J. W. S. Rayleigh, "On the theory of the capillary tube", *Proc. Roy. Soc. London Ser. A* **92** (1915), 184–195.

[Siegel 1980] D. Siegel, "Height estimates for capillary surfaces", *Pacific J. Math.* **88**:2 (1980), 471–515. MR 82h:35037 Zbl 0411.35043

[Siegel 1989] D. Siegel, "Explicit estimates of a symmetric capillary surface for small Bond number", pp. 497–506 in *Continuum mechanics and its applications* (Burnaby, BC, 1988), edited by G. A. C. Graham and S. K. Malik, Hemisphere, New York, 1989. MR 91g:76014 Zbl 0716.76023

[Turkington 1980] B. Turkington, "Height estimates for exterior problems of capillarity type", *Pacific J. Math.* **88**:2 (1980), 517–540. MR 82i:35070 Zbl 0474.76012

Received August 31, 2005.

DAVID SIEGEL
DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO
CANADA N2L 3G1

dsiegel@math.uwaterloo.ca