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NEW EXOTIC CONTAINERS

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We describe the construction of new exotic containers in a gravitational field. These are containers which, for certain volumes of fluid, possess a continuum of noncongruent equilibrium configurations. Unlike the constructions of Gulliver and Hildebrandt (1986) and Concus and Finn (1991), our containers need not be rotationally symmetric. One of their features is that the equilibria seem likely to be local minimizers of energy, in contrast to earlier constructions where the equilibria were always unstable.

1. Introduction

Consider a container Σ, partially filled with a fluid of density ρ, occupying a region T, sitting in a gravitational field of intensity g. We denote the liquid-air interface by Λ, enabling us to write down the potential energy of the configuration,

$$E = \sigma|\Lambda| + \rho g \int_T z\,dv - \sigma\tau|\Sigma'|, \quad |\tau| < 1. \tag{1–1}$$

Here $|\Lambda|$ is the area of the free surface, $|\Sigma'|$ is the wetted area of container wall, and τ is the wetting energy of the fluid in contact with the wall. In equilibrium the configuration will be such that the potential energy is an extremum with respect to the volume constraint, $|T| = V_0$. The case of positive g corresponds to the gravitational force acting downward. However, the cases $g = 0$ and $g < 0$ are also of interest to us.

The Euler–Lagrange equations determine the following conditions for equilibrium. First, the mean curvature of the free surface satisfies $2H = \kappa z + \lambda$, where $\kappa = \rho g/\sigma$ and λ is a constant arising as a Lagrange multiplier. The sign of the mean curvature is determined relative to the unit normal on Λ directed away from the fluid T. Secondly, the boundary conditions stipulate that the free surface meets the container wall at a contact angle γ where $\cos\gamma = \tau$. Here the angle γ is measured interior to the fluid.

An exotic container is a vessel with a smooth wall such that for some particular volume of fluid there will exist a continuum of geometrically distinct equilibrium

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configurations. In the published examples the exotic containers are always taken to be rotationally symmetric about a vertical axis. The corresponding interfaces are also rotationally symmetric and usually contain the flat surface $u \equiv 0$ as one member of the family. The first example appeared in the paper by R. Gulliver and S. Hildebrandt [1986], who considered the case of zero gravity, $g = 0$, and with wetting energy, $\tau = 0$, so that the contact angle was $\gamma = \pi/2$. In this case the free surfaces are spherical caps and the construction of the container wall is very geometric. The remaining cases where $g \geq 0$ and arbitrary contact angle γ, $0 < \gamma < \pi$ were studied in a series of papers by P. Concus and R. Finn.

It turns out that in all of these cases the rotationally symmetric equilibria are not stable. This feature was first discussed in [Concus and Finn 1989]. A complete proof was given in [Wente 1999]. This behavior was verified in a drop tower experiment, discussed in a paper by Concus, Finn, and Weislogel [Concus et al. 1992]. The experiment was also reproduced later in gravity-free conditions in a space lab mission.

This paper shows the construction of many new exotic containers. Consider, first, a round spherical container containing some fluid, with the free surface being a circular planar disk meeting the container wall with contact angle γ. This is an equilibrium configuration in gravity-free conditions with wetting energy $\tau = \cos\gamma$. It is part of a two-parameter family of congruent configurations (not geometrically distinct). They are minimizers of the appropriate energy. What happens if we add gravity? Can we construct exotic containers whose extrema are local minimizers of energy? This is our goal.

Our method of construction is as follows. Start with a one-parameter family of extremal surfaces satisfying the Euler–Lagrange equation. (For the Gulliver–Hildebrandt construction the family consisted of spherical caps symmetric about the z-axis. The Concus–Finn examples use the rotationally symmetric solutions to the sessile drop equation when $g > 0$.) The class of admissible surfaces we shall use are those extrema of cylindrical type. Namely, they are ruled surfaces with a generating curve lying in the xz-plane. The rulings are straight lines parallel to the y-axis. Since at every point one of the two principle curvatures is zero, the surfaces are metrically flat. For $g = 0$, the surfaces we shall use are tilted planes. For $g \neq 0$ the generating curves are determined by the condition that the signed curvature of the generating curve is a linear function of height. Such curves have an interesting history, having first been studied by Euler, and are called elastic curves. We shall refer to the corresponding extremal surfaces as elastic surfaces.

Start with our family, $\{\Lambda_t\}$, parametrized by t. The surface Λ_0 will be the horizontal plane. Λ_t will be that elastic surface whose generating curve passes through the origin, whose curvature k equals κz, and whose inclination angle at $(0, 0)$ is t. These curves will have an inflection point at the origin, bending one way when z is positive and the opposite way when z is negative. We must also allow the surfaces Λ_t to be transported vertically by an amount $h(t)$ to be determined. Call this new family $\tilde{\Lambda}_t$. Pick a region $X_0 \subset \Lambda_0$ with bounding curve Γ_0. One now constructs the rigid bounding surface Σ by analogy with the Monge cone construction. Σ will consist of the union of curves Γ_t lying in $\tilde{\Lambda}_t$. The requirement is that each elastic surface $\tilde{\Lambda}_t$ meet the rigid surface Σ at a prescribed contact angle γ. It is still necessary that the volume enclosed by Σ and $\tilde{\Lambda}_t$ remain fixed. This is achieved by an appropriate choice of $h(t)$. For $\gamma = 90°$ the construction is easier. The volume condition is satisfied by setting $h(t) \equiv 0$ and the surface Σ is generated by taking orthogonal trajectories emanating from the base curve, Γ_0. Generally, we shall assume that Γ_0 is symmetric about the y-axis, but this is not a necessary condition for our construction.

Consider the case $g = 0$, where Λ_t is the family of tilted planes. If $\Gamma_0 \subset \Lambda_0$ is a circle with center at the origin, our construction recovers the round sphere. However, if Γ_0 is some other closed curve, say an ellipse, we obtain some new bounding surfaces which are not so easy to describe.

The construction of exotic containers is also of interest in one lower dimension. This may be imagined as some fluid sitting between two vertical planes. In this case the free surface is a narrow ribbon. We shall analyze this case as well. We refer to this situation as the planar case.

In Section 2 we discuss elastic curves, those for which the signed curvature is a linear function of height. Such curves will generate our elastic surfaces, Λ_t. They were studied by Euler and have other potential applications to capillary theory besides the construction of exotic containers. They determine potential sessile or pendant drops in one lower dimension. In Section 3 we carry out the construction of the exotic containers. We shall analyze the planar case first and then the more physical situation in dimension three.

In Section 4 we discuss the stability question. As mentioned above, for the gravity-free case and where our initial curve Γ_0 is a circle, the corresponding container wall is a sphere. Here the configurations are all energy minimizers (all congruent as well). We would like this to be true more generally. We do show that in the planar case our construction does produce a container for which the initial flat surface Λ_0 is a strong local minimizer. It seems likely that this should be true in the three-dimensional case as well assuming that Γ_0 is chosen properly.

2. Elastic curves

In this section we construct a particular set of solutions $z = u(x, y)$ for the free-surface interface, namely those that depend only on one variable x, so $z = u(x)$. The solutions (regarded as curves in the xz-plane) are called elastic curves; the corresponding ruled surfaces in $\mathbb{R}^3$ will be called elastic surfaces. By a vertical translation we may set the Lagrange multiplier λ to 0. By a rescaling we may suppose that $\kappa = \rho g/\sigma = 1$. This gives us the differential equation

$$u''/(1+(u')^2)^{3/2} = u, \quad z = u(x). \tag{2–1}$$

The signed curvature is precisely u. It is a linear function of position. As noted in the introduction, this is the governing equation for the equilibrium configuration of a planar elastic rod. See [Giaquinta and Hildebrandt 1996] for a nice discussion. Let the curve $\mathscr{C}$ be parametrized by arc length, $\langle x(s), u(s)\rangle$. If we let $\theta(s)$ be the angle of inclination, the differential equation (2–1) can be rewritten as

$$\theta'(s) = k(s) = u(s), \quad k(s) = \text{curvature}. \tag{2–2}$$

Here are some elementary observations concerning solutions to (2–2).

- If $\langle x(s), u(s)\rangle$ solves (2–2) so does $\langle x(s)+x_0, u(s)\rangle$.
- If $\langle x(s), u(s)\rangle$ solves (2–2) so does $\langle x(s), -u(s)\rangle$.
- If we reverse the orientation of the curve by setting $\sigma = -s$, the new curve satisfies the equation $\theta'(\sigma) = -u(\sigma)$.
- Suppose $\theta'(s) = u(s)$. If $u(s) > 0$ the curve is bending counterclockwise, if $u(s) < 0$ it is bending in a clockwise manner, there is an inflection point as the curve crosses the x-axis.
- For $\theta'(s) = u(s)$, $\rho g/\sigma = 1$ the fluid is taken to be on the right in the direction of increasing s.
- The strong touching principle: Let $\mathscr{C}_1$, $\mathscr{C}_2$ be oriented curves both satisfying (2–2), with $\theta_1'(s) = u_1(s)$ and $\theta_2'(s) = u_2(s)$. If $\theta_1 = \theta_2$ at some level $u = \bar{u}$, then the two curves are congruent. If they touch at some point they are identical.

Upon differentiating (2–2) we have $\theta''(s) = u'(s) = \sin\theta$, which can be integrated giving

$$\tfrac{1}{2}\theta'(s)^2 + \cos\theta = E, \qquad u^2(s) = 2(E - \cos\theta). \tag{2–3}$$

We analyze solutions to (2–3) via phase-plane analysis. Clearly the energy E is at least -1. In fact:

(a) If $E = -1$, then $\theta = \pi$ and $u(s) = 0$. The solution is the horizontal line, $u = 0$, being traversed right to left so that the fluid lies above the z-axis.

(b) If $E = 1$, there is the constant solution $\theta(s) = 0$. Again $u(s) = 0$ and the curve is traversed left to right with the fluid below the z-axis.

Now consider the case $-1 < E < 1$. The phase portrait of (2–3) is shown in Figure 1. The solution $\theta(s)$ will be oscillatory with a minimum value $\theta(0) = \theta_0$,

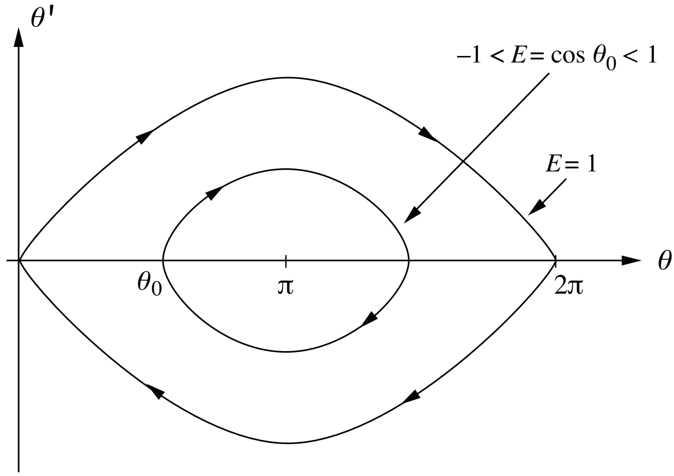

Figure 1. Phase portrait of the solution (2–3).

where $0 < \theta_0 < \pi$ and $E = \cos\theta_0$. We have $\theta_{\min} = \theta_0$ and $\theta_{\max} = 2\pi - \theta_0$. Since $E = \cos\theta_0$, the second equation in (2–3) becomes

$$u^2(s) = 2(\cos\theta_0 - \cos\theta).$$

This expresses u as a function of θ. We can use θ itself as a parameter, $\theta_0 \le \theta \le 2\pi - \theta_0$. Since $dx/d\theta = \cos\theta$ one finds

$$x(\theta) = \int_{\theta_0}^{\theta} \frac{\cos\varphi \, d\varphi}{\sqrt{\cos\theta_0 - \cos\varphi}}.$$

We have a parametrization $\langle x(\theta), y(\theta)\rangle$ valid for $\theta_0 \le \theta \le 2\pi - \theta_0$ with initial condition $\langle x(\theta_0), u(\theta_0)\rangle = \langle 0, 0\rangle$. Let this curve be expressed in terms of arc length, $0 \le s \le \ell$. It may be extended using (2–3) to the interval $-\ell \le s \le \ell$ with $\langle x(-s), u(-s)\rangle = -\langle x(s), u(s)\rangle$. Finally the solution extends for all s by setting $x(s+2\ell) = x(s) + x(2\ell)$, and $u(s+2\ell) = u(s)$. The complete curve bends to the left when u is positive and to the right when u is negative. A complete picture is obtained by observing the curve $\langle x(\theta), u(\theta)\rangle$, $\theta_0 \le \theta \le \pi$.

Theorem 2.1. *Consider solutions to $\theta'(s) = u(s)$ satisfying (2–3) with $-1 < E < 1$. By setting $E = \cos\theta_0$, $0 < \theta_0 < \pi$ we have*

$$u^2(\theta) = 2(\cos\theta_0 - \cos\theta).$$

Consider that portion of the complete curve where $\theta_0 \le \theta \le \pi$; for $\theta = \theta_0$ the curve passes through the origin. One has three types of graphs, depending on whether $0 < \theta_0 < \pi/2$, $\theta_0 = \pi/2$, or $\pi/2 < \theta_0 < \pi$ (Figure 2).

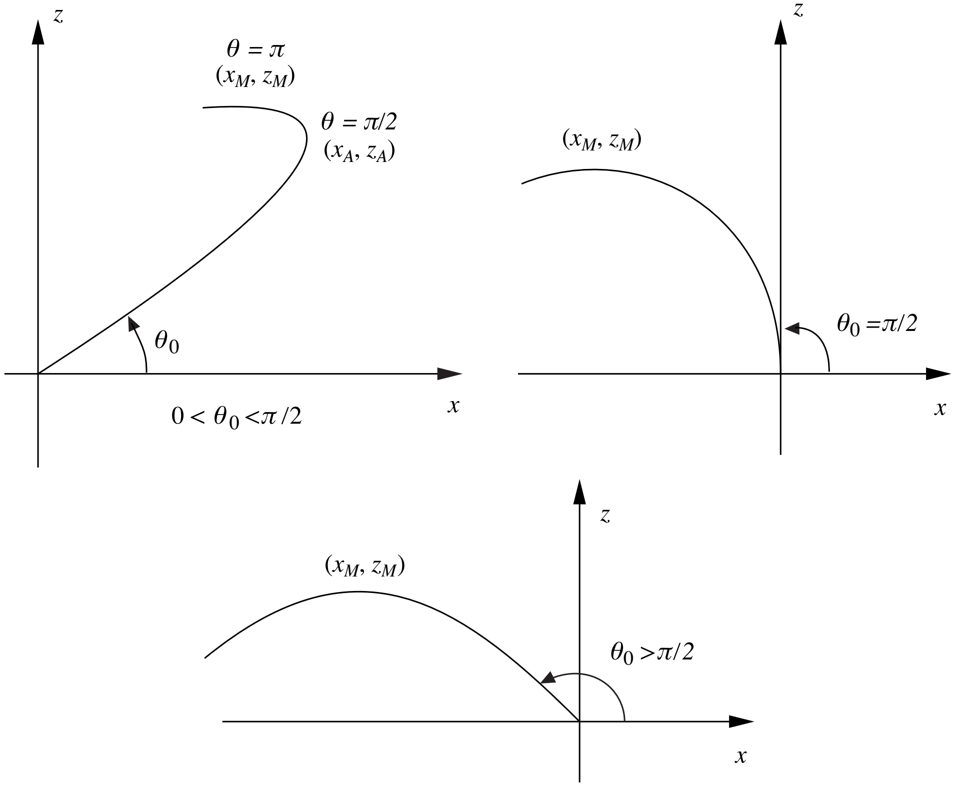

Figure 2. Qualitative appearance of elastic curves.

We set $(x_M, z_M) = \langle x(\pi), u(\pi)\rangle$, *and also* $(x_A, z_A) = \langle x(\pi/2), u(\pi/2)\rangle$ *in the case* $0 < \theta_0 < \pi/2$. *We identify* z_M *with* u_M *and* z_A *with* u_A.

(1) *We have* $u_A^2 = u^2(\pi/2) = 2E$; *equivalently,* $u_A = \sqrt{2E} = \sqrt{2\cos\theta_0}$. *The function* u_A *is strictly increasing in* E, *for* $0 < E < 1$, *with range* $0 < u_A < \sqrt{2}$.

(2) x_A *is strictly increasing in* E, *for* $0 < E < 1$, *with range* $0 < x_A < \infty$, *and* x_A *becomes infinite as* E *approaches* 1.

(3) *We have* $u_M^2 = 2(E+1)$, *or equivalently,* $u_M = \sqrt{2(E+1)}$, *for* $-1 < E < 1$. *Therefore* u_M *is strictly increasing in* E, *for* $-1 < E < 1$, *with range* $0 < u_M < 2$.

(4) x_M *is strictly increasing in* E, *for* $-1 < E < 1$, *with range* $-\pi < x_M < \infty$.

Proof. We need only verify (2) and (4). Suppose $\mathscr{C}_1 : \langle x_1(\theta), u_1(\theta)\rangle$ and $\mathscr{C}_2 : \langle x_2(\theta), u_2(\theta)\rangle$ are two solutions with initial inclination angles $0 < \alpha_2 < \alpha_1 \le \pi/2$ so that $0 \le \cos\alpha_1 = E_1 < E_2 = \cos\alpha_2 < 1$. It follows that $u_{A_1} < u_{A_2}$ by (1). Initially the curve $\mathscr{C}_2$ lies to the right of $\mathscr{C}_1$. I claim that it does so for $0 \le u \le u_{A_1}$.

If this were not the case, the two curves would intersect at some smallest value $\bar{u}$, with $0 < \bar{u} < u_A$. At this level the inclination angles would satisfy $\theta_1 \leq \theta_2$. With equality we are done, since by the touching principle we would have $\mathscr{C}_1 = \mathscr{C}_2$. But $\theta_1 < \theta_2$ is not possible either, for then at some smaller value $\tilde{u}$, $0 < \tilde{u} < \bar{u}$, we would have $\theta_1 = \theta_2$. This would mean that $\mathscr{C}_2$ is a horizontal translate of $\mathscr{C}_1$, which is impossible. It follows that x_A is a strictly increasing function of E, $0 < E < 1$. The integral formula for x_A is

$$x_A = \int_{\theta_0}^{\pi/2} \frac{\cos\varphi\, d\varphi}{\sqrt{2(E - \cos\varphi)}} = \int_{\theta_0}^{\pi/2} \frac{\cos\varphi\, d\varphi}{\sqrt{2(\cos\theta_0 - \cos\varphi)}}.$$

It follows that the range of values for x_A, for $0 < \theta_0 < \pi/2$, is $0 < x_A < +\infty$. Statement (2) follows.

We now consider statement (4).

First take the case $0 < \theta_0 < \pi/2$ or $0 < E < 1$. Here $x_M = x(\pi)$. We find

$$x_M = x_A + \int_{\pi/2}^{\pi} \frac{\cos\varphi\, d\varphi}{\sqrt{2(E - \cos\varphi)}} = x_A - \int_0^{\pi/2} \frac{\cos\psi\, d\psi}{\sqrt{2(E + \cos\psi)}}.$$

Thus, since x_A is strictly motonic in E, so is x_M. Moreover, $\theta_0 \to 0$ and $x_M \to +\infty$ as $E \to 1$.

We remark that $x_M < 0$ for $\theta_0 = \pi/2$. There is exactly one value $\theta_c \in (0, \pi/2)$ with $x_M = 0$. The corresponding complete curve is a closed curve in a figure eight shape.

Now consider the case $\pi/2 \leq \theta_0 < \pi$. Parametrizing by the arc length s, we have a curve traced right to left starting at the origin with $E = \cos\theta_0$, and $-1 < E < 0$. Consider the curve obtained by reflection about the z-axis. This will resemble a sine curve traversed left to right satisfying the curvature equation $\theta'(s) = -u(s)$, with $\theta(0) = \psi_0 = \pi - \theta_0$. The curve bends clockwise and the inclination angle decreases from ψ_0 to 0. For $\theta = 0$ the reflected curve has coördinates $\langle x_R, u_R\rangle$, where $u_R = u_M = 2(1 + E)$ and $x_R = -x_M$ is positive. One finds

$$x_R = \int_0^{\psi_0} \frac{\cos\varphi\, d\varphi}{\sqrt{2(E + \cos\varphi)}} = \int_0^{\psi_0} \frac{\cos\varphi\, d\varphi}{\sqrt{2(\cos\varphi - \cos\psi_0)}}, \quad 0 < \psi_0 < \pi/2.$$

We claim that x_R is a strictly decreasing function of E for $-1 < E < 0$; that is, it is strictly increasing in ψ_0, $0 < \psi_0 < \pi/2$, and $E = -\cos\psi_0$.

We parametrize the curve as a function of u, getting

$$\frac{d}{du}\left(\frac{x'(u)}{\sqrt{1 + x'(u)^2}}\right) = -u.$$

This gives $x'^2/(1+x'^2) = (a^2+u^2)^2/4$ for $0 \le u \le u_M$, where $u_M^2 = 2-a^2$, for a positive constant $a < \sqrt{2}$. We obtain the integral formula

$$x_R = \int_0^{\sqrt{2-a^2}} \frac{(a^2+u^2)\,du}{\sqrt{4-(a^2+u^2)^2}}.$$

Let $u = \sqrt{2}v$ and $a = \sqrt{2}\alpha$. Rewrite the integral and then change variables, setting $\alpha^2+v^2 = t$, to find

$$x_R = \frac{1}{\sqrt{2}} \int_\beta^1 \frac{t\,dt}{\sqrt{1-t^2}\sqrt{t-\beta}}, \quad 0 < \beta = \alpha^2 < 1.$$

Finally, set $t = (1-\beta)x + \beta$, obtaining

$$x_R = \frac{\sqrt{1-\beta}}{\sqrt{2}} \int_0^1 \frac{\big((1-\beta)x+\beta\big)\,dx}{\sqrt{x}\sqrt{1-\big((1-\beta)x+\beta\big)^2}}.$$

This integral is an increasing function of β, $0 < \beta < 1$. As one can see by differentiation,

$$\frac{dx_R}{d\beta} = \frac{(1-\beta)^{3/2}}{2\sqrt{2}} \int_0^1 \frac{(1-x)^2\big(2+((1-\beta)x+\beta)\big)}{\sqrt{x}\big(1-((1-\beta)x+\beta)\big)^{3/2}}\,dx.$$

This is positive for $0 < \beta < 1$.

For $\beta = 0$, one has $\theta_0 = \pi/2$, $u_M = \sqrt{2}$ and

$$x_R = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sqrt{\cos\varphi}\,d\varphi = \frac{1}{\sqrt{2}} \int_0^1 \frac{\sqrt{x}\,dx}{\sqrt{1-x^2}}.$$

For $\beta \approx 1$, $\theta_0 \approx \pi$ and $\psi_0 \approx 0$ and we see that $u_M \to 0$ and $x_R \to \pi/2$; hence $x_M \to -\pi/2$. □

Figure 3 illustrates the family of curves just discussed: solutions $\langle x(s), u(s)\rangle$ of $\theta'(s) = u(s)$ going through $\langle x(0), u(0)\rangle = \langle 0, 0\rangle$ and satisfying $\theta(0) = \theta_0$, $0 < \theta_0 < 1$, where s ranges from 0 to $s_M(\theta_0)$ with $u(s_M) = u_M = \sqrt{2(E+1)}$, for $-1 < E < 1$. These are the generating curves of the family of elastic surfaces we shall use to construct our exotic containers. We see from our discussion that any two curves in this family only intersect at the origin.

The family of curves where $E \ge 1$ are also interesting but are not relevant to our discussion here. For $E = 1$ one also obtains the soliton solution, while for $E > 1$ the curves remain away from the u-axis and always turn in one direction.

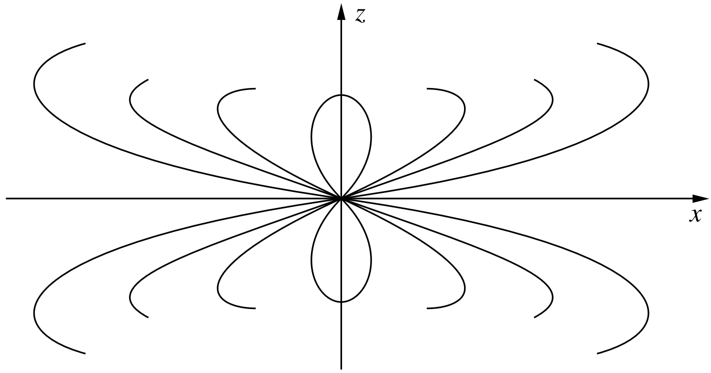

Figure 3. The family of elastic curves, $g > 0$.

3. Construction of exotic containers

There are two theorems in this section. The first carries out the construction of the exotic container in the planar case. Here the free surface is a curve, as is the container wall. Physically, this amounts to the situation of a liquid held between two parallel vertical plates. Though somewhat easier to handle, this construction is worth a separate discussion. The second theorem treats the three-dimensional case.

We start with a one-parameter family Λ_t of immersed surfaces in $\mathbb{R}^3$, each of which has mean curvature $2H = \kappa z$ with $\kappa = \rho g/\sigma$. Specifically we shall choose the family of elastic surfaces described in Section 2, although other choices would work equally well. In the planar case we use the corresponding elastic curves.

We have a map, $F_0 : \Omega \times I$ into $\mathbb{R}^3$ where $\Omega = \mathbb{R}^2$ and I is an open interval centered about $t = 0$. Points in Ω are labeled (u, v), points in I by t, and the target space is (x, y, z). We have

$$F_0(u, v, t) = \langle f(u, t), v, g(u, t)\rangle,$$

where the pair $\langle f(u, t), g(u, t)\rangle$, for a given value of t, describes the generating curve of the elastic family parametrized by arc length with $f(0, 0) = g(0, 0) = 0$ and such that the inclination angle at the origin is t. For each t, the map $F_0(u, v, t)$ is a flat isometric immersion of an elastic surface with

$$|(F_0)_u| = |(F_0)_v| = 1 \quad \text{and} \quad \langle (F_0)_u \cdot (F_0)_v\rangle = 0.$$

We set $\Omega_{t_0} = \Omega \times \{t_0\}$ and call $\Lambda_{t_0} = F_0(\Omega_{t_0})$, our immersed surface with prescribed mean curvature $2H = \kappa z$. For $\kappa = 0$, Λ_t is a tilted plane containing the y-axis with inclination angle t, while for $\kappa > 0$ we have the elastic surfaces described in Section 2, where the fluid lies below the surface (sessile drop case).

For any κ, Λ_0 is the horizontal plane and Λ_{-t} is the reflection of Λ_t in the xy-plane. In particular, $F_0(u, v, 0) = (u, v, 0)$.

We denote by $\xi(u, v, t) = (F_0)_u \wedge (F_0)_v$ the unit normal vector to Λ_t pointing out of the fluid.

Theorem 3.1 (two-dimensional case). *Suppose given a one-parameter family of oriented curves* $F_0(u, t) = \langle f(u, t), g(u, t)\rangle$ *satisfying*

(1) $2H = \kappa z$, *where* $\kappa = \rho g/\sigma$ *and* $2H$ *is the signed curvature*;

(2) *the curves are parametrized by arc length so that* $|(F_0)_u| = 1$ *with* $F_0(u, 0) = \langle u, 0\rangle$; *and*

(3) $F_0(0, t) = \langle 0, 0\rangle$ *and* $(F_0)_u(0, t) = \langle \cos t, \sin t\rangle$. (For $\kappa > 0$ our curves represent the free surface of sessile drops, for $\kappa = 0$ we have $F_0(u, t) = \langle u \cos t, u \sin t\rangle$, while for $\kappa < 0$ we have the pendant drop situation.)

Let initial values for $u_1(t)$, $u_2(t)$ *be given. We want* $u_1(t) < u_2(t)$. *For this reason we assume that* $u_1(0) = u_1$, $u_2(0) = u_2$ *with* $u_2 = -u_1 > 0$. *There exist two smooth functions* $u_1(t)$, $u_2(t)$ *and a function* $h(t)$ *with* $h(0) = 0$, *all defined in a neighborhood of* $t = 0$ *with the following property. Let*

$$F(u, t) = \langle f(u, t), g(u, t)+h(t)\rangle$$

be our family of extremals, characterized by the following properties:

(a) $F(\mathbb{R} \times \{t\}) = \Lambda_t$, *our immersed free curve.*

(b) Σ_1 *is the left bounding wall,* $F(u_1(t), t)$, *while* Σ_2 *is the right bounding wall,* $F(u_2(t), t)$.

We have $F(u_1(t), t) \in \Lambda_t \cap \Sigma_1$ *and* $F(u_2(t), t) \in \Lambda_t \cap \Sigma_2$. *The curve* $F(u, t)$, *with* $u_1(t) \leq u \leq u_2(t)$, *is an extremal curve for the variational problem and each such curve,* Λ_t, *will meet the container walls* Σ_1, Σ_2 *at an interior contact angle* γ, *with* $0 < \gamma < \pi$.

Let $V(t)$ *be the "volume" enclosed by* Λ_t *and the container walls* Σ_1, Σ_2. *We may suppose that the curves* Σ_1, Σ_2 *are connected from below to form a closed container.*

The volume enclosed by the container and any free surface, Λ_t, *is a constant.*

Proof. Given $h(t)$, a smooth function, we shall use the contact angle condition to determine a first-order differential equation for $u_1(t)$, $u_2(t)$ satisfying $u_1(0) = u_1$, $u_2(0) = u_2$, where we shall assume that $u_2 = -u_1 > 0$. Then we use the fixed volume condition to obtain an equation for $h'(t)$ in terms of $u_1(t)$, $u_2(t)$. We end up with a first-order system for the pair $\{u_1(t), u_2(t)\}$ with initial conditions $\langle u_1(0), u_2(0)\rangle = \langle u_1, u_2\rangle$. The existence theorem for ordinary differential equations gives us our solution $\langle u_1(t), u_2(t)\rangle$. We are then able to find $h(t)$ where we set (without loss of generality) $h(0) = 0$.

We have the functions

$$\begin{aligned}F_0(u,t) &= \langle f(u,t), g(u,t)\rangle,\\ F(u,t) &= \langle f(u,t), g(u,t)+h(t)\rangle,\\ F_u &= \langle f_u, g_u\rangle \quad \text{with } f_u^2+g_u^2=1,\\ F_t &= \langle f_t, g_t+h'(t)\rangle,\\ \xi(u,t) &= \langle -g_u, f_u\rangle,\end{aligned}$$

the latter being the unit normal vector. We set $e_1 = F_u$, the unit tangent vector to Λ_t, while $e_2 = \xi(u,t)$ is the unit normal vector. We are to find $u_2(t)$ so that $F(u_2(t), t)$ will describe the right container wall, Σ_2. The tangent vector to this curve is to be parallel to $w = (\cos\gamma)e_1 + (\sin\gamma)e_2 = (\cos\gamma)F_u + (\sin\gamma)\xi$. That is: $\dot u_2 F_u + F_t = \alpha w$ for some scalar α. We write this as

$$F_t = -\dot u F_u + \alpha w.$$

Now F_u and w are linearly independent, $0 < \gamma < \pi$. By taking the inner product of the preceding equation with ξ and F_u successively we find

$$\begin{aligned}\alpha &= \csc\gamma\,(F_t\cdot\xi),\\ \dot u &= \cot\gamma\,(F_t\cdot\xi) - (F_t\cdot F_u).\end{aligned}$$

The second of these equations is our differential equation for $u_2(t)$, which we can rewrite more explicitly as

$$\dot u_2(t) = F_t\cdot(\cot\gamma\,\xi - F_u) = \cot\gamma\,(f_u g_t - f_t g_u) + h'(t)(\cot\gamma\, f_u - g_u). \tag{3–1}$$

The right-hand side is an “explicit” function of (u, t).

The differential equation for $u_1(t)$ is almost identical. The only change is that the outward unit tangent vector to Λ_t is $e_1 = -F_u$. We are led to the following differential equation for $u_1(t)$:

$$\dot u_1(t) = -\cot\gamma\,(F_t\cdot\xi) - (F_t\cdot F_u). \tag{3–2}$$

Given $h(t)$ and initial data for $u_1(t)$ and $u_2(t)$, there exists a unique solution $\langle u_1(t), u_2(t)\rangle$ such that the curves $F(u_1(t), t)$, $F(u_2(t), t)$ describing Σ_1, Σ_2 meet the free surfaces $\Lambda_t : F(u,t)$, $u_1(t) \le u \le u_2(t)$, with the desired interior contact angle γ.

We now use the conservation of volume condition to determine the function $h(t)$. For each t, the free surface Λ_t is given by the map $F(u,t)$, $u_1(t) \le u \le u_2(t)$. As we vary our family, the normal component of the variation is $\varphi = (F_t \cdot \xi)$. The first order variation of "volume" should be zero:

$$\dot{V}(t) = \int_{u_1(t)}^{u_2(t)} (F_t \cdot \xi)\, du = 0.$$

Substituting our formulae for F and ξ we get

$$h'(t) \int_{u_1(t)}^{u_2(t)} f_u du = -\int_{u_1(t)}^{u_2(t)} (f_u g_t - f_t g_u)\, du. \tag{3–3}$$

(For $t = 0$ we have $f_u \equiv 1$ so that the expression for $h'(t)$ is well defined.) Equation (3–3) determines $h'(t)$ as a function of (u_1, u_2). Substitute this expression for $h'(t)$ back into equations (3–1) and (3–2) to obtain a first order system of differential equations for $\{u_1(t), u_2(t)\}$. Given initial conditions $u_1(0) = u_1$, $u_2(0) = u_2$ with $u_2 = -u_1 > 0$ we have solutions $\{u_1(t), u_2(t)\}$. Setting $h(0) = 0$ we obtain $h(t)$ using (3–3). □

Remark. Given symmetric initial data, the generated solutions will have the following properties.

(i) $h(t)$ is an even function of t.

(ii) The boundary curves Σ_1, Σ_2 are symmetric about the z-axis.

(iii) The free surface Λ_t meet the container walls Σ_1, Σ_2 at an interior contact angle γ, $0 < \gamma < \pi$.

(iv) The "volume" enclosed by Λ_t and Σ_1, Σ_2 is a constant independent of t.

(v) For $t = 0$, Λ_t is the horizontal line segment $u_1 \le u \le u_2$.

(vi) For $\gamma = \pi/2$ with $\cot\gamma = 0$ we have $h(t) \equiv 0$ and the curves Σ_1, Σ_2 are simply orthogonal trajectories of the family $F_0(u, t)$, assuming $u_1(0) = -u_2(0)$.

We now consider the three-dimensional case. Our elastic surfaces are described by functions $F_0(u, v, t) = \langle f(u, t), v, g(u, t)\rangle$ with $F_0(u, v, 0) = \langle u, v, 0\rangle$ and $(F_0)_u(0, v, t) = \langle \cos t, 0, \sin t\rangle$. For each t the equilibrium surface $F_0(\Omega \times \{t\})$ has mean curvature $2H = \kappa z$. Now let $h(t)$ be a smooth function of t with $h(0) = 0$ and set

$$F(u, v, t) = \langle f(u, t), v, g(u, t) + h(t)\rangle. \tag{3–4}$$

For each t, $F(\Omega \times \{t\}) = \Lambda_t$ is a potential equilibrium surface, where $h(t)$ is a Lagrange multiplier. Our parameter space has coördinates (u, v, t), while the coördinates in the target space are labeled (x, y, z).

Our construction proceeds as follows. Start with a base curve $C_0 : \langle a(s), b(s), 0\rangle$ with $u = a(s)$, $v = b(s)$. We assume this is a smooth curve parametrized by arc length. Let its length be L, so $a(s)$, $b(s)$ are periodic functions of period L defined on $\mathbb{R}$. Assume also that C_0 is a convex curve in the u-v plane, symmetric about the v-axis. Since $F(u, v, 0) = \langle u, v, 0\rangle$ we see that $\Gamma_0 = F(C_0)$ is identical to C_0. With C_0 as our base curve, we form the cylinder $C_0 \times \mathbb{R}$.

We will consider surfaces S in the parameter space which are normal graphs over this cylinder. The unit normal vector to $C_0 \times \mathbb{R}$ is $n(s) = \langle b'(s), -a'(s), 0\rangle$. Let $\varphi(s,t)$ be a smooth function that is periodic in s of period L and satisfies $\varphi(s, 0) = 0$. Set

$$\begin{aligned} u(s,t) &= a(s) + \varphi(s,t)b'(s) \\ v(s,t) &= b(s) - \varphi(s,t)a'(s). \end{aligned} \tag{3–5}$$

The surface, S, is then described by the map

$$S : \langle u(s,t), v(s,t), t\rangle. \tag{3–6}$$

We set $\Sigma \equiv F(S)$. This is a parametric surface in the target space. We need to determine $h(t)$, $\varphi(s,t)$ so that Σ satisfies the contact angle condition with the surfaces Λ_t. We shall determine $h(t)$ so that the volume enclosed by the container wall Σ and any given free surface Λ_t remains constant. Specifically:

Theorem 3.2 (the three-dimensional case). *Let the base curve $C_0 : \langle a(s), b(s), 0\rangle$ be as described above. It is convex and symmetric about the v-axis, and periodic with period L. There exist*

(a) *a function $\varphi(s,t)$ defined for t in an interval about* 0 *and for all s, periodic in s of period L, satisfying $\varphi(s, 0) = 0$;* and

(b) *a smooth function $h(t)$ that is even in t and satisfies $h(0) = 0$;*

the whole satisfying the following property.

Let S be the surface in the parameter space given as a normal graph over the cylinder $C_0 \times \mathbb{R}$ by (3–5), (3–6), and let $\Sigma = F(S)$ be the image surface in the target space under the map F. For each t, let Λ_t be the equilibrium surface $F(\Omega \times \{t\})$. The wall Σ and the equilibrium surface Λ_t intersect along a curve $\Gamma_t = F\big(u(s,t), v(s,t), t\big)$. The two surfaces intersect transversally with a contact angle γ, where $0 < \gamma < \pi$.

Finally suppose that the container Σ is closed off from below so that Σ and the free surface Λ_t enclose a volume $V(t)$. We can choose $h(t)$ so that the volume remains constant.

Proof. Given the convex base curve C_0 as described, we have the surface S as a normal graph over the cylinder $C_0 \times \mathbb{R}$ given by (3–5) and (3–6) for any function $\varphi(s,t)$. We want $\varphi(s,t)$ to be periodic in s of period L, defined in some interval about $t = 0$, and with $\varphi(s, 0) = 0$. We designate

$$\Sigma = F(S), \quad \Gamma_0 = F(C_0) \text{ (the base curve)}, \quad \Gamma_t = F(C_t).$$

Here Γ_t is a curve lying on Λ_t and on Σ. Given a smooth function $h(t)$ even in t with $h(0) = 0$, we shall derive a first-order partial differential equation for $\varphi(s, t)$ that can be solved by the method of characteristics. This will produce a surface Σ satisfying the correct contact angle condition with Λ_t for any given contact angle γ in the interval $0 < \gamma < \pi$. We then determine $h(t)$ so that the volume condition is satisfied.

Let

$$\mathcal{F}(s, t) \equiv F\big(u(s, t), v(s, t), t\big)$$

describe Σ, where $u(s, t)$ and $v(s, t)$ are given in (3–5). The vectors F_u and F_v are unit tangent vectors to Λ_t with $(F_u \cdot F_v) = 0$. Let $\xi(u, v, t) = F_u \wedge F_v$ be the unit normal vector on Λ_t. We observe that

$$\mathcal{F}_s = u_s F_u + v_s F_v \quad \text{and} \quad \mathcal{F}_t = u_t F_u + v_t F_v$$

are tangent vectors to Σ, and $\mathcal{F}_s$ is tangent to the curve Γ_t as well. We set

$$e_2 = \frac{1}{\sqrt{u_s^2 + v_s^2}} \mathcal{F}_s = \frac{1}{\sqrt{u_s^2 + v_s^2}} (u_s F_u + v_s F_v),$$

the unit tangent vector to $\Gamma_t = \Sigma \cap \Lambda_t$. With ξ as unit normal vector to Λ_t we complete the orthonormal frame along Γ_t by setting

$$e_1 = \frac{1}{\sqrt{u_s^2 + v_s^2}} (v_s F_u - u_s F_v).$$

This is a tangent vector on the surface Λ_t which is a conormal along the bounding curve Γ_t. As in Theorem 3.1 we set

$$w = \cos\gamma\, e_1 + \sin\gamma\, \xi.$$

This vector is to be tangent to Σ. Since $\mathcal{F}_s$ and $\mathcal{F}_t$ span the tangent space we may write

$$\alpha w = \lambda \mathcal{F}_s + \mathcal{F}_t \tag{3–7}$$

for suitable scalers α, λ. Using our expressions above for $\mathcal{F}_s$ and $\mathcal{F}_t$ we rewrite (3–7) to obtain

$$F_t = -(\lambda u_s + u_t) F_u - (\lambda v_s + v_t) F_v + \alpha w.$$

We obtain three equations by taking the inner product of F_t with ξ, F_u, and F_v respectively. First,

$$(F_t \cdot \xi) = \alpha (w \cdot \xi) = \sin\gamma\, \alpha;$$

hence $\alpha = \csc\gamma\ (F_t \cdot \xi)$. For the other two equations one uses the fact that $(F_t \cdot F_v) = 0$ and $(F_u \cdot F_v) = 0$. Taking the inner product of F_t with F_u leads to

$$(\lambda u_s + u_t) = -(F_t \cdot F_u) + \cot\gamma \left(\frac{v_s}{\sqrt{u_s^2 + v_s^2}} (F_t \cdot \xi) \right).$$

Taking the inner product of F_t with F_v yields

$$(\lambda v_s + v_t) = \cot\gamma \left(\frac{-u_s}{\sqrt{u_s^2 + v_s^2}} (F_t \cdot \xi) \right). \tag{3–8}$$

We use these two equations to eliminate λ and find

$$u_s v_t - u_t v_s = (F_t \cdot F_u) v_s - (F_t \cdot \xi) \cot\gamma \sqrt{u_s^2 + v_s^2}. \tag{3–9}$$

Using the expressions for $u(s,t)$, $v(s,t)$ in (3–5) gives us a first order PDE for $\varphi(s,t)$:

$$\begin{aligned} \big(1 + (a'b'' - a''b)\varphi\big)\varphi_t + (F_t \cdot F_u)(b' - a'\varphi_s - a''\varphi) \\ - (F_t \cdot \xi)(\cot\gamma)\sqrt{u_s^2 + v_s^2} = 0, \end{aligned} \tag{3–10}$$

where $u_s^2 + v_s^2 = 1 + \varphi_s^2 + (a''^2 + b''^2)\varphi^2 + 2(a'b'' - a''b')\varphi + 2(a'a'' + b'b'')\varphi\varphi_s$.

This is a first order PDE which we can solve by the method of characteristics subject to the initial condition $\varphi(s,0) = 0$.

As in the two-dimensional case (Theorem 3.1), we use the conservation of volume condition to determine $h'(t)$. Let $V(t)$ be the volume enclosed by the surface Λ_t and the container wall Σ. We may suppose that the bottom of the container is closed off so that the computed volume is finite. The rate of change of volume is obtained by integrating the normal variation over the part of Λ_t that lies inside the container.

$$\dot V(t) = \iint_{F(A_t)} (F_t \cdot \xi)\, dS = 0. \tag{3–11}$$

Here A_t is the domain in the parameter space whose image under F is the desired region. Now $F(u,v,t)$ is given by (3–4). We have $F_t = \langle f_t, 0, g_t + h'(t) \rangle$ and $\xi = F_u \wedge F_v = \langle -g_u, 0, f_u \rangle$. Substitute these expressions into (3–11) and we find

$$\begin{aligned} (F_t \cdot \xi) &= (-f_t g_u + f_u g_t) + h' f_u, \\ (F_t \cdot F_u) &= (f_u f_t + g_u g_t) + h' g_u. \end{aligned} \tag{3–12}$$

We use these expressions to rewrite (3–11) as

$$h'(t) \iint_{A_t} f_u\, du\, dv + \iint_{A_t} (-f_t g_u + f_u g_t)\, du\, dv = 0. \tag{3–13}$$

Here $F(A_t)$ lies on the surface Λ_t, with boundary Γ_t. The identity (3–13) determines $h'(t)$ as the ratio of two integrals over the region A_t, with known integrands. The region A_t in the parameter space is bounded by the curve $\langle u(s,t), v(s,t)\rangle$ as given by (3–5), which is the normal graph over the base curve $\mathscr{C}_0$ and depends solely on the function $\varphi(s,t)$. We use (3–13) to substitute this integral expression for $h'(t)$ into the differential equations (3–9) or (3–10). This occurs in the expressions for $(F_t \cdot F_u)$ and $(F_t \cdot \xi)$ as in (3–12). Our differential equation (3–10) takes the form

$$\varphi_t = \mathscr{G}(s, t, \varphi, \varphi_s, h'), \tag{3–14}$$

where h' is the ratio of two integrals depending only on $\varphi(s,t)$. We can still use the method of characteristics to obtain a solution $\varphi(s,t)$, defined in a neighborhood of $t=0$, periodic in s and satisfying $\varphi(s,0)=0$.

We can show existence of a solution using the Picard iteration process. Insert into (3–14) the expression for $h'(t)$ determined by the fixed volume condition (3–13). The method of characteristics gives a system of differential equations for $s=s(\sigma,t)$, $\varphi=\varphi(\sigma,t)$, $p(\sigma,t)=\varphi_s(\sigma,t)$ and $\varphi_t(\sigma,t)=q(\sigma,t)$, with initial data when $t=0$ determined by the base curve Γ_0. One writes the differential equation, in t, as an integral equation with the initial data built in. This allows us to set up an iteration process. Start with an initial function $\langle s_0(\sigma,t), \varphi_0(\sigma,t), p_0(\sigma,t), q_0(\sigma,t)\rangle$. Use this input to calculate $h_0'(t)$ using (3–14). The Picard process allows us to compute $\langle s_1(\sigma,t), \varphi_1(\sigma,t), p_1(\sigma,t), q_1(\sigma,t)\rangle$. The iteration continues, and convergence to a unique solution follows. Having obtained the solution to (3–14) we use (3–13) to find $h'(t)$ and $h(t)$, setting $h(0)=0$. Finally, we use the map $F(u,v,t)$ to obtain the exotic container Σ.

The solution might be implemented as follows. Let P be a partition of an interval, $[0,T]$ into $0=t_1<t_2<\cdots<t_N=T$. Construct a piecewise linear function $h_N(t)$ by using (3–13) to compute $h_N'(t_k)$ and extending $h_N(t)$ linearly over the interval $[t_k, t_{k+1}]$. Now use (3–10) to evolve $\varphi(s,t)$ through this interval as well.

Our base curve Γ_0 was symmetric about the y-axis. This implies that $h'(0)=0$, so $h(t)\equiv 0$ on the interval $[0,t_1]$. At $t=t_1$ we recompute $h'(t_1)$ using (3–13) and extend linearly onto the next subinterval, $[t_1,t_2]$. The process continues. □

Remarks. (1) If the base curve Γ_0 is symmetric about the y-axis and if the contact angle is $\gamma=\pi/2$, then the volume condition is satisfied by setting $h(t)\equiv 0$. The bounding surface Σ is generated by the set of orthogonal trajectories to the elastic surfaces, $F_0(u,v,t)$, which cut through the base curve Γ_0.

(2) One could set up the surface S in the parameter space as a normal graph over the round cylinder $\mathscr{C}_0 \times \mathbb{R}$, where $\mathscr{C}_0 = \langle \cos s, \sin s, 0\rangle$. This somewhat simplifies the differential equation (3–10), but the initial data $\varphi(s,0)$ will no longer be zero.

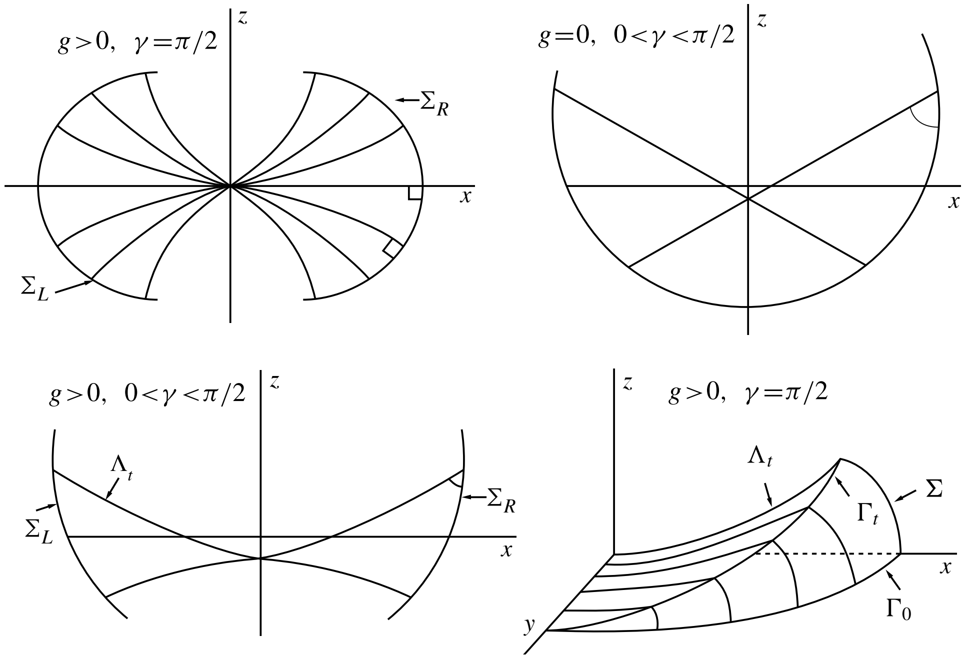

Figure 4. Qualitative appearance of a few exotic containers, showing the relationship of liquid surface and wall. Top row and bottom left: planar case; bottom right: three-dimensional case.

(3) The surface Σ constructed in Theorem 3.2 needs to be filled out. This surface is the union of boundary curves Γ_t. This set of curves Γ_t has an envelope that creates an edge for Σ. The curve Γ_t and the corresponding equilibrium surface Λ_t touch the envelope at points where the normal component of the variation $F_t \cdot \xi$ vanishes. The fixed volume condition (3–11) shows that Λ_t is divided into two regions determined by the sign of $F_t \cdot \xi$. The nodal curve on Λ_t will touch the envelope in two points. Consider the case $g = 0$, so that the extremals are tilted planes. Let Γ_0 be a circle centered at $(0, 0)$. For $\gamma = \pi/2$ the generated surface is a pair of sections of a sphere resembling orange peels (as the referee astutely remarked). In this case the envelope degenerates, becoming two points. To complete the surface one must extend Σ smoothly so that each curve Γ_t lies inside. By continuity the contact angle condition will prevail at the envelope. For $\gamma \neq \pi/2$ the envelope is a curve with each extremal touching the envelope in two points. Again we can extend Σ smoothly so that each extremal surface satisfies the contact angle condition everywhere.

The same discussion applies to the case when $g \neq 0$, with Γ_0 other than a circle. How one fills out the exotic container surface could affect the stability question. Letting the new pieces of surface bulge out increases the chance for stability.

4. Minimization

Given our family of elastic extrema for fixed $\kappa_0 = \rho g/\sigma$ and contact angle γ_0, we have shown how to construct an exotic container with the property that there exists a one-parameter family of equilibria (including the horizontal plane, $u = 0$), each of which meets the container wall at contact angle γ_0 and encloses the same volume V_0. It follows that each member of this family has the same potential energy. Are these equilibria local minimizers of energy subject to the volume constraint? I now outline an argument which indicates that this is the case in the planar case.

4.1. If $\kappa_0 = 0$ the equilibria are tilted lines and the exotic container will be a section of a circle. These are all minimizers.

4.2. Suppose $\kappa = \kappa_0 = \rho g/\sigma$ and $\tau = \tau_0 = \cos\gamma_0$. The corresponding exotic container consists of two curves: Σ_L on the left and Σ_R on the right. Both are symmetric about the z-axis. Let t be a vertical coördinate. Denote by $p(t)$ that point on Σ_L at level t and by $q(t)$ the corresponding point on Σ_R. For $t = 0$ we set $p(0) = p_0$, $q(0) = q_0$ with connecting extremal, $u \equiv 0$. By our exotic container construction this is part of a one-parameter family of extremal curves connecting Σ_L to Σ_R, each enclosing volume V_0 and making a contact angle γ_0 at each end. They are all extremals of the energy functional

$$E = |C| + \kappa_0 \int_T z\,dv - \tau_0|\Sigma'| \equiv E_0 - \tau_0|\Sigma'|. \tag{4–1}$$

Each of these extremals connect some point $p \in \Sigma_L$ to some point q on Σ_R. As p descends the corresponding q will rise.

4.3. There is a continuous map $(p, q) \mapsto C(p, q)$ defined for $p \in \Sigma_L$, $q \in \Sigma_R$ in a neighborhood of (p_0, q_0), where $C(p, q)$ is an extremal for E_0 connecting p to q, enclosing volume V_0, and with $C(p_0, q_0)$ being the extremal $u \equiv 0$. Each $C(p, q)$ will be a strong local minimizer of the energy E_0 for the fixed endpoint problem and subject to the volume constraint.

This is because the solution $u \equiv 0$, with contact angle γ_0, has the property that for the free boundary problem, the second variation is nonnegative for all volume-preserving perturbations with a one-dimensional kernel. The kernel of the corresponding variational problem is our given one-parameter family. The boundary values are not fixed here. For fixed boundary values, the extremal $u = 0$ is a strong local minimizer of energy, subject to the volume constraint. It follows that for any pair of points $p \in \Sigma_L$, $q \in \Sigma_R$ close to p_0, q_0 there will exist exactly one strong local minimizer of energy for the fixed boundary problem and enclosed volume V_0.

4.4. Let $p \in \Sigma_L$ be fixed and let $C(t) = C(p, q(t))$ be the extremal curve that joins p to $q(t)$. Let $\gamma(t)$ be the contact angle of $C(t)$ with Σ_R. We claim that $\gamma(t_1) > \gamma(t_2)$ when $t_1 < t_2$.

Recall that for the extremal $u \equiv 0$, the second variation of the energy was non-negative with a one-dimensional kernel for the free endpoint problem and subject to the volume constraint. Now fix the point $p_0 \in \Sigma_L$ but let $q \in \Sigma_R$ vary. This is a semifree variational problem. The energy is $E = E_0 - \cos\gamma_0\,|\Sigma'_R|$, where Σ'_R is the wetted part of Σ_R. Because of the fixed point restriction the second variation of this functional is positive definite for volume preserving perturbations.

By continuity the same remains true for nearby extremals $C(p, q)$. Let the contact angle of $C(p, q)$ with Σ_R be γ_q. The relevant energy functional now is

$$E = E_0 - \cos\gamma_q\,|\Sigma'_R|.$$

Let $C(t)$ be the extremal connecting p to $q(t)$. Each $C(t)$ is an extremal for its energy

$$\frac{dE}{dt} = \frac{dE_0}{dt} - \cos\gamma(t)\,\frac{d|\Sigma'_R|}{dt} = 0.$$

Now fix $q_1 = q(t_1)$ with extremal $C(t_1)$. We compute the energy

$$E = E_0 - \cos\gamma_1\,|\Sigma'_R|$$

along $C(t)$. The energy functional will have a minimum for $t = t_1$. Let

$$e(t) = E_0 - \cos\gamma_1\,|\Sigma'_R|$$

denote this energy. We have $e'(t_1) = 0$, whereas $e'(t) > 0$ for $t > t_1$ and $e'(t) < 0$ for $t < t_2$. However,

$$e'(t) = \bigl(\cos\gamma(t) - \cos\gamma_1\bigr)\frac{d|\Sigma'_R|}{dt}.$$

Since $d|\Sigma'_R|/dt$ is positive, our assertion follows.

4.5. Let Γ be any curve connecting $p \in \Sigma_L$ to $q \in \Sigma_R$ that encloses volume V_0 and is C^0-close to the extremal $u \equiv 0$. We claim that the energy E of (4–1) applied to Γ is not less than the same energy for the curve, $u \equiv 0$.

First, we may replace Γ by $C(p, q)$. This decreases the energy E_0, and thus the energy E, since the wetted energy is unchanged. Now $C(p, q)$ connects p to $q = q(t)$ for some t. Let t^* be that value such that the extremal $C(p, q(t^*))$ meets Σ_R at contact angle γ_0. Then the contact angle at p for $C(t^*)$ is also γ_0. We apply the discussion in the preceding section to conclude that $E(C(t^*)) \le E(C(t))$, with E given by (4–1).

But $C(t^*)$ is part of our one-parameter family all having the same contact angle, γ_0. It follows that $E(\Gamma) \ge E(C(t^*)) = E(C_0)$. This concludes the argument.

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