Pacific Journal of Mathematics

RINGS THAT ARE ALMOST GORENSTEIN

CRAIG HUNEKE AND ADELA VRACIU

Volume 225 No. 1 May 2006

PACIFIC JOURNAL OF MATHEMATICS
Vol. 225, No. 1, 2006

RINGS THAT ARE ALMOST GORENSTEIN

CRAIG HUNEKE AND ADELA VRACIU

We introduce classes of rings that are close to being Gorenstein and prove they arise naturally as specializations of rings of countable CM type. We study these rings in detail and, inter alia, generalize an old result of Teter characterizing Artinian rings that are Gorenstein rings modulo their socle.

Introduction

This paper began with a desire to better understand Cohen–Macaulay rings of countable or finite representation type. Let $(R, \mathfrak{m})$ be a (commutative Noetherian) local ring of dimension d. Recall that a nonzero R-module M is called *maximal Cohen–Macaulay* (*MCM*) provided it is finitely generated and there exists an M-regular sequence $\{x_1, \ldots, x_d\}$ in the maximal ideal $\mathfrak{m}$.

Definition. A Cohen–Macaulay local ring $(R, \mathfrak{m})$ is said to have *finite* (respectively, *countable*) *Cohen–Macaulay type* if it has only finitely (countably) many isomorphism classes of indecomposable maximal Cohen–Macaulay modules.

A particular question we were interested in answering was the following: what are the possible Hilbert functions of R/I, where R is a Cohen–Macaulay ring of at most countable CM type, and I is generated by a general system of parameters? While we have not answered this question, what we found instead was that such quotients behave much like Gorenstein rings in a very precise sense. This changed the direction of our inquiry to understanding these 'almost' Gorenstein rings. Section 1 presents the basic formulation of the properties of such rings. Section 3 is devoted to proving that Cohen–Macaulay rings having at most countable CM type are almost Gorenstein in the sense of having some of these properties.

Let R be a local Cohen–Macaulay ring with canonical module ω. An example of one of the properties we are considering is that $\omega^*(\omega)$, the set of all elements of the form $f(x)$ where $x \in \omega$ and $f : \omega \to R$ is an R-linear map, contains the maximal ideal of R. If R is Gorenstein, then of course ω is free and $\omega^*(\omega)$ is the whole ring.

MSC2000: 13H10.
Keywords: Gorenstein rings, countable Cohen–Macaulay type, canonical module.
Both authors were partially supported by NSF grant DMS-0098654.

After introducing these rings, it is natural to look for examples. It turns out that Artinian Gorenstein rings modulo their socle are always examples, and this led us to [Teter 1974], which gave an intrinsic characterization of such rings. We are able to improve his result when 2 is a unit, by removing a seemingly important technical assumption of Teter's. This work is in Section 2. We further improve the result when 2 is a unit in the graded case, giving necessary and sufficient conditions for a standard graded Artinian ring to be the homomorphic image of a standard graded Gorenstein ring by its socle element.

The properties that characterize the rings in which we are interested are all closely related and perhaps are all equivalent: we have been unable to decide whether they are equivalent, except in special cases. In Section 4, we prove that all the conditions introduced in the Section 1 are equivalent for Artinian local rings of type 2.

Finally, Section 5 classifies the $\mathfrak{m}$-primary monomial ideals of type three which are 'almost' Gorenstein. An analysis of this classification shows that the conditions under consideration are equivalent in this case.

1. Almost Gorenstein rings

We introduce rings that are almost Gorenstein in a sense identified by the first proposition below. We begin by identifying several Gorenstein-like properties that a ring may have. If M is a module over a ring R, we denote $\mathrm{Hom}_R(M, R)$ by M^*. By $M^*(M)$ we mean the ideal consisting of all $f(x)$ where $x \in M$ and $f \in M^*$.

Proposition 1.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian Cohen–Macaulay local ring with infinite residue field k and a canonical module ω. Consider the conditions*:

(A) $\mathfrak{m} \subseteq \omega^*(\omega)$.

(B) *For all ideals K generated by a system of parameters and for all ideals $I \supseteq K$, we have $K : (K : I) \subseteq I : \mathfrak{m}$.*

(C) *For all ideals K generated by a system of parameters that are not contained in a given finite* (*or countable*, *if R is complete or k is uncountable*) *set of primes not equal to the maximal ideal*, *and for all ideals $I \supseteq K$*, *we have $K : (K : I) \subseteq I : \mathfrak{m}$.*

Then (A) $\Rightarrow$ (B) $\Rightarrow$ (C).

Note. In case R is Artinian, the assumption on K from condition (C) is void, and thus in this case conditions (B) and (C) are automatically the same.

Proof. Obviously (B) $\Rightarrow$ (C), so we need only argue that (A) $\Rightarrow$ (B). We prove the more precise statement that $a \in \omega^*(\omega)$ implies $0 : (0 : I) \subseteq I : a$.

It is well-known that $a \in \omega^*(\omega)$ if and only if the map $m_a : \omega \to \omega$, given by multiplication by a, factors through a free module F [Ding 1993, Lemmas 1.2

and 1.3]. Recall that by Matlis duality we have $0 :_R (0 :_\omega I) = I$ for any ideal I. Consider the image of the submodule $N = 0 :_\omega I \subset \omega$ under m_a. On one hand, it is $a(0 :_\omega I)$. On the other hand, using the factorization, the image in F is contained in $(0 :_R I)F$ and the image of this latter module in ω is contained in $(0 :_R I)\omega$.

Therefore we have

$$a(0 :_\omega I) \subset (0 :_R I)\omega.$$

Taking duals in R, we get

$$0 :_R a(0 :_\omega I) \supset 0 :_R (0 :_R I)\omega.$$

But $0 :_R a(0 :_\omega I) = I : a$, and $0 :_R (0 :_R I)\omega = 0 :_R (0 :_R I)$, since ω is faithful. $\square$

A basic question we are unable to answer (except in some cases) is:

Question. If $(R, \mathfrak{m})$ is an Artinian local ring satisfying (C), does $\omega^*(\omega)$ contain $\mathfrak{m}$? In other words, are the three properties in Proposition 1.1 equivalent?

Observation. We think of all these properties as describing rings which are *almost Gorenstein*. Of course, if R is Gorenstein, then $K : (K : I) = I$, and $\omega^*(\omega) = R$. Ding [1993] studied rings that satisfy property (A); we shall use some of his ideas.

We now consider in some detail what condition (A) means. Let S be a Gorenstein Artinian ring, and $K = (f_1, \dots, f_n) \subseteq S$ an ideal. We want to study the ring $R = S/(0 : K)$. There is no loss of generality in writing R in this way, since every ideal in a Gorenstein Artinian ring is an annihilator ideal. The canonical module for this ring can be identified with $\operatorname{Hom}_S(R, S) \cong 0 :_S (0 :_S K) = K$, so that $\omega_R \cong K$.

To study the trace ideal $\omega^*(\omega)$, consider an R-linear map $\phi : K \to S/(0 : K)$. Let $u_i = \phi(f_i)$, and let v_i denote a lifting of u_i to S. We must have $u_i(0 : f_i) = 0$ in R, thus $v_i(0 : f_i) \subseteq 0 : K$ in S. Since S is Gorenstein, $v_i \in (0 : K) : (0 : f_i) = (f_i) : K$. Hence, the image of any R-linear map ϕ is contained in

$$\frac{(f_1) :_S K + \cdots + (f_n) :_S K}{0 : K}.$$

Thus, a necessary condition for $\omega^*(\omega) \supseteq \mathfrak{m}$ is that

$$(f_1) :_S K + \cdots + (f_n) :_S K \supseteq \mathfrak{m}, \tag{1–1}$$

for any choice of a system of generators $f_1, \dots, f_n$ of K.

In particular, $\omega^*(\omega) \supseteq \mathfrak{m}$ implies that

$$\text{there exists } i \in \{1, \dots, n\} \text{ such that } (f_i) : K \not\subseteq \mathfrak{m}^2. \tag{1–2}$$

This last property also holds under the weaker assumption that R satisfies property (C) of Proposition 1.1:

Proposition 1.2. *Let S be a Gorenstein local Artinian ring. If $R = S/(0{:}K)$ is an Artinian local ring satisfying* (C) *in Proposition 1.1, and if $f_1, \dots, f_n$ is a minimal set of generators for K, there exists $i \in \{1, \dots, n\}$ such that $(f_i){:}K \not\subseteq \mathfrak{m}^2$.*

Proof. Assume R satisfies (C), but $f_i{:}K$ lies in $\mathfrak{m}^2$ for all i. We may assume that $\mathfrak{m}K \neq 0$. For, if not, K is either 0 or has exactly one generator, a representative of the socle of S. In either case, the result is trivial. Property (C) states that for every ideal $I \supset 0{:}K$, we have

$$(0{:}K) : \big((0{:}K){:}I\big) \subseteq I : \mathfrak{m}$$

(all colons are taken in S), or equivalently $IK{:}K \subseteq I{:}\mathfrak{m}$. The equivalence follows because

$$(0{:}K) : \big((0{:}K){:}I\big) = \big(0 : ((0{:}K) : I)\big) : K = I\big(0{:}(0{:}K)\big) : K = IK : K.$$

Take $I_i = 0{:}f_i$. Since $f_i \in K$, we have $I_i \supset 0{:}K$. Then $K(0{:}f_i){:}K \subseteq 0{:}\mathfrak{m}f_i$ for all $i = 1, \dots, n$, and therefore

$$\bigcap\big(K(0{:}f_i)\big) : K \subseteq \bigcap(0{:}\mathfrak{m}f_i) = 0{:}\mathfrak{m}K,$$

or, equivalently (by duality),

$$\begin{aligned}\mathfrak{m}K &\subseteq 0{:}\big(\bigcap\big(K(0{:}f_i)\big){:}K\big) = K\big(0{:}\bigcap\big(K(0{:}f_i)\big)\big)\\ &= K\big(\textstyle\sum\big(0{:}K(0{:}f_i)\big)\big) = K\big(\textstyle\sum(f_i){:}K\big).\end{aligned}$$

This contradicts the assumption that $(f_i){:}K \subseteq \mathfrak{m}^2$ for all i. □

2. Teter's rings

[Teter 1974] characterized the Artinian local rings R that are of the form $S/(\delta)$, where S is a Gorenstein local Artinian ring and δ generates its socle. We shall prove such rings satisfy condition (A). Teter's main theorem gives necessary and sufficient conditions for R to be of this form. His theorem states:

Theorem 2.1. *Suppose that $(R, \mathfrak{m}, k)$ is local Artinian, and let E be an injective hull of k. R is a factor of a local Artinian Gorenstein ring by its socle if and only if there exists an isomorphism $\phi : \mathfrak{m} \to \mathfrak{m}^\vee$ satisfying $\phi(x)(y) = \phi(y)(x)$ for all $x, y \in \mathfrak{m}$, where $(\quad)^\vee$ denotes $\mathrm{Hom}_R(\quad, E)$.*

An immediate corollary of this theorem is:

Corollary 2.2. *If $(R, \mathfrak{m}, k)$ is local Artinian and is the factor of a local Artinian Gorenstein ring by its socle, then R satisfies all the conditions of Proposition 1.1.*

Proof. Taking the Matlis duals of the injection of $\mathfrak{m}$ into R gives a surjective map from E onto $\mathfrak{m}^\vee$. Composing this surjection with the inverse of the isomorphism

ϕ in Teter's theorem gives a homomorphism $f : E \to \mathfrak{m}$ which is onto. As E is isomorphic to the canonical module of R, this proves that $\omega^*(\omega)$ contains $\mathfrak{m}$. $\square$

Our purpose in this section is to show that the condition that ϕ satisfy $\phi(x)(y) = \phi(y)(x)$ for all $x, y \in \mathfrak{m}$ is basically unnecessary if 2 is a unit. To do so, we first need to prove some preliminary remarks concerning an involution on ω^*.

For every $f \in \omega^*$, we can define another linear map $\tilde{f} \in \omega^*$ as follows: Let $x \in \omega$ and consider the map $\phi_{f,x} : \omega \to \omega$ defined by $\phi_{f,x}(y) = f(y)x$. Since $\mathrm{Hom}_R(\omega, \omega) \cong R$, there exists a unique $r_{f,x} \in R$ such that $\phi_{f,x}$ is multiplication by $r_{f,x}$, that is, $f(y)x = r_{f,x}y$ for all $x, y \in \omega$ and $f \in \omega^*$.

Definition. Let $\tilde{f} : \omega \to R$ be defined by $\tilde{f}(x) = r_{f,x}$.

It is not hard to check that $\tilde{f}$ is a linear map. Moreover, the mapping $\Phi : \omega^* \to \omega^*$ with $\Phi(f) = (\tilde{f})$ is linear.

The basic property of $\tilde{f}$ is that, for all $x, y \in \omega$,

$$f(x)\,y = \tilde{f}(y)\,x, \tag{2–1}$$

which follows because by definition $\tilde{f}(y)\,x = r_{f,y}x = f(x)\,y$.

We summarize some of the properties of $\tilde{f}$:

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be a local Artinian ring. Define $\Phi : \omega^* \to \omega^*$ as above. For $f \in \omega^*$, set $I_f = f(\omega)$ and $J_f = \tilde{f}(\omega)$.*

(1) *Φ is an isomorphism.*

(2) *Φ^2 is the identity map on ω^*.*

(3) $\omega^*(\omega) = \sum I_f = \sum J_f$.

(4) $\ker f = 0 :_\omega J_f$.

(5) $\mathrm{Hom}_R(I_f, \omega) \cong J_f$*, that is, J_f is a Matlis dual to I_f.*

Proof. We claim that Φ is injective. Indeed, if $\tilde{f} = \tilde{g}$, then $f(y)x = r_{f,x}y = r_{g,x}y = g(y)x$ for all $x, y \in \omega$. Since ω is faithful, this shows that $f = g$. It follows that Φ is an isomorphism since ω^* has finite length and any injective map is also surjective.

Part (2) follows at once from (2–1), which identifies $r_{\tilde{f},x}y = \tilde{f}(y)x = f(x)\,y$. Hence, $r_{\tilde{f},x} = f(x)$, which means that Φ^2 is the identity map on ω^*.

Part (3) is clear from the definition.

For (4), consider

$$\begin{aligned} 0 :_\omega J_f &= \{y \in \omega \mid r_{f,x}y = 0 \text{ for all } x \in \omega\} \\ &= \{y \in \omega \mid f(y)x = 0 \text{ for all } x \in \omega\} = \ker(f) \end{aligned}$$

(this last equality is because ω is faithful).

Thus,

$$I_f = f(\omega) \cong \frac{\omega}{\ker(f)} = \frac{\omega}{0:_\omega J_f}$$

Finally, we prove (5):

$$\operatorname{Hom}_R\Big(\frac{\omega}{0:_\omega J_f},\ \omega\Big) \cong \big\{\phi:\omega\to\omega \,\big|\, \phi(0:_\omega J_f)=0\big\} \cong 0:_R(0:_\omega J_f) = J_f \quad \square$$

Observation. $0:(0:I)\big/I$ is the kernel of the canonical map $R/I \to (R/I)^{**}$. Thus, property (C) in Proposition 1.1 is satisfied if and only if the kernel of this canonical map is contained in the socle of R/I, for every ideal I. On the other hand, $\omega^*(\omega) = \mathfrak{m}$ if and only if the kernel of the canonical map $\omega \to (\omega)^{**}$ is contained in the socle of ω.

Proof. To see the first claim, note that $(R/I)^* = 0:I$, and consider the short exact sequence

$$0 \longrightarrow 0:I \longrightarrow R \longrightarrow \frac{R}{0:I} \longrightarrow 0.$$

Applying $\operatorname{Hom}_R(\quad, R)$ yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R\Big(\frac{R}{0:I},\ R\Big) \longrightarrow R \longrightarrow \operatorname{Hom}_R\big(0:I,\ R\big).$$

Since the module on the left is $0:(0:I)$ and the module on the right is $(R/I)^{**}$, we obtain an exact sequence

$$0 \longrightarrow 0:(0:I) \longrightarrow R \longrightarrow (R/I)^{**},$$

which proves the claim.

To see the second claim, notice that the kernel of the map $\omega \to (\omega)^{**}$ is

$$\bigcap_{f\in\omega^*} \ker(f) = \bigcap_{f\in\omega^*} \big(0:_\omega I_{\tilde{f}}\big) = 0:_\omega\Big(\sum_{f\in\omega^*} I_f\Big) = 0:_\omega \omega^*(\omega) \qquad \square$$

We now give our improvement of Teter's theorem. The use of the map $\tilde{f}$ makes it possible to avoid the awkward hypothesis that the isomorphism ϕ satisfy $\phi(x)(y) = \phi(y)(x)$.

Theorem 2.4. *Let $(R, \mathfrak{m}, k)$ be an Artinian ring with canonical module ω. Assume that* 2 *is a unit in* R *and* $\operatorname{Soc}(R) \subseteq \mathfrak{m}^2$. *Then R is the quotient of a zero-dimensional Gorenstein ring by its socle if and only if there exists a surjective map $f:\omega\to\mathfrak{m}$.*

Proof. First assume that R is the quotient of a zero-dimensional Gorenstein ring by its socle. The result then follows from Teter's theorem, but the proof is so direct that we give it here for the reader's convenience. Let $R = S/(\delta)$, where S is Gorenstein and δ generates the socle of S. The exact sequence

$$0 \to k \to S \to R \to 0$$

gives, upon dualizing into the injective hull of the residue field of S, the exact sequence,

$$0 \to E \to S \to k \to 0,$$

which proves that the injective hull E of the residue field of R is isomorphic to the maximal ideal of S (which is an R-module), and thus clearly maps surjectively onto the maximal ideal of R.

To prove the harder direction, we need only argue that, given f as in the statement of the theorem, there exists an isomorphism $g : \mathfrak{m} \to \mathfrak{m}^\vee$ such that $g(x)(y) = g(y)(x)$ for all $x, y \in \mathfrak{m}$.

Set $h = (f + \tilde{f})$, which is a homomorphism from $\omega \to R$. We claim that the kernel of this map is the socle of ω. Suppose that $h(x) = 0$. Then $(f + \tilde{f})(x) = 0$. By the definition of $\tilde{f}$ (see (2–1)), it follows that, for all $y \in \omega$, we have $f(y)x = \tilde{f}(x)\,y = -f(x)\,y$. Hence

$$xf(y) + yf(x) = 0.$$

Since f maps ω onto $\mathfrak{m}$, the kernel of f must be the socle of ω (which is one-dimensional). As $f\big(f(x)y - f(y)x\big) = f(x)f(y) - f(y)f(x) = 0$ for all $x, y \in \omega$, it follows that $\mathfrak{m}\big(f(x)y - f(y)x\big) = 0$.

Suppose that $x \in \mathfrak{m}\omega$. Write $x = \sum r_i x_i$ with $r_i \in \mathfrak{m}$. Then

$$xf(y) + yf(x) = \sum r_i \big(x_i f(y) + yf(x_i)\big),$$

but as $r_i \in \mathfrak{m}$, this sum is just $\sum r_i x_i f(y) = 2xf(y)$. Hence $xf(y) = 0$. Since $f(\omega) = \mathfrak{m}$, it follows that, as y varies, x is in the socle of ω. In order to finish the proof of the claim, we need to show that $\ker(h) \subseteq \mathfrak{m}\omega$. Let $x \in \ker(h)$. It follows that $\mathfrak{m}x \subseteq \ker(h) \cap \mathfrak{m}\omega \subseteq \mathrm{Soc}(\omega)$, and thus $x \in 0 :_\omega \mathfrak{m}^2$. The assumption that $\mathrm{Soc}(R) \subseteq \mathfrak{m}^2$ is equivalent, by Matlis duality, to $0 :_\omega \mathfrak{m}^2 \subseteq \mathfrak{m}\omega$.

Since the kernel of h is 1-dimensional, the length of the image of h is exactly one less than the length of R. It follows that h maps ω onto $\mathfrak{m}$.

Next, we prove that, if $x, y \in \omega$, then $h(x)y = h(y)x$. For $h(x)y = (f + \tilde{f})(x)y = \big(f(x)y + \tilde{f}(x)y\big) = \big(f(x)y + f(y)x\big)$. Since this is symmetric with respect to x and y, the claim follows.

Taking Matlis duals, we get a map $h^\vee : \mathfrak{m}^\vee \to R$ that is injective, and so $h^\vee$ is an isomorphism between $\mathfrak{m}^\vee$ and $\mathfrak{m}$. Set g equal to the inverse map of $h^\vee$. Then g is an isomorphism of $\mathfrak{m}$ and $\mathfrak{m}^\vee$. We claim that, for all $x, y \in \mathfrak{m}$, we have $g(x)(y) = g(y)(x)$. We can then apply Teter's theorem to finish the proof. The homomorphism g is the dual of the inverse of h, where we think of h as an isomorphism of $\omega/(\delta)$ with $\mathfrak{m}$. If $u, v \in \mathfrak{m}$, write $u = h(x)$ and $v = h(y)$. Then $h^{-1}(u)\,v = xv = xh(y) = h(x)y = uh^{-1}(v)$, proving that h^{-1} satisfies the same symmetry condition. Taking Matlis duals preserves this condition, proving the theorem. $\square$

In the case when R is graded, we can do better. In particular, the assumption that the socle of R be contained in $\mathfrak{m}^2$ can be removed, in the sense that, if it is not true, then the structure of R is fixed. Recall that a Noetherian graded ring R is standard graded if $R_0 = k$ is a field and $R = k[R_1]$.

Theorem 2.5. *Let R be an Artinian standard graded ring over a field k, not having characteristic 2, with graded canonical module ω. Set $\mathfrak{m}$ equal to the ideal generated by all elements of positive degree. The following are equivalent*:

(1) *Either $R \cong k[X_1, \dots, X_n]/(X_1, \dots, X_n)^2$, or $\mathrm{Soc}(R) \subseteq \mathfrak{m}^2$ and there exists a degree 0 graded surjective homomorphism $f : \omega(t) \to \mathfrak{m}$ for some t.*

(2) *R is the quotient of an Artinian standard graded Gorenstein ring by its socle.*

Proof. First assume (2). If $\mathrm{Soc}(R)$ is not contained in $\mathfrak{m}^2$, there must be a socle element of degree 1, say ℓ. Lifting back to S it follows that $\mathfrak{m}_S \ell \subseteq \mathrm{Soc}(S)$, so the socle of S must live in degree 2. It follows that the Hilbert function of S is $1, n, 1$ for some n. Hence the Hilbert function of R is $1, n$, implying that $R \cong k[X_1, \dots, X_n]/(X_1, \dots, X_n)^2$.

Suppose now that $\mathrm{Soc}(R) \subseteq \mathfrak{m}^2$. The graded canonical module of S is $S(t)$ for some t, and then the graded canonical module of R is $\mathrm{Hom}_S\big(R, S(t)\big) = \mathrm{Hom}_S(R, S)(-t) = \mathfrak{m}_S(-t)$. Hence, there is a graded surjective map onto $\mathfrak{m}_R$ after twisting by t.

Conversely, assuming (1) we construct S explicitly. First, if R is isomorphic to $k[X_1, \dots, X_n]/(X_1, \dots, X_n)^2$, we may take

$$S = k[X_1, \dots, X_n]\big/(X_i X_j,\ X_i^2 - X_j^2),$$

where the indices range over all $1 \le i < j \le n$.

Otherwise, assume that $\mathrm{Soc}(R) \subseteq \mathfrak{m}^2$ and that there exists a degree 0 graded surjective homomorphism $f : \omega(t) \to \mathfrak{m}$ for some t. Define a ring structure on $S = k \oplus \omega(t)$ with multiplication

$$(\alpha_1, x_1)(\alpha_2, x_2) = \big(\alpha_1\alpha_2,\ \alpha_1 x_2 + \alpha_2 x_1 + \tfrac{1}{2}\big(x_1 f(x_2) + x_2 f(x_1)\big)\big).$$

This multiplication is obviously commutative and gives S a graded structure. Moreover, S is standard graded; by counting lengths, the surjection of $\omega(t) \to \mathfrak{m}$ must have a kernel of length 1. Hence, the kernel is exactly the socle of ω, and the socle is never a minimal generator of ω unless $\omega = R = k$. Thus, the minimal generators of $\omega(t)$ correspond to the minimal generators of $\mathfrak{m}$, and all have degree 1. In order to check associativity, observe that the kernel of f must be $\mathrm{Soc}(\omega)$, since it is an R-submodule of ω of length one. This implies that $uxf(y) = uyf(x)$ for any $u \in \mathfrak{m}$ and $x, y \in \omega$.

S is a graded ring with homogeneous maximal ideal $0 \oplus \omega$ (note that $(0, x)^n = \big(0, x^n f(x)\big)$, and $(0, x)$ is nilpotent in S since x is nilpotent in R).

We compute the socle of S. If $(\alpha, x) \in \mathrm{Soc}(S)$, we must have $\alpha = 0$. Thus,

$$\mathrm{Soc}(S) = \{(0, x) \mid x \in \omega \text{ and } xf(y) + yf(x) = 0 \text{ for all } y \in \omega\} = 0 \oplus \mathrm{Soc}(R),$$

as seen in the proof of the theorem.

Note that R is indeed a quotient of S by its socle, since we have a surjective map $S \mapsto R$ given by $(\alpha, u) \to \alpha + f(u)$. It is not hard to check that this is a ring homomorphism; surjectivity is obvious, and by dimension counting it follows that the kernel of the this map is the socle of S. □

This proof works also in the nongraded case, providing a very different approach than Teter's paper.

3. Rings of finite or countable Cohen–Macaulay type

Let $(R, \mathfrak{m})$ be a Cohen–Macaulay ring of countable Cohen–Macaulay type, that is, there are only countably-many isomorphism classes of indecomposable maximal Cohen–Macaulay modules (MCMs). Let d denote the dimension of R.

Let $\{M_i\}_i$ be a complete list of all the nonisomorphic indecomposable MCMs (up to isomorphism). Consider the set Λ consisting of all the annihilators of modules of the type $\mathrm{Ext}^i_R(M_j, M_k)$, for $i \geq 1$.

Assume that the residue field is uncountable. Vector field arguments show that $\mathfrak{m}$ is not contained in any countable union of proper subideals, in particular it is not contained in the union of all ideals in Λ other than $\mathfrak{m}$ itself. Consider an element $x \in R$ not in the union of the ideals in Λ other than possibly $\mathfrak{m}$. We will call the elements $x \in R$ satisfying this condition *general elements*. By a *general system of parameters* $\underline{x}$ we mean a system of parameters such that $(\underline{x})$ contains a general element. If a general x annihilates any one of the modules listed above, it follows that $\mathfrak{m}$ annihilates that module.

Proposition 3.1. *The following modules have annihilator in* Λ:

(1) $\mathrm{Ext}^i_R(M, N)$ *for* $i \geq 1$, *where* M *is MCM*, N *is arbitrary (finitely generated)*;

(2) $\mathrm{Ext}^i_R(N, M)$ *for* $i \geq d + 1$, *where* M, N *are arbitrary (finitely generated)*;

(3) $\mathrm{Ext}^i_{\bar{R}}(N, M/(\underline{x})M)$ *for* $i \geq 1$, *where* M *is MCM*, N *is arbitrary (finitely generated)*, $\underline{x}$ *is a system of parameters such that* $(\underline{x})N = 0$, *and* $\bar{R} = R/(\underline{x})$.

Proof. (1) Let N be an arbitrary module. Consider the Cohen–Macaulay approximation of N (see [Auslander and Buchweitz 1989]):

$$0 \to C \to T \to N \to 0,$$

where C has finite injective dimension and T is MCM. Applying $\mathrm{Hom}_R(M, \quad)$ and using the fact that $\mathrm{Ext}^i_R(M, C) = 0$, we get $\mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_R(M, T)$ for $i \geq 1$.

(2) If $i \geq d+1$, we have $\operatorname{Ext}^i_R(N, M) \cong \operatorname{Ext}^{i-d}_R(\operatorname{syz}^d N, M)$, and the result follows from (a) since $\operatorname{syz}^d N$ is MCM.

(3) We have $\operatorname{Ext}^i_{\bar{R}}\big(N,\ M/(\underline{x})M\big) \cong \operatorname{Ext}^{i+d}_R(N, M)$. □

Proposition 3.2. *If M is a MCM, $(\underline{x}) \subseteq \mathfrak{a} \subseteq I$ are ideals in R, and $\underline{x}$ is a general system of parameters, then $\mathfrak{m}$ annihilates*

$$\frac{(\underline{x})M : (\mathfrak{a} \colon I)}{(\underline{x})M + I(\underline{x}M \colon \mathfrak{a})}.$$

Proof. Let $I = (f_1, \dots, f_n)$. Consider the short exact sequence

$$0 \longrightarrow \frac{R}{\mathfrak{a}\colon I} \longrightarrow \bigoplus \frac{R}{\mathfrak{a}} \longrightarrow N \longrightarrow 0,$$

where the first map is $\bar{u} \mapsto \big(\overline{f_1 u}, \dots, \overline{f_n u}\big)$. Apply $\operatorname{Hom}_{\bar{R}}\big(\quad, M/(\underline{x})M\big)$. By Proposition 3.1(3) we know that $\mathfrak{m}$ annihilates $\operatorname{Ext}^1_{\bar{R}}\big(N,\ M/(\underline{x})M\big)$, and therefore it annihilates the cokernel of the induced map

$$\bigoplus \operatorname{Hom}_{\bar{R}}\left(\frac{R}{\mathfrak{a}}, \frac{M}{\underline{x}M}\right) \longrightarrow \operatorname{Hom}_{\bar{R}}\left(\frac{R}{\mathfrak{a}\colon I}, \frac{M}{\underline{x}M}\right),$$

which is equivalent to

$$\bigoplus \frac{\underline{x}M : \mathfrak{a}}{\underline{x}M} \longrightarrow \frac{\underline{x}M : (\mathfrak{a}\colon I)}{\underline{x}M}.$$

The above map is given by $\big(\bar{u}_1, \dots, \bar{u}_n\big) \mapsto \overline{f_1 u_1 + \cdots + f_n u_n}$, and therefore the cokernel is

$$\bigoplus \operatorname{Hom}_{\bar{R}}\left(\frac{R}{\mathfrak{a}}, \frac{M}{\underline{x}M}\right) \longrightarrow \operatorname{Hom}_{\bar{R}}\left(\frac{R}{\mathfrak{a}\colon I}, \frac{M}{\underline{x}M}\right),$$

which is equivalent to

$$\bigoplus \frac{\underline{x}M : \mathfrak{a}}{\underline{x}M} \longrightarrow \frac{\underline{x}M : (\mathfrak{a}\colon I)}{\underline{x}M}.$$

The map is given by $\big(\bar{u}_1, \dots, \bar{u}_n\big) \mapsto \overline{f_1 u_1 + \cdots + f_n u_n}$, and therefore has cokernel

$$\frac{\underline{x}M : (\mathfrak{a}\colon I)}{\underline{x}M + I(\underline{x}M \colon \mathfrak{a})}.$$ □

Corollary 3.3. *If $\underline{x}$ is a general system of parameters, we have, for any ideal I,*

$$\mathfrak{m}\big(\underline{x} : (\underline{x} \colon I)\big) \subseteq (\underline{x}, I).$$

Proof. Take $M = R$ and $\mathfrak{a} = (\underline{x})$ in Proposition 3.2. □

Corollary 3.3 shows that Cohen–Macaulay local rings of finite or countable CM type satisfy property (C) of Proposition 1.1. We believe that they also satisfy the first condition that $\mathfrak{m} \subseteq \omega^*(\omega)$, but have been unable to prove it in this generality.

4. Ideals of type 2

We prove that the conditions of Proposition 1.1 are equivalent for Artinian rings of type 2. We begin with a general result concerning type n ideals. The following notation will be used throughout the rest of the paper:

Notation. Let $(S, \mathfrak{m})$ be a local Gorenstein ring, and let $R = S/I$, where I is an $\mathfrak{m}$-primary ideal of type n. We represent I as an irredundant intersection of n irreducible ideals $I = J_1 \cap J_2 \cap \cdots \cap J_n$, and choose $J \subseteq I$ to be an irreducible ideal. Then for every $i = 1, \dots, n$ there exists an $f_i \in S$ such that $J_i = J : f_i$. Unless otherwise specified, all colons are computed in S.

Since $I = J_1 \cap \cdots \cap J_n$, we have $J : I = \sum_i J : J_i$. To see this, it suffices to prove equality after computing annihilators into J. But $I = J : (J : I)$, while $J : \left(\sum_i J : J_i\right) = J : (J : J_1) \cap \cdots \cap J : (J : J_n) = J_1 \cap \cdots \cap J_n = I$. It follows that the canonical module of S/I can be computed as:

$$\omega_{S/I} \cong \frac{J : I}{J} = \frac{\sum_i (J : J_i)}{J} = \frac{(J, f_1, \dots, f_n)}{J}. \tag{4–1}$$

Proposition 4.1. *Adopt the notations above. A necessary condition for $R = S/I$ to satisfy condition* (C) *of Proposition 1.1 is that, for all* $1 \le i \le n$,

$$\textstyle\sum_{j \neq i} (J_i : J_j) + (J_1 : J_i) \cap \cdots \cap (J_n : J_i) \supseteq \mathfrak{m}.$$

Proof. Without loss of generality we prove the claim for $i = 1$. Set $K = J_2 \cap \cdots \cap J_n$ and $J_1 = J$. The asserted equality is equivalent to saying that $K : J + J : K \supseteq \mathfrak{m}$. By passing to the quotient, we may assume that $I = 0$. Condition (C) tells us that $0 : (0 : J) \subseteq J : \mathfrak{m}$. We have

$$\begin{aligned} 0 : (0 : J) &= (K \cap J) : \big((K \cap J) : J\big) = (K \cap J) : (K : J) \\ &= K : (K : J) \cap J : (K : J). \end{aligned}$$

The ideal $K : (K : J)$ contains J, and so since J is Gorenstein, it can be written in the form $J : L$ for some ideal $J \subseteq L$. Moreover, since $K : (K : J)$ contains $K + J = J : (J : K)$, it follows that $L \subseteq J + K$. Then

$$0 : (0 : J) = J : L \cap J : (K : J) = J : \big(L + (K : J)\big).$$

The assumption that $0 : (0 : J) \subseteq J : \mathfrak{m}$ then implies that $J : \big(L + (K : J)\big) \subseteq J : \mathfrak{m}$ and so $\mathfrak{m} = J : (J : \mathfrak{m}) \subseteq J : \big(J : (L + (K : J))\big) = L + K : J$. It follows that $J : K + K : J = \mathfrak{m}$, which gives the required formula. □

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a local Artinian ring of type* 2. *Write $R = S/I$, where S is a regular local ring, and write $I = J_1 \cap J_2$, where J_i are irreducible ideals. The three conditions of Proposition 1.1 are equivalent. Furthermore, these conditions are also equivalent to* $J_1 : J_2 + J_2 : J_1 \supseteq \mathfrak{m}$.

Proof. The fact that the weakest condition from Proposition 1.1, that is, that for every ideal I of R one has $0:(0:I) \subseteq I:\mathfrak{m}$, implies that $J_1:J_2 + J_2:J_1 = \mathfrak{m}$ is a particular case of Proposition 4.1.

Assume that $J_2:J_1 + J_1:J_2 = \mathfrak{m}$. We will prove that $\omega^*(\omega)$ contains $\mathfrak{m}$. Using the notation on page 95, let $y \in J_1 : J_2 = (J, f_2) : f_1$, and choose z such that $yf_1 \equiv zf_2 \pmod{J}$.

Recall the description of ω given in (4–1) and define the R-linear map $\Phi : \omega \to \frac{R}{I}$ by $\Phi(f_1) = z$ and $\Phi(f_2) = y$. To check that Φ is well-defined, consider any relation $af_1 + bf_2 \equiv 0 \pmod{J}$. We need to check that $az+by \in I = J:(f_1, f_2)$. Indeed, $af_1z+bf_1y = af_1z+bf_2z = (af_1+bf_2)z \in J$, and $af_2z+bf_2y = af_1y+bf_2y = (af_1+bf_2)y \in J$. We have thus shown that $J_1 : J_2 \subseteq \omega^*(\omega)$. By symmetry, the same holds for $J_2 : J_1$. It follows that $\mathfrak{m} \subseteq \omega^*(\omega)$ and completes the proof the theorem. □

The above proof shows that all of these conditions are equivalent to saying that $I:(I:J_1) \subseteq J_1:\mathfrak{m}$. It is remarkable that this condition implies the strongest condition, namely that $\omega^*(\omega)$ contain $\mathfrak{m}$, especially since it is not obviously symmetric in J_1 and J_2.

Example 4.3. We can analyze type-2 monomial ideals completely. Let $S = k[x_1, \dots, x_n]$ be a polynomial ring, and assume that I is generated by monomials. It follows that I can be represented as an intersection of irreducible monomial ideals. It is well-known that the only irreducible $\mathfrak{m}$-primary monomials ideals are generated by powers of the variables. Hence we may assume that

$$\begin{aligned} J_1 &= \big(x_1^{c_1}, \dots, x_n^{c_n}\big), \quad J_2 = \big(x_1^{d_1}, \dots, x_n^{d_n}\big),\\ J_1 : J_2 &= \big(x_1^{c_1}, \dots, x_n^{c_n}, x_1^{c_1-d_1} \cdots x_n^{c_n-d_n}\big),\\ J_2 : J_1 &= \big(x_1^{d_1}, \dots, x_n^{d_n}, x_1^{d_1-c_1} \cdots x_n^{d_n-c_n}\big), \end{aligned}$$

where by convention if a variable has negative or zero exponent we drop it from the product. Only in the following cases does $J_1 : J_2 + J_2 : J_1$ contain $\mathfrak{m}$:

(1) For every i, either $c_i = 1$ or $d_i = 1$. By relabeling the variables, we can assume in this case that

$$\begin{aligned} J_1 &= \big(x_1, \dots, x_s, x_{s+1}^{c_{s+1}}, \dots, x_n^{c_n}\big),\\ J_2 &= \big(x_1^{d_1}, \dots, x_s^{d_s}, x_{s+1}, \dots, x_n\big),\\ I &= \big(x_1^{d_1}, \dots, x_s^{d_s}, x_{s+1}^{c_{s+1}}, \dots, x_n^{c_n}, x_i x_j \bigm| i \le s, s+1 \le j\big). \end{aligned}$$

(2) One of the products $x_1^{c_1-d_1} \cdots x_n^{c_n-d_n}$ or $x_1^{d_1-c_1} \cdots x_n^{d_n-c_n}$ has all but one nonpositive exponent, while the remaining exponent (say, the exponent of x_n) is equal to one. Moreover, for all $i = 1, \dots, n-1$, either $c_i = 1$ or $d_i = 1$. Assume that

$c_1 \le d_1, \dots, c_{n-1} \le d_{n-1}$ and $c_n = d_n + 1$. Then the second part of the condition forces $c_1 = \cdots = c_{n-1} = 1$. In this case, the ideals are

$$\begin{aligned} J_1 &= \bigl(x_1, \dots, x_{n-1}, x_n^c\bigr),\\ J_2 &= \bigl(x_1^{d_1}, \dots, x_{n-1}^{d_{n-1}}, x_n^{c-1}\bigr),\\ I &= \bigl(x_1^{d_1}, \dots, x_{n-1}^{d_{n-1}}, x_n^c, x_i x_n^{c-1} \bigm| i = 1, \dots, n-1\bigr). \end{aligned}$$

(3) Both products $x_1^{c_1-d_1} \cdots x_n^{c_n-d_n}$ and $x_1^{d_1-c_1} \cdots x_n^{d_n-c_n}$ have all but one nonpositive exponent, while the remaining exponent is equal to one. For instance, $c_n - d_n = 1$ and $c_i \le d_i$ for all $i \ne n$, $d_{n-1} - c_{n-1} = 1$, $d_i \le c_i$ for all $i \ne n-1$. This means that $x_n = x_1^{c_1-d_1} \cdots x_n^{c_n-d_n}$ and $x_{n-1} = x_1^{d_1-c_1} \cdots x_n^{d_n-c_n}$. The conditions stated above imply that $c_i = d_i$ for all $i < n-1$. In order for x_i to be in $J_1 : J_2 + J_2 : J_1$ we need $c_i = d_i = 1$, thus R is isomorphic to S_0/I_0, where $S_0 = k[x, y]$ and $I_0 = (x^c, y^d, x^{c-1}y^{d-1})$ for some c, d (and thus R is a Teter ring).

5. Monomial ideals of type 3

In this section we classify the type-3 primary monomial ideals I in a polynomial ring $S = k[x_1, \dots, x_n]$ such that $R = S/I$ is Artinian and satisfies one of the conditions of Proposition 1.1. We also prove that in this case the three conditions are equivalent.

The following notations will be used throughout this section: $S = k[x_1, \dots, x_n]$ and $I = J_1 \cap J_2 \cap J_3$, where each $J_i = (x_1^{a_{i1}}, \dots, x_n^{a_{in}})$ is a monomial $\mathfrak{m}$-primary Gorenstein ideal (it is well-known that all monomial $\mathfrak{m}$-primary Gorenstein ideals are of this form). We use $x^{a_i - a_j}$ to denote $\prod x_k^{a_{ik}-a_{jk}}$, where the product runs over the indices $k \in \{1, 2, \dots, n\}$ with $a_{ik} \ge a_{jk}$. Note that we have $J_i : J_j = J_i + (x^{a_i - a_j})$.

We begin by establishing a necessary condition for condition (4) (Lemma 5.1), and a sufficient condition for condition (1) (Lemma 5.3).

Lemma 5.1. *Let $I = J_1 \cap J_2 \cap J_3$ be a monomial type-3 ideal. If $x \in R$ is a monomial such that $x J_i \subseteq I : (I : J_i)$ for all $i \in \{1, 2, 3\}$, then*

$$x \in (I : J_1) + (I : J_2) + (I : J_3).$$

Proof. Assume that $x \notin (I : J_1) + (I : J_2) + (I : J_3)$, but that for all $i = 1, 2, 3$ we have $x J_i \subseteq I : (I : J_i)$.

By Proposition 4.1, we have

$$\begin{aligned} x &\in J_1 : J_2 + J_1 : J_3 + I : J_1,\\ x &\in J_2 : J_1 + J_2 : J_3 + I : J_2,\\ x &\in J_3 : J_1 + J_3 : J_2 + I : J_3. \end{aligned}$$

Because x and all the ideals involved are monomial, we must in fact have:

$$x \in J_1 : J_2 + J_1 : J_3, \qquad x \in J_2 : J_1 + J_2 : J_3, \qquad x \in J_3 : J_1 + J_3 : J_2.$$

Since $x \notin I : J_1$, we may assume that either $x \notin J_2 : J_1$ or $x \notin J_3 : J_1$. Assume, for instance, $x \notin J_2 : J_1$. The second equation implies that $x \in J_2 : J_3$, and it follows that $x \notin J_1 : J_3$ (otherwise x would be in $I : J_3$). Thus the first equation implies $x \in J_1 : J_2$, so $x \notin J_3 : J_2$ (otherwise $x \in I : J_2$), and thus $x \in J_3 : J_1$.

Combining these results, we have

$$x \in (J_2 : J_3) \cap (J_3 : J_1) \cap (J_1 : J_2). \tag{5–1}$$

We cannot have $x \in J_1 + J_2 + J_3$, since, for instance, $x \in J_1$ would imply $x \in J_1 : J_3$.

Similarly, the assumption $x \notin J_3 : J_1$ leads to $x \in (J_3 : J_2) \cap (J_2 : J_1) \cap (J_1 : J_3)$ and $x \notin J_1 + J_2 + J_3$.

Each of these situations is impossible: for instance, relation (5–1) implies that $x = x^{a_1 - a_2} = x^{a_2 - a_3} = x^{a_3 - a_1}$. If x_k is a variable which appears in x with exponent $c_k > 0$, we must have $c_k = a_{1k} - a_{2k} = a_{2k} - a_{3k} = a_{3k} - a_{1k}$, which is clearly impossible. □

On a related note, we have the following general fact:

Proposition 5.2. *Let $I = J_1 \cap J_2 \cap \cdots \cap J_n$ be an $\mathfrak{m}$-primary ideal, with $J_1, \ldots, J_n$ $\mathfrak{m}$-primary Gorenstein ideals. Then $\omega^*(\omega) \subseteq (I : J_1 + \cdots + I : J_n)/I$.*

Proof. Pick $J \subseteq I$ a Gorenstein $\mathfrak{m}$-primary ideal. According to relation (4–1), $\omega \cong (J : J_1 + \cdots + J : J_n)/J$. Since $J : J_i$ is annihilated by J_i in ω, its image under any $f \in \omega^*$ is also annihilated by J_i in S/I, and thus it is contained in $(I : J_i)/I$. □

Lemma 5.3. *Let $I = J_1 \cap J_2 \cap \cdots \cap J_n$ be a type n ideal, with $J_1, \ldots, J_n$ $\mathfrak{m}$-primary Gorenstein ideals. Pick $J \subseteq I$ an $\mathfrak{m}$-primary Gorenstein ideal, and write $J_i = J : f_i$ for some $f_i \in R$. If $u_1, \ldots, u_n \in S$ are such that $u_i f_j \equiv u_j f_i \pmod{J}$ for all $i, j = 1, \ldots, n$, then there is an R-linear function $\phi : \omega \to R = S/I$ defined by $\phi(f_i) = u_i$.*

Proof. If $\alpha_1 f_1 + \ldots \alpha_n f_n \in J$ is a relation on $f_1, \ldots, f_n$ as elements in ω, we must show that $\alpha_1 u_1 + \cdots + \alpha_n u_n \in I$ is a relation on the images in S/I. But this is clear, since $f_i(\alpha_1 u_1 + \cdots + \alpha_n u_n) = u_i(\alpha_1 f_1 + \ldots \alpha_n f_n) = 0 \pmod{J}$. □

Theorem 5.4. *Let $S = k[x_1, \ldots, x_n]$ and let $I = J_1 \cap J_2 \cap J_3$ be a type-3 $\mathfrak{m}$-primary monomial ideal, with $J_1 = \left(x_1^{a_{11}}, \ldots, x_n^{a_{1n}}\right)$, $J_2 = \left(x_1^{a_{21}}, \ldots, x_n^{a_{2n}}\right)$ and $J_3 = \left(x_1^{a_{31}}, \ldots, x_n^{a_{3n}}\right)$. The three conditions in Proposition 1.1 are equivalent for $R = S/I$, and they hold if and only if R is isomorphic to the quotient ring obtained in one of the following situations*:

(a) *I is a Teter ideal, that is, $I = \left(x_1^{a_1}, \ldots, x_n^{a_n}, x_1^{a_1 - 1} \cdots x_n^{a_n - 1}\right)$.*

(b) $J_1 = \left(x_1^a, x_2^b, x_3^{c_3}, \ldots, x_n^{c_n}\right)$,
$J_2 = \left(x_1^{a+1}, x_2, x_3, \ldots, x_n\right)$,
$J_3 = \left(x_1, x_2^{b+1}, x_3, \ldots, x_n\right)$, *with* $n \geq 3$ *and* $a, b, c_3, \ldots, c_n > 1$.

(c) $J_1 = \left(x_1^a, x_2^{b+1}, x_3, \ldots, x_n\right)$,
$J_2 = \left(x_1^{a+1}, x_2^b, x_3, \ldots, x_n\right)$,
$J_3 = \left(x_1, x_2, x_3^{c_3}, \ldots, x_n^{c_n}\right)$.

(d) $J_1 = \left(x_1^a, x_2, \ldots, x_s, x_{s+1}^{b_{s+1}}, \ldots, x_n^{b_n}\right)$,
$J_2 = \left(x_1^{a+1}, x_2, \ldots, x_n\right)$,
$J_3 = \left(x_1, x_2^{c_2}, \ldots, x_s^{c_s}, x_{s+1}, \ldots, x_n\right)$.

(e) $J_1 = \left(x_1, \ldots, x_s, x_{s+1}, \ldots, x_t, x_{t+1}^{c_{t+1}}, \ldots, x_n^{c_n}\right)$,
$J_2 = \left(x_1, \ldots, x_s, x_{s+1}^{b_{s+1}}, \ldots, x_t^{b_t}, x_{t+1}, \ldots, x_n\right)$,
$J_3 = \left(x_1^{a_1}, \ldots, x_s^{a_s}, x_{s+1}, \ldots, x_t, x_{t+1}, \ldots, x_n\right)$.

Proof. First we check that in each of these cases we get an ideal $I = J_1 \cap J_2 \cap J_3$ for which condition (A) in Proposition 1.1 holds.

Case (a) is covered by Teter's result.

Case (b). Take $J = \left(x_1^{a+1}, x_2^{b+1}, x_3^{c_3}, \ldots, x_n^{c_n}\right)$, as well as

$$f_1 = x_1 x_2, \qquad f_2 = x_2^b x_3^{c_3-1} \cdots x_n^{c_n-1}, \qquad f_3 = x_1^a x_3^{c_3-1} \cdots x_n^{c_n-1},$$

so that $J_i = J : f_i$. For each $i \geq 3$, define $\phi_i : \omega_R \to R$ by

$$\phi_i(f_1) = x_i, \qquad \phi_i(f_2) = x_2^b, \qquad \phi_i(f_3) = x_1^a.$$

This is well-defined by Lemma 5.3, since we have: $x_i f_2 \equiv x_2^b f_1 \equiv 0 \pmod{J}$, $x_i f_3 \equiv x_1^a f_1 \equiv 0 \pmod{J}$, and $x_1^a f_2 = x_2^b f_3$. Thus, $x_3, \ldots, x_n \in \omega^*(\omega)$. Further, define $\phi_1 : \omega \to R$ and $\phi_2 : \omega_R \to R$ by

$$\phi_1(f_1) = x_1, \qquad \phi_1(f_2) = x_2^{b-1} x_3^{c_3} - 1 \cdots x_n^{c_n-1}, \qquad \phi_1(f_3) = 0;$$
$$\phi_2(f_1) = x_2, \qquad \phi_2(f_2) = 0, \qquad \phi_2(f_3) = x_1^{a-1} x_3^{c_3-1} \cdots c_n^{c_n-1}.$$

It is not hard to check that $x_i \phi(f_j) \equiv x_j \phi(f_i) \pmod{J}$ for all $i, j = 1, 2, 3$, where ϕ is ϕ_1 or ϕ_2. Thus ϕ_1 and ϕ_2 are well-defined, and $x_1, x_2 \in \omega^*(\omega)$.

Case (c). Take $J = \left(x_1^{a+1}, x_2^{b+1}, x_3^{c_3} \ldots, x_n^{c_n}\right)$, as well as

$$f_1 = x_1 x_3^{c_3-1} \cdots x_n^{c_n-1}, \qquad f_2 = x_2 x_3^{c_3-1} \cdots x_n^{c_n-1}, \qquad f_3 = x_1^a x_2^b,$$

so that $J_i = J : f_i$, and $\omega = (J, f_1, f_2, f_3)/J$. For each $i \geq 3$ define $\phi_i : \omega \to R$ by

$$\phi_i(f_1) = x_1, \qquad \phi_i(f_2) = x_2, \qquad \phi_i(f_3) = x_i.$$

We have $x_1 f_2 \equiv x_2 f_1$, $x_i f_1 \equiv x_1 f_3 \equiv 0 \pmod J$, $x_i f_2 \equiv x_2 f_3 \equiv 0 \pmod J$, and thus ϕ_i is well-defined by Lemma 5.3.

Case (d). Take $J = \left(x_1^{a+1}, x_2^{c_2}, \dots, x_s^{c_s}, x_{s+1}^{b_{s+1}}, \dots, x_n^{b_n}\right)$, as well as

$$f_1 = x_1 x_2^{c_2-1} \cdots x_s^{c_s-1}, \qquad f_2 = x_2^{c_2-1} \cdots x_n^{b_n-1}, \qquad f_3 = x_1^a x_{s+1}^{b_{s+1}-1} \cdots x_n^{b_n-1}.$$

For each $i = 2, \dots, s$ and $j = s+1, \dots, n$ define $\phi_{ij} : \omega \to R$ by

$$\phi_{ij}(f_1) = x_j, \qquad \phi_{ij}(f_2) = x_1^a, \qquad \phi_{ij}(f_3) = x_i.$$

We have $x_1^a f_1 \equiv x_j f_2 \equiv 0 \pmod J$, $x_1^a f_3 \equiv x_i f_2 \equiv 0 \pmod J$, and $x_i f_1 \equiv x_j f_3 \equiv 0 \pmod J$, thus ϕ_{ij} is well-defined, and $x_2, \dots, x_n \in \omega^*(\omega)$. Define $\phi : \omega \to R$ by

$$\phi(f_1) = x_1, \qquad \phi(f_2) = x_{s+1}^{b_{s+1}-1} \cdots x_n^{b_n-1}, \qquad \phi(f_3) = 0.$$

We have $x_{s+1}^{b_{s+1}-1} \cdots x_n^{b_n-1} f_1 \equiv x_1 f_2$ and $x_{s+1}^{b_{s+1}-1} \cdots x_n^{b_n-1} f_3 \equiv x_1 f_3 \equiv 0 \pmod J$, thus ϕ is well-defined, and $x_1 \in \omega^*(\omega)$.

Case (e). Take $J = \left(x_1^{a_1}, \dots, x_n^{c_n}\right)$, together with

$$f_1 = x_1^{a_1-1} \cdots x_t^{b_t-1}, \quad f_2 = x_1^{a_1} \cdots x_s^{a_s-1} x_{t+1}^{c_{t+1}-1} \cdots x_n^{c_n-1}, \quad f_3 = x_{s+1}^{b_{s+1}-1} \cdots x_n^{c_n-1}.$$

For each $i = 1, \dots, s$, $j = s+1, \dots, t$, $k = t+1, \dots, n$, we can define a map $\phi_{ijk} : \omega \to R$ by

$$\phi_{ijk}(f_1) = x_k, \qquad \phi_{ijk}(f_2) = x_j, \qquad \phi_{ijk}(f_3) = x_i.$$

We have $x_k f_2 \equiv x_k f_3 \equiv x_j f_1 \equiv x_j f_3 \equiv x_i f_1 \equiv x_i f_2 \equiv 0 \pmod J$, and therefore ϕ_{ijk} is well-defined, and $x_1, \dots, x_n \in \omega^*(\omega)$.

Now, we show that condition (A) in Proposition 1.1 implies one of the cases (a)–(e) listed in the statement of the theorem.

Whenever we have a variable $x_k \in J_i : J_j$, this implies that either $x_k \in J_i$ or $x_k = x^{a_i - a_j}$. The latter is equivalent to

$$\begin{cases} a_{ik} = a_{jk} + 1, \\ a_{il} \leq a_{jl} \quad \text{for } l \neq k. \end{cases}$$

According to Lemma 5.1, every variable x_k is in one of the ideals $I : J_1$, $I : J_2$, or $I : J_3$. Without loss of generality we may assume $x_1 \in I : J_1$. Therefore, one of the following holds:

$$x_1 \in J_2 \cap J_3, \qquad x_1 = x^{a_3-a_1} \in J_2,$$
$$x_1 = x^{a_2-a_1} \in J_3, \qquad x_1 = x^{a_3-a_1} = x^{a_2-a_1}.$$

We claim that $x_1 = x^{a_3-a_1} = x^{a_2-a_1}$ implies that I is a Teter ideal. We have $a_{31} = a_{21} = a_{11}+1$, and $a_{3k}, a_{2k} \le a_{1k}$ for all $k \ne 1$, and therefore we may assume that $x_k \notin J_1$ for all $k \ne 1$. On the other hand, $x_k \in I : J_i$ for some i. If $i = 1$, it follows that $x_k \in J_2 \cap J_3$, since otherwise we would have $x_k = x_1$.

Since I has type 3, there must be at least two variables, say x_2 and x_3, not in $J_2 \cap J_3$ (otherwise we would have a containment between J_2 and J_3). Without loss of generality, assume that $x_2 = x^{a_1-a_2}$ (in $I : J_2$) and $x_3 = x^{a_1-a_3}$ (in $I : J_3$). It follows that $a_{1k} = a_{2k}$ for all $k \ne 1, 2$, and $a_{1k} = a_{3k}$ for all $k \ne 1, 3$. In particular, all the variables x_k with $k > 3$ have $a_{1k} = a_{2k} = a_{3k}$, and therefore they must be in $J_1 \cap J_2 \cap J_3$ and can be omitted. We are in case (a).

Suppose now that I is not a Teter ideal, thus $x_k \in I : J_i$ implies that $x_k \in J_j + J_l$, where $\{i, j, k\} = \{1, 2, 3\}$.

Assume that we have two distinct variables, say $x_1, x_2 \in I : J_1$, that belong to at most one of J_1, J_2, J_3. We must have either $x_1 = x^{a_2-a_1} \in J_3$ and $x_2 = x^{a_3-a_1} \in J_2$, or vice versa. For $k > 2$, we have $a_{2k}, a_{3k} \le a_{1k}$. If $x_k \in I : J_1$, then $x_k \in J_2 \cap J_3$. If $x_k \in I : J_2$, then $x_k = x^{a_1-a_2}$ (since otherwise $x_k \in J_1$ implies $x_k \in J_1 \cap J_2 \cap J_3$ and we may omit the variable x_k from the presentation of S/I). But this implies that $a_{1l} \le a_{2l}$ for all $l \ne k$, contradicting that $x_2 \in J_2 \setminus J_1$. We thus are in case (b).

From now on suppose that, for each $i \in \{1, 2, 3\}$, there is at most one variable in $I : J_i$ that is not in the intersection of two of the ideals J_1, J_2, J_3.

Consider the case when each of $I : J_1$ and $I : J_2$ contains a variable which belongs to at most one of J_1, J_2, J_3. Without loss of generality, assume that these variables are $x_1 \in I : J_1$ and $x_2 \in I : J_2$.

- If $x_1 = x^{a_3-a_1} \in J_2$ and $x_2 = x^{a_1-a_2} \in J_3$, then $a_{1k} \le a_{2k}$ for all $k \ne 2$. On the other hand, $a_{21} = 1$, thus $a_{11} = 1$, contradicting $x_1 \notin J_1$.
- If $x_1 = x^{a_3-a_1} \in J_2, x_2 = x^{a_3-a_2} \in J_1$, then $a_{3k} \le a_{1k}$ for all $k \ne 1$. On the other hand $a_{12} = 1$, thus $a_{32} = 1$, contradicting $x_2 \notin J_3$.
- If $x_1 = x^{a_2-a_1} \in J_3$ and $x_2 = x^{a_3-a_2} \in J_1$, then $a_{2k} \le a_{1k}$ for all $k \ne 1$. On the other hand, $a_{12} = 1$, thus $a_{22} = 1$, contradicting $x_2 \notin J_2$.
- If $x_1 = x^{a_2-a_1} \in J_3$ and $x_2 = x^{a_1-a_2} \in J_3$, then $a_{1k} = a_{2k}$ for all $k > 2$. Thus, $x_k \in J_1 : J_2$ or $x_k \in J_2 : J_1$ implies that $x_k \in J_1 \cap J_2$. If $x_k \in I : J_3$, it follows that $x_1 x_k \in J_2 \cap J_3$. Since $x_1 \notin (J_2 + J_3)$, we must have $x_k \in J_2 \cap J_3$. Therefore $x_k \in J_1 \cap J_2 \cap J_3$, and we may omit these variables.

Therefore we are in case (c).

Suppose that there is only one variable that belongs to exactly one of J_1, J_2, J_3, say $x_1 = x^{a_2-a_1} \in J_3$, and that all other variables belong to exactly two of J_1, J_2, J_3. Since $a_{2k} \le a_{1k}$ for all $k \ne 1$, we have either $x_k \in J_2 \cap J_1$ or $x_k \in J_2 \cap J_3$. Thus we must be in case (d).

Finally, if every variable belongs to exactly two of J_1, J_2, J_3, we are in case (e). $\square$

References

[Auslander and Buchweitz 1989] M. Auslander and R.-O. Buchweitz, "The homological theory of maximal Cohen–Macaulay approximations", pp. 5–37 in *Colloque en l'honneur de Pierre Samuel* (Orsay, 1987), Mém. Soc. Math. France (N.S.) **38**, Soc. mat. de France, Paris, 1989. MR 91h:13010 Zbl 0697.13005

[Ding 1993] S. Ding, "A note on the index of Cohen–Macaulay local rings", *Comm. Algebra* **21**:1 (1993), 53–71. MR 94b:13014 Zbl 0766.13003

[Teter 1974] W. Teter, "Rings which are a factor of a Gorenstein ring by its socle", *Invent. Math.* **23** (1974), 153–162. MR 51 #3157 Zbl 0276.13018

Received August 24, 2004.

CRAIG HUNEKE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KANSAS
LAWRENCE, KS 66045
UNITED STATES

huneke@math.ku.edu

ADELA VRACIU
DEPARTMENT OF MATHEMATICS
1523 GREENE STREET
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, SC 29208
UNITED STATES

vraciu@math.sc.edu