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## CHARACTERIZATION OF CHEEGER SETS FOR CONVEX SUBSETS OF THE PLANE

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**Given a planar convex domain  $\Omega$ , its Cheeger set  $\mathcal{C}_\Omega$  is defined as the unique minimizer of  $|\partial X|/|X|$  among all nonempty open and simply connected subsets  $X$  of  $\bar{\Omega}$ . We prove an interesting geometric property of  $\mathcal{C}_\Omega$ , characterize domains  $\Omega$  which coincide with  $\mathcal{C}_\Omega$  and provide a constructive algorithm for the determination of  $\mathcal{C}_\Omega$ .**

### 1. Introduction

We consider the following problem: given an open nonempty bounded subset  $\Omega$  of  $\mathbb{R}^2$ , determine

$$(1) \quad h(\Omega) := \min_{X \subset \bar{\partial}\Omega} \frac{|\partial X|}{|X|},$$

where  $|X|$  is the 2-dimensional area (Lebesgue measure) of  $X$  and  $|\partial X|$  is the perimeter in  $\mathbb{R}^2$  (1-dimensional Hausdorff measure) of  $\partial X$ . The minimum is sought among all nonempty *open* and *simply connected* subsets of  $|\partial\Omega|$ .

If the class of admissible domains  $X$  is restricted to smoothly bounded and simply connected ones that are compactly contained in  $\Omega$ , this is known as the *Cheeger problem* for  $\Omega$ , after [Cheeger 1970], and then  $h$  has an infimum but no minimum. It is well-known that the minimum in (1) is attained by a subset of  $\bar{\Omega}$  that touches the boundary  $\partial\Omega$ . Its value  $h(\Omega)$  is called the *Cheeger constant of  $\Omega$* . To fix notation we will call any minimizer of (1) *Cheeger set of  $\Omega$* . For planar convex domains it is known (see the beginning of Section 2) that there is only one open Cheeger set, which is again convex. Therefore we may speak of *the* Cheeger set of  $\Omega$  and denote it by  $\mathcal{C}_\Omega$ .

Finding the Cheeger sets of a given  $\Omega$  is usually considered a difficult task. Only some particular cases seem to be explicitly known. If  $\Omega$  is a disk or an annulus, then  $\mathcal{C}_\Omega = \Omega$  and this equality is true also for an ellipse of moderate eccentricity (until recently there was only numerical evidence [Demengel et al. 2002]). If  $\Omega$  is

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a triangle or rectangle,  $\mathcal{C}_\Omega$  can be obtained from  $\Omega$  by “rounding all the corners”. However, this is not true for a more general polygon, even a convex one, except in special cases which we shall call *Cheeger-regular*.

In this paper, we give a constructive method to compute the Cheeger set of any convex polygon. Our direct algebraic algorithm has a finite number of steps not exceeding the number of sides of the polygon. In particular we give a direct characterization of those convex polygons whose Cheeger set can be obtained by rounding all of their corners. For those polygons we derive an explicit formula for the Cheeger constant  $h(\Omega)$ .

Finally we derive a characterization of those convex sets  $\Omega$  for which  $\mathcal{C}_\Omega = \Omega$ . These are exactly the sets satisfying  $\bar{\kappa} |\Omega| \leq |\partial\Omega|$ , where  $\bar{\kappa}$  is the maximum value of the curvature of the boundary. From this criterion one deduces easily the well known fact that  $\mathcal{C}_\Omega \neq \Omega$ , if  $\partial\Omega$  is not of class  $C^1$  (since then  $\bar{\kappa}$  is infinite). On the other hand our characterization also proves  $\mathcal{C}_\Omega = \Omega$  for the disk, for a stadium domain, for an ellipse of moderate eccentricity, etc.

At the end of the paper we discuss the case of more general sets  $\Omega$ , where the situation is much more complicated.

## 2. The inner Cheeger set

In this section  $\Omega$  is convex. In this situation it follows from [Stredulinsky and Ziemer 1997, Theorem 3.32(i)] that  $C := \mathcal{C}_\Omega$  is convex and uniquely defined as the union of a set of disks of suitable radius. In [Kawohl and Fridman 2003, Remark 12] this theorem was erroneously referred to as Theorem 3.14, a number from a preprint version of [Stredulinsky and Ziemer 1997], and the suitable radius was identified as  $1/h(\Omega)$ .

From a scaling argument one can easily see that the boundary of  $C$  touches the boundary of  $\Omega$ . Moreover, a variational argument shows that the inner part (if any)  $\partial C \cap \Omega$  of  $\partial C$  is composed of circular arcs, all of them of radius  $r := 1/h(\Omega)$ .

For a given  $\Omega$ , and any  $x \in |\partial\Omega|$ , we denote the distance function to the boundary by  $\text{dist}(x, \partial\Omega)$ , and for  $t \geq 0$  we denote points of distance at least  $t$  by

$$\Omega^t := \{ x \in \Omega \mid \text{dist}(x, \partial\Omega) > t \}.$$

So  $\partial\Omega^t$  is the inner parallel set to  $\partial\Omega$  at distance  $t$ . With these definitions we have:

**Theorem 1.** *There exists a unique value  $t = t^* > 0$  such that  $|\Omega^t| = \pi t^2$ . Then  $h(\Omega) = 1/t^*$  and the Cheeger set of  $\Omega$  is  $\mathcal{C}_\Omega = \Omega^{t^*} + t^* B_1$ , with  $B_1$  denoting the unit disk.*

In this statement and throughout this paper the addition of sets is the Minkowski addition, that is  $A + B := \{a + b; a \in A, b \in B\}$ .

The theorem states that the parallel set  $\Omega^{t^*}$  must have the same area as the ball of radius  $t^*$ . We will see in the proof that  $\Omega^{t^*}$  is also the parallel set  $C^{t^*}$  of the Cheeger set. Since the set  $\Omega^{t^*}$  plays such a crucial role we give it a name: the *inner Cheeger set* of  $\Omega$ .

*Proof.* (After this manuscript was completed we were kindly informed by V. Caselles that Theorem 1 can also be deduced from [Alter et al. 2005b, Lemma 6, Theorem 3, Proposition 2 and Corollary 1]. For the reader's convenience we present our proof anyway.)

If we define  $f(t) := |\Omega^t|$  as a function of  $t \geq 0$ , then we have  $f(0) = |\Omega| > 0$  and  $f(t) \equiv 0$  for  $t \geq R$ , where  $R$  is the inradius of  $\Omega$ . Moreover, since  $\Omega^t \subset \Omega^{t'}$  if  $t > t'$ ,  $f$  is decreasing on  $[0, R]$ . Consequently, since the function  $g(t) = \pi t^2$  is increasing, there is a unique  $t = t^* \in (0, R)$  such that  $f(t) = g(t)$ .

Set  $r := 1/h(\Omega)$ . We prove that  $r = t^*$ . We consider now  $C^r := \{x \in C \mid \text{dist}(x, \partial C) > r\}$ , that is the inner set  $C^r$  of distance  $r$  to  $\partial C = \partial \mathcal{C}_\Omega$ . Since  $C$  is a union of disks of radius  $r$  contained in  $\Omega$  (as mentioned at the beginning of this section), we have  $C = C^r + rB_1$ . In particular this implies the relations for (exterior) parallel sets at distance  $r$  known as *Steiner's formulae*:

$$(2) \quad \begin{aligned} |C| &= |C^r| + r|\partial C^r| + \pi r^2, \\ |\partial C| &= |\partial C^r| + 2\pi r. \end{aligned}$$

Since  $C$  is the Cheeger set of  $\Omega$ , we have on the other hand by definition  $\frac{1}{r} = h(\Omega) = |\partial C|/|C|$  or equivalently  $|C| = r|\partial C|$ . Inserting (2) here yields  $|C^r| = \pi r^2$ . It remains to show that  $|\Omega^r| = \pi r^2$ .

Now since  $C = C^r + rB_1 \subset \Omega$ , one sees for any  $x \in C^r$  that  $\delta(x) \geq \text{dist}(x, \partial C) > r$ , therefore  $C^r \subset \Omega^r$ . Conversely, if  $x \in \Omega^r$ , then the disk centered at  $x$  with radius  $r$  is contained in  $\Omega$  and hence in  $C$ . But this implies that  $x \in C^r$ . Therefore we have shown that  $C^r = \Omega^r$  and thus  $|\Omega^r| = |C^r| = \pi r^2$ .  $\square$

### 3. Characterization of trivial cheeger sets

We are now in a position to characterize those convex sets  $\Omega$  for which  $\Omega = \mathcal{C}_\Omega$ . In this case the explicit determination of the Cheeger set is trivial.

We recall that for any convex set  $\Omega$ , we can define the curvature  $\kappa$  on  $\partial\Omega$  as a Radon measure. We will denote by  $\bar{\kappa}$  its maximum value in  $[0, +\infty]$  and note that  $\bar{\kappa}$  is finite only if  $\Omega$  is of class  $C^1$  (or has no corners). To make things more precise, we describe how  $\kappa$  and  $\bar{\kappa}$  can be precisely defined. For a convex domain  $\Omega$  with  $C^2$  boundary  $\partial\Omega$ , the boundary is a simple curve homeomorphic to the circle  $\mathbf{S}^1$ . It can be parametrized as an application  $t \in \mathbf{S}^1 \mapsto x(t) = (x_1(t), x_2(t))$ :

$$(3) \quad x_1(t) = p(t) \cos t - p'(t) \sin t, \quad x_2(t) = p(t) \sin t + p'(t) \cos t.$$

Here  $p(t)$  is the support function of  $\Omega$ , that is,

$$(4) \quad p(t) = \sup_{x \in \Omega} (x_1 \cos t + x_2 \sin t).$$

It follows from the definition of  $p$  that the vectors  $x'(t)$  and  $(-\sin t, \cos t)$  are collinear. Hence the curvature radius satisfies  $\rho(t) = p(t) + p''(t) = |x'(t)| > 0$ . The curvature is given by  $\kappa(t) = 1/\rho(t)$ . This is how the curvature is defined in the regular case.

Conversely, given a map  $p : \mathbf{S}^1 \rightarrow \mathbb{R}$  satisfying  $p + p'' \geq 0$  we can find a convex domain  $\Omega$  whose support function is  $p$ . Moreover this can be done even if  $p$  is not twice differentiable, assuming only that the inequality  $p + p'' \geq 0$  is satisfied in a distributional sense (which implies that it is a Radon measure on  $\mathbf{S}^1$ ). Indeed consider  $\Omega := \bigcap_{t \in \mathbf{S}^1} H_t$  where  $H_t$  is the half-plane  $\{x : x_1 \cos t + x_2 \sin t \leq p(t)\}$ . Then  $p$  satisfies (4) and, for any given  $t$ , the supremum is attained at a unique point  $x(t)$  if and only if  $p$  is differentiable for this  $t$ ; and then  $x(t)$  satisfies (3).

Thus there is a one-to-one relationship between convex domains and Radon measures  $\rho = p + p''$  on  $\mathbf{S}^1$ . This is the generalized curvature radius that we consider here. Again we define  $\kappa(t) = 1/\rho(t)$ , with values in  $[0, +\infty]$ . If it is not a bounded function, we define  $\bar{\kappa} := +\infty$ ; otherwise  $\bar{\kappa}$  is the pointwise supremum of the Lebesgue precise representative of  $\kappa$ . One of the more crucial properties we will need later on is the following: if  $\bar{\kappa}$  is finite, then for every point  $x \in \partial\Omega$ , there is a ball of radius  $1/\bar{\kappa}$  contained in  $\Omega$ , with  $x$  on its boundary. This follows easily from the fact that  $p + p'' \geq 1/\bar{\kappa}$  in the sense of measures implies  $p(t) \geq 1/\bar{\kappa} + a \sin(t - t_0)$  on  $\mathbf{S}^1$ , for some constants  $a, t_0$ . But the right hand side here is just the support function of a ball with radius  $1/\bar{\kappa}$ .

Having specified what we mean by the maximum of the curvature, we return to Cheeger sets.

**Theorem 2.** *Let  $\Omega$  be any convex set,  $\bar{\kappa}$  the maximum value of its curvature. Then  $\mathcal{C}_\Omega = \Omega$  if and only if*

$$(5) \quad \bar{\kappa} |\Omega| \leq |\partial\Omega|.$$

**Remark.** Recently we learned from V. Caselles and A. Chambolle, that in a different context, the sets  $\Omega$  for which  $\Omega = \mathcal{C}_\Omega$  are called *calibrable sets*. In fact, a characterization of these sets was given in [Bellettini et al. 2002], although the word Cheeger did not appear there. Theorem 4 in that reference can be interpreted as a variant of our Theorem 2, but its proof uses different and more technical methods. Moreover we learned from A. A. Kosmodem'yanskii that the “only if” part of our Theorem 2 was already stated in [Finn and Kosmodem'yanskii 2002, Theorem 3].

*Proof.* Let  $\rho := 1/\bar{\kappa}$  be the smallest curvature radius.

Assume first that (5) holds, so that  $\rho \geq |\Omega|/|\partial\Omega| > 0$ . For each  $x \in \partial\Omega$ , there is a ball  $B_\rho$  of radius  $\rho$  contained in  $\Omega$  with  $x$  on  $\partial B_\rho$ . It is well known that formulas similar to (2) actually hold for any  $t \in [0, \rho]$  (see [Bandle 1980, Section I.1.4], for example), namely

$$(6) \quad |\Omega^t| = |\Omega| - t|\partial\Omega| + \pi t^2, \quad |\partial\Omega^t| = |\partial\Omega| - 2\pi t.$$

Note that these formulas are no longer true for  $t > \rho$ . Define  $t_0 := |\Omega|/|\partial\Omega|$ . From (5) we have  $t_0 \leq \rho$ . This implies that the first equation in (6) holds in particular for  $t_0$ , so that  $|\Omega^{t_0}| = |\Omega| - t_0|\partial\Omega| + \pi t_0^2 = \pi t_0^2$  by definition of  $t_0$ . So from the previous theorem,  $t_0 = 1/h(\Omega)$ . Hence  $\Omega = \mathcal{C}_\Omega$ .

Conversely, assume that  $\Omega = \mathcal{C}_\Omega$ : Then again from the previous theorem we get  $\Omega = \Omega^{t^*} + t^*B_1$  with  $t^* = 1/h(\Omega) = |\Omega|/|\partial\Omega|$ . But this implies that for each  $x \in \partial\Omega$ , there is a ball  $B_{t^*}$  of radius  $t^*$  contained in  $\Omega$  with  $x$  on  $\partial B_{t^*}$ . In other words,  $\rho \geq t^*$  or  $\bar{\kappa} \leq 1/t^* = h(\Omega)$  in this case, that is (5) holds.  $\square$

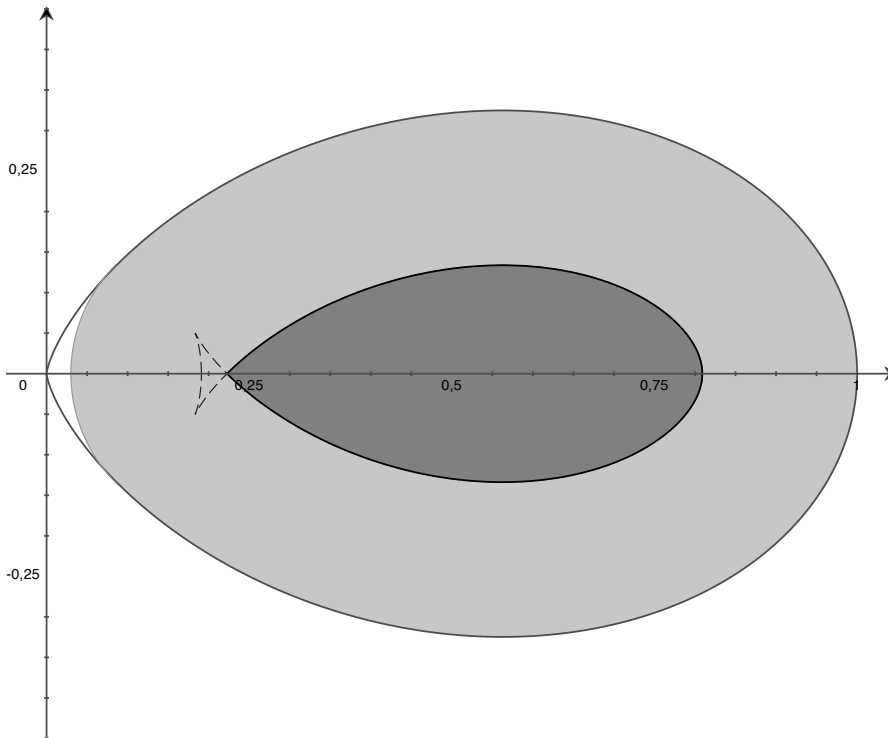
As an application of Theorem 2, we check that for an ellipse  $\mathcal{E}$  with sufficient small eccentricity, we have  $\mathcal{E} = \mathcal{C}_\mathcal{E}$ . Indeed, if  $2a$  and  $2b$  are the lengths of the axes of  $\mathcal{E}$ , with  $a \geq b$ , then  $\bar{\kappa} = a/b^2$  and  $|\mathcal{E}| = \pi ab$ . So (5) is equivalent to  $|\partial\mathcal{E}| \geq \pi a/b^2$ . This is true for an ellipse with eccentricity smaller than a critical value of about 0.79117.

Here is a more complicated application, to show how one can use the results in Theorem 1. We consider the “ovoid curve” with equation, in polar coordinates  $(\rho, \theta)$ ,  $\rho = a \cos^3 \theta$ ,  $\theta \in [-\pi/2, \pi/2]$ ,  $a > 0$  given; it encloses an egg-shaped convex domain  $\Omega$  (see Figure 1 where  $a = 1$ ).

Even though the ovoid is  $C^1$ , its curvature is infinite at 0 (actually the curvature is increasing from right to left), so we know from Theorem 2 that  $\mathcal{C}_\Omega \subsetneq \Omega$  in that case.

The curve is easily parametrized as  $x(\theta) = (a \cos^4 \theta, a \cos^3 \theta \sin \theta)$ , so we can get the outward unit normal vector  $\vec{v}(\theta)$  in a straightforward way. (We do not give the details of the calculations here, but a Maple worksheet is available [Lachand-Robert  $\geq$  2006].)

Now for  $t > 0$ ,  $y(\theta) = x(\theta) - t\vec{v}(\theta)$  is a parametrization of the parallel curve to  $\partial\Omega$  at distance  $t$ , which in this case has the shape of a stylized fish, as seen in the figure. Only the “body” of the fish is the inner parallel set  $\Omega^t$ ; so it is given by the parametrization above, with  $\theta \in [-\theta_0, \theta_0]$ , where  $\theta_0 > 0$  is the first root of the equation  $y_2(\theta) = 0$ . This equation reduces to a third order polynomial one with respect to  $\cos^2 \theta$ , so  $\theta_0$  is given explicitly from the Cardano formula. Now computing the area of  $\Omega^t$  amounts to integrating  $\frac{1}{2}y \wedge y'$  for  $\theta \in [-\theta_0, \theta_0]$ . This gives a (complicated) integral of elliptic type, which can be evaluated numerically



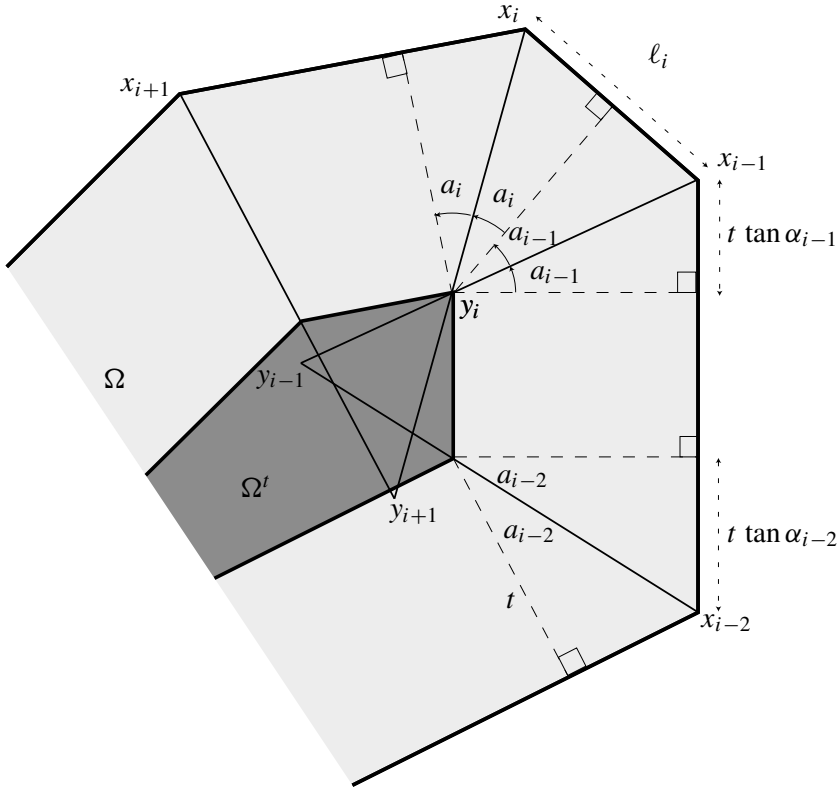
**Figure 1.** The ovoid domain  $(x^2 + y^2)^2 < x^3$ , its Cheeger set (light gray), its inner Cheeger set (dark gray), and the curve parallel to its boundary (dashed).

with arbitrary precision. Now finding numerically the root of  $|\Omega^t| = \pi t^2$  is easy, and we get  $t^* \simeq 0.190648$ ; that is  $h(\Omega) \simeq 5.245270$  (for  $a = 1$ ). This is to be compared to the ratio  $|\partial\Omega|/|\Omega|$  which is given by an explicit formula (with elliptic functions) and turns out to be approximately 5.258384.

#### 4. Convex polygons

We restrict ourselves to convex polygons in this section, and see some consequences of Theorem 1. We denote the vertices of the polygon  $\Omega$  by  $x_0, x_1, \dots, x_{n-1}, x_n = x_0$ , ordered counterclockwise. The outward normal unit vector on the side  $[x_{i-1}, x_i]$  has a constant value  $(\cos \phi_i, \sin \phi_i)$  in an orthonormal basis. The angle  $\alpha_i$  denotes half the jump of the angle  $\phi$  at  $x_i$ , that is  $\alpha_i := (\phi_{i+1} - \phi_i)/2$ ; hence  $\alpha_i \in (0, \pi/2)$  and the inner angle of the polygon at  $x_i$  is  $\pi - 2\alpha_i$  (see Figure 2). To shorten the notation, we will also call  $\ell_i := |x_i - x_{i-1}|$  the length of the  $i$ -th side of the polygon (so  $|\partial\Omega| = \sum_{i=1}^n \ell_i$ ). Most of the computations in what follows will





**Figure 2.** (Part of) a polygon  $\Omega$ , and its parallel inner set  $\Omega^t$ , here in the critical case  $t = \ell_i / (\tan \alpha_{i-1} + \tan \alpha_i)$ .

involve the perimeter and area of  $\Omega$ , but also the quantity

$$(7) \quad T(\Omega) := \sum_{i=1}^n \tan \alpha_i.$$

We recall that  $\sum_{i=1}^n \alpha_i = \pi$  (since  $\phi_n = 2\pi + \phi_0$ ). This implies  $T(\Omega) > \pi$ , taking into account that  $\alpha_i \in (0, \pi/2)$  and  $\tan x > x$  for any  $x \in (0, \pi/2)$ .

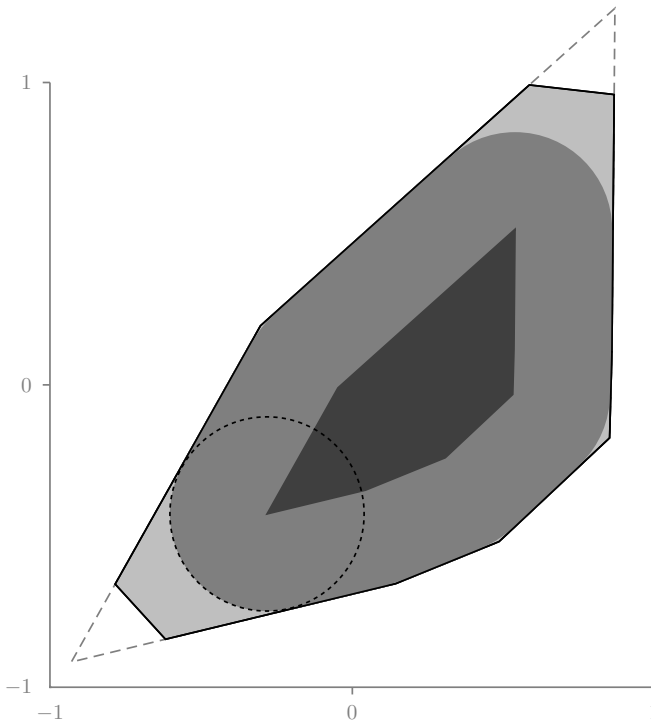
Since the Cheeger set  $\mathcal{C}_\Omega$  has a  $C^{1,1}$  boundary consisting of circular arcs and line-segments, it is certainly different from the polygon. It turns out that there are two possibilities.

The first possibility is that  $\mathcal{C}_\Omega$  can be obtained from  $\Omega$  by rounding all the corners (a process we will describe more precisely later on in this section). In that case the boundary of the Cheeger set touches each side of  $\Omega$ , and the inner Cheeger set is a polygon having the same number of sides as  $\Omega$  (with possibly some sides of length zero, though). This is what happens e.g. for a triangle or a rectangle.

A polygon  $\Omega$  with this property will be called *Cheeger-regular*, and the Cheeger constant can be computed explicitly for these, using the area, perimeter and  $T(\Omega)$ .

The second possibility is that the Cheeger set  $\mathcal{C}_\Omega$  does not touch all the sides of  $\Omega$ , and then the inner Cheeger set is a polygon with  $p < n$  sides. This happens even for quadrilaterals when one side is considerably smaller than the others, and it turns out that in that case the Cheeger constant cannot be directly computed from the characteristic values for  $\Omega$ , since  $\mathcal{C}_\Omega$  is also the Cheeger set of many other  $n$ -sided polygons; see Figure 3. For this polygon  $h(\Omega) = 3.086741237$  while  $|\partial\Omega|/|\Omega| = 3.212325451$ . According to Theorem 1, the area of the dashed disc is the same as the area of the inner Cheeger set. Note that only after we enlarge the polygon does it become Cheeger-regular.

Such an  $\Omega$  will be called *Cheeger-irregular*, and we can easily distinguish between regular and irregular domains by using the following theorem, which provides also an explicit computation of the Cheeger constant and a construction of the Cheeger set, provided  $\Omega$  is Cheeger-regular. The construction of Cheeger sets



**Figure 3.** A polygon  $\Omega$  (light gray), its Cheeger set  $\mathcal{C}_\Omega$  (medium gray) and its inner Cheeger set  $\Omega^{t^*}$  (dark gray), as computed by the algorithm given in Section 5. For this polygon  $h(\Omega) = 3.086741237$  while  $|\partial\Omega|/|\Omega| = 3.212325451$ .

for Cheeger-irregular sets will be reduced to the one for Cheeger-regular sets in the subsequent section.

**Theorem 3.** *A polygon  $\Omega$  is Cheeger-regular (that is, its Cheeger set touches every side of  $\Omega$ ) if and only if*

$$(8) \quad |\Omega| - r_0 |\partial\Omega| + r_0^2 (T(\Omega) - \pi) \leq 0$$

where  $T(\Omega)$  was defined in (7) and  $r_0$  is given by

$$(9) \quad r_0 := \min_{1 \leq i \leq n} \frac{\ell_i}{\tan \alpha_i + \tan \alpha_{i-1}}.$$

In that case, the area and perimeter of  $\mathcal{C}_\Omega$  are given by

$$(10) \quad \begin{aligned} |\partial\mathcal{C}_\Omega| &= |\partial\Omega| - 2(T(\Omega) - \pi)r, \\ |\mathcal{C}_\Omega| &= |\Omega| - (T(\Omega) - \pi)r^2 = r|\partial\mathcal{C}_\Omega|, \end{aligned}$$

where  $r = 1/h(\Omega)$  is the smaller root of  $(T(\Omega) - \pi)r^2 - r|\partial\Omega| + |\Omega| = 0$ . Thus

$$(11) \quad \begin{aligned} h(\Omega) &= \frac{1}{r} = \frac{2(T(\Omega) - \pi)}{|\partial\Omega| - \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}} \\ &= \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 - 4(T(\Omega) - \pi)|\Omega|}}{2|\Omega|}. \end{aligned}$$

This formula for the Cheeger constant of these particular polygons is not new — it can be found in [Brooks and Waksman 1987] — but the characterization (8) was apparently unknown.

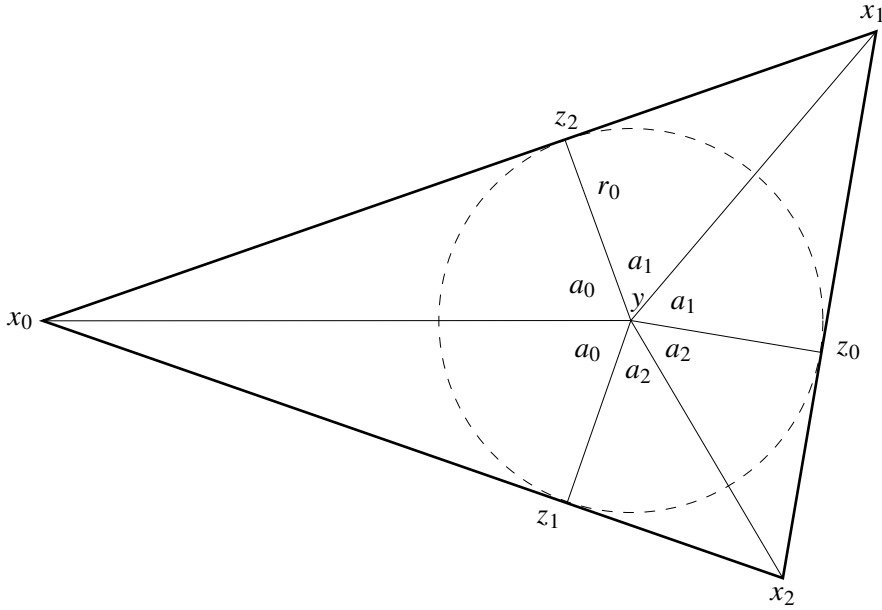
It is easy to deduce from this theorem that the rectangle  $(-a, a) \times (-b, b)$  (with  $b \geq a$ ) is Cheeger-regular, since in that case  $T(\Omega) = 4$ ,  $r_0 = b$ , and the left hand side of (8) is  $(3 - \pi)b^2 < 0$ . This gives

$$h((-a, a) \times (-b, b)) = \frac{4 - \pi}{a + b - \sqrt{(a - b)^2 + \pi ab}}.$$

In particular for a square with side  $2a$  we get the well-known result (see [Kawohl 1990] or [Kawohl and Fridman 2003], for example)

$$(12) \quad h((-a, a)^2) = \frac{2 + \sqrt{\pi}}{a}.$$

For a triangle, the three bisecting lines cross at the center  $y$  of the inscribed disk. Hence all three quantities defining  $r_0$  in (9) are the same, and  $r_0$  is the radius of this disk (Figure 4). Using the three projections  $z_i$  of  $y$  on the sides, we divide the triangle into six smaller right triangles  $x_0yz_2$ , etc. Since  $|x_0 - z_2| + |z_2 - x_1| =$



**Figure 4.** A triangle and its inscribed circle.

$|x_0 - x_1|$ , etc., we see that  $|\Omega| = r_0|\partial\Omega|/2$ , a well-known formula in geometry, and also that

$$|\partial\Omega| = \sum_{i=0}^2 (|x_i - z_{i+2}| + |z_{i+2} - x_{i+1}|) = \sum_{i=0}^2 r_0(\tan \alpha_i + \tan \alpha_{i+1}) = 2r_0T(\Omega).$$

Hence  $T(\Omega) = |\partial\Omega|/(2r_0) = |\Omega|/r_0^2$ . Substituting in (8) yields  $-\pi r_0^2 \leq 0$  which is obviously true. Substituting in (11) yields for the Cheeger constant

$$h(\Omega) = \frac{|\partial\Omega| + \sqrt{4\pi|\Omega|}}{2|\Omega|},$$

a formula already given in [Brooks and Waksman 1987]. Incidentally, the proof above for a triangle works for any polygon circumscribed on a circle, except that the number of sides is larger, and then it recovers known results from [Brooks and Waksman 1987].

*Proof of Theorem 3.* Write  $A := |\Omega|$ ,  $L := |\partial\Omega|$  and  $T := T(\Omega)$  for short.

A bit of geometry with the help of Figure 2 shows that for a sufficiently small  $t$ , the inner parallel set  $\Omega^t$  is a  $n$ -sided polygon, whose sides are parallel to those of  $\Omega$ , and that

$$(13) \quad A = |\Omega^t| + t|\partial\Omega^t| + t^2T, \quad L = |\partial\Omega^t| + 2tT.$$

If  $y_i$  is the intersection of the bisecting lines of the inner angles at  $x_i$  and at  $x_{i-1}$  then  $\Omega^t$  has exactly  $n$  sides if and only if  $\text{dist}(y_i, [x_{i-1}, x_i]) > t$  for all  $i$ . The latter distance is easily computed to be  $\ell_i/(\tan \alpha_i + \tan \alpha_{i-1})$ , so this condition gives  $t < r_0$ , where  $r_0$  was defined in (9). Under this condition the formulas above for  $A$  and  $L$  are valid, and they are still true for the limit case  $t = r_0$ . For  $t > r_0$ , by contrast, one of the sides is “lost”, and one should write strict inequalities instead.

So if we assume that  $t \leq r_0$ , we have  $|\Omega^t| = A - tL + t^2T$  using (13). From Theorem 1, we know that if this value is not larger than  $\pi t^2$ , then  $t \geq t^* = 1/h(\Omega)$ . So if the inequality (8) holds, then  $t^* \leq r_0$ . We then deduce the area and perimeter of the inner set from (13) and then we get (10) using (2).

The value of  $r = t^* = 1/h(\Omega)$  can now be deduced from the equation  $|\mathcal{C}_\Omega| = r|\partial\mathcal{C}_\Omega|$  which gives the second order equation stated in the Theorem. Since the map  $f(t) := (T(\Omega) - \pi)t^2 - t|\partial\Omega| + |\Omega|$  satisfies  $f(r_0) \leq 0$  by (8), has two positive roots because  $T(\Omega) > \pi$  and satisfies  $\lim_{t \rightarrow \infty} f(t) = +\infty$ , we see that one of the roots is larger than  $r_0$ . Therefore  $r$  must be the smaller root. This yields (11).  $\square$

### 5. An algorithm to find the Cheeger set of a convex polygon

We now consider a general (possibly Cheeger-irregular) convex polygon  $\Omega$  again. We shall deduce from Theorem 3 an explicit algorithm to find the Cheeger set and the Cheeger constant for such general sets. Indeed, if condition (8) is satisfied, then we are done and the Cheeger set is given by rounding each corner with radius  $r = 1/h(\Omega)$ , given by (11).

On the other hand, if (8) does not hold for  $\Omega$ , we define a new polygon  $\tilde{\Omega}$  as in Figure 3. In this example it was necessary to extend the polygon twice (dashed lines) since two of the sides were too short to be touched by  $\mathcal{C}_\Omega$ . Let  $i_0$  be some index where the minimum in (9) is attained. For each  $i$  we call  $E_i$  the half space containing  $\Omega$  whose boundary is the straight line through  $x_{i-1}$  and  $x_i$ . Therefore  $\Omega = \bigcap_{i=1}^n E_i$ . Now we define  $\tilde{\Omega} := \bigcap_{i \neq i_0} E_i$ . It is then easy to check that the  $n - 1$  vertices  $\tilde{x}_i$  of  $\tilde{\Omega}$  are given as follows:  $\tilde{x}_i = x_i$  if  $i < i_0$ ,  $\tilde{x}_i = x_{i+1}$  if  $i > i_0$ , and  $\tilde{x}_{i_0} \notin \Omega$  is such that  $x_{i-1} \in [x_{i-2}, \tilde{x}_{i_0}]$ ,  $x_i \in [\tilde{x}_{i_0}, x_{i+1}]$  (see Figure 3, where  $\tilde{x}_{i_0}$  is the intersection of dashed lines). Similarly,  $\tilde{\alpha}_i = \alpha_i$  or  $\tilde{\alpha}_{i+1}$  if  $i \neq i_0$ , but  $\tilde{\alpha}_{i_0} = \alpha_{i-1} + \alpha_i$ . This implies in particular

$$(14) \quad \tilde{T} = T - \tan \alpha_{i_0} - \tan \alpha_{i_0+1} + \tan(\alpha_{i_0} + \alpha_{i_0+1}) > T,$$

taking into account that  $\alpha_i \in (0, \pi/2)$  for all  $i$  and

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} > \tan x + \tan y$$

for any  $x, y \in (0, \pi/2)$ . (Here and in the following,  $T := T(\Omega)$ ,  $A := |\Omega|$ , and  $L := |\partial\Omega|$ , while  $\tilde{T}$  and so on are the corresponding values for  $\tilde{\Omega}$ .)

The lengths  $\tilde{\ell}_i$  are also the same as the  $\ell_i$ , except that

$$\begin{aligned}\tilde{\ell}_{i_0-1} &= \ell_{i_0-1} - r_0 \tan \alpha_{i_0-1} + r_0 \tan \tilde{\alpha}_{i_0} > \ell_{i_0-1}, \\ \tilde{\ell}_{i_0} &= \ell_{i_0} - r_0 \tan \alpha_{i_0} + r_0 \tan \tilde{\alpha}_{i_0} > \ell_{i_0}.\end{aligned}$$

This implies in particular that  $\tilde{r}_0 \geq r_0$ .

The characteristic property of  $\tilde{\Omega}$  is that it is an  $(n-1)$ -sided polygon containing  $\Omega$ , and such that  $\Omega^{r_0} = \tilde{\Omega}^{r_0}$ . Consequently we have

$$\begin{aligned}\tilde{A} &= |\Omega^{r_0}| - r_0 |\partial \Omega^{r_0}| + r_0^2 \tilde{T} = A + r_0^2 (\tilde{T} - T), \\ \tilde{L} &= |\partial \Omega^{r_0}| + 2r_0 \tilde{T} = L + 2r_0 (\tilde{T} - T).\end{aligned}$$

Finally,  $\mathcal{C}_{\tilde{\Omega}} = \mathcal{C}_{\Omega}$  by construction. So we can now check whether (8) holds for the new polygon (every tilded value being computed from the initial one as explained). If it holds, we are done; otherwise we restart the same process again.

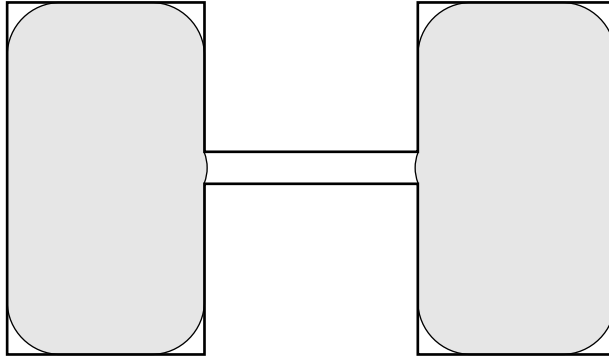
Note that the number of steps is limited to at most  $n-3$ , since in each step the number of corners is reduced by one and every triangle is Cheeger-regular. Also every step can be done with arbitrary precision since all the calculations are algebraic. There are also related numerical experiments that use total variation flow and methods from partial differential equations; see [Alter et al. 2005b] for details.

One can think of this algorithm also in terms of the distance function  $\delta(x) := \text{dist}(x, \partial \Omega)$ , whose graph consists of planar faces. The set of points where  $\delta$  is not differentiable is usually called *ridge of  $\Omega$* . For convex polygons  $\Omega$  the ridge contains line segments that emanate from each corner. They merge in at most  $n-3$  points at certain heights. Whenever they merge at a height below the peak, the set  $\Omega$  has to be enlarged by dropping a face. In that case  $r_0$  is the height of  $\delta$  at the merging point.

This algorithm can also be used to find an approximation for the Cheeger set of an arbitrary planar convex set  $\Omega$ . It suffices to replace  $\Omega$  by an approximating polygon  $P$  such that  $P \subset \Omega \subset (1+\varepsilon)P$  for  $\varepsilon > 0$  small enough. Note that for planar convex domains the Cheeger set depends monotonically on  $\Omega$ , but that the monotone dependence is not strict in the sense that  $\Omega \subset \tilde{\Omega}$  implies  $\mathcal{C}_{\Omega} \subset \mathcal{C}_{\tilde{\Omega}}$ , but  $\mathcal{C}_{\Omega} = \mathcal{C}_{\tilde{\Omega}}$  does not necessarily imply  $\Omega = \tilde{\Omega}$ .

## 6. Remarks on the general case

Many of the statements in this paper break down if  $\Omega$  is no longer convex or two-dimensional. For convex polyhedra in higher dimensions  $\mathcal{C}_{\Omega}$  is bounded by planar faces and by surfaces of constant mean curvature  $h(\Omega)$ . So in contrast to the planar case a maximal union of balls inside a cube will not give us the Cheeger set.

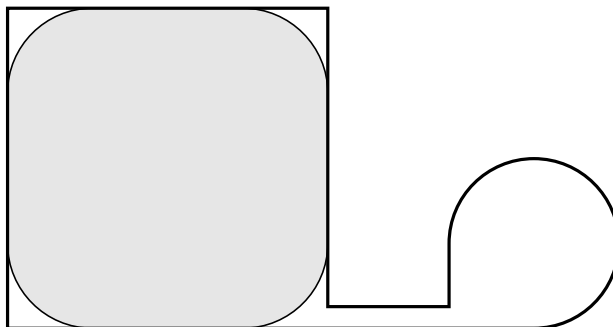


**Figure 5.** An example of nonuniqueness; the shaded set and each component are all Cheeger sets of the barbell.

Some numerical results in this direction have been obtained in [Lachand-Robert and Oudet 2005].

Formulas (2) as well as the  $C^1$  regularity of Cheeger sets  $\mathcal{C}_\Omega$  can break down if  $\Omega$  is no longer convex and has reentrant corners as in an  $L$ -shaped domain. The uniqueness and monotone dependence of Cheeger sets is in general violated when  $\Omega$  is not convex, as can be seen from barbells like in Figure 5. The small channel joining the two parts forbids any Cheeger set to pass through, since this would imply a much higher perimeter with only a small gain in area. Therefore there exists more than one connected Cheeger set, and in that case it is easy to check that their union is also a (disconnected) Cheeger set.

For convex planar domains the Cheeger set  $\mathcal{C}_\Omega$  is the union of all balls of radius  $\rho := 1/h(\Omega)$  contained in  $\Omega$ , and in Figure 5 this is still true. Figure 6 illustrates that for nonconvex planar domains the union of balls of radius  $\rho := 1/h(\Omega)$  is not always a Cheeger set. The domain it illustrates is essentially the union of a unit



**Figure 6.** A counterexample to the union of balls property.

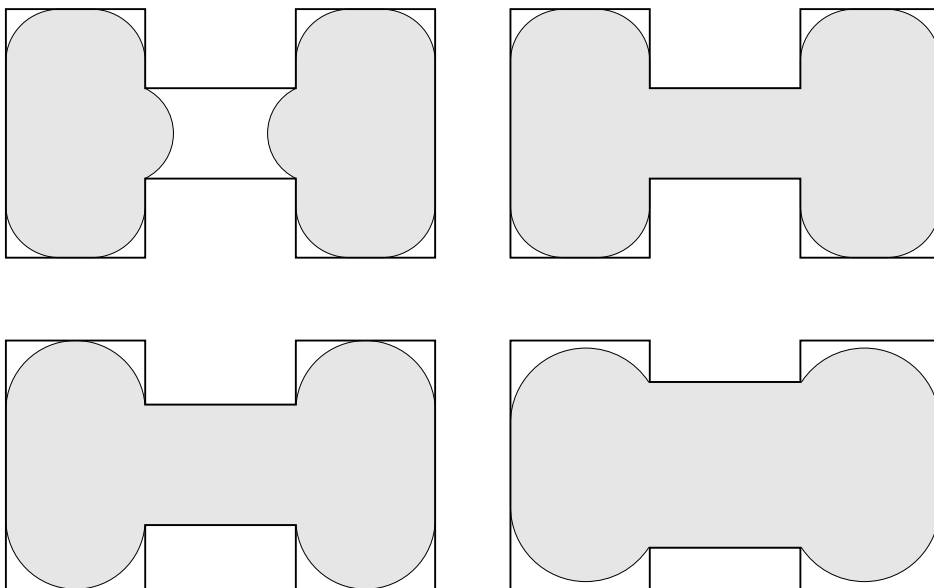
square and a ball, with again a small channel joining them. For the same reason as in the barbell case, any Cheeger set will not intersect the channel (assuming its width is small enough). Therefore connected Cheeger sets have to be contained either in the square, or in the (enlarged) disk. For the unit square we already know that the Cheeger set is a rounded square with radius  $r_0 = 1/(2 + \sqrt{\pi})$  from (12). This implies  $\rho \geq r_0$  because the domain is larger. Assume that the disk has the same radius  $r_0$ , or even a slightly larger one, say  $r$ . The union of balls of radius  $\rho$  in the disk part is quite similar to the disk. In fact, it bulges slightly into the lower left corner. Therefore the ratio of perimeter over area is approximately  $2/r$ , and thus larger than  $1/r_0$ . We conclude that all Cheeger set are contained in the square. Hence they must be only the rounded square, and there exists balls of radius  $\rho$  contained in  $\Omega$  that do not even intersect the Cheeger set.

For completeness we remark that, aside from the ball and other convex domains, even some nonconvex domains  $\Omega \subset \mathbb{R}^N$  satisfy the relation  $\Omega = \mathcal{C}_\Omega$ : this is the case for an annulus [Demengel et al. 2002].

For  $n \geq 3$  it is still an open problem, if the Cheeger set of a convex  $n$ -dimensional domain is unique or, in case of nonuniqueness, if all of its Cheeger sets are convex. Until recently the convexity of *any* Cheeger set seemed to be known only under a great circle condition, in which case  $\Omega$  can be cut into two parts with an  $(n - 1)$ -dimensional ball as cross-section, see [Stredulinsky and Ziemer 1997]. For  $n = 3$  one can think of each part of the Cheeger set as a convex (or concave) capillary surface in zero gravity, defined on a planar disc, lying above a convex (or below a concave) obstacle and touching it on the boundary with zero contact angle. In [Rosales 2003] the great circle condition was recently replaced by the assumption that  $\Omega$  has rotational symmetry. In [Kawohl and Fridman 2003, Remark 10] (and implicitly in [Alter et al. 2005a], under a regularity assumption on  $\Omega$ ) the existence of a convex Cheeger set is shown. However the uniqueness argument in [Kawohl and Fridman 2003] still contains a gap, because it does not rule out the possibility to have two convex Cheeger sets, one contained in the other, with nonempty intersection of their boundaries.

Finally the continuous dependence of  $\mathcal{C}_\Omega$  on  $\Omega$  can break down if one slightly perturbs Figure 5 for instance by increasing only one of the squares. Other examples are shown in Figure 7, where similar barbells with larger channels are shown. As the width of the channel increases, the domain becomes larger, so the radii of the boundary arcs increase as well. If the width of the channel is very large (almost the height of the enclosing rectangle), the Cheeger sets will contain all the channel. So there must be a critical value of this width where two different symmetric Cheeger sets exist, one connected and the other not (Figure 7, top). After that, if the width of the channel is increased again, the radii of the boundary arcs will equal half the width of the lateral rectangles, assuming these are not too large (Figure 7, bottom





**Figure 7.** Discontinuous dependency on the domain.

left). For larger values of the width, these rounded arcs cannot touch the top or bottom of the rectangles anymore. This means that the Cheeger set will actually decrease (Figure 7, bottom right).

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