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We classify rank-2 vector bundles with no intermediate cohomology on the general prime Fano threefold of index 1 and genus 12. The structure of their moduli spaces is given by means of a monad-theoretic resolution in terms of exceptional bundles.

1. Introduction

The study of vector bundles with no intermediate cohomology, also called *arithmetically Cohen–Macaulay* bundles (see Definition 2.1), has been taken up by several authors. The well-known splitting criterion for projective spaces showed by Horrocks [1964] has been generalized by Ottaviani [1987; 1989] to Grassmannians and quadrics. Knörrer [1987] proved that line bundles and spinor bundles are the only ACM bundles on quadrics, while Buchweitz, Greuel and Schreyer showed in [1987] that only projective spaces and quadrics admit a finite number of equivalence classes of ACM bundles, up to twists.

On the other hand, the problem of classifying ACM bundles on special classes of varieties has been studied in several papers. Arrondo and Costa [2000] took up the case of prime Fano threefolds of index 2, while Faenzi [2005] considered the case of the index-2 prime threefold V_5 .

Madonna classified rank-2 ACM bundles on the quartic threefold [2000], and got a numerical classification [2002] of the invariants of these bundles on any prime Fano threefold V_{2g-2} of index 1 and genus g , with $2 \leq g \leq 12$ and $g \neq 11$. In particular, he conjectured that all the cases of this classification occur on every such threefold V_{2g-2} .

For higher dimensional varieties, the case of $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^4)$ has been studied in [Arrondo and Graña 1999].

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In this paper we consider rank-2 ACM bundles on the general prime Fano threefold X of index 1 and genus 12 (see Definition 2.3). Write the Chern classes of a sheaf \mathcal{F} on X as integers (see Section 2), and we denote by \mathcal{F}_{c_1, c_2} a rank-2 sheaf \mathcal{F} on X with $c_1(\mathcal{F}) = c_1$ and $c_2(\mathcal{F}) = c_2$. The main result of this paper states:

Theorem. *On the general X as above, there exist the following vector bundles with no intermediate cohomology:*

- (1) *The bundle $\mathcal{F}_{-1,1}$ associated to a line contained in X ;*
- (2) *The bundle $\mathcal{F}_{0,2}$ associated to a conic contained in X ;*
- (3) *The bundle $\mathcal{F}_{-1,d}(1)$ associated to an elliptic curve of degree d , with $7 \leq d \leq 14$;*
- (4) *The bundle $\mathcal{F}_{0,4}(1)$ associated to a canonical curve of degree 26 and genus 14 contained in X ;*
- (5) *The bundle $\mathcal{F}_{-1,15}(2)$ associated to a half-canonical curve C_{60}^{59} of degree 59 and genus 60 contained in X .*

These are the only possible indecomposable vector bundles with no intermediate cohomology on X , up to isomorphism and twists by line bundles.

The moduli space of semistable vector bundles with no intermediate cohomology is generically smooth, of dimension equal to 2 in Case (2), $2d-14$ in Case (3), 5 in Case (4), and 16 in Case (5).

This gives a complete classification of ACM rank-2 bundles on the general Fano threefold X , together with a description of their moduli spaces. The main tools for proving the theorem are the study of elliptic curves in X and the resolution of the diagonal on $X \times X$ obtained in [Faenzi 2006].

The paper is structured as follows: In Section 2 we state basic definitions and review some known facts concerning the threefold X . We also recall for the reader's convenience the available descriptions of X , which we will use frequently.

In Section 3 we briefly consider lines and conics contained in X . We also give a monad-theoretic interpretation of the Hilbert scheme of lines and conics in X . In Sections 4 and 5 we take up the analysis of elliptic, canonical, and half-canonical curves in X that give rise to vector bundles with no intermediate cohomology, proving their existence and describing their associated moduli spaces.

2. Preliminaries

Let Y be a smooth projective threefold with $\text{Pic}(Y) \simeq \mathbb{Z} = \langle \mathcal{O}_Y(1) \rangle$ and $H^1(\mathcal{O}(t)) = H^2(\mathcal{O}(t)) = 0$ for any $t \in \mathbb{Z}$. Following standard terminology, we have:

Definition 2.1. Given a sheaf \mathcal{F} over Y , we say that \mathcal{F} is *ACM (arithmetically Cohen–Macaulay)* if $H^p(Y, \mathcal{F}(t)) = 0$ for all $t \in \mathbb{Z}$ and $0 < p < 3$. Equivalently, we say that \mathcal{F} *has no intermediate cohomology*.

We denote the dual of a vector bundle \mathcal{F} by \mathcal{F}^* , and recall that if \mathcal{F} has rank 2 then $\mathcal{F}^* \simeq \mathcal{F}(-c_1(\mathcal{F}))$.

We now review the Hartshorne–Serre correspondence between codimension-2 subvarieties and rank-2 vector bundles, originally introduced in [Serre 1963] and later considered by many authors; see for example [Hartshorne 1974; Vogelaar 1978; Okonek et al. 1980].

Definition 2.2. A complete subvariety Z of Y is called *subcanonical* if there exists a line bundle $\mathcal{O}(r)$ on Y such that $\mathcal{O}(r)|_Z \cong \omega_Z$. Let Z be a subcanonical locally complete intersection codimension-2 subvariety of Y . By [Okonek et al. 1980, Theorem 5.1.1], there exist a rank-2 vector bundle \mathcal{F}_Z over Y and a section $s_Z \in H^0(Y, \mathcal{F}_Z^*)$ such that $Z = \{s_Z=0\}$, that is, Z is the zero locus of s_Z . We say in this case that \mathcal{F}_Z is *associated* to Z . We denote by $N_{Z,Y}$ the normal bundle of Z in Y and by $J_{Z,Y}$ the ideal sheaf of Z in Y .

Under these hypotheses, we have the fundamental exact sequence

$$(1) \quad 0 \longrightarrow \det \mathcal{F}_Z \longrightarrow \mathcal{F}_Z \longrightarrow J_{Z,Y} \longrightarrow 0$$

and the adjunction isomorphism

$$(2) \quad (\mathcal{F}_Z^*)|_Z \simeq N_{Z,Y}.$$

Definition 2.3. A *prime Fano threefold of index 1 and genus 12* is a 3-dimensional algebraic variety X with $\text{Pic}(X) \simeq \mathbb{Z} = \langle \mathcal{O}_X(1) \rangle$ and $\omega_X \cong \mathcal{O}_X(-1)$, and with $\deg \mathcal{O}_X(1) = 22$. Any such X is rational. We have $h^0(\mathcal{O}_X(1)) = 14$, while $\text{CH}^i(X)$, the i -th Chow group of X , is isomorphic to \mathbb{Z} for $i = 1, 2, 3$.

From now on, X will denote a prime Fano threefold of index 1 and genus 12. We denote the Chern classes of a sheaf \mathcal{F} on X by integers $c_i \in \mathbb{Z}$, meaning that $c_i(\mathcal{F}) = c_i \xi_i$, where ξ_i is the generator of $\text{CH}^i(X) \simeq \mathbb{Z}$ for $i = 1, 2, 3$. Recall that ξ_2 is the class of a line in X .

Further, we define $\mu(\mathcal{F})$ as the rational number $c_1(\mathcal{F})/\text{rk } \mathcal{F}$. We say that a vector bundle \mathcal{F} is *normalized* if $-\text{rk } \mathcal{F} < c_1(\mathcal{F}) \leq 0$. Equivalently, \mathcal{F} is normalized if $-1 < \mu(\mathcal{F}) \leq 0$. Clearly, $\mu(\mathcal{F}_1 \otimes \mathcal{F}_2) = \mu(\mathcal{F}_1) + \mu(\mathcal{F}_2)$.

We refer to [Huybrechts and Lehn 1997] for the definition of (semi)stability (in the sense of Mumford and Takemoto). A stable bundle \mathcal{F} with $\mu(\mathcal{F}) < 0$ satisfies $h^0(\mathcal{F}) = 0$. Recall that, by Hoppe’s criterion, since $\text{Pic}(X)$ is generated by $\mathcal{O}_X(1)$, a rank-2 bundle \mathcal{F} on X is stable if $h^0(\mathcal{F}(t)) = 0$, for the only integer t such that $c_1(\mathcal{F}(t)) = 0$ or $c_1(\mathcal{F}(t)) = -1$. See for example [Okonek et al. 1980, Lemma 1.2.5].

From the Hirzebruch–Riemann–Roch formula for a vector bundle on X of rank r and Chern classes c_i , we obtain:

$$6\chi(\mathcal{F}(s)) = 22s^3r + 11s^2(3r+6c_1) + s(23r+66c_1-6c_2+66c_1^2) \\ + 6r + 23c_1 - 3c_2 - 3c_1c_2 + 33c_1^2 + 3c_3 + 22c_1^3.$$

Given a smooth projective variety Y equipped with a very ample line bundle $\mathcal{O}_Y(1)$, for any integer r and string c with $c_i \in \text{CH}^i(Y)$ (identified with integers whenever possible), we write $M_Y(r; c)$ for the moduli space of rank- r semistable sheaves on Y with Chern classes c_i .

By virtue of the exact sequence (1), the Hilbert polynomial and the Chern classes of \mathcal{F}_Z are determined by the Hilbert polynomial of Z . Denote by $P_{[Z]}$ the Hilbert polynomial of Z with respect to the polarization $\mathcal{O}_Y(1)$, and by $\text{Hilb}_{[Z]}(Y)$ the Hilbert scheme of closed subschemes of Y with Hilbert polynomial $P_{[Z]}$ (see [Huybrechts and Lehn 1997, Page 41]). Further, if Z is a curve of degree d and genus g contained in X , we denote $\text{Hilb}_{[Z]}(X)$ by $\mathcal{H}_{d,g}(X)$.

If the bundle \mathcal{F}_Z is stable, the Hartshorne–Serre correspondence provides a morphism

$$(3) \quad \tau : \text{Hilb}_{[Z]}(X) \rightarrow M_X(2; c_1(\mathcal{F}_Z), c_2(\mathcal{F}_Z)), \quad [Z] \mapsto [\mathcal{F}_Z].$$

We next recall some of the available constructions of the threefold X . We also sketch the description of four fundamental vector bundles E, U, Q, K of respective ranks 2, 3, 4, 5 and defined over X .

We refer to [Mukai 1992, 2004; Schreyer 2001; Faenzi 2006] for proofs and more details.

Nets of dual quadrics and twisted cubics. Let k be an algebraically closed field. Let $A \simeq k^4$ and $B \simeq k^3$ be k -vector spaces, and let $R(A) = k[A]$ and $R(B) = k[B]$ be polynomial algebras. Let $S^d A = R(A)_d$ be the d -th symmetric power of the vector space A .

Given a twisted cubic Γ , we have $P_{[\Gamma]}(t) = \chi(\mathcal{O}_\Gamma(t)) = 3t + 1$. We consider the Hilbert scheme $\text{Hilb}_{3t+1}(\mathbb{P}(A))$ of closed subschemes of $\mathbb{P}(A)$ with Hilbert polynomial $3t + 1$, and we define the variety H to be the irreducible component of $\text{Hilb}_{3t+1}(\mathbb{P}(A))$ containing the rational normal cubics in $\mathbb{P}(A)$, as constructed in [Ellingsrud et al. 1987]. Given a twisted cubic $[\Gamma] \in H$, we denote by J_Γ the ideal sheaf of Γ in $\mathbb{P}(A)$. The open subset H_c consisting of points that are arithmetically Cohen–Macaulay embeds in $\mathbb{G}(k^3, S^2 A)$ by means of the vector bundle U_H whose fiber over $[\Gamma] \in H_c$ is $\text{Tor}_1^{R[A]}(R[A]/J_\Gamma, k)_2 \simeq k^3$. Equivalently, we associate to any $[\Gamma] \in H$ the net of quadrics in $\mathbb{P}(A)$ vanishing on Γ .

Definition 2.4. A net of dual quadrics Ψ (parametrized by B) in $\mathbb{P}(A)$ is defined as a surjective map $\Psi : S^2 A \rightarrow B$. Let $V_\Psi = \ker \Psi$. Given a general net Ψ , we define:

$$\begin{aligned} X_\Psi &= \{[\Gamma] \in \mathbb{H} \subset \text{Hilb}_{3t+1}(\mathbb{P}^3) \mid \Psi(\mathbf{H}^0(J_\Gamma(2))) = 0\} \\ &= \{[\Gamma] \in \mathbb{H} \subset \text{Hilb}_{3t+1}(\mathbb{P}^3) \mid \mathbf{H}^0(J_\Gamma(2)) \subset V_\Psi\}. \end{aligned}$$

We define the bundle U on $X = X_\Psi$ as the restriction of $U_{\mathbb{H}}$ to X .

Definition 2.5. Let Ψ be a general net of dual quadrics and $X = X_\Psi$. There is a rank-2 vector bundle E on X defined by $E_{[\Gamma]} = \text{Tor}_2^{R[A]}(R[A]/J_\Gamma, k)_3 \simeq k^2$. Equivalently, we associate to any $[\Gamma] \in \mathbb{H}$ its space of first-order syzygies.

Lemma 2.6 [Faenzi 2006, Lemma 6.3]. *The bundle E^* is globally-generated and ACM, with $h^0(E^*) = 8$. Consider the rank-6 bundle $E' = \ker(\mathbf{H}^0(E^*) \otimes \mathbb{O} \rightarrow E^*)$. The bundle E' is also stable and ACM.*

Plane quartics. Let B be a 3-dimensional k -vector space and $F \in S^4 B$ a plane quartic. Set $\check{\mathbb{P}}^2 = \mathbb{P}(B^*)$. Take the Hilbert scheme $\text{Hilb}_6(\check{\mathbb{P}}^2)$ of zero-dimensional length 6 closed subschemes of $\check{\mathbb{P}}^2$. Define the subvariety of $\text{Hilb}_6(\check{\mathbb{P}}^2)$ consisting of polar hexagons to F ,

$$X_F = \{\Lambda = (f_1, \dots, f_6) \in \text{Hilb}_6(\check{\mathbb{P}}^2) \mid f_1^4 + \dots + f_6^4 = F\}.$$

Lemma 2.7 [Mukai 2004; Schreyer 2001]. *For a general F , the variety X_F is a prime Fano threefold of index 1 and genus 12. Given a net of dual quadrics Ψ , there exists a quartic form F such that $X_F \simeq X_\Psi$.*

Definition 2.8. Let F be a general plane quartic and let $X = X_F$. There is a rank-5 vector bundle K on X_F defined over an element $\Lambda = (f_1, \dots, f_6) \in X_F$ by $K_\Lambda = \langle f_1^4, \dots, f_6^4 \rangle / F$. The bundle K^* is stable and ACM, with $h^0(K^*) = 14$ and $c_1(K) = -2$ [Faenzi 2006, Lemma 6.1 and 6.2].

Remark 2.9. Under the hypothesis of Lemma 2.7, there is a natural isomorphism $V_\Psi \simeq S^3 B / F(B^*)$, where we consider F as a map $B^* \rightarrow S^3 B$ taking an element $\partial \in B^*$ to the cubic form $\partial(F)$ (apolarity action). We set $V_F = S^3 B / F(B^*)$. The fiber of U over an element $\Lambda = (f_1, \dots, f_6) \in X_F$ is naturally identified with $\langle f_1^3, \dots, f_6^3 \rangle / F(B^*)$. The global sections of U^* and K^* are then identified with $V_F = S^3 B / F(B^*)$ and $S^4 B / F$, respectively. An element ∂ of B^* gives a map $S^4 B \rightarrow S^3 B$ by the apolarity action and, therefore, a homomorphism $\partial : K \rightarrow U$.

Nets of alternating 2-forms. Let V be a 7-dimensional k -vector space and B a 3-dimensional one. Let G be the Grassmannian $\mathbb{G}(k^3, V)$. Define U_G as the universal rank-3 subbundle, and Q_G as the universal rank-4 quotient bundle on G . Let σ

be a section of $B^* \otimes \wedge^2 U_G^*$. Equivalently, σ is a net of alternating 2-forms $\sigma \in B^* \otimes \wedge^2 V^*$.

Definition 2.10. Define X_σ as the zero locus in G of $\sigma \in B^* \otimes \wedge^2 V^*$. For a general σ , the variety X_σ is a prime Fano threefold of index 1 and genus 12.

Lemma 2.11 [Mukai 2004]. *Given a general plane quartic F , there is a net of alternating 2-forms σ_F such that $X_\sigma \simeq X_F$.*

From now on, we identify X with $X_\Psi \simeq X_F \simeq X_\sigma$, where Ψ is a general net of dual quadrics, F is the quartic form provided by Lemma 2.7, and σ is the net of alternating 2-forms given by Lemma 2.11. In particular, we fix the 3- and 4-dimensional k -vector spaces B and A . Recall that, by Remark 2.9, we have $V \simeq V_F \simeq V_\Psi$. We also notice that, under our hypotheses, $(U_G)|_X \simeq (U_H)|_X$. Thus, we denote also by U the restriction of the vector bundle U_G to X_σ . We set $Q = (Q_G)|_X$.

Lemma 2.12. *There are natural isomorphisms*

$$(4) \quad \text{Hom}(U, Q^*) \simeq B, \quad \text{Hom}(E, U) \simeq A^*,$$

$$(5) \quad \text{Hom}(K, U) \simeq B^*, \quad \text{Hom}(E, K) \simeq A.$$

Moreover, there are exact sequences

$$(6) \quad 0 \longrightarrow U \longrightarrow V \otimes \mathbb{C} \longrightarrow Q \longrightarrow 0,$$

$$(7) \quad 0 \longrightarrow K \longrightarrow B \otimes U \longrightarrow Q^* \longrightarrow 0,$$

$$(8) \quad 0 \longrightarrow \wedge^2 U \longrightarrow A \otimes E \longrightarrow K \longrightarrow 0,$$

$$(9) \quad 0 \longrightarrow E \longrightarrow \mathbb{C}^{\oplus 8} \longrightarrow (E')^* \longrightarrow 0.$$

The Chern classes of these bundles are

$$\begin{array}{lll} c_1(E) = -1, & c_2(E) = 7, & \\ c_1(U) = -1, & c_2(U) = 10, & c_3(U) = -2, \\ c_1(Q^*) = -1, & c_2(Q^*) = 12, & c_3(Q^*) = -4, \\ c_1(K) = -2, & c_2(K) = 40, & c_3(K) = -20, \\ c_1(E') = -1, & c_2(E') = 15, & c_3(E') = -8. \end{array}$$

Proof. The exact sequences (6) and (7), together with (4) and the first isomorphism in (5), are proved in [Faenzi 2006, Lemma 6.1]. The sequence (8) follows from [Faenzi 2006, Proposition 6.4], while (9) is Lemma 2.6. The second isomorphism in (5) follows from [Faenzi 2006, Corollary 6.8]. The Chern classes of U , Q^* and $\wedge^2 U$ are easily computed by restriction from $\mathbb{G}(k^3, V)$. Finally, the Chern classes of K , E and E' follow from the exact sequences (7), (8) and (9). \square

Birational geometry. We briefly sketch the birational geometry of X following [Iskovskih 1978, 1989]. Fano’s double projection from a line is used, and we refer to [Iskovskih and Prokhorov 1999] for a complete treatment.

Let V_5 be the del Pezzo threefold obtained by cutting $\mathbb{G}(\mathbb{P}^1, \mathbb{P}^4) \subset \mathbb{P}^9$ with a general $\mathbb{P}^6 \subset \mathbb{P}^9$. Denote by S_5 a general hyperplane section of V_5 .

It turns out that our X is birational to V_5 under the double projection from a line contained in X . We will use this map to embed in X some elliptic curves contained in V_5 .

The divisor S_5 is a degree 5 del Pezzo surface, hence it is isomorphic to the blow up of \mathbb{P}^2 at 4 points B_1, \dots, B_4 . Further, we have $\omega_{S_5}^* \simeq \mathcal{O}_{S_5}(1) \simeq \mathcal{O}(3\ell - \sum b_i)$, where ℓ is the class of a line in \mathbb{P}^2 and b_i is the exceptional divisor over the point B_i .

Recall that, by [Iskovskih and Prokhorov 1999], the threefold V_5 contains a rational normal curve C_0^5 of degree 5 (restrict to S_5 and take the divisor $2\ell - b_1$). Furthermore, C_0^5 has exactly 3 chords T_1, T_2, T_3 . Indeed, any chord of C_0^5 is contained in S_5 , and the only lines in S_5 meeting C_0^5 in two points are of the form $\ell - b_i - b_j$ for $1 < i < j$.

Denote by H_{V_5} the divisor associated to $\mathcal{O}_{V_5}(1)$. The linear system $3H_{V_5} - 2C_0^5$ defines a birational map $\varphi : V_5 \dashrightarrow X$. Let \tilde{X} be the variety obtained by blowing up V_5 along C_0^5 and then along the proper preimages of T_1, T_2, T_3 . Denote by ψ_1 the contraction to V_5 . There also exists a contraction $\psi_2 : \tilde{X} \rightarrow X$, and we have $\varphi \circ \psi_1 = \psi_2$.

Definition 2.13. Fix a general hyperplane section S_5 of V_5 and an isomorphism $S_5 \rightarrow \text{Bl}_{B_1, \dots, B_4}(\mathbb{P}^2)$ (there is a finite number of such isomorphisms). Let b_i be the exceptional divisors of S_5 over B_i . For a given rational normal curve $C_0^5 \subset V_5$ with chords $\{T_1, T_2, T_3\}$, let $\{e_1, \dots, e_5\} = S_5 \cap C_0^5$ and $f_i = S_5 \cap T_i$. On S_5 , define $\mathcal{L} = 9\ell - 3\sum b_i - 2\sum e_j - \sum f_k$. We have $\varphi|_{S_5} = \varphi|_{\mathcal{L}}$, where $\varphi|_{\mathcal{L}}$ is the map associated to the linear system $|\mathcal{L}|$.

Resolution of the diagonal. We recall here the resolution of the diagonal on X and the induced Beilinson theorem. We refer to [Gorodentsev 1990; Rudakov 1990; Drezet 1986] for the general setup on exceptional collections and mutations.

Define the collection $(G_3, \dots, G_0) = (E, U, Q^*, \mathcal{O})$. This collection is strongly exceptional, that is, $\text{Ext}^p(G_j, G_i) = 0$ if $p > 0$ or $i > j$, as proved in [Kuznetsov 1996]. Further, we define the collection $(G^3, \dots, G^0) = (E, K, U, \mathcal{O})$. The following lemma, proved in [Faenzi 2006, Theorem 7.2], states that these two collections fit together to give a resolution of \mathcal{O}_Δ over $X \times X$.

Lemma 2.14. *For a general X , there exists a resolution of \mathcal{O}_Δ on $X \times X$ of the form:*

$$0 \longrightarrow G_3 \boxtimes G^3 \longrightarrow \dots \longrightarrow G_0 \boxtimes G^0 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

Any coherent sheaf \mathcal{F} on X is functorially isomorphic to the cohomology of a complex $\mathcal{C}_{\mathcal{F}}$ whose terms are

$$\mathcal{C}_{\mathcal{F}}^k = \bigoplus_{i-j=k} \mathbf{H}^i(\mathcal{F} \otimes G^j) \otimes G_j.$$

Alternatively, \mathcal{F} is functorially isomorphic to the cohomology of a complex $\mathcal{D}_{\mathcal{F}}$ whose terms are

$$\mathcal{D}_{\mathcal{F}}^k = \bigoplus_{i-j=k} \mathbf{H}^i(\mathcal{F} \otimes G_j) \otimes G^j.$$

The following Castelnuovo–Mumford regularity for the collection (G_3, \dots, G_0) is a consequence of Lemma 2.14; see [Faenzi 2006, Corollary 7.4].

Corollary 2.15. *Let \mathcal{F} be a coherent sheaf on X . If $\mathbf{H}^p(G_p \otimes \mathcal{F}) = 0$ for $p > 0$, then \mathcal{F} is globally-generated.*

Vector bundles with no intermediate cohomology. Recall from the introduction that a rank-2 vector bundle \mathcal{F} with $c_1(\mathcal{F}) = c_1$ and $c_2(\mathcal{F}) = c_2$ is denoted by \mathcal{F}_{c_1, c_2} . Similarly, a curve of genus g and degree d is denoted by C_g^d .

Lemma 2.16 (Madonna). *The only possible classes of indecomposable normalized rank-2 ACM vector bundles on X are, up to isomorphism:*

- (1) the unstable bundle $\mathcal{F}_{-1,1}$ associated to a line in X ;
- (2) the semistable bundle $\mathcal{F}_{0,2}$ associated to a conic in X ;
- (3) the stable bundle $\mathcal{F}_{-1,d}(1)$ associated to an elliptic curve C_1^d contained in X , with $7 \leq d \leq 14$;
- (4) the stable bundle $\mathcal{F}_{0,4}(1)$ associated to a canonical curve C_{14}^{26} in X ;
- (5) the stable bundle $\mathcal{F}_{-1,15}(2)$ associated to a half-canonical curve C_{60}^{59} contained in X .

In each case, the smallest $t \in \mathbb{Z}$ with $\mathbf{h}^0(\mathcal{F}(t)) \neq 0$ is the one stated.

Proof. We refer to [Madonna 2002] for the full proof, with the exception of the condition $d \geq 7$ in (3), which we prove at the end of Section 4. We nonetheless sketch here the Madonna’s main argument.

Considering the first twist \mathcal{F}_{c_1, c_2} of \mathcal{F} by a nonzero global section s , one proves easily that $Z = \{s=0\}$ is a connected curve of arithmetic genus $1 + 1/2(c_1c_2 - c_2)$ and degree c_2 . Therefore $c_1 \geq 1 - 2/c_2 \geq -1$, so \mathcal{F} is stable except for $c_1 = -1$ or $c_1 = 0$, which correspond, respectively, to Cases (1) and (2).

For $c_1 = 1$, we end up in Case (3) and, making use of (1), it is easy to check that $d \leq 14$.

For $c_1 > 1$, we find $h^p(\mathcal{F}_{c_1, c_2}(-1)) = 0$ and $h^p(\mathcal{F}_{c_1, c_2}(-2)) = 0$ for any p . Take the following polynomial equations in the variables c_1 and c_2 :

$$\begin{cases} \chi(\mathcal{F}_{c_1, c_2}(-1)) = 0, \\ \chi(\mathcal{F}_{c_1, c_2}(-2)) = 0. \end{cases}$$

When $c_1 > 1$, we find Cases (4) and (5) as the only solutions. □

3. Lines and conics

It is classically known that X contains a one-dimensional family of lines and a two-dimensional family of smooth conics; see [Iskovskikh and Prokhorov 1999, Propositions 4.2.2 and 4.2.5] and references therein. Denote a line in X by C_0^1 and a conic in X by C_0^2 . We will just provide resolutions of the sheaf $\mathcal{O}_{C_0^1}(-1)$ and of the bundle $\mathcal{F}_{C_0^2}$ with respect to the collection (G_3, \dots, G_0) . This will give a straightforward description of the Hilbert schemes of lines and conics in X .

Lemma 3.1. *The sheaf $\mathcal{O}_{C_0^1}(-1)$ admits the resolution*

$$(10) \quad 0 \longrightarrow E \longrightarrow K \xrightarrow{\alpha_{C_0^1}} U \longrightarrow \mathcal{O}_{C_0^1}(-1) \longrightarrow 0.$$

The map $\alpha_{C_0^1} \in \text{Hom}(K, U) \simeq B^$ degenerates along a line C_0^1 if and only if it lies in the discriminant quartic curve $\det \Psi^\top \subset \check{\mathbb{P}}^2 = \mathbb{P}(B^*)$. In particular, the Hilbert scheme of lines in X is isomorphic to the curve $\det \Psi^\top$.*

Proof. Clearly, we have $(G_j)_{C_0^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}^{4-j}$. Hence, $h^1(G_j \otimes \mathcal{O}_{C_0^1}(-1)) = 1$ for $j = 1, 2, 3$. By Lemma 2.14, the sheaf $\mathcal{O}_{C_0^1}(-1)$ admits the resolution (10).

It is known from [Schreyer 2001, Theorem 6.1] that the Hilbert scheme of lines in X is isomorphic to the curve $\det \Psi^\top$. We nonetheless sketch here a simpler argument.

From (5) follows the isomorphism $\text{Hom}(K, U) \simeq B^*$. The application of the functor $\text{Hom}(E, -)$ to a morphism $\alpha : K \rightarrow U$ corresponds, under the morphism $\Psi^\top : B^* \rightarrow S^2 A^*$, to the linear map $\alpha \mapsto \Psi^\top(\alpha)$. That is, α is taken by $\text{Hom}(E, -)$ to a linear map $\Psi^\top(\alpha) : A \rightarrow A^*$. Since both $\text{Hom}(E, K) \otimes E \rightarrow K$ and $\text{Hom}(E, U) \otimes E \rightarrow U$ are epimorphisms, it follows that $\text{Hom}(E, \alpha)$ is surjective if and only if α is surjective. This fails to hold precisely when α lies in the discriminant curve $\det \Psi^\top$, in which case there is a unique map $E \rightarrow \ker \alpha$. This map is an isomorphism. By a Hilbert polynomial computation, $\text{coker } \alpha$ is isomorphic to $\mathcal{O}_{C_0^1}(-1)$. □

Lemma 3.2 (Takeuchi). *Through any point in X there exists a finite number of conics contained in X . The Hilbert scheme of conics in X is isomorphic to $\mathbb{P}(B)$.*

Proof. The first statement is proved in [Takeuchi 1989]. One may also consult [Iskovskikh and Prokhorov 1999, Lemma 4.2.6].

For any conic C_0^2 in X , there exists an exact sequence

$$(11) \quad 0 \longrightarrow U \longrightarrow Q^* \longrightarrow J_{C_0^2, X} \longrightarrow 0.$$

Any homomorphism $U \rightarrow Q^*$ degenerates along a conic. Since $\text{Hom}(U, Q^*) \simeq B$, the lemma is proved. \square

Corollary 3.3. *The set of stable points in the moduli space $M_X(2; 0, 2)$ is empty. The set of semistable points is isomorphic to $\mathbb{P}^2 = \mathbb{P}(B)$. The bundle $\mathcal{F}_{0,2}$ of Lemma 2.16, Case (2), admits the resolution*

$$0 \longrightarrow U \longrightarrow Q^* \oplus \mathbb{O} \longrightarrow \mathcal{F}_{0,2} \longrightarrow 0.$$

Proof. Since the bundle $\mathcal{F}_{0,2}$ admits a unique global section s , and since s vanishes along a conic C_0^2 , there exists an isomorphism between $M_X(2; 0, 2)$ and $\text{Hilb}_{2t+1}(X) \simeq \mathbb{P}^2$, the Hilbert scheme of conics contained in X . The bundle $\mathcal{F}_{0,2}$ is strictly semistable for $c_1(\mathcal{F}) = 0$.

In this case, the exact sequence (1) reads

$$(12) \quad 0 \longrightarrow \mathbb{O} \longrightarrow \mathcal{F}_{0,2} \longrightarrow J_{C_0^2, X} \longrightarrow 0.$$

Since $\text{Ext}^1(Q^*, \mathbb{O}) = 0$, any morphism $Q^* \rightarrow J_{C_0^2, X}$ lifts to a morphism $Q^* \rightarrow \mathcal{F}_{0,2}$. Considering the map $\mathbb{O} \rightarrow \mathcal{F}_{0,2}$ in the exact sequence (12) and lifting the projection $Q^* \rightarrow J_{C_0^2, X}$ in the exact sequence (11), we obtain a surjective bundle map $Q^* \oplus \mathbb{O} \rightarrow \mathcal{F}_{0,2}$ whose kernel is isomorphic to U . This provides the desired resolution. \square

4. Elliptic curves

In this section we prove the existence in X of elliptic curves with the properties required by Case (3) of Lemma 2.16. In particular, the degree of these curves varies from 7 to 14. The case $7 \leq d \leq 13$ is considered in Proposition 4.1, while the case $d = 14$ is considered in Proposition 4.4. In the latter we also deal with the case $d = 15$, which we will need in Section 5.

Proposition 4.1. *On the general variety X , there exist smooth elliptic curves C_1^d of any degree d , for $7 \leq d \leq 13$. The curve C_1^d is contained in exactly $14 - d$ independent hyperplanes.*

We will construct smooth elliptic curves in X by means of the birational map $\varphi : V = V_5 \dashrightarrow X$ of page 207.

Lemma 4.2. *Let $S = S_5$ be a fixed hyperplane section of V , and fix the notations from page 207. The irreducible component H_{5t+1} of the Hilbert scheme*

$\text{Hilb}_{5\ell+1}(V)$ containing smooth rational normal quintics in V has dimension 10 at a general $[C_0^5]$. There is a dominant map $\zeta : \text{H}_{5\ell+1}(V) \rightarrow \text{Hilb}_5(\mathbb{P}^2)$ defined by $\zeta : [C_0^5] \mapsto e_1 + \cdots + e_5$.

Proof. Set $C = C_0^5$. First, notice that by the Riemann–Roch formula we have $\text{expdim}(\mathcal{J}_{\text{H}_{5\ell+1}(V), [C]}) = 10$, because $\deg N_{C,V} = 10$ and so $\chi(N_{C,V}) = 10$. Since $C \subset S$, we have the exact sequence of normal bundles

$$0 \longrightarrow N_{C,S} \longrightarrow N_{C,V} \longrightarrow (N_{S,V})|_C \longrightarrow 0.$$

Now, by computing $(2\ell - b_1)^2 = 3$, after the identification $C \simeq \mathbb{P}^1$ we get $N_{C,S} \simeq \mathcal{O}_{\mathbb{P}^1}(3)$ and obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(3) \longrightarrow N_{C,V} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(5) \longrightarrow 0.$$

Therefore $h^0(N_{C,V}) = \chi(N_{C,V}) = 10$, so $\text{H}_{5\ell+1}(V)$ is smooth and 10-dimensional.

Let $\mathbb{P}(\text{H}^0(V, \mathcal{O}_V(1))) = \mathbb{P}^6$. Notice that, once we fix the hyperplane section S , for any curve C the intersection $C \cap S$ gives 5 points spanning $\mathbb{P}^4 \subset \mathbb{P}^6$. Conversely, given any $\mathbb{P}^4 \subset \mathbb{P}^6$, there is a curve C such that the spaces $\langle C \rangle$ and $\langle S \rangle$ span \mathbb{P}^6 . Fixing S thus provides a birational map $\text{H}_{5\ell+1}(V) \dashrightarrow \mathbb{G}(\mathbb{P}^4, \mathbb{P}^6)$.

Since $\dim \text{H}_{5\ell+1}(V) = \dim \text{Hilb}_5(\mathbb{P}^2) = 10$, we have to prove that the map ζ is generically finite. So we fix $\underline{e} = (e_1, \dots, e_5)$ and consider the space $\mathbb{P}_{\underline{e}}^4 = \langle e_1, \dots, e_5 \rangle$. Varying a hyperplane section S' of V in the pencil of hyperplanes containing $\mathbb{P}_{\underline{e}}^4$, we obtain a ruled surface $S_{\underline{e}}^j$ consisting of exceptional lines in S' of type b_j' . The ruled surface $S_{\underline{e}}^j$ is not a cone, for there are finitely many lines through any point in V (see [Iskovskikh and Prokhorov 1999, page 64] and [Furushima and Nakayama 1989]). Thus, its dual variety is a hypersurface in $\check{\mathbb{P}}^6$.

Given a curve $C \subset S'$, write $C = 2\ell - b_1'$. We have $\zeta(C) = e_1 + \cdots + e_5$ if and only if there is a hyperplane section $S' = \mathbb{P}^5 \cap V$ with $\mathbb{P}^5 \supset \mathbb{P}_{\underline{e}}^4$ and such that \mathbb{P}^5 contains the curve of class $2\ell - b_1'$. This happens if and only if the hyperplane \mathbb{P}^5 is tangent to the ruled surface $S_{\underline{e}}^1$. Being the dual variety of the hypersurface $S_{\underline{e}}^1$, it intersects the general pencil of \mathbb{P}^5 's containing $\mathbb{P}_{\underline{b}}^4$ in a finite set of points. \square

Lemma 4.3. *Let S be a fixed hyperplane section and fix notation as in Definition 2.13. Define the linear systems*

$$\mathcal{L}_9 = 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - e_2 - e_3 - \sum f_j,$$

$$\mathcal{L}_{10} = 5\ell - 2\sum b_i - 2e_1 - e_2 - e_3 - \sum f_j,$$

$$\mathcal{L}_{11} = 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - e_2 - \sum f_j,$$

$$\mathcal{L}_{12} = 5\ell - 2\sum b_i - 2e_1 - e_2 - \sum f_j,$$

$$\mathcal{L}_{13} = 4\ell - 2b_1 - 2b_2 - b_3 - b_4 - e_1 - \sum f_j.$$

Each \mathcal{L}_d has positive dimension and contains a smooth element \tilde{C}_1^d . The curve $\varphi(\tilde{C}_1^d)$ is a smooth elliptic curve in X of degree d , contained in precisely $14 - d$ independent hyperplanes.

Proof. The linear systems \mathcal{L}_j have positive dimension, as can be seen by counting parameters. Indeed, it suffices to compute the expected dimension of the linear system of curves in \mathbb{P}^2 passing through assigned points and with prescribed nodes.

For odd (even) d , the system \mathcal{L}_d contains a smooth element \tilde{C}_1^d if and only if there exists an irreducible plane quartic with nodes only at B_1 and B_2 (respectively, an irreducible plane quintic with nodes only at B_1, \dots, B_6 and at the point in \mathbb{P}^2 corresponding to e_1). It suffices to project an elliptic normal quartic (quintic) in \mathbb{P}^3 (\mathbb{P}^4) from a general point (line) to obtain such a curve.

The degree of $\varphi(\tilde{C}_1^d)$ is easily computed to be $d = \mathcal{L}_d \cdot \mathcal{L}$, where \mathcal{L} is the linear system of Definition 2.13.

Since any elliptic curve of degree $d \leq 13$ is contained in a hyperplane section S_{22} of X , we have that

$$h^0(J_{C_1^d, X}(1)) = h^0(J_{C_1^d, S_{22}}(1)) + 1.$$

Using the map φ and the fixed isomorphism $S \rightarrow \text{Bl}_{B_1, \dots, B_4}(\mathbb{P}^2)$, we get

$$h^0(J_{C_1^d, S_{22}}(1)) = h^0(\mathbb{P}^2, \mathcal{L} - \mathcal{L}_d).$$

It is then enough to compute the dimension of the following linear systems on \mathbb{P}^2 :

$$(13) \quad \mathcal{L} - \mathcal{L}_9 = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - e_3 - 2e_4 - 2e_5,$$

$$(14) \quad \mathcal{L} - \mathcal{L}_{10} = 4\ell - \sum b_i - e_2 - e_3 - 2e_4 - 2e_5,$$

$$(15) \quad \mathcal{L} - \mathcal{L}_{11} = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - e_2 - 2e_3 - 2e_4 - 2e_5,$$

$$(16) \quad \mathcal{L} - \mathcal{L}_{12} = 4\ell - \sum b_i - e_2 - 2e_3 - 2e_4 - 2e_5,$$

$$(17) \quad \mathcal{L} - \mathcal{L}_{13} = 5\ell - b_1 - b_2 - 2b_3 - 2b_4 - e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5.$$

Using Lemma 4.2, we can compute the dimension of these linear systems by choosing the points corresponding to the e_i 's in a Zariski open set of $\text{Hilb}_5(\mathbb{P}^2)$. Notice that $\text{expdim}(\mathcal{L} - \mathcal{L}_d) = 13 - d$, so we need only check that $\text{expdim}(\mathcal{L} - \mathcal{L}_d) = \dim(\mathcal{L} - \mathcal{L}_d)$. This we can do using Cremona transformations on \mathbb{P}^2 .

For Case (13), consider the Cremona transformation γ_9 associated to the linear system $2\ell - b_3 - b_4 - e_4$. Any curve in $\mathcal{L} - \mathcal{L}_9$ touches a conic through $b_3 - b_4 - e_4$ in 4 points. Further, any curve in $\mathcal{L} - \mathcal{L}_9$ touches the line $\langle B_3, B_4 \rangle$ (or, respectively, $\langle B_4, e_4 \rangle$ or $\langle B_3, e_4 \rangle$) in a single further point e'_4 (or, respectively, b'_3 or b'_4). Therefore, the linear system $\mathcal{L} - \mathcal{L}_9$ is mapped under γ_9 to $4\ell - b_1 - b_2 - b'_3 - b'_4 - e_1 - e_2 - e_3 - e'_4 - 2e_5$. By Lemma 4.2, the points e_1, \dots, e_5 lie in general position. The points b_i can be chosen generic, for we can define S to

be the blow-up of \mathbb{P}^2 at a general 4-tuple of points. Since we now have a linear system of plane quartics with only one node and passing through 8 general points, we conclude that $h^0(\mathbb{P}^2, \mathcal{L} - \mathcal{L}_9) = 4$.

In Case (15), define γ_{11} as the Cremona transformation associated to $2\ell - b_3 - b_4 - e_3$, sending $\mathcal{L} - \mathcal{L}_9$ to $4\ell - b_1 - b_2 - b'_3 - b'_4 - e_1 - e_2 - e'_3 - 2e_4 - 2e_5$. Take $\gamma'_{11} = \gamma_{2\ell - e_3 - e_4 - b_1}$. Then $\gamma'_{11} \circ \gamma_{11}$ sends $\mathcal{L} - \mathcal{L}_9$ to $3\ell - b_2 - b'_3 - b'_4 - e_1 - e_2 - e'_3 - e'_4 - e'_5$. And 8 general points impose 8 linearly independent conditions on the 10-dimensional space of plane cubics.

In Case (17), put $\gamma_{13} = \gamma_{2\ell - b_3 - b_4 - e_2}$ and $\gamma'_{13} = \gamma_{2\ell - e_3 - e_4 - e_5}$. The linear system $\mathcal{L} - \mathcal{L}_{13}$ is mapped by $\gamma'_{13} \circ \gamma_{13}$ to $2\ell - b_2 - b'_3 - b'_4 - e_1 - e'_2$. Since there is no conic through 6 general points, we are done.

In Case (14), set $\gamma_{10} = \gamma_{2\ell - e_3 - e_4 - e_5}$. The lines $\langle e_3, e_4 \rangle$ and $\langle e_3, e_5 \rangle$ give rise to two extra points e'_4 and e'_5 , so we compute $h^0(3\ell - \sum b_i - e_2 - e'_4 - e'_5) = 3$.

In Case (16), put $\gamma_{12} = \gamma_{2\ell - e_3 - e_4 - e_5}$. Here we have no extra points, and the statement follows since $h^0(2\ell - \sum b_i - e_2) = 1$. □

Proof of Proposition 4.1. The curve C_1^7 exists according to [Kuznetsov 1996; Faenzi 2006]. In fact, it is just the zero locus of a general global section s from $H^0(E^*) \simeq k^8$.

For C_1^8 , consider a homomorphism $\alpha : K \rightarrow U$, with $\alpha \in \text{Hom}(K, U) \simeq B^*$. This morphism is surjective whenever α lies outside the discriminant curve $\det \Psi^T \subset \mathbb{P}(B^*)$ (see Lemma 3.1), so for a general α we get a rank-2 locally free sheaf $F_8 = \ker \alpha$. It follows easily from Lemma 2.12 that $c_1(F_8) = -1$ and $c_2(F_8) = 8$. Taking global sections of F_8^* and using the identifications of Lemma 2.7, we get

$$H^0(F_8^*) \simeq \ker(\alpha : S^4 B/F \rightarrow S^3 B/F(B^*)).$$

For a general α , this map is surjective, so $h^0(F_8^*) = 7$. Further, F_8^* is globally-generated since K^* is. Therefore, a general section of F_8^* vanishes along the required curve C_1^8 .

Finally, for $9 \leq d \leq 13$ the statement follows from Lemma 4.3. □

Proposition 4.4. *On the general variety X , there exists a smooth elliptic curve C_1^d of degree d for $d = 14$ or $d = 15$. In both cases, C_1^d is nondegenerate.*

Proof. It is well-known that there exist smooth elliptic normal curves of degree 7 in V . Nonetheless, we sketch a quick proof.

Denote by U_V and Q_V the universal rank-2 subbundle and the universal rank-3 quotient bundle on $\mathbb{G}(k^2, k^5)$, restricted to V . One proves that, for a general map $\alpha : U_V^{\oplus 2} \rightarrow (Q_V^*)^{\oplus 2}$, the sheaf $\text{coker } \alpha \otimes \mathcal{O}_V(1)$ is a globally-generated rank-2 bundle on V , whose general section vanishes on the required curve D_7 .

Take now a hyperplane section S and denote by d_1, \dots, d_7 the intersection points of D_7 with S . Recall the notation from Definition 2.13. Choose a smooth curve

C_0^5 in the linear system $2\ell - b_1 - d_1 - d_2 - d_3$. Clearly, this linear system has positive dimension. The curve D_7 is mapped by $\varphi_{|\mathcal{L}|}$ to a smooth elliptic curve of degree 15, for it intersects C_0^5 at 3 points with normal crossings. This curve is nondegenerate, since D_7 is nondegenerate as well.

Moving the hyperplane section S in $\check{\mathbb{P}}^6$, we can suppose that the point d_4 coincides with the point f_1 . Taking again $C_0^5 \in |2\ell - b_1 - d_1 - d_2 - d_3|$, we have that D_7 is now mapped by $\varphi_{|\mathcal{L}|}$ to a nondegenerate smooth elliptic curve of degree 14; indeed, it intersects C_0^5 at 3 points and T_1 at 1 point, with normal crossings. \square

Proposition 4.5. *Consider d with $7 \leq d \leq 15$ and let F_d be the rank-2 vector bundle over X associated to the elliptic curve C_1^d constructed above. We have $c_1(F_d) = -1$ and $c_2(F_d) = d$. Furthermore, F_d is stable for any d , is ACM when $7 \leq d \leq 14$, and has $h^0(F_{15}^*) = h^1(F_{15}^*) = 1$ when $d = 15$.*

Proof. Set $C = C_1^d$. The numerical invariants of the bundle F_d are obvious, while its stability follows at once from Hoppe’s criterion.

By Serre duality and (1), one has $h^2(F_d^*) = h^1(F_d(-1)) = h^1(F_d^*(-2)) = 0$.

Taking twisted sections in the sequence (1), we get that F_d is ACM if and only if $h^1(F_d(1)) = 0$, that is, if and only if $h^1(J_{C,X}(1)) = 0$. Indeed, in this case the map $H^0(\mathcal{O}_X(1)) \rightarrow H^0(\mathcal{O}_C(1))$ is surjective. This implies that $H^0(\mathcal{O}_X(t)) \rightarrow H^0(\mathcal{O}_C(t))$ is surjective for all $t \geq 1$, so $h^1(J_{C,X}(t)) = 0$ for $t \geq 1$. After using (1), we get $h^1(F_d(t)) = 0$ for $t \geq 1$. For $t \leq 0$ this trivially holds as well, so, by Serre duality, F_d is ACM.

This happens precisely when $h^0(J_{C,X}(1)) = 14 - d$, so the conclusion follows from Propositions 4.1 and 4.4. \square

Theorem 4.6. *For d with $8 \leq d \leq 15$, the bundle F_d of Proposition 4.5 is isomorphic to the cohomology of a monad*

$$(18) \quad E^{\oplus d-8} \xrightarrow{\beta_d} K^{\oplus d-7} \xrightarrow{\alpha_d} U^{\oplus d-7}.$$

For $d = 7$, the bundle F_7 is isomorphic to E .

Proof. From Hirzebruch–Riemann–Roch we get the equalities

$$\begin{aligned} \chi(Q^* \otimes F_d) &= d - 7, \\ \chi(U \otimes F_d) &= d - 7, \\ \chi(E \otimes F_d) &= d - 8. \end{aligned}$$

Recall that the vector bundles U , Q^* , E and F_d are stable. Hence, by [Maruyama 1981, Theorem 1.14], any tensor product between them is also stable. This implies at once the vanishings

$$h^0(Q^* \otimes F_d) = 0, \quad h^0(U \otimes F_d) = 0, \quad h^0(E \otimes F_d) = 0.$$

Serre duality also yields

$$(19) \quad h^3(Q^* \otimes F_d) = h^0(Q \otimes F_d) = 0 \quad \text{because } \mu(Q \otimes F_d) = -1/4,$$

$$(20) \quad h^3(U \otimes F_d) = h^0(U^* \otimes F_d) = 0 \quad \text{because } \mu(U^* \otimes F_d) = -1/6,$$

$$(21) \quad h^3(E \otimes F_d) = h^0(E^* \otimes F_d) = 0 \quad \text{because } c_2(E) \neq c_2(F_d).$$

Here, (21) follows since $\mu(E) = \mu(F_d) = -1/2$, but $c_2(E) = 7 \neq d = c_2(F_d)$, so $\text{Hom}(E, F_d) = 0$.

Consider the tensor product of the bundle F_d by the sequences (6), (9), and the dual of sequence (6). Since $h^0(F_d) = 0$ and $h^1(F_d) = 0$, we have

$$\begin{aligned} h^1(Q^* \otimes F_d) &= h^0(U^* \otimes F_d) = 0 && \text{by (20),} \\ h^1(U \otimes F_d) &= h^0(Q \otimes F_d) = 0 && \text{by (19),} \\ h^1(E \otimes F_d) &= h^0((E')^* \otimes F_d). \end{aligned}$$

The group $H^0((E')^* \otimes F_d)$ vanishes as well, because E' is a stable bundle as well, and we have $\mu((E')^* \otimes F_d) = -1/3$. Summing up:

$$h^2(Q^* \otimes F_d) = d - 7, \quad h^2(U \otimes F_d) = d - 7, \quad h^2(E \otimes F_d) = d - 8.$$

This implies that F_d is isomorphic to the cohomology of a monad of form (18). Clearly, for $d = 7$ the above argument implies $E \simeq F_7$. \square

Theorem 4.7. *Consider d with $7 \leq d \leq 15$, and let X be general. Take the Hilbert scheme $\mathcal{H}_{d,1}(X)$ of curves in X of degree d and arithmetic genus 1. At generic points, $\mathcal{H}_{d,1}(X)$ is smooth of dimension d and the moduli space $M_X(2; -1, d)$ is smooth of dimension $2d - 14$.*

Proof. Let $Z = C_1^d$ be a curve of degree d and arithmetic genus 1, contained in X . Consider the vector bundle F_d associated to Z .

Tensoring by F_d both the exact sequence (1) and the exact sequence defining $Z \subset X$, we get, after using the isomorphism (2), the exact sequences

$$(22) \quad 0 \longrightarrow F_d \longrightarrow \mathcal{E}nd(F_d) \longrightarrow F_d^* \otimes J_{Z,X} \longrightarrow 0,$$

$$(23) \quad 0 \longrightarrow F_d^* \otimes J_{Z,X} \longrightarrow F_d^* \longrightarrow N_{Z,X} \longrightarrow 0.$$

Taking global sections, we get $h^2(X, \mathcal{E}nd(F_d)) = h^1(Z, N_{Z,X})$. This means that $M_X(2; -1, d)$ is unobstructed at $[F_d]$ if and only if $\mathcal{H}_{d,1}(X)$ is unobstructed at $[Z]$.

Consider now the monad (18) given by Theorem 4.6. Denote by W_d^1 the vector space $H^2(Q^* \otimes F_d) \simeq k^{d-7}$, and by W_d^2 the space $H^2(U \otimes F_d) \simeq k^{d-7}$. An element (m, n) of the group $\text{SL}(W_d^1) \times \text{SL}(W_d^2)$ acts on $\mathbb{P}(\text{Hom}(K, U) \otimes \text{Hom}(W_d^1, W_d^2))$ by taking α_d to $n \circ \alpha_d \circ m^{-1}$. For a general α_d , this action is free. Taking now the

functor $\text{Hom}(E, -)$, we get a morphism:

$$(24) \quad \text{Hom}(K, U) \otimes \text{Hom}(W_d^1, W_d^2) \longrightarrow A^* \otimes A^* \otimes \text{Hom}(W_d^1, W_d^2).$$

Recall from (5) that $\text{Hom}(K, U) \simeq B^*$. Hence, any element α_d in the vector space $\text{Hom}(K, U) \otimes \text{Hom}(W_d^1, W_d^2)$ can be seen as a map $W_d^1 \rightarrow W_d^2$ with entries in B^* . The morphism (24) takes the map α_d to a $4(d-7) \times 4(d-7)$ square matrix $W_d^1 \otimes A \rightarrow W_d^2 \otimes A^*$, whose entries are given by $\Psi^\top \otimes \text{id}_{(W_d^1)^*} \otimes \text{id}_{W_d^2}$. Denote this matrix by $\Psi^\top(\alpha_d)$ (see Lemma 3.1).

Consider the sheaf $\ker(\alpha_d : W_d^1 \otimes K \rightarrow W_d^2 \otimes U)$. The above discussion implies that there exists an injective map $\beta_d : E^{d-8} \hookrightarrow \ker \alpha_d$ if and only if $\text{rk } \Psi^\top(\alpha_d) \leq 4(d-7) - (d-8) = 3d - 20$. Since F_d is stable and $h^2(E \otimes F_d) = d - 8$, there is a unique β_d up to isomorphisms.

Summing up, around $[F_d]$ there exists an open neighborhood of an irreducible component of the moduli space $M_X(2; -1, d)$, isomorphic to the set

$$M(d) = \{[\alpha_d] \in \mathbb{P}(B^* \otimes \text{Hom}(W_d^1, W_d^2)) \mid \text{rk } \Psi^\top(\alpha_d) = 3d - 20\} \\ / \text{SL}(d-7) \times \text{SL}(d-7).$$

For a sufficiently general $\Psi^\top : B^* \rightarrow A^* \otimes A^*$, the variety $M(d)$ admits smooth points, indeed it is obtained by cutting the smooth subset of the variety of $(3d-20)$ -secant $(3d-19)$ -spaces to the Segre image of $\mathbb{P}^{4d-27} \times \mathbb{P}^{4d-27}$ by a sufficiently general linear space.

It is easy to check that the dimension of $M(d)$ at a smooth point $[\alpha'_d]$ is $2d - 14$. At the bundle $[F'_d]$ corresponding to $[\alpha'_d]$, the dimension of $M_X(2; -1, d)$ is also $2d - 14$. Thus, taking a section of the general bundle F'_d , we obtain a curve $(Z)'$ with $h^1(N_{(Z)', X}) = 0$, so $h^0(N_{(Z)', X}) = d$. Therefore, the Hilbert scheme $\mathcal{H}_{d,1}(X)$ is d -dimensional and smooth at $[(Z)']$. \square

End of the proof of Lemma 2.16. Consider a general hyperplane section S_{22} of X . It is a K3 surface of Picard number $\rho(S_{22}) = 1$. Take F_d as defined in Proposition 4.5. Restricting F_d to S_{22} , we get a stable rank-2 vector bundle on S_{22} . The moduli space $M_{S_{22}}(2; -1, d)$ is then smooth and projective, of dimension $-\chi(\text{End}(S_{22}, F_d)) - 2$. It is immediate to check that $\dim M_{S_{22}}(2; -1, d) = 4d - 28$. Hence $d \geq 7$. \square

5. Canonical and half-canonical curves

We now prove the existence of the bundles from Cases (4) and (5) of Lemma 2.16. We deal with the latter case first.

Half-canonical curves. We prove the existence of a smooth half-canonical curve C_{60}^{59} by a deformation argument.

Lemma 5.1. *There exists a smooth curve $Z = C_{60}^{59}$ in X of degree 59 and genus 60, given as the zero locus of a section of an ACM vector bundle $\mathcal{F}_{-1,15}(2)$. We have $\omega_Z \simeq \mathcal{O}_X(2)|_Z$. The ACM bundle $\mathcal{F}_{-1,15}$ specializes to the non-ACM bundle F_{15} .*

Proof. Recall from Proposition 4.4 that there exists an elliptic curve $C = C_1^{15}$ such that $h^1(J_{C,X}) = 1$ and C is not contained in any hyperplane. According to Proposition 4.5, the vector bundle F_{15}^* has a unique section vanishing along C .

By Theorem 4.6, the moduli space $M_X(2; -1, 15)$ is smooth and 16-dimensional at a general $[F_{15}]$. Consider the irreducible component of $M_X(2; -1, 15)$ that contains $[F_{15}]$ and take an open neighborhood of $[F_{15}]$ contained in this component. Pick a point $[F'_{15}]$ belonging to this neighborhood and represented by a stable bundle F'_{15} not isomorphic to F_{15} .

Suppose $F'_{15}(1)$ has a nontrivial global section s . Recall that $h^0(F_{15}) = 0$ by stability. The zero locus of s would be a curve C' of degree 15 and arithmetic genus 1. Therefore, s would give a point $[C']$ in $\mathcal{H}_{15,1}(X)$. The point $[C']$ could not coincide with $[C]$, for otherwise $J_{C',X} \simeq J_{C,X}$ would yield $F'_{15} \simeq F_{15}$.

Since $\mathcal{H}_{15,1}(X)$ is smooth of dimension 15 at $[C]$, the above discussion proves that the map $\tau : \mathcal{H}_{15,1}(X) \rightarrow M_X(2; -1, 15)$ is an open embedding at $[C]$, and that its image is the codimension-1 locus $\{[F'_{15}] \in M_X(2; -1, 15) \mid h^0(F'_{15}(1)) \neq 0\}$. Thus, for a general $[F'_{15}]$ we must have $h^0(F'_{15}(1)) = 0$.

Now, since $\chi(F'_{15}(1)) = 0$, we also get $h^1(F'_{15}(1)) = 0$. We set $\mathcal{F}_{-1,15} = F'_{15}$, and then $\mathcal{F}_{-1,15}$ is ACM. Finally, by Castelnuovo–Mumford regularity, $\mathcal{F}_{-1,15}(2)$ is globally generated, so a general section vanishes along a smooth curve Z with the required invariants. \square

Remark 5.2. Any ACM stable bundle of type $\mathcal{F}_{-1,15}$ is the cohomology of a monad of type (18) with $d = 15$. Indeed, it suffices to apply the proof of Theorem 4.6 to $\mathcal{F}_{-1,15}$.

Canonical curves. Here we will prove the existence of a smooth canonical curve in X by exhibiting the bundle $\mathcal{F}_{0,4}$ of Lemma 2.16.

Lemma 5.3. *Given a general homomorphism $\alpha : U^{\oplus 2} \rightarrow (Q^*)^{\oplus 2}$, the sheaf coker α is a vector bundle of type $\mathcal{F}_{0,4}$.*

Proof. Let W_1 and W_2 be 2-dimensional vector spaces such that the domain and codomain of α are $W_1 \otimes U$ and $W_2 \otimes Q^*$. Let $p_1 : k \rightarrow W_1$ be an element of $\check{\mathbb{P}}(W_1)$ and $p_2 : W_2 \rightarrow k$ an element of $\mathbb{P}(W_2)$. To the pair (p_1, p_2) we associate a map $U \rightarrow Q^*$ via the morphism

$$\eta_\alpha : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2 = \mathbb{P}(B), \quad (p_1, p_2) \mapsto (p_2 \otimes \text{id}_{Q^*}) \circ \alpha \circ (p_1 \otimes \text{id}_{U^*}).$$

For a general α , the map η_α is a 2 : 1 cover. Suppose now that α is not injective, as a bundle map, at some given point x of X . Then there exists $p_1 : k \rightarrow W_1$ such that,

for any $p_2 : W_2 \rightarrow k$, the map $\eta_\alpha(p_1, p_2)$ is zero over x . Equivalently, x lies in the conic whose ideal is $\text{coker } \eta_\alpha(p_1, p_2)$. Since η_α is a finite map, this means that x lies in the pencil of conics parameterized by $p_2 \in \mathbb{P}(W_2)$, thus contradicting Lemma 3.2. Therefore $\text{coker } \alpha$ is locally free and, by a straightforward computation, it has the required Chern classes.

From the exact sequence $0 \rightarrow U^{\oplus 2} \rightarrow (Q^*)^{\oplus 2} \rightarrow \mathcal{F}_{0,4} \rightarrow 0$, we see immediately that $h^0(\mathcal{F}_{0,4}) = 0$ and $h^1(\mathcal{F}_{0,4}(t)) = 0$ for any $t \in \mathbb{Z}$; indeed, U and Q^* are ACM bundles.

Therefore $\mathcal{F}_{0,4}$ is stable and ACM. Indeed, Serre duality gives $h^2(\mathcal{F}_{0,4}(t)) = h^1(\mathcal{F}_{0,4}(-1-t)) = 0$ for all $t \in \mathbb{Z}$. Finally, one can compute

$$h^1(Q^* \otimes \mathcal{F}_{0,4}(1)) = 0, \quad h^2(U \otimes \mathcal{F}_{0,4}(1)) = 0, \quad h^3(E \otimes \mathcal{F}_{0,4}(1)) = 0.$$

By Corollary 2.15, we get that $\mathcal{F}_{0,4}(1)$ is globally generated, hence the zero locus of its general global section is the required canonical curve. \square

Lemma 5.4. *Any ACM stable vector bundle of type $\mathcal{F}_{0,4}$ is the cokernel of a map $\alpha : U^{\oplus 2} \rightarrow (Q^*)^{\oplus 2}$.*

Proof. The argument is analogous to that of Theorem 4.6. We find $h^p(U \otimes \mathcal{F}_{0,4}) = 0$ for $p \neq 1$, $h^p(K \otimes \mathcal{F}_{0,4}) = 0$ for $p \neq 1$, and $h^p(E \otimes \mathcal{F}_{0,4}) = 0$ for all p . We conclude that $h^1(U \otimes \mathcal{F}_{0,4}) = -\chi(U \otimes \mathcal{F}_{0,4}) = 2$ and that $h^1(K \otimes \mathcal{F}_{0,4}) = -\chi(K \otimes \mathcal{F}_{0,4}) = 2$, so the statement follows from Lemma 2.14. \square

Remark 5.5. Summing up, we found that an open subset of a component of $M_X(2; 0, 4)$ is isomorphic to an open subset of the variety of Kronecker modules

$$\mathbb{P}(W_1^* \otimes W_2 \otimes B) / \text{SL}(W_1) \times \text{SL}(W_2),$$

where W_1 and W_2 are 2-dimensional vector spaces. In particular, it is unirational and generically smooth of dimension 5.

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