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It is known that the $\mathbb{Q}$-factoriality of a nodal quartic 3-fold in $\mathbb{P}^4$ implies its nonrationality. We prove that a nodal quartic 3-fold with at most 8 nodes is $\mathbb{Q}$-factorial, while one with 9 nodes is not $\mathbb{Q}$-factorial if and only if it contains a plane. There are nonrational non-$\mathbb{Q}$-factorial nodal quartic 3-folds. In particular, we prove the nonrationality of a general non-$\mathbb{Q}$-factorial nodal quartic 3-fold that contains either a plane or a smooth del Pezzo surface of degree 4.

1. Introduction

All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

Let $X \subset \mathbb{P}^4$ be a nodal quartic 3-fold, that is, a hypersurface of degree 4 whose singular points are ordinary double points. Then X is a Fano 3-fold with terminal singularities and satisfies $-K_X \sim \mathcal{O}_{\mathbb{P}^4}(1)|_X$. The following result is Theorem 2 in [Mella 2004] (see also [Iskovskih and Manin 1971; Pukhlikov 1988]):

Theorem 1. *Suppose X is $\mathbb{Q}$-factorial. Then X is* not *birational to either a conic bundle, a fibration in rational surfaces, or a Fano* 3*-fold of Picard rank* 1 *with terminal $\mathbb{Q}$-factorial singularities that is not biregular to X.*

In this paper we prove:

Theorem 2. *If* $|\mathrm{Sing}\, X| \leq 8$, *then X is $\mathbb{Q}$-factorial.*

Corollary 3. *Nodal quartic* 3*-folds with at most* 8 *nodes are nonrational.*

The conditions of Theorem 2 cannot be weakened:

Example 4. If X is a sufficiently general quartic 3-fold containing a two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$, then X is nodal, non-$\mathbb{Q}$-factorial, and has 9 nodes, which are the intersection of two cubic curves in the plane Π.

However, we prove:

Theorem 5. *Suppose that* $|\mathrm{Sing}\, X| = 9$. *The quartic X is $\mathbb{Q}$-factorial if and only if it does not contain any two-dimensional linear subspace of* $\mathbb{P}^4$.

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A general nodal quartic 3-fold with 9 nodes is $\mathbb{Q}$-factorial. A posteriori, the non-$\mathbb{Q}$-factoriality of the quartic X does not necessarily imply its rationality. Indeed, we prove the following result (compare with [Mella 2004, Remark 3]):

Theorem 6. *A very general quartic* 3*-fold X containing a two-dimensional linear subspace of $\mathbb{P}^4$ is nonrational.*

Nonetheless, rational nodal quartic 3-folds do exist:

Example 7 [Petterson 1998]. Any general determinantal quartic 3-fold X is nodal, rational, non-$\mathbb{Q}$-factorial, and satisfies $|\mathrm{Sing}\, X| = 20$.

Remark 8. The quartic X cannot have more than 45 nodes [Varchenko 1983; Friedman 1986]. It is shown in [de Jong et al. 1990] that there is a unique nodal quartic 3-fold $\mathcal{B}_4$ with 45 nodes, which can be given by the equation

$$w^4 - w(x^3+y^3+z^3+t^3) + 3xyzt = 0$$

in $\mathbb{P}^4 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big)$. It is known as the *Burkhardt quartic* [Burkhardt 1891; 1892; Todd 1936; Baker 1946; Finkelberg 1989; Petterson 1998]. This quartic is determinantal and, moreover, is the unique invariant of degree 4 of the simple group $\mathrm{PSp}(4, \mathbb{Z}_3)$ of order 25920 [van der Geer 1987; Hunt 1996; Hoffman and Weintraub 2001; Hulek and Sankaran 2002]. The nodes of $\mathcal{B}_4$ correspond to the 45 tritangents of a smooth cubic surface, and the Weyl group of E_6 is a nontrivial extension of $\mathrm{PSp}(4, \mathbb{Z}_3)$ by $\mathbb{Z}_2$.

For a given variety, one of the most substantial questions is to decide whether it is rational. This question has been considered in depth for smooth 3-folds [Iskovskih and Manin 1971; Clemens and Griffiths 1972; Beauville 1977; Tyurin 1980; Sarkisov 1980; Shokurov 1983; Alekseev 1987; Corti 1995; Pukhlikov 1998; Iskovskikh and Prokhorov 1999; Corti 2000]. However, relatively mild singularities can force a 3-fold to be rational. For example, with a few exceptions, all canonical Gorenstein Fano 3-folds having a non-cDV point are rational [Prokhorov 2004]. In the non-Gorenstein case the situation is different [Corti et al. 2000; Cheltsov 1997, 2004]. Hence, the rationality of nodal 3-folds is a natural topic [Pukhlikov 1988; Grinenko 1998a, 1998b; Mella 2004; Cheltsov and Park 2004].

Remark 9. Every nodal hypersurface in $\mathbb{P}^4$ of degree at least 5 is nonrational. Every quadric 3-fold in $\mathbb{P}^4$ is rational. A nodal cubic 3-fold in $\mathbb{P}^4$ is nonrational if and only if it is smooth. See [Clemens and Griffiths 1972, Theorem 13.12].

There are non-$\mathbb{Q}$-factorial nodal quartic 3-folds that do not contain any two-dimensional linear subspaces of $\mathbb{P}^4$; see [Mella 2004; Ellia and Franco 2000].

Example 10. Consider a sufficiently general quartic 3-fold $X \subset \mathbb{P}^4$, passing through a smooth quadric surface $Q \subset \mathbb{P}^4$. The quartic X can be given by the equation

$$a_2(x, y, z, t, w)\, h_2(x, y, z, t, w) = b_3(x, y, z, t, w)\, g_1(x, y, z, t, w)$$

in $\mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big)$. Here, a_2, h_2, b_3 and g_1 are homogeneous polynomials of degrees 2, 2, 3 and 1 respectively; the quadric $Q \subset \mathbb{P}^4$ is given by the equations $h_2 = g_1 = 0$. The 3-fold X is nodal and non-$\mathbb{Q}$-factorial, and it has 12 nodes given by $h_2 = g_1 = a_2 = b_3 = 0$. Introducing a new variable $\alpha = a_2/g_1$, we obtain a commutative diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\psi}\swarrow & & \searrow{\scriptstyle\phi} \\ X & \overset{\rho}{\dashrightarrow} & V, \end{array}$$

where ρ is a birational map, ϕ is an extremal divisorial contraction [Corti 1995, §3.3.1], the morphism $\psi : Y \to X$ is a flopping contraction [Kollár 1989], and V is a complete intersection

$$\alpha\, g_1(x, y, z, t, w) - a_2(x, y, z, t, w) = \alpha\, h_2(x, y, z, t, w) - b_3(x, y, z, t, w) = 0$$

in $\mathbb{P}^5$. Often ρ is called an *unprojection* of X [Reid 2001]. The variety V is smooth outside a point $P = (0:0:0:0:0:1)$, which is a node. The morphism ϕ contracts the surface $\mathbb{P}^1 \times \mathbb{P}^1$ to P, while ψ contracts the images of 12 lines on V passing through P to the nodes of X. It is unknown whether X is rational [Iskovskikh and Pukhlikov 1996; Corti 2000].

There exist nonrational non-$\mathbb{Q}$-factorial nodal quartic 3-folds in $\mathbb{P}^4$ that contain neither planes nor quadric surfaces. In particular, we will prove:

Theorem 11. *If $X \subset \mathbb{P}^4$ is a sufficiently general quartic 3-fold that contains a smooth del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4, then X is nodal, non-$\mathbb{Q}$-factorial and nonrational, and has* $|\mathrm{Sing}\, X| = 16$.

The varieties in Theorems 6 and 11 are the only known examples of nodal, nonrational and non-$\mathbb{Q}$-factorial quartic 3-folds. We note that the degeneration technique [Clemens 1975; Beauville 1977; Tyurin 1980; Kollár 1996], together with either Theorem 6 or Theorem 11, gives another proof that a very general smooth quartic 3-fold is nonrational.

Remark 12. There are only a few examples of unirational smooth quartic 3-folds [Iskovskih and Manin 1971; Marchisio 2000], and it is unknown whether a generic quartic 3-fold is unirational. However, the quartics in Theorems 6 and 11 are birational to del Pezzo fibrations of degree 3 or 4, which implies that the quartics in Theorems 6 and 11 are unirational [Manin 1967].

Theorems 2 and 5 can be considered part of the following conjecture:

Conjecture 13. *Let $V \subset \mathbb{P}^4$ be a nodal hypersurface. If either*

- $|\mathrm{Sing}\, V| < (\deg V - 1)^2$,
- $|\mathrm{Sing}\, V| < 2(\deg V - 1)(\deg V - 2)$ *and V does not contain planes*, *or*
- $|\mathrm{Sing}\, V| \leq 2(\deg V - 1)(\deg V - 2)$ *and V does not contain planes or quadrics*,

then V is $\mathbb{Q}$-factorial.

We note that an analogue to Conjecture 13 for smooth surfaces on a nodal hypersurface in $\mathbb{P}^4$ is proved in [Ciliberto and Di Gennaro 2004]. It is easy to see that Conjecture 13 holds for quadrics and cubics [Finkelnberg and Werner 1989].

2. The proof of Theorems 2 and 5

The $\mathbb{Q}$-factoriality of nodal 3-folds is studied in [Clemens 1983; Schoen 1985; Werner 1987; Dimca 1990; Borcea 1990; Endraß 1999; Cynk 2001].

Let $X \subset \mathbb{P}^4$ be a nodal quartic 3-fold. It is well known [Werner 1987; Dimca 1990; Cynk 2001] that the following conditions are equivalent:

- The quartic X is $\mathbb{Q}$-factorial.
- Every Weil divisor on X is a Cartier divisor.
- Every Zariski local ring of the quartic X is UFD, that is, X is factorial.
- The group $H_4(X, \mathbb{Z})$ is generated by the class of a hyperplane section.
- $\dim H_4(X, \mathbb{Z}) = \dim H^2(X, \mathbb{Z}) = 1$.
- The nodes of X impose independent linear conditions on cubic hypersurfaces in $\mathbb{P}^4$.

Suppose X does not contain planes and $|\mathrm{Sing}\, X| \leq 9$. We show that the nodes of the quartic X impose independent linear conditions on cubic hypersurfaces in $\mathbb{P}^4$.

Definition 14. The points of a set $\Gamma \subset \mathbb{P}^4$ are in *general position* if

- at most 3 points of Γ can lie on a line,
- at most 6 points of Γ can lie on a conic, and
- at most 8 points of Γ can lie on a plane.

Proposition 15. *The nodes of the quartic X are in general position.*

Proof. Let $L \subset \mathbb{P}^4$ be a line and $\Pi \subset \mathbb{P}^4$ a sufficiently general two-dimensional linear subspace passing through L. We have $\Pi \not\subset X$, and $\Pi \cap X = L \cup S$ for some plane cubic curve S. Moreover,

$$\operatorname{Sing} X \cap L \subset L \cap S,$$

but $|L \cap S| \leq 3$. Thus, at most 3 nodes of the quartic X can lie on a line in $\mathbb{P}^4$.

Let $C \subset \mathbb{P}^4$ be a smooth conic and $Y \subset \mathbb{P}^4$ a sufficiently general two-dimensional quadric cone over the conic C. We have $Y \not\subset X$, and $Y \cap X = C \cup R$ for some curve R of degree 6. As above, we have the inclusion

$$\operatorname{Sing} X \cap C \subset C \cap R.$$

However, the curves C and R lie in the smooth locus of Y and the intersection $C \cdot R$ in Y equals 6. Thus, the inequality $|C \cap R| \leq 6$ holds, and at most 6 nodes of the 3-fold X can lie on a smooth conic in $\mathbb{P}^4$.

Let $\Sigma \subset \mathbb{P}^4$ be a plane. The intersection $T = \Sigma \cap X$ is a possibly reducible and nonreduced plane quartic, and $\operatorname{Sing} X \cap \Sigma \subset \operatorname{Sing} T$. In case T is nonreduced, we have $|\operatorname{Sing} X \cap \Sigma| \leq 6$, as we already proved that at most 3 nodes of X can lie on a line and at most 6 nodes can lie on a conic. Moreover, $|\operatorname{Sing} T| \leq 6$ whenever T is reduced. Therefore, at most 6 nodes of X can lie on a plane in $\mathbb{P}^4$. □

Proposition 16. *Let $\Pi \subset \mathbb{P}^4$ be a two-dimensional linear subspace such that* $\operatorname{Sing} X$ *is contained in* Π. *The nodes of X impose independent linear conditions both on cubic curves in $\Pi \cong \mathbb{P}^2$ and on cubic hypersurfaces in $\mathbb{P}^4$.*

Proof. We must show that, for any subset $\Sigma \subsetneq \operatorname{Sing} X$ and any point $p \in \operatorname{Sing} X \setminus \Sigma$, there exist a cubic curve in Π and a cubic hypersurface in $\mathbb{P}^4$ passing through the points of Σ but not through p. Let $\pi : V \to \Pi$ be a blow-up of points in Σ. Then, owing to Proposition 15, V is a weak del Pezzo surface of degree $9 - |\Sigma| \geq 2$.

The linear system $|-K_V|$ does not have base points [Demazure 1980, §IV, Theorem 1; Bese 1983, Theorem 2]. Thus, there exists a curve C in $|-K_V|$ that does not pass through the point $\pi^{-1}(p)$. The cubic curve $\pi(C)$ passes through all the points of Σ but not through p. Let Y be a cone in $\mathbb{P}^4$ over $\pi(C)$, with vertex on a sufficiently general line of $\mathbb{P}^4$. The cubic hypersurface Y passes through all the points of Σ but not through p. □

Lemma 17 [Cheltsov and Park 2004]. *Let $\Delta \subset \mathbb{P}^n$ be a subset and $p \in \mathbb{P}^n \setminus \Delta$ a point such that $\{p\} \cup \Delta \subset \mathbb{P}^n$ is not contained in any linear subspace of dimension r. There exists a linear subspace $H \subset \mathbb{P}^n$ of dimension r that contains at least $r+1$ points of Δ but not p.*

Proof. We prove the claim by induction on n. For $n = 2$ the claim is trivial. Suppose that $n > 2$ and $r < n$. By assumption, there are $r+1$ points $\{q_1, \dots, q_{r+1}\} \subset \Delta$ such that their linear span T has dimension r.

We assume $p \in T$, since otherwise we are done. There is a point $q \in \Delta \setminus T$, because $\{p\} \cup \Delta \subset \mathbb{P}^n$ is not contained in any linear subspace of dimension r. By induction, there exists a linear subspace $S \subset T$ of dimension $r-1$ that contains r points from among $\{q_1, \dots, q_{r+1}\}$ but not p.

Consider a cone $H \subset \mathbb{P}^n$ over T with vertex q. The cone H is a linear subspace of dimension r that contains at least $r+1$ points of Δ but not p. □

Proposition 18. *Let $\Gamma \subset \mathbb{P}^4$ be a hyperplane such that* $\operatorname{Sing} X$ *is contained in* Γ. *The nodes of X impose independent linear conditions both on cubic surfaces in $\Gamma \cong \mathbb{P}^3$ and on cubic hypersurfaces in $\mathbb{P}^4$.*

Proof. Let $\Sigma \subsetneq \operatorname{Sing} X$ be any subset and let $p \in \operatorname{Sing} X \setminus \Sigma$ be a point. We must show that there is a cubic surface in Γ and a cubic hypersurface in $\mathbb{P}^4$ passing through Σ and not passing through p. As in the proof of Proposition 16, it is enough to find a cubic surface in Γ that passes through all the points of Σ but not through p. A general cone over such a cubic surface will then give a cubic hypersurface in $\mathbb{P}^4$ passing through all the points of Σ and not passing through p.

Without loss of generality, we can assume that $|\operatorname{Sing} X| = |\Sigma| + 1 = 9$.

Let $r \geq 2$ be the maximal number of points of Σ that, together with p, belong to a two-dimensional linear subspace Π in Γ. Then, by Proposition 15, $r \leq 7$. Write

$$\Sigma = \{p_1, \dots, p_8\}$$

so that the points $p_1, \dots, p_r$, together with p, are contained in the plane Π. The points p and $p_1, \dots, p_r$ do not lie on any one line, since otherwise we could find a hyperplane in Γ containing more than r points of Σ. We prove the statement case by case.

Suppose $r = 2$. Divide the set Σ into three, possibly overlapping, subsets such that each subset contains three points of Σ and that their union is the whole Σ. The hyperplane in Γ generated by each subset does not contain p, because $r = 2$. Hence the union of these three hyperplanes is the required cubic surface.

Suppose $r = 3$. By Lemma 17, we can find three points of Σ outside Π, say p_4, p_5, p_6, such that they generate a hyperplane in Γ not passing though p. By Proposition 15, the four points $\{p, p_1, p_2, p_3\}$ do not lie on any one line. Therefore there is a line passing through two points of the set $\{p_1, p_2, p_3\}$, say through p_1 and p_2, and not passing through p. The union of a hyperplane passing through p_4, p_5, p_6, a hyperplane passing through p_7, p_1, p_2, and a sufficiently general hyperplane passing through p_3 and p_8 gives a cubic surface in $\Gamma \cong \mathbb{P}^3$ that passes through all the points of Σ but not through p.

Suppose $r = 4$. There are two lines in Π, say L_1 and L_2, such that the line L_1 contains p_1 and p_2, the line L_2 contains p_3 and p_4, and neither line passes through p. At most two points among $\{p_5, p_6, p_7, p_8\}$ lie on a line passing through p.

Therefore, there are two points, say p_5 and p_6, such that the line passing through p_5 and p_6 does not pass through p. The union of a hyperplane passing through L_1 and p_7, a hyperplane passing through L_2 and p_8, and a sufficiently general hyperplane passing through p_5 and p_6 gives the required cubic surface.

Suppose $r = 5$. There are two lines in Π, say L_1 and L_2, such that $p \notin L_1 \cup L_2$ and $L_1 \cup L_2$ contains four points of $\Sigma \cap \Pi$, say p_1, p_2, p_3 and p_4. The union of a hyperplane passing through L_1 and p_7, a hyperplane passing through L_2 and p_8, and a sufficiently general hyperplane passing through p_5 and p_6 gives a cubic surface in Γ passing through all the points of Σ but not through p.

Suppose $r = 6$. Now we have six points in $\Sigma \cap \Pi$ and two points, say p_7 and p_8, of Σ outside Π. By Proposition 16, we can find a cubic curve C on Π that passes through the points of $\Sigma \cap \Pi$ but not through p. A sufficiently general hyperplane in Γ passing through the points p_7 and p_8 meets the curve C in three points. Let q and q' be two of those points, and let O be an intersection point of the lines $\langle p_7, q\rangle$ and $\langle p_8, q'\rangle$. Then the cone in Γ over the cubic curve C with vertex O is a cubic surface that passes through all the points of Σ but not through p.

Suppose $r = 7$. By Proposition 16, we can find a cubic curve C on Π that passes through the seven points of $\Sigma \cap \Pi$ and does not pass through the point p. The cone in $\Gamma \cong \mathbb{P}^3$ over the cubic curve C with vertex p_8 is a cubic surface that passes through Σ but not through p. □

Proposition 19. *The nodes of X impose independent linear conditions on cubic curves in $\mathbb{P}^4$.*

Proof. We must show that, for any subset $\Sigma \subsetneq \operatorname{Sing} X$ and any point $p \in \operatorname{Sing} X \setminus \Sigma$, there exists a cubic hypersurface in $\mathbb{P}^4$ passing through Σ but not through p.

Without loss of generality, we may assume that $|\operatorname{Sing} X| = |\Sigma| + 1 = 9$.

Let $r \geq 3$ be the maximal number of points in Σ that, together with p, belong to a hyperplane $\Xi \subset \mathbb{P}^4$. By Proposition 18, we may assume $r \leq 7$. Write

$$\Sigma = \{p_1, \dots, p_8\}$$

so that the points $p_1, \dots, p_r$, together with the point p, are contained in Ξ. We prove the claim case by case.

The points p and $p_1, \dots, p_r$ do not belong to a two-dimensional linear subspace in $\mathbb{P}^4$, since otherwise we could find a hyperplane passing through $r+1$ points of the set Σ.

Suppose $r = 3$. Divide the set Σ into three, possibly overlapping, subsets such that each subset contains exactly four points of Σ. The hyperplane generated by each subset does not contain the point p, because $r = 3$. The union of these three hyperplanes is the required cubic hypersurface.

Suppose $r = 4$. There are two lines L_1 and L_2 in Ξ such that the line L_1 passes through p_1 and p_2, the line L_2 passes through p_3 and p_4, and neither line passes through p. There are at most two points of $\{p_5, p_6, p_7, p_8\}$ that lie on a line containing p. Hence, there are two points, say p_5 and p_6, such that the line passing through p_5 and p_6 does not pass through p. The union of two sufficiently general hyperplanes, one passing through L_1 and p_7, the other through L_2 and p_8, and a sufficiently general hyperplane passing through the points p_5 and p_6 gives the required cubic hypersurface in $\mathbb{P}^4$.

Suppose $r = 5$. As in the previous case, there are two lines L_1 and L_2 in Ξ such that the line L_1 passes through p_1 and p_2, the line L_2 passes through p_3 and p_4, and neither line passes through p. The union of two general hyperplanes, one passing through the L_1 and p_7, one through L_2 and p_8, and a sufficiently general hyperplane passing through the points p_5 and p_6 gives a cubic hypersurface in $\mathbb{P}^4$ that passes through all the points of Σ but not through p.

Suppose $r = 6$. There are six points in $\Sigma \cap \Xi$ and two points, say p_7 and p_8, of Σ outside Ξ. By Proposition 18, there is a cubic surface $S \subset \Xi$ that passes through the six points of $\Sigma \cap \Xi$ and does not pass through p. A general two-dimensional linear subspace passing through the points p_7 and p_8 meets S in three different points. From these, choose two points q and q'. Let O be an intersection of the lines $\langle p_7, q\rangle$ and $\langle p_8, q'\rangle$. The required cubic hypersurface is the cone in $\mathbb{P}^4$ over the cubic surface S with vertex O.

Suppose $r = 7$. By Proposition 18, we can find a cubic surface $S \subset \Xi$ that passes through the seven points of $\Sigma \cap \Pi$ and does not pass through p. The cone in $\mathbb{P}^4$ over S with vertex p_8 passes through all the points of Σ but not through p. $\square$

This concludes the proof of Theorems 2 and 5. The same method can be applied to any nodal hypersurface in $\mathbb{P}^4$. The following result is implied by Theorem 24 (see [Werner 1987; Dimca 1990; Ciliberto and Di Gennaro 2004]).

Theorem 20. *A nodal hypersurface $V \subset \mathbb{P}^4$ is $\mathbb{Q}$-factorial whenever*

$$|\operatorname{Sing} V| \leq 2 \deg V - 4.$$

This bound on nodes is not sharp, except for hyperquadrics.

3. The proof of Theorem 11

Let $X \subset \mathbb{P}^4$ be a sufficiently general (that is, from the complement of a Zariski closed subset in moduli) quartic 3-fold containing a smooth del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4. The quartic X can be given by the equation

$$a_2(x, y, z, t, w)\, h_2(x, y, z, t, w) + b_2(x, y, z, t, w)\, g_2(x, y, z, t, w) = 0$$

in $\mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big)$. Here, a_2, b_2, h_2 and g_2 are homogeneous polynomials of degree 2 such that S is defined by the equations $h_2 = g_2 = 0$. The quartic X is nodal and non-$\mathbb{Q}$-factorial; it has 16 nodes given by the equations $h_2 = g_2 = a_2 = b_2 = 0$.

Lemma 21. *The divisor class group* $\mathrm{Cl}(X)$ *is* $\mathbb{Z} \oplus \mathbb{Z}$.

Proof. Let $f : U \to \mathbb{P}^4$ be a blow-up of the surface S, let E be an exceptional divisor of the birational map f, and let $H = f^*\big(\mathcal{O}_{\mathbb{P}^4}(1)\big)$. The linear system $|2H - E|$ does not have base points, because the del Pezzo surface $S \subset \mathbb{P}^4$ is a complete intersection of two quadrics. In particular, the divisor $2H - E$ is nef and the divisor $4H - E$ is ample.

Let $\tilde{X} \subset U$ be a proper transform of the quartic X. Then $\tilde{X}$ is rationally equivalent to the divisor $4H - E$ on the 4-fold U. The restriction $f|_{\tilde{X}} : \tilde{X} \to X$ is a small resolution and $\tilde{X}$ is smooth. Therefore, by the Lefschetz theorem [Andreotti and Frankel 1959; Bott 1959],

$$H^2\big(\tilde{X}, \mathbb{Z}\big) \cong H^2(U, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

which implies the claim of the lemma. □

The pencil generated by the quadrics $a_2 = 0$ and $b_2 = 0$ cuts on X the del Pezzo surface S together with a pencil $\mathcal{M}$ whose general element is a smooth del Pezzo surface of degree 4. Let $\tau : V \to X$ be a small resolution such that the pencil $\mathcal{H} = \tau^{-1}(\mathcal{M})$ does not have base points. We have

$$V = \mathrm{Proj}\big(\textstyle\bigoplus_{i \geq 0} \mathcal{O}_X(-S)^{\otimes i}\big),$$

and τ is a natural projection to X [Kawamata 1988]. The 3-fold V is smooth and projective, we have $\mathrm{Pic}(V) = \mathbb{Z} \oplus \mathbb{Z}$, and the pencil $\mathcal{H}$ gives a morphism

$$\xi : V \to \mathbb{P}^1$$

whose general fiber is a del Pezzo surface of degree 4.

Corollary 22 [Alekseev 1987; Iskovskikh 1996a]. *The* 3*-fold* V *is birational to a conic bundle.*

The generality of the choice of X implies that ξ is standard in the sense of [Alekseev 1987]; that is, every fiber of ξ is normal and $\mathrm{Pic}(V) = \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 23 [Alekseev 1987, Theorem 2]. *Let* $\gamma : Y \to \mathbb{P}^1$ *be a standard del Pezzo fibration of degree* 4*. If the topological Euler characteristic of* Y *is not* 0, -4 *or* -8, *then* Y *is nonrational.*

Therefore, in order to prove Theorem 11, we must calculate the topological Euler characteristic of the 3-fold V.

Theorem 24 [Cynk 2001, Theorem 2]. *Let W be a projective smooth 4-fold, and Y an ample reduced and irreducible divisor on W such that the only singularities of Y are nodes and such that*

$$h^2(\Omega^1_W) = h^3\big(\Omega^1_W \otimes \mathcal{O}_W(-Y)\big) = h^1(\mathcal{O}_W) = h^2(\mathcal{O}_W) = 0.$$

If $\tilde{Y}$ is a small resolution of Y, then

$$\begin{aligned} &h^1(\mathcal{O}_{\tilde{Y}}) = h^2(\mathcal{O}_{\tilde{Y}}) = 0,\\ &h^1(\Omega^1_{\tilde{Y}}) = h^1(\Omega^1_W) + \delta,\\ &h^2(\Omega^1_{\tilde{Y}}) = h^0\big(K_W \otimes \mathcal{O}_W(2Y)\big) + h^3(\mathcal{O}_W) - h^0\big(K_W \otimes \mathcal{O}_W(Y)\big)\\ &\qquad\qquad - h^3(\Omega^1_W) - h^4\big(\Omega^1_W \otimes \mathcal{O}_W(-Y)\big) - |\mathrm{Sing}\, Y| + \delta, \end{aligned}$$

where δ is the number of dependent equations that are imposed on the global sections of the line bundle $K_W \otimes \mathcal{O}_W(2Y)$ by the vanishing at the nodes of Y; that is, δ is the defect of the 3-fold Y.

The topological Euler characteristic $\chi(V)$ of the 3-fold V is $6 - 2h^2(\Omega^1_V)$. The twisted Euler exact sequence and Serre duality imply that $h^3\big(\Omega^1_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(-4)\big) = 0$ and $h^4\big(\Omega^1_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(-4)\big) = 5$. Thus, by Theorem 24,

$$h^2(\Omega^1_V) \;=\; h^0\big(\mathcal{O}_{\mathbb{P}^4}(3)\big) - h^3(\Omega^1_{\mathbb{P}^4}) - h^4\big(\Omega^1_{\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(-4)\big) - |\mathrm{Sing}\, X| + 1 \;=\; 15.$$

It follows that $\chi(V) = -24$. By Theorem 23, the quartic 3-fold X is nonrational, which proves Theorem 11.

4. The proof of Theorem 6

Let $X \subset \mathbb{P}^4$ be a very general (that is, from the complement of a countable union of Zariski closed subsets in moduli) quartic 3-fold containing a plane $\Pi \subset \mathbb{P}^4$. The quartic X can be given by the equation

$$x\,h_3(x, y, z, t, w) + y\,g_3(x, y, z, t, w) = 0$$

in $\mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big)$. Here, h_3 and g_3 are homogeneous polynomials of degree 3, while the plane Π is defined by the equations $x = y = 0$. The quartic X is nodal and has 9 nodes given by

$$x = y = h_3 = g_3 = 0.$$

Lemma 25. *The divisor class group* $\mathrm{Cl}(X)$ *is* $\mathbb{Z} \oplus \mathbb{Z}$.

Proof. By Theorem 24, the lemma's assertion is equivalent to the statement nodes of the quartic X imposes 8 independent linear conditions on cubic hypersurfaces in $\mathbb{P}^4$. That is, we must show that the defect of X is 1.

During the proof of Theorem 2, we showed that any 8 nodes of X impose 8 independent linear conditions on cubic hypersurfaces in $\mathbb{P}^4$. The nodes of X cannot impose 9 independent linear conditions on cubic hypersurfaces in $\mathbb{P}^4$, because in this case, by Theorem 24, the 3-fold X must be $\mathbb{Q}$-factorial, which is absurd. □

Theorem 26 [Kollár 1996, §IV, Theorem 1.8.3]. *Let $\xi : Y \to Z$ be a flat proper morphism with irreducible and reduced geometric fibers. There are countably many closed subvarieties $Z_i \subset Z$ such that, for an arbitrary closed point $s \in Z$, the fiber $\xi^{-1}(s)$ is ruled if and only if $s \in \bigcup Z_i$.*

Consider a sufficiently general quartic 3-fold $V \subset \mathbb{P}^4$, given by the equation

$$x\bar{h}_3(x, y, z, t, w) + y\bar{g}_3(x, y, z, t, w) = 0,$$

such that

$$\bar{h}_3(x, y, z, t, w) = xa_2(x, y, z, t, w) + yb_2(x, y, z, t, w) + f_1(z, t, w)h_2(z, t, w)$$

and

$$\bar{g}_3(x, y, z, t, w) = xc_2(x, y, z, t, w) + yd_2(x, y, z, t, w) + f_1(z, t, w)g_2(z, t, w),$$

where a_2, b_2, c_2, d_2, h_2 and g_2 are homogeneous polynomials of degree 2, and f_1 is a homogeneous polynomial of degree 1. The quartic V contains the plane Π. The singularities of the 3-fold V consist of 4 nodes given by the equations

$$x = y = h_2 = g_2 = 0$$

and a single double line $L \subset \Pi$ given by the equations $x = y = f_1 = 0$.

Remark 27. The resolution of singularities of V is rationally connected by [Kollár 1996, §IV, Corollary 5.7.1]. Hence, the 3-fold V is rational if and only if it is ruled. However, the 3-fold V is a flat degeneration of the 3-fold X. Thus, by Theorem 26, the nonrationality of V implies the nonrationality of X.

To prove Theorem 6 it is enough to prove the nonrationality of V.

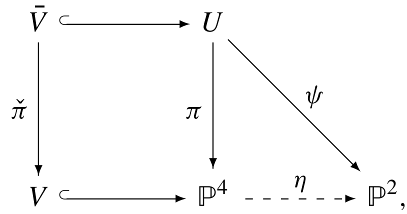
Remark 28. The nonrationality of a sufficiently general quartic 3-fold with a double line is proved in [Conte and Murre 1977] by using the method of the intermediate Jacobian [Clemens and Griffiths 1972; Beauville 1977; Tyurin 1980].

Let $\pi : U \to \mathbb{P}^4$ be a blow-up of the line $L \subset \mathbb{P}^4$, let E be an exceptional divisor of the birational morphism π, and let $\bar{V} \subset U$ be a proper transform of the 3-fold V. The linear system

$$\left|\pi^*\big(\mathcal{O}_{\mathbb{P}^4}(1)\big) - E\right|$$

does not have base points and gives a $\mathbb{P}^2$-bundle $\psi : U \to \mathbb{P}^2$.

We have the commutative diagram



where η is the projection from the line L. The 3-fold $\bar{V}$ is smooth in the neighborhood of the exceptional divisor E, while the singularities of $\bar{V}$ consist of 4 nodes that are the images of the nodes of V.

For a point $x \in L$, the intersection

$$\pi^{-1}(x) \cap \bar{V} \subset \pi^{-1}(x) \cong \mathbb{P}^2$$

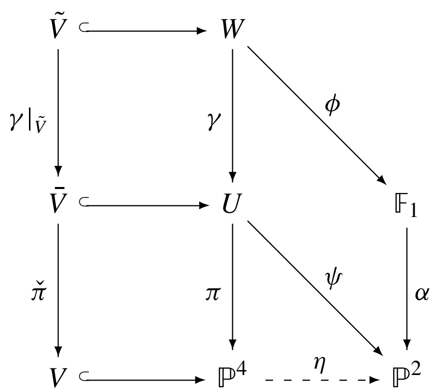
is a smooth conic if x is not a zero of h_2 or g_2, and is a union of two different lines otherwise; that is, there are 4 reducible fibers of the morphism $\pi|_{E\cap\bar{V}}$.

Let $\bar{\Pi} \subset U$ be a proper transform of Π. Then $\psi(\bar{\Pi}) = O$ is a point. The restriction

$$\psi|_{\bar{V}} : \bar{V} \to \mathbb{P}^2$$

is a morphism whose fibers over the points in $\mathbb{P}^2 \setminus O$ are conics, while over O is the surface $\bar{\Pi} \subset \bar{V}$.

Let $\gamma : W \to U$ be a blow-up of $\bar{\Pi}$, let G be a γ-exceptional divisor, and let $\tilde{V} \subset W$ be a proper transform of $\bar{V}$. The linear system $\left|\gamma^*\big(\pi^*(\mathcal{O}_{\mathbb{P}^4}(1)) - E\big) - G\right|$ has no base points, while the linear system $\left|\gamma^*\big(2\big(\pi^*(\mathcal{O}_{\mathbb{P}^4}(1)) - E\big)\big) - G\right|$ gives a morphism $\phi : W \to \mathbb{F}^1$ such that the diagram



is commutative, where $\alpha : \mathbb{F}^1 \to \mathbb{P}^2$ is the blow-up of the point O. The 3-fold $\tilde{V}$ is smooth, and the birational morphism $\gamma|_{\tilde{V}}$ is a small resolution of the 3-fold $\bar{V}$.

Lemma 29. *The Picard group of the* 3*-fold* $\tilde{V}$ *is* $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Proof. The divisor $\tilde{V} \subset W$ is rationally-equivalent to a divisor

$$\gamma^*\big(\pi^*(\mathbb{O}_{\mathbb{P}^4}(4)) - 2E\big) - G \sim \gamma^*\big(\pi^*(\mathbb{O}_{\mathbb{P}^4}(1)) - E\big) - G + \gamma^*\big(\pi^*(\mathbb{O}_{\mathbb{P}^4}(1)) - E\big) + (\pi \circ \gamma)^*\big(\mathbb{O}_{\mathbb{P}^4}(2)\big)$$

that is ample on the 4-fold W. Hence,

$$H^2(\tilde{V}, \mathbb{Z}) \cong H^2(W, \mathbb{Z})$$

by the Lefschetz theorem [Andreotti and Frankel 1959; Bott 1959], which implies the claim of the lemma. □

Corollary 30 [Sarkisov 1980]. *The restriction $\tilde{\phi} = \phi|_{\tilde{V}} : \tilde{V} \to \mathbb{F}_1$ is a standard conic bundle.*

Let $\Delta \subset \mathbb{F}_1$ be a degeneration divisor of the standard conic bundle $\tilde{\phi}$. Then Δ is a reduced divisor with at most simple normal crossings [Beauville 1977; Tyurin 1980; Sarkisov 1980, 1982; Shokurov 1983; Corti 2000].

Lemma 31. *Let s_∞ be an exceptional section of the ruled surface $\mathbb{F}_1$, and let ℓ be a fiber of the natural projection of the surface $\mathbb{F}_1$ to $\mathbb{P}^1$. We have*

$$\Delta \sim 5s_\infty + 8\ell \qquad \textit{and} \qquad 2K_{\mathbb{F}_1} + \Delta \sim s_\infty + 2\ell.$$

Proof. Set $\Delta \sim as_\infty + b\ell$ for some integers a and b. Consider a general divisor H in the linear system $|\tilde{\phi}^*(\ell)|$ and take the surface $\tilde{\Pi} = \psi^{-1}(s_\infty)$. By Bertini's theorem, H is smooth. The surface $\tilde{\Pi}$ is smooth as well, because

$$\gamma|_{\tilde{\Pi}} : \tilde{\Pi} \to \bar{\Pi} \cong \Pi$$

is a blow-up of the four points on $\Pi \cong \mathbb{P}^2$ given by $h_2 = g_2 = 0$.

The birational map $\gamma|_{\tilde{\Pi}}$ resolves the base points of the pencil of conics generated by the conics $h_2 = 0$ and $g_2 = 0$, which induces the restriction morphism $\phi|_{\tilde{\Pi}}$. The surface H is a cubic surface whose image on the quartic V is a cubic surface residual to the plane Π. Hence, $K_H^2 = 3$ and $K_{\tilde{\Pi}}^2 = 5$, and thus $\Delta \cdot \ell = 5$ and $\Delta \cdot s_\infty = 3$. □

The following result is a special case of a conjectured rationality criterion for standard three-dimensional conic bundles [Iskovskikh 1987; 1991; 1996b].

Theorem 32 [Shokurov 1983, Theorem 10.2]. *Let $\xi : Y \to Z$ be a conic bundle such that Y is a smooth 3-fold, Z is either $\mathbb{P}^2$ or a ruled surface $\mathbb{F}_r$, and $\mathrm{Pic}(Y/Z) = \mathbb{Z}$. If Y is rational and D is a degeneration divisor of ξ, then the linear system $|2K_Z + D|$ is empty.*

Therefore, by Theorem 32, the 3-fold $\tilde{V}$ is nonrational, which proves Theorem 6.

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