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NONRATIONAL NODAL QUARTIC THREEFOLDS

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It is known that the \mathbb{Q} -factoriality of a nodal quartic 3-fold in \mathbb{P}^4 implies its nonrationality. We prove that a nodal quartic 3-fold with at most 8 nodes is \mathbb{Q} -factorial, while one with 9 nodes is not \mathbb{Q} -factorial if and only if it contains a plane. There are nonrational non- \mathbb{Q} -factorial nodal quartic 3-folds. In particular, we prove the nonrationality of a general non- \mathbb{Q} -factorial nodal quartic 3-fold that contains either a plane or a smooth del Pezzo surface of degree 4.

1. Introduction

All varieties are assumed to be projective, normal and defined over \mathbb{C} .

Let $X \subset \mathbb{P}^4$ be a nodal quartic 3-fold, that is, a hypersurface of degree 4 whose singular points are ordinary double points. Then X is a Fano 3-fold with terminal singularities and satisfies $-K_X \sim \mathcal{O}_{\mathbb{P}^4}(1)|_X$. The following result is Theorem 2 in [Mella 2004] (see also [Iskovskih and Manin 1971; Pukhlikov 1988]):

Theorem 1. *Suppose X is \mathbb{Q} -factorial. Then X is not birational to either a conic bundle, a fibration in rational surfaces, or a Fano 3-fold of Picard rank 1 with terminal \mathbb{Q} -factorial singularities that is not biregular to X .*

In this paper we prove:

Theorem 2. *If $|\text{Sing } X| \leq 8$, then X is \mathbb{Q} -factorial.*

Corollary 3. *Nodal quartic 3-folds with at most 8 nodes are nonrational.*

The conditions of [Theorem 2](#) cannot be weakened:

Example 4. If X is a sufficiently general quartic 3-fold containing a two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$, then X is nodal, non- \mathbb{Q} -factorial, and has 9 nodes, which are the intersection of two cubic curves in the plane Π .

However, we prove:

Theorem 5. *Suppose that $|\text{Sing } X| = 9$. The quartic X is \mathbb{Q} -factorial if and only if it does not contain any two-dimensional linear subspace of \mathbb{P}^4 .*

MSC2000: [14E08](#), [14J30](#), [14J45](#), [14J70](#), [14M20](#).

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A general nodal quartic 3-fold with 9 nodes is \mathbb{Q} -factorial. A posteriori, the non- \mathbb{Q} -factoriality of the quartic X does not necessarily imply its rationality. Indeed, we prove the following result (compare with [Mella 2004, Remark 3]):

Theorem 6. *A very general quartic 3-fold X containing a two-dimensional linear subspace of \mathbb{P}^4 is nonrational.*

Nonetheless, rational nodal quartic 3-folds do exist:

Example 7 [Pettersen 1998]. Any general determinantal quartic 3-fold X is nodal, rational, non- \mathbb{Q} -factorial, and satisfies $|\text{Sing } X| = 20$.

Remark 8. The quartic X cannot have more than 45 nodes [Varchenko 1983; Friedman 1986]. It is shown in [de Jong et al. 1990] that there is a unique nodal quartic 3-fold \mathcal{B}_4 with 45 nodes, which can be given by the equation

$$w^4 - w(x^3 + y^3 + z^3 + t^3) + 3xyzt = 0$$

in $\mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, t, w])$. It is known as the *Burkhardt quartic* [Burkhardt 1891; 1892; Todd 1936; Baker 1946; Finkelberg 1989; Pettersen 1998]. This quartic is determinantal and, moreover, is the unique invariant of degree 4 of the simple group $\text{PSp}(4, \mathbb{Z}_3)$ of order 25920 [van der Geer 1987; Hunt 1996; Hoffman and Weintraub 2001; Hulek and Sankaran 2002]. The nodes of \mathcal{B}_4 correspond to the 45 tritangents of a smooth cubic surface, and the Weyl group of E_6 is a nontrivial extension of $\text{PSp}(4, \mathbb{Z}_3)$ by \mathbb{Z}_2 .

For a given variety, one of the most substantial questions is to decide whether it is rational. This question has been considered in depth for smooth 3-folds [Iskovskih and Manin 1971; Clemens and Griffiths 1972; Beauville 1977; Tyurin 1980; Sarkisov 1980; Shokurov 1983; Alekseev 1987; Corti 1995; Pukhlikov 1998; Iskovskikh and Prokhorov 1999; Corti 2000]. However, relatively mild singularities can force a 3-fold to be rational. For example, with a few exceptions, all canonical Gorenstein Fano 3-folds having a non-cDV point are rational [Prokhorov 2004]. In the non-Gorenstein case the situation is different [Corti et al. 2000; Cheltsov 1997, 2004]. Hence, the rationality of nodal 3-folds is a natural topic [Pukhlikov 1988; Grinenko 1998a, 1998b; Mella 2004; Cheltsov and Park 2004].

Remark 9. Every nodal hypersurface in \mathbb{P}^4 of degree at least 5 is nonrational. Every quadric 3-fold in \mathbb{P}^4 is rational. A nodal cubic 3-fold in \mathbb{P}^4 is nonrational if and only if it is smooth. See [Clemens and Griffiths 1972, Theorem 13.12].

There are non- \mathbb{Q} -factorial nodal quartic 3-folds that do not contain any two-dimensional linear subspaces of \mathbb{P}^4 ; see [Mella 2004; Ellia and Franco 2000].

Example 10. Consider a sufficiently general quartic 3-fold $X \subset \mathbb{P}^4$, passing through a smooth quadric surface $Q \subset \mathbb{P}^4$. The quartic X can be given by the equation

$$a_2(x, y, z, t, w) h_2(x, y, z, t, w) = b_3(x, y, z, t, w) g_1(x, y, z, t, w)$$

in $\text{Proj}(\mathbb{C}[x, y, z, t, w])$. Here, a_2, h_2, b_3 and g_1 are homogeneous polynomials of degrees 2, 2, 3 and 1 respectively; the quadric $Q \subset \mathbb{P}^4$ is given by the equations $h_2 = g_1 = 0$. The 3-fold X is nodal and non- \mathbb{Q} -factorial, and it has 12 nodes given by $h_2 = g_1 = a_2 = b_3 = 0$. Introducing a new variable $\alpha = a_2/g_1$, we obtain a commutative diagram

$$\begin{array}{ccc} & Y & \\ \psi \swarrow & & \searrow \phi \\ X & \overset{\rho}{\dashrightarrow} & V, \end{array}$$

where ρ is a birational map, ϕ is an extremal divisorial contraction [Corti 1995, §3.3.1], the morphism $\psi : Y \rightarrow X$ is a flopping contraction [Kollár 1989], and V is a complete intersection

$$\alpha g_1(x, y, z, t, w) - a_2(x, y, z, t, w) = \alpha h_2(x, y, z, t, w) - b_3(x, y, z, t, w) = 0$$

in \mathbb{P}^5 . Often ρ is called an *unprojection* of X [Reid 2001]. The variety V is smooth outside a point $P = (0 : 0 : 0 : 0 : 0 : 1)$, which is a node. The morphism ϕ contracts the surface $\mathbb{P}^1 \times \mathbb{P}^1$ to P , while ψ contracts the images of 12 lines on V passing through P to the nodes of X . It is unknown whether X is rational [Iskovskih and Pukhlikov 1996; Corti 2000].

There exist nonrational non- \mathbb{Q} -factorial nodal quartic 3-folds in \mathbb{P}^4 that contain neither planes nor quadric surfaces. In particular, we will prove:

Theorem 11. *If $X \subset \mathbb{P}^4$ is a sufficiently general quartic 3-fold that contains a smooth del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4, then X is nodal, non- \mathbb{Q} -factorial and nonrational, and has $|\text{Sing } X| = 16$.*

The varieties in Theorems 6 and 11 are the only known examples of nodal, nonrational and non- \mathbb{Q} -factorial quartic 3-folds. We note that the degeneration technique [Clemens 1975; Beauville 1977; Tyurin 1980; Kollár 1996], together with either Theorem 6 or Theorem 11, gives another proof that a very general smooth quartic 3-fold is nonrational.

Remark 12. There are only a few examples of unirational smooth quartic 3-folds [Iskovskih and Manin 1971; Marchisio 2000], and it is unknown whether a generic

quartic 3-fold is unirational. However, the quartics in Theorems 6 and 11 are birational to del Pezzo fibrations of degree 3 or 4, which implies that the quartics in Theorems 6 and 11 are unirational [Manin 1967].

Theorems 2 and 5 can be considered part of the following conjecture:

Conjecture 13. *Let $V \subset \mathbb{P}^4$ be a nodal hypersurface. If either*

- $|\text{Sing } V| < (\deg V - 1)^2$,
- $|\text{Sing } V| < 2(\deg V - 1)(\deg V - 2)$ and V does not contain planes, or
- $|\text{Sing } V| \leq 2(\deg V - 1)(\deg V - 2)$ and V does not contain planes or quadrics,

then V is \mathbb{Q} -factorial.

We note that an analogue to Conjecture 13 for smooth surfaces on a nodal hypersurface in \mathbb{P}^4 is proved in [Ciliberto and Di Gennaro 2004]. It is easy to see that Conjecture 13 holds for quadrics and cubics [Finkelberg and Werner 1989].

2. The proof of Theorems 2 and 5

The \mathbb{Q} -factoriality of nodal 3-folds is studied in [Clemens 1983; Schoen 1985; Werner 1987; Dimca 1990; Borcea 1990; Endraß 1999; Cynk 2001].

Let $X \subset \mathbb{P}^4$ be a nodal quartic 3-fold. It is well known [Werner 1987; Dimca 1990; Cynk 2001] that the following conditions are equivalent:

- The quartic X is \mathbb{Q} -factorial.
- Every Weil divisor on X is a Cartier divisor.
- Every Zariski local ring of the quartic X is UFD, that is, X is factorial.
- The group $H_4(X, \mathbb{Z})$ is generated by the class of a hyperplane section.
- $\dim H_4(X, \mathbb{Z}) = \dim H^2(X, \mathbb{Z}) = 1$.
- The nodes of X impose independent linear conditions on cubic hypersurfaces in \mathbb{P}^4 .

Suppose X does not contain planes and $|\text{Sing } X| \leq 9$. We show that the nodes of the quartic X impose independent linear conditions on cubic hypersurfaces in \mathbb{P}^4 .

Definition 14. The points of a set $\Gamma \subset \mathbb{P}^4$ are in *general position* if

- at most 3 points of Γ can lie on a line,
- at most 6 points of Γ can lie on a conic, and
- at most 8 points of Γ can lie on a plane.

Proposition 15. *The nodes of the quartic X are in general position.*

Proof. Let $L \subset \mathbb{P}^4$ be a line and $\Pi \subset \mathbb{P}^4$ a sufficiently general two-dimensional linear subspace passing through L . We have $\Pi \not\subset X$, and $\Pi \cap X = L \cup S$ for some plane cubic curve S . Moreover,

$$\text{Sing } X \cap L \subset L \cap S,$$

but $|L \cap S| \leq 3$. Thus, at most 3 nodes of the quartic X can lie on a line in \mathbb{P}^4 .

Let $C \subset \mathbb{P}^4$ be a smooth conic and $Y \subset \mathbb{P}^4$ a sufficiently general two-dimensional quadric cone over the conic C . We have $Y \not\subset X$, and $Y \cap X = C \cup R$ for some curve R of degree 6. As above, we have the inclusion

$$\text{Sing } X \cap C \subset C \cap R.$$

However, the curves C and R lie in the smooth locus of Y and the intersection $C \cdot R$ in Y equals 6. Thus, the inequality $|C \cap R| \leq 6$ holds, and at most 6 nodes of the 3-fold X can lie on a smooth conic in \mathbb{P}^4 .

Let $\Sigma \subset \mathbb{P}^4$ be a plane. The intersection $T = \Sigma \cap X$ is a possibly reducible and nonreduced plane quartic, and $\text{Sing } X \cap \Sigma \subset \text{Sing } T$. In case T is nonreduced, we have $|\text{Sing } X \cap \Sigma| \leq 6$, as we already proved that at most 3 nodes of X can lie on a line and at most 6 nodes can lie on a conic. Moreover, $|\text{Sing } T| \leq 6$ whenever T is reduced. Therefore, at most 6 nodes of X can lie on a plane in \mathbb{P}^4 . \square

Proposition 16. *Let $\Pi \subset \mathbb{P}^4$ be a two-dimensional linear subspace such that $\text{Sing } X$ is contained in Π . The nodes of X impose independent linear conditions both on cubic curves in $\Pi \cong \mathbb{P}^2$ and on cubic hypersurfaces in \mathbb{P}^4 .*

Proof. We must show that, for any subset $\Sigma \subsetneq \text{Sing } X$ and any point $p \in \text{Sing } X \setminus \Sigma$, there exist a cubic curve in Π and a cubic hypersurface in \mathbb{P}^4 passing through the points of Σ but not through p . Let $\pi : V \rightarrow \Pi$ be a blow-up of points in Σ . Then, owing to [Proposition 15](#), V is a weak del Pezzo surface of degree $9 - |\Sigma| \geq 2$.

The linear system $|-K_V|$ does not have base points [[Demazure 1980](#), §IV, Theorem 1; [Bese 1983](#), Theorem 2]. Thus, there exists a curve C in $|-K_V|$ that does not pass through the point $\pi^{-1}(p)$. The cubic curve $\pi(C)$ passes through all the points of Σ but not through p . Let Y be a cone in \mathbb{P}^4 over $\pi(C)$, with vertex on a sufficiently general line of \mathbb{P}^4 . The cubic hypersurface Y passes through all the points of Σ but not through p . \square

Lemma 17 [[Cheltsov and Park 2004](#)]. *Let $\Delta \subset \mathbb{P}^n$ be a subset and $p \in \mathbb{P}^n \setminus \Delta$ a point such that $\{p\} \cup \Delta \subset \mathbb{P}^n$ is not contained in any linear subspace of dimension r . There exists a linear subspace $H \subset \mathbb{P}^n$ of dimension r that contains at least $r+1$ points of Δ but not p .*

Proof. We prove the claim by induction on n . For $n=2$ the claim is trivial. Suppose that $n > 2$ and $r < n$. By assumption, there are $r+1$ points $\{q_1, \dots, q_{r+1}\} \subset \Delta$ such that their linear span T has dimension r .

We assume $p \in T$, since otherwise we are done. There is a point $q \in \Delta \setminus T$, because $\{p\} \cup \Delta \subset \mathbb{P}^n$ is not contained in any linear subspace of dimension r . By induction, there exists a linear subspace $S \subset T$ of dimension $r-1$ that contains r points from among $\{q_1, \dots, q_{r+1}\}$ but not p .

Consider a cone $H \subset \mathbb{P}^n$ over T with vertex q . The cone H is a linear subspace of dimension r that contains at least $r+1$ points of Δ but not p . \square

Proposition 18. *Let $\Gamma \subset \mathbb{P}^4$ be a hyperplane such that $\text{Sing } X$ is contained in Γ . The nodes of X impose independent linear conditions both on cubic surfaces in $\Gamma \cong \mathbb{P}^3$ and on cubic hypersurfaces in \mathbb{P}^4 .*

Proof. Let $\Sigma \subsetneq \text{Sing } X$ be any subset and let $p \in \text{Sing } X \setminus \Sigma$ be a point. We must show that there is a cubic surface in Γ and a cubic hypersurface in \mathbb{P}^4 passing through Σ and not passing through p . As in the proof of [Proposition 16](#), it is enough to find a cubic surface in Γ that passes through all the points of Σ but not through p . A general cone over such a cubic surface will then give a cubic hypersurface in \mathbb{P}^4 passing through all the points of Σ and not passing through p .

Without loss of generality, we can assume that $|\text{Sing } X| = |\Sigma| + 1 = 9$.

Let $r \geq 2$ be the maximal number of points of Σ that, together with p , belong to a two-dimensional linear subspace Π in Γ . Then, by [Proposition 15](#), $r \leq 7$. Write

$$\Sigma = \{p_1, \dots, p_r\}$$

so that the points p_1, \dots, p_r , together with p , are contained in the plane Π . The points p and p_1, \dots, p_r do not lie on any one line, since otherwise we could find a hyperplane in Γ containing more than r points of Σ . We prove the statement case by case.

Suppose $r = 2$. Divide the set Σ into three, possibly overlapping, subsets such that each subset contains three points of Σ and that their union is the whole Σ . The hyperplane in Γ generated by each subset does not contain p , because $r = 2$. Hence the union of these three hyperplanes is the required cubic surface.

Suppose $r = 3$. By [Lemma 17](#), we can find three points of Σ outside Π , say p_4, p_5, p_6 , such that they generate a hyperplane in Γ not passing through p . By [Proposition 15](#), the four points $\{p, p_1, p_2, p_3\}$ do not lie on any one line. Therefore there is a line passing through two points of the set $\{p_1, p_2, p_3\}$, say through p_1 and p_2 , and not passing through p . The union of a hyperplane passing through p_4, p_5, p_6 , a hyperplane passing through p_7, p_1, p_2 , and a sufficiently general hyperplane passing through p_3 and p_8 gives a cubic surface in $\Gamma \cong \mathbb{P}^3$ that passes through all the points of Σ but not through p .

Suppose $r = 4$. There are two lines in Π , say L_1 and L_2 , such that the line L_1 contains p_1 and p_2 , the line L_2 contains p_3 and p_4 , and neither line passes through p . At most two points among $\{p_5, p_6, p_7, p_8\}$ lie on a line passing through p .

Therefore, there are two points, say p_5 and p_6 , such that the line passing through p_5 and p_6 does not pass through p . The union of a hyperplane passing through L_1 and p_7 , a hyperplane passing through L_2 and p_8 , and a sufficiently general hyperplane passing through p_5 and p_6 gives the required cubic surface.

Suppose $r = 5$. There are two lines in Π , say L_1 and L_2 , such that $p \notin L_1 \cup L_2$ and $L_1 \cup L_2$ contains four points of $\Sigma \cap \Pi$, say p_1, p_2, p_3 and p_4 . The union of a hyperplane passing through L_1 and p_7 , a hyperplane passing through L_2 and p_8 , and a sufficiently general hyperplane passing through p_5 and p_6 gives a cubic surface in Γ passing through all the points of Σ but not through p .

Suppose $r = 6$. Now we have six points in $\Sigma \cap \Pi$ and two points, say p_7 and p_8 , of Σ outside Π . By [Proposition 16](#), we can find a cubic curve C on Π that passes through the points of $\Sigma \cap \Pi$ but not through p . A sufficiently general hyperplane in Γ passing through the points p_7 and p_8 meets the curve C in three points. Let q and q' be two of those points, and let O be an intersection point of the lines $\langle p_7, q \rangle$ and $\langle p_8, q' \rangle$. Then the cone in Γ over the cubic curve C with vertex O is a cubic surface that passes through all the points of Σ but not through p .

Suppose $r = 7$. By [Proposition 16](#), we can find a cubic curve C on Π that passes through the seven points of $\Sigma \cap \Pi$ and does not pass through the point p . The cone in $\Gamma \cong \mathbb{P}^3$ over the cubic curve C with vertex p_8 is a cubic surface that passes through Σ but not through p . \square

Proposition 19. *The nodes of X impose independent linear conditions on cubic curves in \mathbb{P}^4 .*

Proof. We must show that, for any subset $\Sigma \subsetneq \text{Sing } X$ and any point $p \in \text{Sing } X \setminus \Sigma$, there exists a cubic hypersurface in \mathbb{P}^4 passing through Σ but not through p .

Without loss of generality, we may assume that $|\text{Sing } X| = |\Sigma| + 1 = 9$.

Let $r \geq 3$ be the maximal number of points in Σ that, together with p , belong to a hyperplane $\Xi \subset \mathbb{P}^4$. By [Proposition 18](#), we may assume $r \leq 7$. Write

$$\Sigma = \{p_1, \dots, p_r\}$$

so that the points p_1, \dots, p_r , together with the point p , are contained in Ξ . We prove the claim case by case.

The points p and p_1, \dots, p_r do not belong to a two-dimensional linear subspace in \mathbb{P}^4 , since otherwise we could find a hyperplane passing through $r+1$ points of the set Σ .

Suppose $r = 3$. Divide the set Σ into three, possibly overlapping, subsets such that each subset contains exactly four points of Σ . The hyperplane generated by each subset does not contain the point p , because $r = 3$. The union of these three hyperplanes is the required cubic hypersurface.

Suppose $r = 4$. There are two lines L_1 and L_2 in Ξ such that the line L_1 passes through p_1 and p_2 , the line L_2 passes through p_3 and p_4 , and neither line passes through p . There are at most two points of $\{p_5, p_6, p_7, p_8\}$ that lie on a line containing p . Hence, there are two points, say p_5 and p_6 , such that the line passing through p_5 and p_6 does not pass through p . The union of two sufficiently general hyperplanes, one passing through L_1 and p_7 , the other through L_2 and p_8 , and a sufficiently general hyperplane passing through the points p_5 and p_6 gives the required cubic hypersurface in \mathbb{P}^4 .

Suppose $r = 5$. As in the previous case, there are two lines L_1 and L_2 in Ξ such that the line L_1 passes through p_1 and p_2 , the line L_2 passes through p_3 and p_4 , and neither line passes through p . The union of two general hyperplanes, one passing through the L_1 and p_7 , one through L_2 and p_8 , and a sufficiently general hyperplane passing through the points p_5 and p_6 gives a cubic hypersurface in \mathbb{P}^4 that passes through all the points of Σ but not through p .

Suppose $r = 6$. There are six points in $\Sigma \cap \Xi$ and two points, say p_7 and p_8 , of Σ outside Ξ . By [Proposition 18](#), there is a cubic surface $S \subset \Xi$ that passes through the six points of $\Sigma \cap \Xi$ and does not pass through p . A general two-dimensional linear subspace passing through the points p_7 and p_8 meets S in three different points. From these, choose two points q and q' . Let O be an intersection of the lines $\langle p_7, q \rangle$ and $\langle p_8, q' \rangle$. The required cubic hypersurface is the cone in \mathbb{P}^4 over the cubic surface S with vertex O .

Suppose $r = 7$. By [Proposition 18](#), we can find a cubic surface $S \subset \Xi$ that passes through the seven points of $\Sigma \cap \Pi$ and does not pass through p . The cone in \mathbb{P}^4 over S with vertex p_8 passes through all the points of Σ but not through p . \square

This concludes the proof of [Theorems 2 and 5](#). The same method can be applied to any nodal hypersurface in \mathbb{P}^4 . The following result is implied by [Theorem 24](#) (see [[Werner 1987](#); [Dimca 1990](#); [Ciliberto and Di Gennaro 2004](#)]).

Theorem 20. *A nodal hypersurface $V \subset \mathbb{P}^4$ is \mathbb{Q} -factorial whenever*

$$|\text{Sing } V| \leq 2 \deg V - 4.$$

This bound on nodes is not sharp, except for hyperquadrics.

3. The proof of [Theorem 11](#)

Let $X \subset \mathbb{P}^4$ be a sufficiently general (that is, from the complement of a Zariski closed subset in moduli) quartic 3-fold containing a smooth del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4. The quartic X can be given by the equation

$$a_2(x, y, z, t, w) h_2(x, y, z, t, w) + b_2(x, y, z, t, w) g_2(x, y, z, t, w) = 0$$

in $\text{Proj}(\mathbb{C}[x, y, z, t, w])$. Here, a_2, b_2, h_2 and g_2 are homogeneous polynomials of degree 2 such that S is defined by the equations $h_2 = g_2 = 0$. The quartic X is nodal and non- \mathbb{Q} -factorial; it has 16 nodes given by the equations $h_2 = g_2 = a_2 = b_2 = 0$.

Lemma 21. *The divisor class group $\text{Cl}(X)$ is $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Let $f : U \rightarrow \mathbb{P}^4$ be a blow-up of the surface S , let E be an exceptional divisor of the birational map f , and let $H = f^*(\mathcal{O}_{\mathbb{P}^4}(1))$. The linear system $|2H - E|$ does not have base points, because the del Pezzo surface $S \subset \mathbb{P}^4$ is a complete intersection of two quadrics. In particular, the divisor $2H - E$ is nef and the divisor $4H - E$ is ample.

Let $\tilde{X} \subset U$ be a proper transform of the quartic X . Then \tilde{X} is rationally equivalent to the divisor $4H - E$ on the 4-fold U . The restriction $f|_{\tilde{X}} : \tilde{X} \rightarrow X$ is a small resolution and \tilde{X} is smooth. Therefore, by the Lefschetz theorem [Andreotti and Frankel 1959; Bott 1959],

$$H^2(\tilde{X}, \mathbb{Z}) \cong H^2(U, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

which implies the claim of the lemma. \square

The pencil generated by the quadrics $a_2 = 0$ and $b_2 = 0$ cuts on X the del Pezzo surface S together with a pencil \mathcal{M} whose general element is a smooth del Pezzo surface of degree 4. Let $\tau : V \rightarrow X$ be a small resolution such that the pencil $\mathcal{H} = \tau^{-1}(\mathcal{M})$ does not have base points. We have

$$V = \text{Proj}\left(\bigoplus_{i \geq 0} \mathcal{O}_X(-S)^{\otimes i}\right),$$

and τ is a natural projection to X [Kawamata 1988]. The 3-fold V is smooth and projective, we have $\text{Pic}(V) = \mathbb{Z} \oplus \mathbb{Z}$, and the pencil \mathcal{H} gives a morphism

$$\xi : V \rightarrow \mathbb{P}^1$$

whose general fiber is a del Pezzo surface of degree 4.

Corollary 22 [Alekseev 1987; Iskovskikh 1996a]. *The 3-fold V is birational to a conic bundle.*

The generality of the choice of X implies that ξ is standard in the sense of [Alekseev 1987]; that is, every fiber of ξ is normal and $\text{Pic}(V) = \mathbb{Z} \oplus \mathbb{Z}$.

Theorem 23 [Alekseev 1987, Theorem 2]. *Let $\gamma : Y \rightarrow \mathbb{P}^1$ be a standard del Pezzo fibration of degree 4. If the topological Euler characteristic of Y is not 0, -4 or -8 , then Y is nonrational.*

Therefore, in order to prove Theorem 11, we must calculate the topological Euler characteristic of the 3-fold V .

Theorem 24 [Cynk 2001, Theorem 2]. *Let W be a projective smooth 4-fold, and Y an ample reduced and irreducible divisor on W such that the only singularities of Y are nodes and such that*

$$h^2(\Omega_W^1) = h^3(\Omega_W^1 \otimes \mathbb{C}_W(-Y)) = h^1(\mathbb{C}_W) = h^2(\mathbb{C}_W) = 0.$$

If \tilde{Y} is a small resolution of Y , then

$$\begin{aligned} h^1(\mathbb{C}_{\tilde{Y}}) &= h^2(\mathbb{C}_{\tilde{Y}}) = 0, \\ h^1(\Omega_{\tilde{Y}}^1) &= h^1(\Omega_W^1) + \delta, \\ h^2(\Omega_{\tilde{Y}}^1) &= h^0(K_W \otimes \mathbb{C}_W(2Y)) + h^3(\mathbb{C}_W) - h^0(K_W \otimes \mathbb{C}_W(Y)) \\ &\quad - h^3(\Omega_W^1) - h^4(\Omega_W^1 \otimes \mathbb{C}_W(-Y)) - |\text{Sing } Y| + \delta, \end{aligned}$$

where δ is the number of dependent equations that are imposed on the global sections of the line bundle $K_W \otimes \mathbb{C}_W(2Y)$ by the vanishing at the nodes of Y ; that is, δ is the defect of the 3-fold Y .

The topological Euler characteristic $\chi(V)$ of the 3-fold V is $6 - 2h^2(\Omega_V^1)$. The twisted Euler exact sequence and Serre duality imply that $h^3(\Omega_{\mathbb{P}^4}^1 \otimes \mathbb{C}_{\mathbb{P}^4}(-4)) = 0$ and $h^4(\Omega_{\mathbb{P}^4}^1 \otimes \mathbb{C}_{\mathbb{P}^4}(-4)) = 5$. Thus, by [Theorem 24](#),

$$h^2(\Omega_V^1) = h^0(\mathbb{C}_{\mathbb{P}^4}(3)) - h^3(\Omega_{\mathbb{P}^4}^1) - h^4(\Omega_{\mathbb{P}^4}^1 \otimes \mathbb{C}_{\mathbb{P}^4}(-4)) - |\text{Sing } X| + 1 = 15.$$

It follows that $\chi(V) = -24$. By [Theorem 23](#), the quartic 3-fold X is nonrational, which proves [Theorem 11](#).

4. The proof of [Theorem 6](#)

Let $X \subset \mathbb{P}^4$ be a very general (that is, from the complement of a countable union of Zariski closed subsets in moduli) quartic 3-fold containing a plane $\Pi \subset \mathbb{P}^4$. The quartic X can be given by the equation

$$x h_3(x, y, z, t, w) + y g_3(x, y, z, t, w) = 0$$

in $\text{Proj}(\mathbb{C}[x, y, z, t, w])$. Here, h_3 and g_3 are homogeneous polynomials of degree 3, while the plane Π is defined by the equations $x = y = 0$. The quartic X is nodal and has 9 nodes given by

$$x = y = h_3 = g_3 = 0.$$

Lemma 25. *The divisor class group $\text{Cl}(X)$ is $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. By [Theorem 24](#), the lemma's assertion is equivalent to the statement nodes of the quartic X imposes 8 independent linear conditions on cubic hypersurfaces in \mathbb{P}^4 . That is, we must show that the defect of X is 1.

During the proof of [Theorem 2](#), we showed that any 8 nodes of X impose 8 independent linear conditions on cubic hypersurfaces in \mathbb{P}^4 . The nodes of X cannot impose 9 independent linear conditions on cubic hypersurfaces in \mathbb{P}^4 , because in this case, by [Theorem 24](#), the 3-fold X must be \mathbb{Q} -factorial, which is absurd. \square

Theorem 26 [[Kollár 1996](#), §IV, Theorem 1.8.3]. *Let $\xi : Y \rightarrow Z$ be a flat proper morphism with irreducible and reduced geometric fibers. There are countably many closed subvarieties $Z_i \subset Z$ such that, for an arbitrary closed point $s \in Z$, the fiber $\xi^{-1}(s)$ is ruled if and only if $s \in \bigcup Z_i$.*

Consider a sufficiently general quartic 3-fold $V \subset \mathbb{P}^4$, given by the equation

$$x\bar{h}_3(x, y, z, t, w) + y\bar{g}_3(x, y, z, t, w) = 0,$$

such that

$$\bar{h}_3(x, y, z, t, w) = xa_2(x, y, z, t, w) + yb_2(x, y, z, t, w) + f_1(z, t, w)h_2(z, t, w)$$

and

$$\bar{g}_3(x, y, z, t, w) = xc_2(x, y, z, t, w) + yd_2(x, y, z, t, w) + f_1(z, t, w)g_2(z, t, w),$$

where a_2, b_2, c_2, d_2, h_2 and g_2 are homogeneous polynomials of degree 2, and f_1 is a homogeneous polynomial of degree 1. The quartic V contains the plane Π . The singularities of the 3-fold V consist of 4 nodes given by the equations

$$x = y = h_2 = g_2 = 0$$

and a single double line $L \subset \Pi$ given by the equations $x = y = f_1 = 0$.

Remark 27. The resolution of singularities of V is rationally connected by [[Kollár 1996](#), §IV, Corollary 5.7.1]. Hence, the 3-fold V is rational if and only if it is ruled. However, the 3-fold V is a flat degeneration of the 3-fold X . Thus, by [Theorem 26](#), the nonrationality of V implies the nonrationality of X .

To prove [Theorem 6](#) it is enough to prove the nonrationality of V .

Remark 28. The nonrationality of a sufficiently general quartic 3-fold with a double line is proved in [[Conte and Murre 1977](#)] by using the method of the intermediate Jacobian [[Clemens and Griffiths 1972](#); [Beauville 1977](#); [Tyurin 1980](#)].

Let $\pi : U \rightarrow \mathbb{P}^4$ be a blow-up of the line $L \subset \mathbb{P}^4$, let E be an exceptional divisor of the birational morphism π , and let $\tilde{V} \subset U$ be a proper transform of the 3-fold V . The linear system

$$|\pi^*(\mathcal{O}_{\mathbb{P}^4}(1)) - E|$$

does not have base points and gives a \mathbb{P}^2 -bundle $\psi : U \rightarrow \mathbb{P}^2$.

We have the commutative diagram

$$\begin{array}{ccccc}
 \bar{V} & \hookrightarrow & U & & \\
 \downarrow \check{\pi} & & \downarrow \pi & \searrow \psi & \\
 V & \hookrightarrow & \mathbb{P}^4 & \dashrightarrow \eta & \mathbb{P}^2,
 \end{array}$$

where η is the projection from the line L . The 3-fold \bar{V} is smooth in the neighborhood of the exceptional divisor E , while the singularities of \bar{V} consist of 4 nodes that are the images of the nodes of V .

For a point $x \in L$, the intersection

$$\pi^{-1}(x) \cap \bar{V} \subset \pi^{-1}(x) \cong \mathbb{P}^2$$

is a smooth conic if x is not a zero of h_2 or g_2 , and is a union of two different lines otherwise; that is, there are 4 reducible fibers of the morphism $\pi|_{E \cap \bar{V}}$.

Let $\bar{\Pi} \subset U$ be a proper transform of Π . Then $\psi(\bar{\Pi}) = O$ is a point. The restriction

$$\psi|_{\bar{V}} : \bar{V} \rightarrow \mathbb{P}^2$$

is a morphism whose fibers over the points in $\mathbb{P}^2 \setminus O$ are conics, while over O is the surface $\bar{\Pi} \subset \bar{V}$.

Let $\gamma : W \rightarrow U$ be a blow-up of $\bar{\Pi}$, let G be a γ -exceptional divisor, and let $\tilde{V} \subset W$ be a proper transform of \bar{V} . The linear system $|\gamma^*(\pi^*(\mathbb{C}_{\mathbb{P}^4}(1)) - E) - G|$ has no base points, while the linear system $|\gamma^*(2(\pi^*(\mathbb{C}_{\mathbb{P}^4}(1)) - E)) - G|$ gives a morphism $\phi : W \rightarrow \mathbb{F}^1$ such that the diagram

$$\begin{array}{ccccc}
 \tilde{V} & \hookrightarrow & W & & \\
 \downarrow \gamma|_{\tilde{V}} & & \downarrow \gamma & \searrow \phi & \\
 \bar{V} & \hookrightarrow & U & & \mathbb{F}^1 \\
 \downarrow \check{\pi} & & \downarrow \pi & \searrow \psi & \downarrow \alpha \\
 V & \hookrightarrow & \mathbb{P}^4 & \dashrightarrow \eta & \mathbb{P}^2
 \end{array}$$

is commutative, where $\alpha : \mathbb{F}^1 \rightarrow \mathbb{P}^2$ is the blow-up of the point O . The 3-fold \tilde{V} is smooth, and the birational morphism $\gamma|_{\tilde{V}}$ is a small resolution of the 3-fold \bar{V} .

Lemma 29. *The Picard group of the 3-fold \tilde{V} is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. The divisor $\tilde{V} \subset W$ is rationally-equivalent to a divisor

$$\begin{aligned} & \gamma^*(\pi^*(\mathcal{O}_{\mathbb{P}^4}(4)) - 2E) - G \\ & \sim \gamma^*(\pi^*(\mathcal{O}_{\mathbb{P}^4}(1)) - E) - G + \gamma^*(\pi^*(\mathcal{O}_{\mathbb{P}^4}(1)) - E) + (\pi \circ \gamma)^*(\mathcal{O}_{\mathbb{P}^4}(2)) \end{aligned}$$

that is ample on the 4-fold W . Hence,

$$H^2(\tilde{V}, \mathbb{Z}) \cong H^2(W, \mathbb{Z})$$

by the Lefschetz theorem [Andreotti and Frankel 1959; Bott 1959], which implies the claim of the lemma. \square

Corollary 30 [Sarkisov 1980]. *The restriction $\tilde{\phi} = \phi|_{\tilde{V}} : \tilde{V} \rightarrow \mathbb{F}_1$ is a standard conic bundle.*

Let $\Delta \subset \mathbb{F}_1$ be a degeneration divisor of the standard conic bundle $\tilde{\phi}$. Then Δ is a reduced divisor with at most simple normal crossings [Beauville 1977; Tyurin 1980; Sarkisov 1980, 1982; Shokurov 1983; Corti 2000].

Lemma 31. *Let s_∞ be an exceptional section of the ruled surface \mathbb{F}_1 , and let ℓ be a fiber of the natural projection of the surface \mathbb{F}_1 to \mathbb{P}^1 . We have*

$$\Delta \sim 5s_\infty + 8\ell \quad \text{and} \quad 2K_{\mathbb{F}_1} + \Delta \sim s_\infty + 2\ell.$$

Proof. Set $\Delta \sim as_\infty + b\ell$ for some integers a and b . Consider a general divisor H in the linear system $|\tilde{\phi}^*(\ell)|$ and take the surface $\tilde{\Pi} = \psi^{-1}(s_\infty)$. By Bertini's theorem, H is smooth. The surface $\tilde{\Pi}$ is smooth as well, because

$$\gamma|_{\tilde{\Pi}} : \tilde{\Pi} \rightarrow \tilde{\Pi} \cong \Pi$$

is a blow-up of the four points on $\Pi \cong \mathbb{P}^2$ given by $h_2 = g_2 = 0$.

The birational map $\gamma|_{\tilde{\Pi}}$ resolves the base points of the pencil of conics generated by the conics $h_2 = 0$ and $g_2 = 0$, which induces the restriction morphism $\phi|_{\tilde{\Pi}}$. The surface H is a cubic surface whose image on the quartic V is a cubic surface residual to the plane Π . Hence, $K_H^2 = 3$ and $K_{\tilde{\Pi}}^2 = 5$, and thus $\Delta \cdot \ell = 5$ and $\Delta \cdot s_\infty = 3$. \square

The following result is a special case of a conjectured rationality criterion for standard three-dimensional conic bundles [Iskovskikh 1987; 1991; 1996b].

Theorem 32 [Shokurov 1983, Theorem 10.2]. *Let $\xi : Y \rightarrow Z$ be a conic bundle such that Y is a smooth 3-fold, Z is either \mathbb{P}^2 or a ruled surface \mathbb{F}_r , and $\text{Pic}(Y/Z) = \mathbb{Z}$. If Y is rational and D is a degeneration divisor of ξ , then the linear system $|2K_Z + D|$ is empty.*

Therefore, by Theorem 32, the 3-fold \tilde{V} is nonrational, which proves Theorem 6.

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