CLASSIFICATION OF SINGULARITIES FOR A SUBCRITICAL FULLY NONLINEAR PROBLEM

María del Mar González
We study isolated singularities for a fully nonlinear elliptic PDE of subcritical type. This equation appears in conformal geometry when dealing with the $k$-curvature of a locally conformally-flat manifold. (The $k$-curvature generalizes scalar curvature.) We give a classification result: either the function is bounded near the singularity, or it has a specific asymptotic behavior.

1. Introduction

The study of singularities for the subcritical problem

\[(1-1) \quad -\Delta u = u^\beta \quad \text{in} \ B\setminus\{0\}, \quad \beta \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right),\]

has received a lot of attention. In particular, Gidas and Spruck [1981] gave a classification result: a positive solution of (1-1) with a nonremovable singularity at zero must behave like

\[u(x) = (1 + o(1)) \frac{c_0}{|x|^{2/(-\beta+1)}} \quad \text{near} \ x = 0,\]

for some $c_0 = c_0(\beta, n)$. In this paper, we deal with a more general subcritical equation, of the form

\[(1-2) \quad \sigma_k(A^{v}) = v^\alpha \quad \text{in} \ B\setminus\{0\}, \quad \alpha > 0,\]

where $g_v = v^{-2} |dx|^2$ for $v > 0$ is a locally conformally-flat metric on the unit ball $B \subset \mathbb{R}^n$, with an isolated singularity at the origin. For a general metric $g$, the matrix $A^{g}$ is given by $A^{g} = g^{-1} \tilde{A}^{g}$, where $\tilde{A}^{g}$ is the Schouten tensor

\[\tilde{A}^{g}_{ij} = \frac{1}{n-2} \left( Ric_{ij} - \frac{1}{2(n-1)} R g_{ij} \right).\]
while \( Ric \) and \( R \) denote the Ricci tensor and the scalar curvature of \( g \). In the metric \( g_v \), the Schouten tensor becomes

\[ A^g_v = v(D^2 v) - \frac{1}{2}|\nabla v|^2 I. \]

The curvatures \( \sigma_k \) are defined as symmetric functions of the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of the \((1, 1)\)-tensor \( A^g \),

\[ \sigma_k := \sigma_k(A^g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \]

The scalar curvature is simply

\[ \sigma_1 = \lambda_1 + \cdots + \lambda_n = \frac{1}{2(n-1)} R. \]

Problem (1-2) for \( k = 1 \) becomes the well known (1-1): if we write

\[ u^4/(n-2) = v - 2 \]

and

\[ 1 + (n/2) - \beta(n-2)/2 = \alpha, \]

the two problems are equivalent. Note that the critical exponent is

\[ \beta = (n+2)/(n-2), \] or \( \alpha = 0. \]

For a general \( k \), we are dealing with a fully nonlinear equation of second order. The problem is elliptic in the positive cone

\[ \Gamma^+_k = \{ v \mid \sigma_1(A^g_v), \ldots, \sigma_k(A^g_v) > 0 \}, \]

but, in general, not uniformly elliptic. However, it still carries an “almost” divergence structure

\[ m \sigma_m = v \partial_j (v_i T^{m-1}_{ij} - n T^{m-1}_{ij} v_i v_j + \frac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2), \]

where \( T^{m}_{ij} \) denotes the Newton tensor (2-1). This was explored in [González 2005b].

Our main result is a classification of the isolated singularities of (1-2):

**Theorem 1.1.** Let \( \alpha \in (0, k) \) and \( n > 2(k+1) \). If \( v \) is a solution of

\[ (1-3) \quad \sigma_k(v) = v^{\alpha} \quad \text{in } B \setminus \{0\}, \]

with \( \alpha > 0 \), \( v \in \Gamma^+_k \), and \( v^{-1} \in C^3(B \setminus \{0\}) \), then

\[ v^{-1}(x) \leq \frac{C}{|x|^{2k/(2k-\alpha)}} \quad \text{near } x = 0. \]

**Theorem 1.2.** Let \( v \) be a solution of (1-3) for \( \alpha \in (0, 2k/(k+1)) \) and \( n > 2(k+1) \), with \( v^{-1} \in C^3(B \setminus \{0\}) \). If the function \( v^{-1} \) is not bounded near the origin, then there exist \( c_1, c_2 > 0 \) such that

\[ \frac{c_1}{|x|^{2k/(2k-\alpha)}} \leq v^{-1}(x) \leq \frac{c_2}{|x|^{2k/(2k-\alpha)}} \quad \text{near } x = 0. \]
The local behavior of singularities for the critical problem \( \sigma_k(v) = 1 \) has been addressed in [González 2005a]. There, we gave a sufficient condition for the function to be bounded near the singularity: the finiteness of the volume of the metric \( g_v \) (when \( n > 2k \)). The same result was obtained by Han [2004] for \( n = 2k \). For the Laplacian problem (\( k = 1 \)), a complete classification of solutions was obtained by Caffarelli, Gidas, and Spruck [1989].

At the time this paper was submitted, it was conjectured that a similar classification result was true also for \( \sigma_k \), where \( n > 2k \). This has now been proved [Li 2006]. In the case \( n < 2k \), all the singularities are removable [Gursky and Viaclovsky 2005].

One of the motivations for the study of (1-1) is that it appears in the resolution of the Yamabe problem (for a very good survey, see [Lee and Parker 1987]). We can establish an analogous \( k \)-Yamabe problem: find the infimum over all the metrics \( g_v = v^{-2}g_0 \) with \( v > 0 \) of the functional

\[
\mathcal{F}_k(g) = \frac{1}{\text{vol}(g)^{(n-2k)/n}} \int_M \sigma_k(A^g) \, d\text{vol}_g.
\]

This functional was first introduced by Viaclovsky [2000], and it generalizes the Yamabe functional. Its Euler equation is precisely \( \sigma_k(v) = 1 \).

The global subcritical problem has been understood by Li and Li [2003]. Indeed, if \( v \) is a positive solution of

\[
\sigma_k(v) = v^\alpha \quad \text{in } \mathbb{R}^n, \quad \alpha \geq 0,
\]

with \( v^{-1} \in \mathcal{C}^2(\mathbb{R}^n) \), then either \( v \) is constant, or \( \alpha = 0 \) and

\[
v^{-1}(x) = \frac{a}{1 + b^2|x - \bar{x}|^2}
\]

for some \( \bar{x} \in \mathbb{R}^n \) and some positive constants \( a \) and \( b \).

The methods of Gidas and Spruck [1981] for the problem with \( k = 1 \) can be generalized to our case. The key ingredient in the present paper is to understand the structure of \( \sigma_k \) and, in particular, to replace the traceless Ricci tensor by the traceless \( k \)-Newton tensor (2-2).

The paper is structured as follows: in Section 2 we give some properties of \( \sigma_k \) that will be crucial in the proofs. We use the divergence structure of \( \sigma_k \) (2-5), an inductive process (2-7), and the properties of the traceless Newton tensor (2-2).

In Section 3 we establish the expression that will allow us to obtain the necessary \( L^p \) estimates, through a generalization of an argument due to Obata and very successfully used by Chang, Gursky, and Yang [2002] and by Li and Li [2002]. In particular, we give a more refined formula (3-1) that is precisely the missing
ingredient for the critical problem. The $L^p$ estimates are in Section 4, while in the last two sections we prove the theorems.

**Remark 1.3.** We believe that the theorems are also true for $n = 2k + 1$, but, as in the case of [Gidas and Spruck 1981], one needs different estimates in (4-12).

**Remark 1.4.** We make the regularity assumption $v^{-1} \in C^3(B \setminus \{0\})$. However, many of the arguments use integral estimates and only require that $v^{-1}$ is in some suitable Sobolev space; for instance, the whole of Section 4.

### 2. Algebraic properties of $\sigma_k$

For a general $n \times n$ matrix $A$, take its eigenvalues $\lambda_1, \ldots, \lambda_n$ and construct the symmetric functions $\sigma_k$, as well as the $k$-th Newton tensor

$$T^k := \sigma_k - \sigma_{k-1}A + \cdots + (-1)^k A^k = \sigma_k I - T^{k-1}A$$

and the traceless Newton tensor

$$L^k := \frac{n-k}{n} \sigma_k I - T^k.$$  

**Remark 2.1.** Take $\sigma_0 := 1$ and $T^0_{ij} := \delta_{ij}$. Although the standard notation for a $(1,1)$-tensor is $A^i_j$, we write both indices as subindices without risking confusion.

**Lemma 2.2** [Gårding 1959; Reilly 1973].

1. $(n-k)\sigma_k = \text{trace } T^k$;
2. $(k+1)\sigma_{k+1} = \text{trace}(AT^k)$;
3. $\text{trace } L^k = 0$;
4. if $\sigma_1, \ldots, \sigma_k > 0$, then $T^m$ is positive definite for $m = 1, \ldots, k-1$;
5. if $\sigma_1, \ldots, \sigma_k > 0$, then $\sigma_k \leq C_{n,k}(\sigma_1)^k$.

In particular, if $A = A^s_g$ for $g_v = v^{-2}|dx|^2$, then the Schouten tensor becomes

$$A_{ij} = vv_i v - \frac{1}{2} |\nabla v|^2 \delta_{ij},$$

while the traceless Ricci tensor (strictly speaking, a constant multiple of the actual traceless Ricci tensor) is now

$$E_{ij} := L^1_{ij} = vv_{ij} - \frac{1}{n} v \Delta v \delta_{ij}.$$  

**Lemma 2.3** [Viaclovsky 2000]. Let $g_v = v^{-2}|dx|^2$. The Newton tensor $T^m$ for $m \leq n - 1$ is divergence-free with respect to this metric; that is,

$$\sum_j \hat{\partial}_j T^m_{ij} = 0, \quad \text{for all } i.$$
As a consequence,
\[ \sum_j \tilde{\partial}_j L_{ij}^m = \frac{n-m}{n} \partial_i \sigma_m(A^v), \]
where \( \tilde{\partial}_j \) is the \( j \)-th covariant derivative with respect to the metric \( g_v \), while \( \partial_j \) denotes the usual Euclidean derivative.

The following two lemmas were proved in [González 2005b]. Expression (2-6) shows the ‘almost’ divergence structure of \( \sigma_m \), while (2-7) is an inductive formula allowing us to handle the nondivergence terms (of order \( m-1 \)) that appear in (2-6).

**Lemma 2.4.** In this setting,
(2-5) \[ \sum_j \partial_j T_{ij}^m = -(n-m) \sigma_m v_i v^{-1} + n \sum_i T_{ij}^m v_i v^{-1} \quad \text{for each } i; \]
(2-6) \[ m \sigma_m(A^v) = v \sum_{i,j} \partial_j (v_i T_{ij}^{m-1}) - n \sum_{i,j} T_{ij}^{m-1} v_i v_j + \frac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2. \]

**Lemma 2.5.** Let \( U \) be a domain in \( \mathbb{R}^n \), \( v^{-1} \in \mathcal{C}^\infty(U) \), and \( \varphi \in \mathcal{C}^\infty_0(U) \) a smooth cutoff function. For any integers \( 1 \leq s \leq k \leq n \) and real number \( \gamma \),
(2-7) \[ \int_U \sum_{i,j} \partial_j T_{ij}^{k-s} v_i v_j |\nabla v|^{2(s-1)} \varphi^{2k} v^{-\gamma} dx = \left( 1 + \frac{k-s}{2s} \right) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} v^{-\gamma} dx \]
\[ + \frac{s+n+1-\gamma}{2s} \int_U \sum_{i,j} T_{ij}^{k-s-1} v_i v_j |\nabla v|^{2s} \varphi^{2k} v^{-\gamma} dx \]
\[ - \frac{n-k+s+1}{4s} \int_U \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} v^{-\gamma} dx \]
\[ + \frac{k}{s} \int_U \sum_{i,j} T_{ij}^{k-s} \varphi_i v_j |\nabla v|^{2(s-1)} \varphi^{2k-1} v^{-\gamma} dx. \]

In Section 3 we will need a similar formula for the traceless Newton tensor:

**Corollary 2.6.** For any fixed \( i \),
(2-8) \[ \sum_j \partial_j (L_{ij}^m) = \frac{n-m}{n} \partial_i \sigma_k + n \sum_j L_{ij}^m v_j v^{-1}. \]

**Proof.** Follows easily from (2-5) and (2-2). \( \square \)

**Lemma 2.7.** If \( \sigma_1, \ldots, \sigma_m > 0 \) and \( m \leq n \), then
\[ \| T_{ij}^{m-1} \| \leq C_{m,n} \sigma_{m-1}. \]

**Proof.** Because of Lemma 2.2, \( T^{m-1} \) is positive definite. To estimate its norm we just need to look at its biggest eigenvalue. We are done, because
\[ \text{trace } T^{m-1} = (n-m) \sigma_{m-1}. \] \( \square \)
Lemma 2.8. For any $1 \leq k \leq n - 1$, if we have a metric $g = v^{-2}|dx|^2$ in the positive cone $\Gamma_k^+$, then
\[
\sum_{i,j} L^k_{ij} E_{ij} \geq 0,
\]
with equality if and only if $E = 0$.

Proof. Because $E_{ij}$ is traceless,
\[
\sum_{i,j} L^k_{ij} E_{ij} = -\sum_{i,j} T^k_{ij} E_{ij}.
\]
Using
\[
E_{ij} = -\frac{1}{n} \sigma_1 \delta_{ij} + A_{ij}, \quad (k + 1) \sigma_{k+1} = T^k_{ij} A_{ij}, \quad T^k_{ij} \delta_{ij} = (n - k) \sigma_k,
\]
we see that
\[
\sum_{i,j} T^k_{ij} E_{ij} = -\frac{n-k}{n} \sigma_k \sigma_1 + (k + 1) \sigma_{k+1}.
\]
The result follows by the general inequality for matrices in the positive cone $\Gamma_k^+$:
\[
\sigma_{k+1} \leq \frac{n-k}{n(k+1)} \sigma_1 \sigma_k,
\]
with equality if and only if $E \equiv 0$. \qed

3. An Obata-type formula

Obata’s original result [1962] states that, if we have a metric $g$ on the unit sphere $\mathbb{S}^n$ that is conformal to the standard metric $g_c$ and of constant scalar curvature, then $E \equiv 0$; that is, $g$ is the standard metric $g_c$ or is obtained from it by a conformal diffeomorphism of the sphere. His method uses crucially the traceless Ricci tensor $E_{ij} = vv_{ij} - (1/n) v \Delta v \delta_{ij}$ and the Bianchi identity $\nabla^i E_{ij} = \nabla^j R$. Indeed, his main step is to prove that
\[
\int_{\mathbb{S}^n} \sum_{i,j} E_{ij} E_{ij} v^{-1} d\text{vol}_{g_c} = 0,
\]
and thus establish that $g$ is an Einstein metric on $\mathbb{S}^n$.

This same argument was generalized for constant $\sigma_k$ (instead of constant $R$) by Viaclovsky [2000], with the role of $E$ played now by $L^k$ and the Bianchi identity replaced by (2-8). If the metric is defined on $\mathbb{R}^n$ instead of $\mathbb{S}^n$, an analogous argument works; however, a cutoff function $\eta$ is introduced and, in order to get the same conclusion, a careful estimate of the error terms is needed. We should also mention the work of Chang, Gursky, and Yang [2002; 2003] and of Li and Li [2002].
However, we are interested in the subcritical-problem approach of Gidas and Spruck [1981]; they have refined the computation of

$$0 \leq \int_B \sum_{i,j} E_{ij} E_{ij} v^{-\delta} \eta \, dx = \cdots$$

for any $\delta \in \mathbb{R}$. The main result of this section is the corresponding refinement for $\sigma_k$:

**Proposition 3.1.** Let $\alpha > 0$ and $n > 2k$. Take a solution $v$ of $\sigma_k(v) = v^\alpha$ in $U$, with $v \in \Gamma^+_k$, $v > 0$, and $v^{-1} \in (C^3(U)$, where $U$ is a domain in $\mathbb{R}^n$. Pick $\eta \in (C^\infty_0(U)$ and a big positive integer $\theta$. There exist constants $d_{k-s}$ such that

$$(3-1) \quad \int_U \sum_{i,j} L^k_{ij} E_{ij} v^{-\delta} \eta^\theta + \left( \frac{n-k}{n} \alpha - (1+n-\delta) \frac{k(n+2)}{2n} \right) \int_U v^\alpha |\nabla v|^2 v^{-\delta} \eta^\theta$$

$$+ (1+n-\delta) \sum_{s=1}^k d_{k-s} \int_U \sigma_{k-s} |\nabla v|^{2s+1} v^{-\delta} \eta^\theta = E_1(\eta),$$

where

$$(3-2) \quad E_1(\eta) \lesssim \left| \int_U \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \eta^{\theta-1} \right| + \sum_{s=1}^k \left| \int_U \sum_{i,j} T^k_{ij} v_i \eta_j |\nabla v|^{2s} v^{1-\delta} \eta^{\theta-1} \right|.$$  

In addition, if $\delta$ is smaller than but close enough to $n + 1$, all the coefficients in front of the integrals in (3-1) are positive.

**Proof.** One uses the inductive method developed in [González 2005b; 2005a] and the properties of $L^k$. In view of (2-4), integrate over $U$ to get

$$\int \sum_{i,j} L^k_{ij} E_{ij} v^{-\delta} \eta^\theta = \int \sum_{i,j} L^k_{ij} v_i v_j v^{1-\delta} \eta^\theta - \frac{1}{n} \int \sum_{i,j} L^k_{ij} (\Delta v) v^{1-\delta} \delta_{ij} \eta^\theta.$$  

The last term vanishes since $L^k$ is trace-free. Integrating by parts and using (2-8),

$$\int \sum_{i,j} L^k_{ij} E_{ij} v^{-\delta} \eta^\theta$$

$$= - \int \sum_{i,j} (\partial_i L^k_{ij}) v_j v^{1-\delta} \eta^\theta - (1-\delta) \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta - \int \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \eta^{\theta-1}$$

$$= - \frac{n-k}{n} \int \sum_i (\partial_i \sigma_k) v_i v^{1-\delta} \eta^\theta - (1+n-\delta) \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta$$

$$- \int \sum_{i,j} L^k_{ij} v_i \eta_j v^{1-\delta} \eta^{\theta-1}.$$
Group in $E_1(\eta)$ all the terms containing derivatives of $\eta$. Now compute, using (2-1), (2-2), and (2-3):

$$
(3-3) \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta = \frac{n-k}{n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta - \int \sum_{i,j} T^k_{ij} v_i v_j v^{-\delta} \eta^\theta
= -\frac{k}{n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j} T^{k-1}_{il} A_{ij} v_i v_j v^{-\delta} \eta^\theta
= -\frac{k}{n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j} T^{k-1}_{il} v_i v_j v^{1-\delta} \eta^\theta
- \frac{1}{2} \int \sum_{i,j} T^{k-1}_{ij} v_i v_j v^{-\delta} \eta^\theta.
$$

The middle term can be handled similarly to [González 2005b, Section 4]:

$$
(3-4) \int \sum_{i,j} T^{k-1}_{il} v_i v_j v^{1-\delta} \eta^\theta = \frac{1}{2} \int \sum_{i,l} \partial_i (|\nabla v|^2) T^{k-1}_{il} v_i v^{1-\delta} \eta^\theta
= -\frac{\delta-1}{2} \int \sum_{i,l} T^{k-1}_{il} v_i v_j |\nabla v|^2 v^{-\delta} \eta^\theta - \frac{1}{2} \int \sum_{i,l} \partial_i (T^{k-1}_{il} v_i) |\nabla v|^2 v^{1-\delta} \eta^\theta
- \frac{1}{2} \int \sum_{i,l} T^{k-1}_{il} v_i v_j |\nabla v|^2 v^{1-\delta} \eta^{\theta-1}.
$$

To eliminate the term $\partial_i (T^{k-1}_{il} v_i)$ from (3-4), just use the equality (2-5) and then substitute (3-4) into (3-3):

$$
(3-5) \int \sum_{i,j} L^k_{ij} v_i v_j v^{-\delta} \eta^\theta
= -k \frac{n+2}{2n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta - \frac{2+n-\delta}{2} \int \sum_{i,j} T^{k-1}_{ij} v_i v_j |\nabla v|^2 v^{-\delta} \eta^\theta
+ \frac{n-k+1}{4} \int \sigma_{k-1} |\nabla v|^4 v^{-\delta} \eta^\theta + E_1(\eta)
= -k \frac{n+2}{2n} \int \sigma_k |\nabla v|^2 v^{-\delta} \eta^\theta + B_{k-1} + E_1(\eta),
$$

where we have defined, for $k$ fixed and $s = 1, \ldots, k - 1$,

$$
B_{k-s} = -\frac{s+1+n-\delta}{s+1} \int \sum_{i,j} T^{k-s}_{ij} v_i v_j |\nabla v|^2 v^{-\delta} \eta^\theta + \frac{n-k+s}{2(s+1)} \int \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta^\theta.
$$
The computations in (3-4) can be redone for $T^{k-s}$, and thus

(3-6) \[ \mathcal{B}_{k-s} = \tilde{d}_{k-s} \int \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta^\theta + \tilde{c}_{k-s} \mathcal{B}_{k-s-1} + E_1(\eta), \]

with

$$\tilde{d}_{k-s} = -\frac{s+n+1-\delta}{s+1} \left( 1 + \frac{k-s}{2(s+1)} \right) + \frac{n-k+s}{2(s+1)}, \quad \tilde{c}_{k-s} = \frac{(s+n+1-\delta)(s+2)}{2(s+1)^2}.$$  

The last step is

$$\mathcal{B}_1 = d_1 \int \sigma_1 |\nabla v|^{2k} v^{-\delta} \eta + c_0 \int |\nabla v|^{2(k+1)} v^{-\delta} \eta.$$  

Substitute (3-6) into (3-5), inductively. This proves (3-1) for some constants $c_{k-s}$ and $d_{k-s}$ obtained from $\tilde{c}_{k-s}$ and $\tilde{d}_{k-s}$. Note that $c_{k-s} > 0$ if $\delta < n+1$. We also want $d_{k-s} > 0$ for $s = 1, \ldots, k$, and this is achieved when $\delta$ is close enough to $n+1$ because $n > 2k$.  

**Lemma 3.2.** With the same hypothesis as in the previous lemma,

(3-7) \[ \int_U v^{\alpha/k-\gamma} \eta^\theta \lesssim \left( -1 + \gamma - \frac{n}{2} \right) \int_U \sigma_k |\nabla v|^2 v^{-\gamma} \eta^\theta + E_2(\eta), \text{ where} \]

$$E_2(\eta) \lesssim \int_U \sum_i v_i \eta_i v^{1-\gamma} \eta^\theta.$$  

**Proof.** Since $\sigma_k(v) = v^\alpha$ and $\sigma_k \leq C(n, k) \sigma_1^k$ (Lemma 2.2), we get $\sigma_1(v) \gtrsim v^{\alpha/k}$. It is easy to see that

$$\int \sigma_1 v^{-\gamma} \eta^\theta = \left( -1 + \gamma - \frac{n}{2} \right) \int |\nabla v|^2 v^{-\gamma} \eta^\theta + E_2(\eta),$$  

and the lemma is proved.

**4. Main estimates**

Here we obtain the needed $L^p$ estimate, as a consequence of (3-1). The terms on the left-hand side of (3-1) will be “good” terms, and we will give an estimate of the error terms.

**Proposition 4.1.** Take $n > 2k$, $\alpha \in (0, k)$, and let $v$ be a solution of (1-3). We have

(4-1) \[ \int_{\rho < |x| < M \rho} v^{(k+1)/k-\delta} \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M \rho}} v^{2(k+1) - \delta} + \frac{1}{\rho^2} \int_{A_\rho \cup A_{M \rho}} v^{2+k-\delta}, \]

for $\delta$ smaller than but close enough to $n+1$, and for $A_\rho = \{ \frac{1}{2} \rho < |x| < \rho \}$ and $A_{M \rho} = \{ M \rho < |x| < 2M \rho \}$; the constants depend on $M$ but not on $\rho$. 
Proof. If we take $\alpha - \delta = -\gamma$, then $-1 - \frac{1}{2}n + \gamma > 0$, and the preceding lemma allows us, in (3-1), to replace

$$\int |\nabla v|^2 v^{\alpha - \delta} \eta^\theta \, \text{by} \, \int v^{\alpha(k+1)/k - \delta} \eta^\theta + E_2(\eta).$$

Let $\eta$ be a smooth cutoff function such that

$$\eta = \begin{cases} 1 & \text{if } \rho < |x| < M\rho, \\ 0 & \text{if } 0 < |x| < \frac{1}{2}\rho \text{ and } 2M\rho < |x|, \end{cases}$$

$|\nabla \eta| \lesssim 1/\rho$, and $|D^2 \eta| \lesssim 1/\rho^2$. The error $E_1(\eta)$ in (3-2) is of one of these two types:

$$E_{11}(\eta) \lesssim \left| \int_{A_i \cup A_{M_i}} \sum_{i,j} L_{ij} \eta_i \eta_j v^{1-\delta} \eta^{\theta-1} \right|,$$

or

$$E_{12}(\eta) \lesssim \sum_{s=1}^{k} \left| \int_{A_i \cup A_{M_i}} \sum_{i,j} T_{ij}^{k-s} \eta_i \eta_j |\nabla v|^{2s} v^{1-\delta} \eta^{\theta-1} \right|.$$

These will be handled as in the proof of [González 2005a, Theorem 1.1], but here we present a clearer proof for this particular cutoff.

To understand $E_{11}$, substitute $L_{ij} = (1 - k/n) \sigma_k I - T_{ij}$, so that

$$E_{11}(\eta) \lesssim \int_{A_i \cup A_{M_i}} \sigma_k v_i \eta_i v^{1-\delta} \eta^{\theta-1} + \int_{A_i \cup A_{M_i}} T_{ij}^{k} \eta_i \eta_j v^{1-\delta} \eta^{\theta-1}. \tag{4-2}$$

We cannot use the standard trick — to estimate the norm $\|T^k\| \lesssim \sigma_k$ as in Lemma 2.7 — because we cannot conclude that $T^k$ is positive definite from the information on $\sigma_1, \ldots, \sigma_k$, and we need to write everything in terms of smaller $T^{k-s}$’s. An inductive process is needed.

Substitute $T_{ij}^k = \sigma_k \delta_{ij} - A_i T_{ij}^{k-1}$ and $A_{ij} = v_i v_j - \frac{1}{2} |\nabla v|^2 \delta_{ij}$ in (4-2). Together with Lemma 2.7, we have

$$E_{11}(\eta) \lesssim \int \sigma_k |\nabla v| |\nabla \eta| v^{1-\delta} \eta^{\theta-1} + \int \sigma_k |\nabla v|^3 |\nabla \eta| v^{1-\delta} \eta^{\theta-1}$$

$$+ \left| \int T_{ij}^{k-1} v_i v_j v^{2-\delta} \eta^{\theta-1} \right|.$$

For the last term, proceed as in (3-4):

$$\int T_{ij}^{k-1} v_i v_j v^{2-\delta} \eta^{\theta-1} = \frac{1}{2} \int \delta_i (|\nabla v|^2) T_{ij}^{k-1} \eta_j v^{2-\delta} \eta^{\theta-1}$$

$$= -\frac{1}{2} \int (\delta_i T_{ij}^{k-1}) |\nabla v|^2 \eta_j v^{2-\delta} \eta^{\theta-1} - \frac{1}{2} \int T_{ij}^{k-1} |\nabla v|^2 \eta_j v^{2-\delta} \eta^{\theta-2}$$

$$- \frac{2-\delta}{2} \int T_{ij}^{k-1} \eta_j v |\nabla v|^2 v^{1-\delta} \eta^{\theta-1}.$$
Note that (2-5) helps to compute $\partial T_{ij}^{k-1}$, and thus, from (4-4) and Lemma 2.7,

\begin{equation}
\int T_{ij}^{k-1} v_{il} v_{lj} v^{2-\delta} \eta^{\theta-1} \lesssim \int \sigma_{k-1} |D^2 \eta| |\nabla v|^2 v^{2-\delta} \eta^{\theta-2} + \int \sigma_{k-1} |\nabla \eta| v^{1-\delta} \eta^{\theta-1}.
\end{equation}

Young’s inequality for a small $\epsilon$, together with (4-3) and (4-5), gives

\begin{equation}
E_{11}(\eta) \lesssim \epsilon \int \sigma_k |\nabla v|^2 \eta^{\theta} v^{-\delta} + \frac{C_\epsilon}{\rho^2} \int_{A_{\rho} \cup A_{M_\rho}} \sigma_k v^{2-\delta} \eta^{\theta-2} + \epsilon \int \sigma_{k-1} |\nabla v|^4 \eta^{\theta} v^{-\delta} + \frac{C_\epsilon}{\rho^4} \int_{A_{\rho} \cup A_{M_\rho}} \sigma_{k-1} v^{4-\delta} \eta^{\theta-4}.
\end{equation}

To finish the estimate, we just need (4-7) from the next lemma, applied iteratively:

\begin{equation}
E_{11}(\eta) \lesssim \epsilon \sum_{s=0}^{k} \int \sigma_{k-s} |\nabla v|^{2(s+1)} \eta^{\theta} v^{-\delta} + \frac{C_\epsilon}{\rho^{2(s+1)}} \int_{A_{\rho} \cup A_{M_\rho}} v^{2(s+1)-\delta}.
\end{equation}

The estimate for $E_{12}(\eta)$ follows in a similar manner. For the error in $E_2(\eta)$, defined in 3-7, we use Young’s inequality with $p = q = 2$:

\begin{equation}
E_2(\eta) \lesssim \int |\nabla v| |\nabla \eta| v^{1-\nu} \eta^{\theta-1} \lesssim \epsilon \int |\nabla v|^2 v^{\alpha-\delta} \eta^{\theta} + \frac{C_\epsilon}{\rho^2} \int_{A_{\rho} \cup A_{M_\rho}} v^{2+\alpha-\delta}.
\end{equation}

Putting it all together in (3-1), and taking into account that $\sum L_{ij} \geq 0$,

\begin{equation}
\int_{B < |x| < M_\rho} v^{\alpha(k+1)/k-\delta} \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_{\rho} \cup A_{M_\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^{2}} \int_{A_{\rho} \cup A_{M_\rho}} v^{2+\alpha-\delta}.
\end{equation}

Lemma 4.2. For all $\epsilon > 0$ and $s = 0, \ldots, k - 1$, and for $\theta$ a big positive integer,

\begin{equation}
\frac{1}{\rho^{2(s+1)}} \int \sigma_{k-s} v^{2(s+1)-\delta} \eta^{\theta-2(s+1)} \leq \epsilon \int \sigma_{k-s} |\nabla v|^{2(s+2)} \eta^{\theta} v^{-\delta} + \frac{C_\epsilon}{\rho^{2(s+2)}} \int |\nabla \eta| \eta^{\theta-2(s+2)} v^{2(s+2)-\delta}.
\end{equation}
Proof. First use the “divergence” formula (2-6) for $\sigma_{k-s}$ with integration by parts:

\[(4-8) \quad (k-s) \int \sigma_{k-s} v^{2(s+1)-\delta} \eta^{\theta-2(s+1)}
\]

\[= \frac{n-k+s+1}{2} \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta}
\]

\[- (n+2(s+1)-\delta+1) \int T^{k-s-1}_{ij} v_i v_j \eta^{\theta-2(s+1)} v^{2(s+1)-\delta}
\]

\[- \int T^{k-s-1}_{ij} v_i \eta^{\theta-2(s+1)-1} v^{2(s+1)-\delta+1}.
\]

Use Lemma 2.7 again to bound the norm of the Newton tensor in (4-8):

\[(4-9) \quad \int \sigma_{k-s} v^{2(s+1)-\delta} \eta^{\theta-2(s+1)} v^{2(s+1)-\delta}
\]

\[\lesssim \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta}
\]

\[+ \frac{1}{\rho} \int \sigma_{k-s-1} |\nabla v| \eta^{\theta-2(s+1)-1} v^{2(s+1)-\delta+1}.
\]

Young’s inequality with $\varepsilon$ and $p = s+2$, $q = (s+2)/(s+1)$ now reads

\[(4-10) \quad \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta}
\]

\[\lesssim \varepsilon \rho^{2(s+1)} \int \sigma_{k-s-1} |\nabla v|^{2(s+1)} \eta^{\theta-\delta} v^{-\delta} + C \varepsilon \rho^2 \int \sigma_{k-s-1} \eta^{\theta-2(s+2)} v^{2(s+2)-\delta}.
\]

For the second part in (4-9), take $p = 2(s+2)$ and $q = \frac{2(s+2)}{2(s+2)-1}$:

\[(4-11) \quad \frac{1}{\rho} \int \sigma_{k-s-1} |\nabla v| \eta^{\theta-2(s+2)-1} v^{2(s+1)-\delta+1}
\]

\[\lesssim \varepsilon \rho^{2(s+1)} \int \sigma_{k-s-1} |\nabla v|^{2(s+2)} \eta^{\theta-\delta} v^{-\delta} + C \varepsilon \rho^2 \int \sigma_{k-s-1} \eta^{\theta-2(s+2)} v^{2(s+2)-\delta}.
\]

The lemma is proved by substituting (4-10) and (4-11) into (4-9). 

\[\square\]

Proposition 4.3. For $n \geq 2(k+1)$, $\alpha \in (0, k)$, and $v$ a solution of (1-3), we have

\[(4-12) \quad \int_{\rho < |x| < M \rho} v^{\alpha(k+1)/k-\delta} \leq C \rho^{n-(\delta-\alpha(k+1)/k)(1-\alpha/2k)},
\]

where $C$ depends on $M$ and $\delta$, but not on $\rho$.

Proof. Use Hölder’s inequality with

\[p = \frac{\delta - \alpha(k+1)/k}{\delta - 2(k+1)} \quad \text{and} \quad q = \frac{p}{p-1}
\]

to get

\[(4-13) \quad \frac{1}{\rho^{2(k+1)}} \int_{A_{\rho \cup A_{M \rho}}} v^{2(k+1)-\delta} \leq \varepsilon \int_{A_{\rho \cup A_{M \rho}}} v^{\alpha(k+1)/k-\delta} + C \varepsilon \rho^{n-2(k+1)q},
\]
for some small $\varepsilon$, to be chosen later. Also, a Hölder estimate with

$$
\hat{p} = \frac{\delta - \alpha(k + 1)/k}{\delta - 2 - \alpha} \quad \text{and} \quad \tilde{q} = \frac{\hat{p}}{\hat{p} - 1}
$$

gives

$$
\frac{1}{\rho^2} \int_{A_{\rho} \cup A_{M_{\rho}}} v^{2 + \alpha - \delta} \leq \varepsilon \int_{A_{\rho} \cup A_{M_{\rho}}} v^{\alpha(k + 1)/k - \delta} + C\varepsilon \rho^{n - 2\tilde{q}}.
$$

When $\alpha \in (0, k)$ and $\delta$ is close enough to $n + 1$, then $p, \hat{p} > 1$. Look at the powers of $\rho$ in (4-13) and (4-14):

$$
n - 2(k + 1)q = n - 2\tilde{q} = n - \frac{\delta - \alpha(k + 1)/k}{1 - \alpha/2k}.
$$

Choosing $\varepsilon$ small enough, we conclude from (4-1) that

$$
\int_{\rho < |x| < M\rho} v^{\alpha(k + 1)/k - \delta} \leq C\rho^{n - (\delta - \alpha(k + 1)/k)/(1 - \alpha/2k)}.
$$

5. Proof of Theorem 1.1

The next proposition is similar to the study of the critical problem in [González 2005a]. In particular, a volume finiteness condition gives regularity near the singularity.

**Proposition 5.1.** Take $\alpha \in (0, k)$ and $n > 2k$, and let $v$ be a solution of (1-3) on $B_\rho(x_0) \subset B$, with $v > 0$ and $v \in \Gamma_+^k$. If

$$
\int_{B_\rho(x_0)} v^{(\alpha - 2k)n/(2k)} \leq a
$$

for some small enough $a$ (not depending on $\rho$), then

$$
\sup_{B_{\rho/2}(x_0)} |v^{-1}| \leq \frac{C}{\rho^{n/p}} \|v^{-1}\|_{L^p(B_\rho(x_0))}
$$

for all $p > (n - 2k)k/(k + 1)$. In particular, if

$$
\int_{\varepsilon < |x| < 1} v^{(\alpha - 2k)n/(2k)} < C < \infty
$$

for some constant $C$ independent of $\varepsilon$, the function $v$ is bounded near the origin.

**Proof.** The argument is similar to [González 2005a, Theorem 1.2] for the critical problem. Condition (5-2) is analogous to its volume smallness condition.

**Proof of Theorem 1.1.** Fix $x_0$ small enough and take $2R = |x_0|$. First, note that Hölder estimates with

$$
r = \frac{\delta - (k + 1)/k}{(2k - \alpha)n/(2k)} > 1 \quad \text{and} \quad 1 = \frac{1}{r} + \frac{1}{s}
$$


give, independently of $x_0$,

\begin{equation}
\int_{B_R(x_0)} v^{(\alpha-2k)n/(2k)} \leq \left( \int_{R \leq |x| \leq 3R} v^{(k+1)/k-\delta} \right)^{1/r} \varepsilon^{n/s} \lesssim R \left( \frac{n-\delta-\alpha(k+1)/k}{1-\alpha/2k} \right)^{1/r} \varepsilon^{n/s} \lesssim R^0 < \infty.
\end{equation}

We cannot apply Proposition 5.1 directly to $v$. However, we could have started with the function $\tilde{v}(y) = A^{2k/(2k-\alpha)}v(y/A)$ that still satisfies the same equation $\sigma_k(\tilde{v}) = \tilde{v}^{\alpha}$, for some $A$ big enough and of the form $A = (\text{constant}) \int_{R \leq |x| \leq 3R} v^{(\alpha-2k)n/(2k)}$.

Since we are interested only in the local behavior near zero, we can assume that (5-1) gives an estimate for $v$,

$$\sup_{B_{R/2}(x_0)} |v^{-1}| \leq \frac{C |x_0|^n}{R^{n/p}} \|v^{-1}\|_{L^p(B_R(x_0))}$$

for all $p > (n-2k)k/(k+1)$, and with $C$ depending on

$$\int_{R \leq |x| \leq 3R} v^{(\alpha-2k)n/(2k)}.$$

This estimate is uniformly bounded by a constant, independently of $R$, because of (5-3). It is also true that

\begin{equation}
\sup_{B_{R/2}(x_0)} |v^{-1}| \leq \frac{C |x_0|^n}{R^{n/p}} \|v^{-1}\|_{L^p(\{R \leq |x| \leq 3R\})}
\end{equation}

for all $p > (n-2k)k/(k+1)$. Set $p = \delta - \alpha(k+1)/k$; this choice is valid when $\alpha \in (0, k)$ and $n > 2k$. Use (4-12) again:

$$\int_{R \leq |x| \leq 3R} v^{-p} \leq C |x_0|^{-\frac{p}{1-\alpha(2k)}},$$

and thus, from (5-4), we arrive at

$$v^{-1}(x_0) \leq \frac{C}{|x_0|^{2k/(2k-\alpha)}},$$

as desired. \qed

**Corollary 5.2 (Harnack).** Under these hypotheses, there exists $M_0 > 0$ such that, for all $\rho > 0$ and $M \leq M_0$,

\begin{equation}
\sup_{\rho \leq |x| \leq \rho M} v^{-1} \leq C \inf_{\rho \leq |x| \leq \rho M} v^{-1},
\end{equation}

where $C$ is independent of $v$, $\rho$, and $M$. 

Proof. Once we get a supremum estimate (5-4) for a ball, standard elliptic theory yields the infimum estimate. If we write \( v^{-2} = u^{2/(n-2)} \), then \( u \) is a superharmonic function. To finish, use a covering argument for the annulus \( \{ \rho \leq |x| \leq \rho M \} \).

**Corollary 5.3.** If \( v \) is a solution of (1-3), then either \( v^{-1} \) is bounded near the origin, or \( v^{-1}(x) \to \infty \) as \( x \to 0 \).

**Proof.** The argument follows the steps of [Gidas and Spruck 1981, Corollary 3.3], by using the second part of Proposition 5.1. □

### 6. Proof of Theorem 1.2

We have proved the estimate

\[
(6-1) \quad v^{-1}(x) \leq \frac{C}{|x|^{2k/(2k-\alpha)}}
\]

Now we would like to get the opposite inequality. Suppose that

\[
\lim_{x \to 0} \inf |x|^{2k/(2k-\alpha)} v^{-1}(x) = 0.
\]

From the Harnack estimate (5-5) follows that

\[
(6-2) \quad \lim_{x \to 0} |x|^{2k/(2k-\alpha)} v^{-1}(x) = 0.
\]

We want to see that in this case the function \( v^{-1} \) is bounded near the origin and thus that the theorem follows. It suffices to establish (5-2).

Let’s review two results from [González 2005a]:

**Proposition 6.1.** Let \( v \) be a solution with \( v^{-1} \in \mathcal{C}^3(U) \), \( v > 0 \), \( v \in \Gamma_k^+ \), and \( n > 2k \). For all \( \varphi \in \mathcal{C}^\infty_0(U) \) and \( \theta \) a big positive integer,

\[
(6-3) \quad \int_U \sigma_k \varphi^{\theta} v^{-\gamma} \geq \sum_{s=1}^k c_{k-s}(\gamma) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^{\theta} v^{-\gamma} + E(\varphi),
\]

where

\[
(6-4) \quad E(\varphi) \lesssim \sum_{s=1}^k \left| \int_U \sum_{i,j} T_{ij}^{k-s} v \varphi_i |\nabla v|^{2(s-1)} \varphi^{\theta-1} v^{1-\gamma} \right|,
\]

and where the coefficients \( c_{k-s}(\gamma) \) are positive for all \( \gamma \) with

\[
(6-5) \quad \gamma > n - \frac{n - 2k}{k+1},
\]

**Proposition 6.2.** For all \( \varepsilon > 0 \), the error term (6-4) can be estimated by

\[
E(\varphi) \leq \varepsilon \sum_{s=1}^k \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^{\theta} v^{-\gamma} + C_\varepsilon \sum_{U_k} \int_U \sigma_{k-s}(\varphi) \frac{\theta^{\theta-1} v^{2k-\gamma}}{
\]

This completes the proof of Theorem 1.2.
where the $U_k(\varphi)$'s are groups of derivatives of $\varphi$ of order $2k$, and $\alpha_k \in \mathbb{R}$ are constants depending on each of the $U_k$'s. These concepts are defined inductively in the following manner:

- For a fixed $s = 1, \ldots, k$, the starting point is
  \[ U_s(\varphi)\varphi^{\alpha_s} = |\nabla \varphi|^{2s} \varphi^{-2s}. \]

- For each integer $l = 0, 1, \ldots$ and $m = s + l$, and once given $U_m \varphi^{\alpha_m}$, the following step is of one of these three shapes:

\[
U_{m+1} \varphi^{-\alpha_{m+1}} = \begin{cases} 
U^{(m+1)/m}_m \varphi^{-\alpha_m(m+1)/m}, \\
|\nabla U_m|^{2(m+1)-1} \varphi^{-\alpha_m^{2(m+1)-1}}, \\
(|\nabla \varphi|^2 U_m) \varphi^{-\alpha_m-2}.
\end{cases}
\]  

- The ending point is when $m = s + l$ reaches $k$.

We will use (6-3) for a suitable cutoff function. Take $\varphi = \eta r$ with $\eta \in \mathcal{C}_0^\infty(B\setminus\{0\})$, such that

\[
\eta = \begin{cases} 
1 & \text{if } \varepsilon < |x| < R, \\
0 & \text{if } |x| < \frac{1}{2}\varepsilon \text{ and } |x| > 2R,
\end{cases}
\]

and so that the derivatives have a good bound on $\frac{1}{2}\varepsilon < |x| < \varepsilon$ and $R < |x| < 2R$.

The value of $\gamma$ will be chosen later. Rewrite (6-3) as

\[
(6-7) \quad \int \sigma_k v^{-\gamma} \varphi^\theta \gtrsim \sum_{s=1}^k \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \varphi^\theta - \int T_{ij}^{k-1} v_i \varphi_j \varphi^\theta v^{1-\gamma} + \tilde{E}(\varphi),
\]

with

\[
\tilde{E}(\varphi) \lesssim \sum_{s=2}^k \left| \int T_{ij}^{k-s} v_i \varphi_j |\nabla v|^{2(s-1)} \varphi^\theta v^{1-\gamma} \right|,
\]

since we will look more carefully at the term in $T^{k-1}$. Integration by parts gives

\[
-\int \sum_{i,j} T_{ij}^{k-1} v_i \varphi_j \varphi^{\theta-1} v^{1-\gamma} = -\frac{1}{2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \partial_i (v^{2-\gamma}) \varphi_j \varphi^{\theta-1}
\]

\[
= \frac{1}{2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi^{\theta-1} v^{2-\gamma} + \frac{1}{2-\gamma} \int \sum_{i,j} \partial_i (T_{ij}^{k-1}) \varphi_j \varphi^{\theta-1} v^{2-\gamma}
\]

\[
+ \frac{\theta-1}{2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi^{\theta-2} v^{2-\gamma}.
\]
Substituting (2-5) into this, we get
\[-(n+2-\gamma) \int \sum_{i,j} T_{ij}^{k-1} v_i \phi_j \phi^{\theta-1} v^{1-\gamma} = \int \sum_{i,j} T_{ij}^{k-1} \phi_i \phi_j \phi^{\theta-1} v^{2-\gamma} - (n-k+1) \int \sum_i \sigma_{k-1} v_i \phi_i \phi^{\theta-1} v^{1-\gamma} + (\theta-1) \int \sum_{i,j} T_{ij}^{k-1} \phi_i \phi_j \phi^{\theta-2} v^{2-\gamma}.
\]

Now substitute this into (6-7):
\[(6-8) \int \sigma_k \phi^\theta v^{-\gamma} \gtrsim \sum_s^{k} \int \sigma_{k-s} \phi |\nabla v|^{2s} \phi^\theta v^{-\gamma} + \frac{1}{n+2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \phi_i \phi_j \phi^{\theta-2} v^{2-\gamma} + \frac{\theta-1}{n+2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \phi_i \phi_j \phi^{\theta-2} v^{2-\gamma} - \frac{n-k+1}{n+2-\gamma} \int \sum_i \sigma_{k-1} v_i \phi_i \phi^{\theta-1} v^{1-\gamma} + 1 \phi.
\]

Group all the error terms into
\[E(\phi) \lesssim \sum_s^{k} \int \sigma_{k-s} \phi |\nabla v| \phi^{\theta-1} v^{1-\gamma}.
\]

Compute
\[\phi_i = \frac{x_i}{r} \eta + E_1(\phi),
\]
\[\phi_{ij} = r^{-1} \left( -\frac{x_i x_j}{r^2} + \delta_{ij} \right) \eta + E_1(\phi),
\]
\[\sum_{i,j} T_{ij}^{k-1} \phi_{ij} = r^{-1} \left( -\sum_{i,j} T_{ij}^{k-1} \frac{x_i x_j}{r^2} + (n-k+1) \sigma_{k-1} \right) \eta + E_1(\phi).
\]

Since \( T^{k-1} \) is positive definite and trace \( T^{k-1} = (n-k+1) \sigma_{k-1} \), as long as we keep
\[1 < \theta \]
we have
\[\sum_{i,j} T_{ij}^{k-1} \phi_{ij} + (\theta-1) \phi_i \phi_j r^{-\theta} \geq C(\theta) \sigma_{k-1} r^{-1} \eta^2 + E_1(\phi),
\]
for some \( C(\theta) > 0 \). If we keep \( \gamma < n+2 \), we can conclude from (6-8) that
\[(6-9) \ E(\phi) + E_1(\phi) + \int \sigma_k \phi^\theta v^{-\gamma} \gtrsim \sum_{s=1}^{k} \int \sigma_{k-s} \phi |\nabla v|^{2s} \phi^\theta v^{-\gamma} + \int \sigma_{k-1} r^{-2} \phi^\theta v^{2-\gamma}.
\]
We have not been very precise with the errors $E_1(\varphi)$; however, they are of a similar type to $E(\varphi)$ and can be treated in the same manner. Note that, in the positive cone,
\[ \sigma_{k-1} \gtrsim \sigma_k^{(k-1)/k} = v^{\alpha(k-1)/k}, \]
so, with (6-9) we have actually proved
\[ (6-10) \quad E(\varphi) \gtrsim \int (v^{\alpha(k-1)/k+2-\gamma} r^{-2} - v^{\alpha-\gamma}) \varphi^\theta + \sum_{s=1}^k \int \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma}. \]

To handle $E(\varphi)$, we need to control the error terms that appear in Proposition 6.2. Using Lemma 6.3 below,
\[ (6-11) \int U_k(\varphi) \varphi^{\theta-\alpha} v^{2k-\gamma} \]
\[ \lesssim \int r^{-2k} \varphi^\theta v^{2k-\gamma} + \frac{1}{\varepsilon^{2k}} \int_{|\varepsilon/2| < |x| < \varepsilon} r^\theta v^{2k-\gamma} + \frac{1}{R^{2k}} \int_{R < |x| < 2R} r^\theta v^{2k-\gamma} \]
Looking one-by-one at the terms above, notice that
\[ \frac{1}{\varepsilon^{2k}} \int_{|\varepsilon/2| < |x| < \varepsilon} r^\theta v^{2k-\gamma} \to 0 \quad \text{as } \varepsilon \to 0, \]
by using the previous estimate (6-1) and the definition of $\eta$, and as soon as
\[ (6-12) \quad \gamma > n - \alpha \left( \frac{n-2k}{2k} \right). \]
A similar argument gives
\[ \frac{1}{R^{2k}} \int_{R < |x| < 2R} r^\theta v^{2k-\gamma} \leq C. \]
The other integral in (6-11) is bounded by
\[ \int r^{-2k} \varphi^\theta v^{2k-\gamma} \lesssim \int (v^{\alpha(k-1)/k+2-\gamma} r^{-2} - v^{\alpha-\gamma}) (v^{-\alpha(k-1)/k+2-\gamma} r^{2-2k}) \varphi^\theta. \]
Our assumption (6-2) yields
\[ v^{-\alpha(k-1)/k+2-\gamma} r^{2-2k} = o(1), \]
and thus from (6-10) we obtain
\[ C \gtrsim \int (v^{\alpha(k-1)/k+2-\gamma} r^{-2} - v^{\alpha-\gamma}) \varphi^\theta. \]
Again, because of (6-2), we have
\[ r^{2k} v^{\alpha/2-k} = o(1). \]
Theorem 1.1 gives
Comparing the orders, we quickly obtain
\begin{equation}
(6-13) \quad \int v^{\alpha(k-1)/k} r^{-2} v^{\alpha - \gamma} \phi^\theta < \infty.
\end{equation}

This is precisely the term (5-2) that we need to estimate, because
\begin{equation}
(6-14) \quad \int_{\varepsilon \leq |x| \leq R} v^{(\alpha - 2) n/(2k)} = \int v^{(\alpha - 2) n/(2k)} v^{-\alpha(k-1)/k - n + an/(2k) - 2 + \gamma} \eta^\theta \leq \int v^{(\alpha - k)/(k - 1)} r^{-2} v^{\alpha - \gamma} \phi^\theta,
\end{equation}

after using Theorem 1.1 and choosing \( \theta \) and \( \gamma \) so that
\begin{equation}
(6-15) \quad \left(-\alpha \frac{k-1}{k} - n + \alpha \frac{n}{2k} - 2 + \gamma\right) \left(\frac{2k}{2k - \alpha}\right) = -2 + \theta,
\end{equation}
that is, picking
\[ \gamma = n - \alpha \left(\frac{n - 2k}{2k}\right) + \theta \left(1 - \frac{\alpha}{2k}\right). \]
This is an admissible value for \( \gamma \) because, when \( \alpha < 2k/(k + 1) \), it can be chosen to satisfy (6-5), (6-12), \( \gamma < n + 2 \), and \( \theta > 1 \).

**Lemma 6.3.** For the cutoff \( \varphi = r \eta \) constructed in the previous proof,
\[ U_k(\varphi) \varphi^{\theta - a_k} \leq r^{-2k} \varphi^\theta + e^{-2k} r^\theta \chi_{[\varepsilon/2 \leq |x| < \varepsilon]} + R^{-2k} r^\theta \chi_{[R \leq |x| < 2R]} \]

**Proof.** The definition of the \( U_k \) was given in Proposition 6.2. We are just interested in the orders of \( r \) and \( \varepsilon \). For fixed \( s = 1, \ldots, k \), the initial step is
\[ U_s(\varphi) \varphi^{\theta - 2s} = |\nabla \varphi|^{2s} \varphi^{\theta - 2s} \leq |\nabla \varphi|^{2s} \varphi^{\theta - 2s} \eta^{2s} + |\nabla \eta|^{2s} r^{2s} \varphi^{\theta - 2s} \leq r^{-2s} \varphi^\theta + e^{-2s} r^\theta \eta^{2s}.\]
Next, assume that the result is true for \( m = s + l \):
\[ U_m(\varphi) \varphi^{\theta - am} \leq r^{-2m} \varphi^\theta + e^{-2m} r^\theta \eta^{2m}.\]
The proof for \( m + 1 \) follows easily from (6-6).

**References**


A. Li and Y. Li, “Further results on Liouville type theorems for some conformally invariant fully nonlinear equations”, 2003. math.AP/0301254


MARÍA DEL MAR GONZÁLEZ
UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS
1 UNIVERSITY STATION C1200
AUSTIN, TX 78712–0257
UNITED STATES
mgonzale@math.utexas.edu
http://www.ma.utexas.edu/users/mgonzale/