CLASSIFICATION OF SINGULARITIES
FOR A SUBCRITICAL FULLY NONLINEAR PROBLEM

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We study isolated singularities for a fully nonlinear elliptic PDE of subcritical type. This equation appears in conformal geometry when dealing with the $k$-curvature of a locally conformally-flat manifold. (The $k$-curvature generalizes scalar curvature.) We give a classification result: either the function is bounded near the singularity, or it has a specific asymptotic behavior.

1. Introduction

The study of singularities for the subcritical problem

\[(1-1) \quad -\Delta u = u^\beta \quad \text{in } B\{0\}, \quad \beta \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right), \]

has received a lot of attention. In particular, Gidas and Spruck [1981] gave a classification result: a positive solution of (1-1) with a nonremovable singularity at zero must behave like

\[ u(x) = \left(1 + o(1)\right) \frac{c_0}{|x|^{2/(\beta-1)}} \quad \text{near } x = 0, \]

for some $c_0 = c_0(\beta, n)$. In this paper, we deal with a more general subcritical equation, of the form

\[(1-2) \quad \sigma_k(A^g) = v^\alpha \quad \text{in } B\{0\}, \quad \alpha > 0, \]

where $g_v = v^{-2}|dx|^2$ for $v > 0$ is a locally conformally-flat metric on the unit ball $B \subset \mathbb{R}^n$, with an isolated singularity at the origin. For a general metric $g$, the matrix $A^g$ is given by $A^g = g^{-1} \tilde{A}^g$, where $\tilde{A}^g$ is the Schouten tensor

\[ \tilde{A}^g_{ij} = \frac{1}{n-2} \left( \text{Ric}_{ij} - \frac{1}{2(n-1)} R g_{ij} \right), \]


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while $Ric$ and $R$ denote the Ricci tensor and the scalar curvature of $g$. In the metric $g_v$, the Schouten tensor becomes

$$A^{g_v} = v(D^2 v) - \frac{1}{2} |\nabla v|^2 I.$$  

The curvatures $\sigma_k$ are defined as symmetric functions of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the $(1, 1)$-tensor $A^g$,

$$\sigma_k := \sigma_k(A^g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$  

The scalar curvature is simply

$$\sigma_1 = \lambda_1 + \cdots + \lambda_n = \frac{1}{2(n-1)} R.$$  

Problem (1-2) for $k = 1$ becomes the well known (1-1): if we write $u^{4/(n-2)} = v^{-2}$ and $1 + (n/2) - \beta(n-2)/2 = \alpha$, the two problems are equivalent. Note that the critical exponent is $\beta = (n+2)/(n-2)$, or $\alpha = 0$.

For a general $k$, we are dealing with a fully nonlinear equation of second order. The problem is elliptic in the positive cone

$$\Gamma_1^+ = \{ v \mid \sigma_1(A^{g_v}), \ldots, \sigma_k(A^{g_v}) > 0 \},$$

but, in general, not uniformly elliptic. However, it still carries an “almost” divergence structure

$$m \sigma_m = v \partial_j (v_i T^m_{ij}^{m-1}) - n T^m_{ij}^{m-1} v_i v_j + \frac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2,$$

where $T^m_{ij}$ denotes the Newton tensor (2-1). This was explored in [González 2005b].

Our main result is a classification of the isolated singularities of (1-2):

**Theorem 1.1.** Let $\alpha \in (0, k)$ and $n > 2(k+1)$. If $v$ is a solution of

$$(1-3) \quad \sigma_k(v) = v^\alpha \quad \text{in } B \backslash \{0\},$$

with $v > 0$, $v \in \Gamma_1^+$, and $v^{-1} \in C^3(B \backslash \{0\})$, then

$$v^{-1}(x) \leq \frac{C}{|x|^{2k/(2k-\alpha)}} \quad \text{near } x = 0.$$  

**Theorem 1.2.** Let $v$ be a solution of (1-3) for $\alpha \in (0, 2k/(k+1))$ and $n > 2(k+1)$, with $v^{-1} \in C^3(B \backslash \{0\})$. If the function $v^{-1}$ is not bounded near the origin, then there exist $c_1, c_2 > 0$ such that

$$\frac{c_1}{|x|^{2k/(2k-\alpha)}} \leq v^{-1}(x) \leq \frac{c_2}{|x|^{2k/(2k-\alpha)}} \quad \text{near } x = 0.$$
The local behavior of singularities for the critical problem \( \sigma_k(v) = 1 \) has been addressed in [González 2005a]. There, we gave a sufficient condition for the function to be bounded near the singularity: the finiteness of the volume of the metric \( g_v \) (when \( n > 2k \)). The same result was obtained by Han [2004] for \( n = 2k \). For the Laplacian problem \( (k = 1) \), a complete classification of solutions was obtained by Caffarelli, Gidas, and Spruck [1989].

At the time this paper was submitted, it was conjectured that a similar classification result was true also for \( \sigma_k \), where \( n > 2k \). This has now been proved [Li 2006]. In the case \( n < 2k \), all the singularities are removable [Gursky and Viaclovsky 2005].

One of the motivations for the study of (1-1) is that it appears in the resolution of the Yamabe problem (for a very good survey, see [Lee and Parker 1987]). We can establish an analogous \( k \)-Yamabe problem: find the infimum over all the metrics \( g_v = v^{-2}g_0 \) with \( v > 0 \) of the functional

\[
\mathcal{F}_k(g) = \frac{1}{\text{vol}(g)^{(n-2k)/n}} \int_M \sigma_k(A^k) \, d\text{vol}_g.
\]

This functional was first introduced by Viaclovsky [2000], and it generalizes the Yamabe functional. Its Euler equation is precisely \( \sigma_k(v) = 1 \).

The global subcritical problem has been understood by Li and Li [2003]. Indeed, if \( v \) is a positive solution of

\[
\sigma_k(v) = v^\alpha \quad \text{in} \quad \mathbb{R}^n, \quad \alpha \geq 0,
\]

with \( v^{-1} \in \mathcal{C}^2(\mathbb{R}^n) \), then either \( v \) is constant, or \( \alpha = 0 \) and

\[
v^{-1}(x) = \frac{a}{1 + b^2 |x - \bar{x}|^2},
\]

for some \( \bar{x} \in \mathbb{R}^n \) and some positive constants \( a \) and \( b \).

The methods of Gidas and Spruck [1981] for the problem with \( k = 1 \) can be generalized to our case. The key ingredient in the present paper is to understand the structure of \( \sigma_k \) and, in particular, to replace the traceless Ricci tensor by the traceless \( k \)-Newton tensor (2-2).

The paper is structured as follows: in Section 2 we give some properties of \( \sigma_k \) that will be crucial in the proofs. We use the divergence structure of \( \sigma_k \) (2-5), an inductive process (2-7), and the properties of the traceless Newton tensor (2-2).

In Section 3 we establish the expression that will allow us to obtain the necessary \( L^p \) estimates, through a generalization of an argument due to Obata and very successfully used by Chang, Gursky, and Yang [2002] and by Li and Li [2002]. In particular, we give a more refined formula (3-1) that is precisely the missing
ingredient for the critical problem. The \( L^p \) estimates are in Section 4, while in the last two sections we prove the theorems.

**Remark 1.3.** We believe that the theorems are also true for \( n = 2k + 1 \), but, as in the case of [Gidas and Spruck 1981], one needs different estimates in (4-12).

**Remark 1.4.** We make the regularity assumption \( v^{-1} \in C^3(B\backslash\{0\}) \). However, many of the arguments use integral estimates and only require that \( v^{-1} \) is in some suitable Sobolev space; for instance, the whole of Section 4.

## 2. Algebraic properties of \( \sigma_k \)

For a general \( n \times n \) matrix \( A \), take its eigenvalues \( \lambda_1, \ldots, \lambda_n \) and construct the symmetric functions \( \sigma_k \), as well as the \( k \)-th Newton tensor

\[
T^k := \sigma_k - \sigma_{k-1} A + \cdots + (-1)^k A^k = \sigma_k I - T^{k-1} A
\]

and the traceless Newton tensor

\[
L^k := \frac{n-k}{n} \sigma_k I - T^k.
\]

**Remark 2.1.** Take \( \sigma_0 := 1 \) and \( T^0_{ij} := \delta_{ij} \). Although the standard notation for a \((1,1)\)-tensor is \( A^i_j \), we write both indices as subindices without risking confusion.

**Lemma 2.2** [Gårding 1959; Reilly 1973].

1. \((n - k) \sigma_k = \text{trace} \ T^k\);
2. \((k + 1) \sigma_{k+1} = \text{trace} (AT^k)\);
3. \( \text{trace} \ L^k = 0 \);
4. if \( \sigma_1, \ldots, \sigma_k > 0 \), then \( T^m \) is positive definite for \( m = 1, \ldots, k - 1 \);
5. if \( \sigma_1, \ldots, \sigma_k > 0 \), then \( \sigma_k \leq C_{n,k}(\sigma_1)^k \).

In particular, if \( A = A^g \) for \( g_v = v^{-2}|dx|^2 \), then the Schouten tensor becomes

\[
A_{ij} = v_{ij} v - \frac{1}{2} |\nabla v|^2 \delta_{ij},
\]

while the traceless Ricci tensor (strictly speaking, a constant multiple of the actual traceless Ricci tensor) is now

\[
E_{ij} := L^1_{ij} = v_{ij} - \frac{1}{n} v \Delta v \delta_{ij}.
\]

**Lemma 2.3** [Viaclovsky 2000]. Let \( g_v = v^{-2}|dx|^2 \). The Newton tensor \( T^m \) for \( m \leq n - 1 \) is divergence-free with respect to this metric; that is,

\[
\sum_j \partial_j T^m_{ij} = 0, \quad \text{for all } i.
\]
As a consequence,
\[ \sum_j \tilde{\partial}_j L^m_{ij} = \frac{n-m}{n} \partial_i \sigma_m(A^{gs}), \]
where \( \tilde{\partial}_j \) is the \( j \)-th covariant derivative with respect to the metric \( g_v \), while \( \partial_j \) denotes the usual Euclidean derivative.

The following two lemmas were proved in [González 2005b]. Expression (2-6) shows the ‘almost’ divergence structure of \( \sigma_m \), while (2-7) is an inductive formula allowing us to handle the nondivergence terms (of order \( m-1 \)) that appear in (2-6).

**Lemma 2.4.** In this setting,
\[
(2-5) \quad \sum_j \partial_j T^m_{ij} = -(n-m)\sigma_m v_i v^{-1} + n \sum_i T^m_{ij} v_i v^{-1} \quad \text{for each } i;
\]
\[
(2-6) \quad m \sigma_m(A^{gs}) = v \sum_{i,j} \partial_j(v_i T^{m-1}_{ij}) - n \sum_{i,j} T^{m-1}_{ij} v_i v_j + \frac{n-m+1}{2} \sigma_{m-1} |\nabla v|^2.
\]

**Lemma 2.5.** Let \( U \) be a domain in \( \mathbb{R}^n \), \( v^{-1} \in C^\infty(U) \), and \( \varphi \in C^\infty_0(U) \) a smooth cutoff function. For any integers \( 1 \leq s \leq k \leq n \) and real number \( \gamma \),
\[
(2-7) \quad \int_U \sum_{i,j} T^{k-s}_{ij} v_i v_j |\nabla v|^{2(s-1)} \varphi^{2k} v^{-\gamma} dx
= \left(1 + \frac{k-s}{2s}\right) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^{2k} v^{-\gamma} dx
\]
\[+ \frac{s+n+1-\gamma}{2s} \int_U \sum_{i,j} T^{k-s}_{ij} v_i v_j |\nabla v|^{2s} \varphi^{2k} v^{-\gamma} dx
- \frac{n-k+s+1}{4s} \int_U \sigma_{k-s-1} |\nabla v|^{2(s+1)} \varphi^{2k} v^{-\gamma} dx
+ \frac{k}{s} \int_U \sum_{i,j} T^{k-s}_{ij} v_i \varphi_i |\nabla v|^{2(s-1)} \varphi^{2k-1} v^{-\gamma} dx.
\]

In Section 3 we will need a similar formula for the traceless Newton tensor:

**Corollary 2.6.** For any fixed \( i \),
\[
(2-8) \quad \sum_j \partial_j(L^m_{ij}) = \frac{n-m}{n} \partial_i \sigma_k + n \sum_j L^m_{ij} v_i v^{-1}
\]

**Proof.** Follows easily from (2-5) and (2-2). \( \square \)

**Lemma 2.7.** If \( \sigma_1, \ldots, \sigma_m > 0 \) and \( m \leq n \), then
\[
\|T^{m-1}_{ij}\| \leq C_{m,n} \sigma_{m-1}.
\]

**Proof.** Because of **Lemma 2.2**, \( T^{m-1} \) is positive definite. To estimate its norm we just need to look at its biggest eigenvalue. We are done, because
\[
\text{trace } T^{m-1} = (n-m) \sigma_{m-1}.
\]
Lemma 2.8. For any $1 \leq k \leq n - 1$, if we have a metric $g = v^{-2} |dx|^2$ in the positive cone $\Gamma_k^+$, then

$$\sum_{i,j} L^k_{ij} E_{ij} \geq 0,$$

with equality if and only if $E = 0$.

Proof. Because $E_{ij}$ is traceless,

$$\sum_{i,j} L^k_{ij} E_{ij} = -\sum_{i,j} T^k_{ij} E_{ij}.$$

Using

$$E_{ij} = -\frac{1}{n} \sigma_1 \delta_{ij} + A_{ij}, \quad (k + 1) \sigma_{k+1} = T^k_{ij} A_{ij}, \quad T^k_{ij} \delta_{ij} = (n - k) \sigma_k,$$

we see that

$$\sum_{i,j} T^k_{ij} E_{ij} = -\frac{n-k}{n} \sigma_k \sigma_1 + (k + 1) \sigma_{k+1}.$$

The result follows by the general inequality for matrices in the positive cone $\Gamma_k^+$:

$$\sigma_{k+1} \leq \frac{n-k}{n(k+1)} \sigma_1 \sigma_k,$$

with equality if and only if $E \equiv 0$. \qed

3. An Obata-type formula

Obata’s original result [1962] states that, if we have a metric $g$ on the unit sphere $\mathbb{S}^n$ that is conformal to the standard metric $g_c$ and of constant scalar curvature, then $E \equiv 0$; that is, $g$ is the standard metric $g_c$ or is obtained from it by a conformal diffeomorphism of the sphere. His method uses crucially the traceless Ricci tensor $E_{ij} = vv_{ij} - (1/n) \Delta v \delta_{ij}$ and the Bianchi identity $\nabla^i E_{ij} = \nabla^j R$. Indeed, his main step is to prove that

$$\int_{\mathbb{S}^n} \sum_{i,j} E_{ij} E_{ij} v^{-1} d\text{vol}_{g_c} = 0,$$

and thus establish that $g$ is an Einstein metric on $\mathbb{S}^n$.

This same argument was generalized for constant $\sigma_k$ (instead of constant $R$) by Viaclovsky [2000], with the role of $E$ played now by $L^k$ and the Bianchi identity replaced by (2-8). If the metric is defined on $\mathbb{R}^n$ instead of $\mathbb{S}^n$, an analogous argument works; however, a cutoff function $\eta$ is introduced and, in order to get the same conclusion, a careful estimate of the error terms is needed. We should also mention the work of Chang, Gursky, and Yang [2002; 2003] and of Li and Li [2002].
However, we are interested in the subcritical-problem approach of Gidas and Spruck [1981]; they have refined the computation of

$$0 \leq \int_B \sum_{i,j} E_{ij} E_{ij} v^{-\delta} \eta \, dx = \cdots$$

for any $\delta \in \mathbb{R}$. The main result of this section is the corresponding refinement for $\sigma_k$:

**Proposition 3.1.** Let $\alpha > 0$ and $n > 2k$. Take a solution $v$ of $\sigma_k(v) = v^\alpha$ in $\Omega$, with $v \in \Gamma_k^+$, $v > 0$, and $v^{-1} \in C^3(\Omega)$, where $\Omega$ is a domain in $\mathbb{R}^n$. Pick $\eta \in C_0^\infty(\Omega)$ and a big positive integer $\theta$. There exist constants $d_{k-s}$ such that

\begin{equation}
(3-1) \quad \int_\Omega \sum_{i,j} L_{ij}^k E_{ij} v^{-\delta} \eta^\theta + \left( \frac{n-k}{n} \alpha - (1+n-\delta) \frac{k(n+2)}{2n} \right) \int_\Omega v^\alpha |\nabla v|^2 v^{-\delta} \eta^\theta \\
+ (1+n-\delta) \sum_{s=1}^k d_{k-s} \int_\Omega \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta^\theta = E_1(\eta),
\end{equation}

where

\begin{equation}
(3-2) \quad E_1(\eta) \lesssim \left| \int_\Omega \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta} \eta^\theta - 1 \right| + \sum_{s=1}^k \left| \int_\Omega \sum_{i,j} T_{ij}^{k-s} v_j \eta_j |\nabla v|^{2s} v^{1-\delta} \eta^\theta - 1 \right|.
\end{equation}

In addition, if $\delta$ is smaller than but close enough to $n+1$, all the coefficients in front of the integrals in (3-1) are positive.

**Proof:** One uses the inductive method developed in [González 2005b; 2005a] and the properties of $L^k$. In view of (2-4), integrate over $\Omega$ to get

$$\int \sum_{i,j} L_{ij}^k E_{ij} v^{-\delta} \eta^\theta = \int \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta} \eta^\theta - \frac{1}{n} \int \sum_{i,j} L_{ij}^k (\Delta v) v^{1-\delta} \delta_i \eta^\theta.$$}

The last term vanishes since $L^k$ is trace-free. Integrating by parts and using (2-8),

$$\int \sum_{i,j} L_{ij}^k E_{ij} v^{-\delta} \eta^\theta$$

$$= - \int \sum_{i,j} (\partial_i L_{ij}^k) v_j v^{1-\delta} \eta^\theta - (1-\delta) \int \sum_{i,j} L_{ij}^k v_i v_j v^{-\delta} \eta^\theta - \int \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta} \eta^\theta - 1$$

$$= - \frac{n-k}{n} \int \sum_i (\partial_i \sigma_k) v_j v^{1-\delta} \eta^\theta - (1+n-\delta) \int \sum_{i,j} L_{ij}^k v_i v_j v^{-\delta} \eta^\theta - \int \sum_{i,j} L_{ij}^k v_i \eta_j v^{1-\delta} \eta^\theta - 1.$$
Group in $E_1(\eta)$ all the terms containing derivatives of $\eta$. Now compute, using (2-1), (2-2), and (2-3):

\begin{equation}
(3-3) \int \sum_{i,j} L_{ij}^k v_i v_j v^{-\delta} \eta^\theta = \frac{n-k}{n} \int \mathcal{L}_{k} |\nabla v|^2 v^{-\delta} \eta^\theta - \int \sum_{i,j} T_{ij}^k v_i v_j v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= -\frac{k}{n} \int \mathcal{L}_{k} |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j} T_{ij}^{k-1} A_{ij} v_i v_j v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= -\frac{k}{n} \int \mathcal{L}_{k} |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= -\frac{k}{n} \int \mathcal{L}_{k} |\nabla v|^2 v^{-\delta} \eta^\theta + \int \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= \frac{1}{2} \sum_{i,j} T_{ij}^{k-1} v_i v_j v^{-\delta} \eta^\theta.
\end{equation}

The middle term can be handled similarly to [González 2005b, Section 4]:

\begin{equation}
(3-4) \int \sum_{i,j,l} T_{il}^{k-1} v_i v_j v^{-\delta} \eta^\theta = \frac{1}{2} \int \sum_{i,l} \partial_l (|\nabla v|^2) T_{il}^{k-1} v_i v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= -\frac{\delta-1}{2} \int \sum_{i,l} T_{il}^{k-1} v_i v_l |\nabla v|^2 v^{-\delta} \eta^\theta - \frac{1}{2} \int \sum_{i,l} \partial_l (T_{il}^{k-1} v_i) |\nabla v|^2 v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= -\frac{1}{2} \int \sum_{i,l} T_{il}^{k-1} v_i v_l |\nabla v|^2 v^{-\delta} \eta^\theta - \frac{1}{2} \int \sum_{i,l} |\nabla v|^4 v^{-\delta} \eta^\theta - E_1(\eta).
\end{equation}

To eliminate the term $\partial_l (T_{il}^{k-1} v_i)$ from (3-4), just use the equality (2-5) and then substitute (3-4) into (3-3):

\begin{equation}
(3-5) \int \sum_{i,j} L_{ij}^k v_i v_j v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
= -k \frac{n+2}{2n} \int \mathcal{L}_{k} |\nabla v|^2 v^{-\delta} \eta^\theta - \frac{2+n-\delta}{2} \int \sum_{i,j} T_{ij}^{k-1} v_i v_j |\nabla v|^2 v^{-\delta} \eta^\theta
\end{equation}

\begin{equation}
+ \frac{n-k+1}{4} \int \sigma_{k-1} |\nabla v|^4 v^{-\delta} \eta^\theta + E_1(\eta)
\end{equation}

\begin{equation}
= -k \frac{n+2}{2n} \int \mathcal{L}_{k} |\nabla v|^2 v^{-\delta} \eta^\theta + \mathcal{B}_{k-1} + E_1(\eta).
\end{equation}

where we have defined, for $k$ fixed and $s = 1, \ldots, k-1$,

\begin{equation}
\mathcal{B}_{k-s} =
\end{equation}

\begin{equation}
-\frac{s+1+n-\delta}{s+1} \int \sum_{i,j} T_{ij}^{k-s} v_i v_j |\nabla v|^{2s} v^{-\delta} \eta^\theta + \frac{n-k+s}{2(s+1)} \int \sigma_{k-s} |\nabla v|^{2(s+1)} v^{-\delta} \eta^\theta.
\end{equation}
The computations in (3-4) can be redone for $T^{k-s}$, and thus

$$B_{k-s} = \tilde{d}_{k-s} (1, \tilde{c}_{k-s} B_{k-s-1} + E_1(\eta),$$

with

$$\tilde{d}_{k-s} = \frac{s+n+1-\delta}{s+1} \left(1 + \frac{k-s}{2(s+1)}\right) + \frac{n-k+s}{2(s+1)} + \frac{(s+n+1-\delta)(s+2)}{2(s+1)^2}.$$

The last step is

$$B_1 = d_1 \int \sigma_1 |\nabla v|^{2k} v^{-\delta} \eta + \tilde{c}_1 d_0 \int |\nabla v|^{2(k+1)} v^{-\delta} \eta.$$

Substitute (3-6) into (3-5), inductively. This proves (3-1) for some constants $c_{k-s}$ and $d_{k-s}$ obtained from $\tilde{c}_{k-s}$ and $\tilde{d}_{k-s}$. Note that $c_{k-s} > 0$ if $\delta < n + 1$. We also want $d_{k-s} > 0$ for $s = 1, \ldots, k$, and this is achieved when $\delta$ is close enough to $n + 1$ because $n > 2k$.

**Lemma 3.2.** With the same hypothesis as in the previous lemma,

$$\int_U v^{\alpha/k} \gamma v^{\theta} \lesssim \left(-1 + \gamma - \frac{n}{2}\right) \int_U \sigma_k |\nabla v|^{2} v^{-\gamma} \eta^{\theta} + E_2(\eta),$$

where

$$E_2(\eta) \lesssim \left|\int_U \sum_i v_i \eta_i v^{1-\gamma} \eta^{\theta-1}\right|.$$

**Proof.** Since $\sigma_k(v) = v^{\alpha}$ and $\sigma_k \leq C(n,k)\sigma_1^k$ (Lemma 2.2), we get $\sigma_1(v) \gtrsim v^{\alpha/k}$. It is easy to see that

$$\int \sigma_1 v^{-\gamma} \eta^{\theta} = \left(-1 + \gamma - \frac{n}{2}\right) \int |\nabla v|^{2} v^{-\gamma} \eta^{\theta} + E_2(\eta),$$

and the lemma is proved.

**4. Main estimates**

Here we obtain the needed $L^p$ estimate, as a consequence of (3-1). The terms on the left-hand side of (3-1) will be “good” terms, and we will give an estimate of the error terms.

**Proposition 4.1.** Take $n > 2k$, $\alpha \in (0,k)$, and let $v$ be a solution of (1-3). We have

$$\int_{|x| < M\rho} v^{\alpha(k+1)/k} \lesssim \frac{1}{\rho^{2(k+1)}} \int_{A_{\rho} \cup A_{M\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^\alpha} \int_{A_{\rho} \cup A_{M\rho}} v^{2+\alpha-\delta},$$

for $\delta$ smaller than but close enough to $n + 1$, and for $A_{\rho} = \{\frac{1}{2}\rho < |x| < \rho\}$ and $A_{M\rho} = \{M\rho < |x| < 2M\rho\}$; the constants depend on $M$ but not on $\rho$.  


Proof: If we take \( \alpha - \delta = -\gamma \), then \(-1 - \frac{1}{2} n + \gamma > 0\), and the preceding lemma allows us, in (3-1), to replace
\[
\int |\nabla v|^2 v^{\alpha - \delta} \eta^\theta \quad \text{by} \quad \int v^{\alpha(k+1)/k - \delta} \eta^\theta + E_2(\eta).
\]
Let \( \eta \) be a smooth cutoff function such that
\[
\eta = \begin{cases} 
1 & \text{if } \rho < |x| < M\rho, \\
0 & \text{if } 0 < |x| < \frac{1}{2} \rho \text{ and } 2M\rho < |x|,
\end{cases}
\]
|\( \nabla \eta | \lesssim 1/\rho \) and |\( D^2 \eta | \lesssim 1/\rho^2 \). The error \( E_1(\eta) \) in (3-2) is of one of these two types:
\[
E_{11}(\eta) \lesssim \left| \int_{A_0 \cup \cup A_{M\rho}} \sum_{i,j} T_{ij}^k v_i \eta_j v^{1-\delta} \eta^{\theta - 1} \right|, \quad \text{or}
\]
\[
E_{12}(\eta) \lesssim \sum_{s=1}^k \left| \int_{A_0 \cup \cup A_{M\rho}} \sum_{i,j} \eta_j \sum_{s} T_{ij}^{k-s} v_i |\nabla v|^{2s} v^{1-\delta} \eta^{\theta - 1} \right|.
\]
These will be handled as in the proof of [González 2005a, Theorem 1.1], but here we present a clearer proof for this particular cutoff.

To understand \( E_{11} \), substitute \( T^k = (1 - k/n) \sigma_k I - T^k \), so that
\[
E_{11}(\eta) \lesssim \int_{A_0 \cup \cup A_{M\rho}} \sigma_k v_i \eta_j v^{1-\delta} \eta^{\theta - 1} + \int_{A_0 \cup \cup A_{M\rho}} T_{ij}^k v_i \eta_j v^{1-\delta} \eta^{\theta - 1}.
\]
We cannot use the standard trick — to estimate the norm \( ||T^k|| \lesssim \sigma_k \) as in Lemma 2.7 — because we cannot conclude that \( T^k \) is positive definite from the information on \( \sigma_1, \ldots, \sigma_k \), and we need to write everything in terms of smaller \( T^{k-s} \)'s. An inductive process is needed.

Substitute \( T_{ij}^k = \sigma_k \delta_{ij} - A_{ij} T_{ij}^{k-1} \) and \( A_{ij} = vv_{ij} - \frac{1}{2} |\nabla v|^2 \delta_{ij} \) in (4-2). Together with Lemma 2.7, we have
\[
E_{11}(\eta) \lesssim \int_{A_0 \cup \cup A_{M\rho}} \sigma_k |\nabla v| |\nabla \eta| v^{1-\delta} \eta^{\theta - 1} + \int_{A_0 \cup \cup A_{M\rho}} \sigma_{k-1} |\nabla v|^3 |\nabla \eta| v^{1-\delta} \eta^{\theta - 1}
\]
\[
+ \int_{A_0 \cup \cup A_{M\rho}} T_{ij}^{k-1} v_i v_j \eta_j v^{2-\delta} \eta^{\theta - 1}.
\]
For the last term, proceed as in (3-4):
\[
T_{ij}^{k-1} v_i v_j \eta_j v^{2-\delta} \eta^{\theta - 1} = \frac{1}{2} \int \partial_l (|\nabla v|^2) T_{ij}^{k-1} \eta_j v^{2-\delta} \eta^{\theta - 1}
\]
\[
= -\frac{1}{2} \int (\partial_l T_{ij}^{k-1}) |\nabla v|^2 \eta_j v^{2-\delta} \eta^{\theta - 1} - \frac{1}{2} \int T_{ij}^{k-1} |\nabla v|^2 \eta_j v^{2-\delta} \eta^{\theta - 2}
\]
\[
- \frac{2-\delta}{2} \int T_{ij}^{k-1} \eta_j v_j |\nabla v|^2 v^{1-\delta} \eta^{\theta - 1}.
\]
Note that (2-5) helps to compute $\partial T_{ij}^{k-1}$, and thus, from (4-4) and Lemma 2.7,

(4-5) $\left| \int T_{ij}^{k-1} v_{il} v_{j} v^{2-\delta} \eta^{\theta-1} \right| \lesssim \int \sigma_{k-1} |D^2 \eta| |\nabla v|^2 v^{2-\delta} \eta^{\theta-2} + \int \sigma_{k-1} |\nabla \eta| v^{1-\delta} \eta^{\theta-1}$.

Young’s inequality for a small $\varepsilon$, together with (4-3) and (4-5), gives

(4-6) $E_{11}(\eta) \lesssim \varepsilon \int \sigma_k |\nabla v|^2 \eta^{\theta} v^{\delta} + \frac{C_\varepsilon}{\rho^2} \int_{A_\rho \cup A_{M_\rho}} \sigma_k v^{2-\delta} \eta^{\theta-2} + \varepsilon \int \sigma_{k-1} |\nabla v|^4 \eta^{\theta} v^{\delta} + \frac{C_\varepsilon}{\rho^2} \int_{A_\rho \cup A_{M_\rho}} \sigma_{k-1} v^{4-\delta} \eta^{\theta-4}$.

To finish the estimate, we just need (4-7) from the next lemma, applied iteratively:

(4-7) $E_{11}(\eta) \lesssim \varepsilon \sum_{s=0}^{k} \int \sigma_{k-s} |\nabla v|^{2(s+1)} \eta^{\theta} v^{\delta} + \frac{C_\varepsilon}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M_\rho}} v^{2(k+1)-\delta}$.

The estimate for $E_{12}(\eta)$ follows in a similar manner. For the error in $E_2(\eta)$, defined in 3-7, we use Young’s inequality with $p = q = 2$:

$E_2(\eta) \lesssim \int |\nabla v| |\nabla \eta| v^{1-\gamma} \eta^{\theta-1} \lesssim \varepsilon \int |\nabla v|^2 v^{\alpha-\delta} \eta^{\theta} + \frac{C_\varepsilon}{\rho^2} \int_{A_\rho \cup A_{M_\rho}} v^{2+\alpha-\delta}$.

Putting it all together in (3-1), and taking into account that $\sum_{i,j} L_{ij}^k E_{ij} \geq 0$,

$\int_{\rho < |x| < M_\rho} v^{\alpha(k+1)/k-\delta} \lesssim \varepsilon \int v^{\alpha(k+1)/k-\delta} \eta^{\theta} + \frac{1}{\rho^{2(k+1)}} \int_{A_\rho \cup A_{M_\rho}} v^{2(k+1)-\delta} + \frac{1}{\rho^2} \int_{A_\rho \cup A_{M_\rho}} v^{2+\alpha-\delta}$. $\square$

**Lemma 4.2.** For all $\varepsilon > 0$ and $s = 0, \ldots, k - 1$, and for $\theta$ a big positive integer,

(4-7) $\frac{1}{\rho^{2(s+1)}} \int \sigma_{k-s} v^{2(s+1)-\delta} \eta^{\theta-2(s+1)} \lesssim \varepsilon \int \sigma_{k-s-1} |\nabla v|^{2(s+2)} \eta^{\theta} v^{\delta} + \frac{C_\varepsilon}{\rho^{2(s+2)}} \int_{|\nabla \eta| \neq 0} \sigma_{k-s-1} \eta^{\theta-2(s+2)} v^{2(s+2)-\delta}$.
Proof. First use the “divergence” formula (2.6) for $\sigma_{k-s}$ with integration by parts:

\begin{align}
(4.8) \quad & (k-s) \int \sigma_{k-s} v^{2(n+1)-\delta} \eta^{\theta-2(s+1)} \\
& = \frac{n-k+s+1}{2} \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta} \\
& \quad - (n+2(s+1)-\delta+1) \int T_{ij}^{k-s-1} v_i v_j \eta^{\theta-2(s+1)} v^{2(s+1)-\delta} \\
& \quad - \int T_{ij}^{k-s-1} v_i \eta_j \eta^{\theta-2(s+1)-1} v^{2(s+1)-\delta+1}. 
\end{align}

Use Lemma 2.7 again to bound the norm of the Newton tensor in (4.8):

\begin{align}
(4.9) \quad & \int \sigma_{k-s} v^{2(s+1)-\delta} \eta^{\theta-2(s+1)} \lesssim \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta} \\
& \quad + \frac{1}{\rho} \int \sigma_{k-s-1} |\nabla v| \eta^{\theta-2(s+1)-1} v^{2(s+1)-\delta+1}. 
\end{align}

Young’s inequality with $\varepsilon$ and $p = s+2$, $q = (s+2)/(s+1)$ now reads

\begin{align}
(4.10) \quad & \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta} \\
& \lesssim \varepsilon \rho^{2(s+1)} \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+1)} v^{2(s+1)-\delta} + \frac{C_\varepsilon}{\rho^2} \int \sigma_{k-s-1} \eta^{\theta-2(s+2)} v^{2(s+2)-\delta}. 
\end{align}

For the second part in (4.9), take $p = 2(s+2)$ and $q = \frac{2(s+2)}{2(s+2)-1}$:

\begin{align}
(4.11) \quad & \frac{1}{\rho} \int \sigma_{k-s-1} |\nabla v| \eta^{\theta-2(s+2)-1} v^{2(s+1)-\delta+1} \\
& \lesssim \varepsilon \rho^{2(s+1)} \int \sigma_{k-s-1} |\nabla v|^2 \eta^{\theta-2(s+2)} v^{2(s+2)-\delta} + \frac{C_\varepsilon}{\rho^2} \int \sigma_{k-s-1} \eta^{\theta-2(s+2)} v^{2(s+2)-\delta}. 
\end{align}

The lemma is proved by substituting (4.10) and (4.11) into (4.9). \qed

Proposition 4.3. For $n \geq 2(k+1)$, $\alpha \in (0, k)$, and $v$ a solution of (1.3), we have

\begin{align}
(4.12) \quad & \int_{|\rho| < |x| < M \rho} v^\alpha (k+1)/k - \delta \leq C \rho^{n-(\delta-\alpha(k+1)/k)/(1-\alpha/2k)}, 
\end{align}

where $C$ depends on $M$ and $\delta$, but not on $\rho$.

Proof. Use Hölder’s inequality with

\begin{align*}
\rho = \frac{\delta - \alpha(k+1)/k}{\delta - 2(k+1)} \quad \text{and} \quad q = \frac{p}{p-1},
\end{align*}

to get

\begin{align}
(4.13) \quad & \frac{1}{\rho^{2(k+1)}} \int_{A_{\rho} \cup A_{M \rho}} v^{2(k+1)-\delta} \leq \varepsilon \int_{A_{\rho} \cup A_{M \rho}} v^\alpha (k+1)/k - \delta + C_\varepsilon \rho^{n-2(k+1)q},
\end{align}
for some small $\varepsilon$, to be chosen later. Also, a Hölder estimate with
\[
\hat{p} = \frac{\delta - \alpha(k + 1)/k}{\delta - 2 - \alpha} \quad \text{and} \quad \hat{q} = \frac{\hat{p}}{\hat{p} - 1}
\]
gives

\[(4-14) \quad \frac{1}{\rho^n} \int_{A_\rho \cup A_{\hat{\rho}} \cup A_{\hat{\rho}}} v^{2+\alpha-\delta} \leq \varepsilon \int_{A_\rho \cup A_{\hat{\rho}}} v^{\alpha(k+1)/k - \delta} + C \rho^{n-2\hat{q}}.\]

When $\alpha \in (0, k)$ and $\delta$ is close enough to $n + 1$, then $p, \tilde{p} > 1$. Look at the powers of $\rho$ in (4-13) and (4-14):
\[
n - 2(k+1)q = n - 2\hat{q} = n - \frac{\delta - \alpha(k + 1)/k}{1 - \alpha/2k}.
\]
Choosing $\varepsilon$ small enough, we conclude from (4-1) that
\[
\int_{\rho < |x| < M\rho} v^{(\alpha(k+1)/k - \delta)} \leq C \rho^{n-(\delta-\alpha(k+1)/k)\left(1-\alpha/2k\right)}. \quad \square
\]

5. Proof of Theorem 1.1

The next proposition is similar to the study of the critical problem in [González 2005a]. In particular, a volume finiteness condition gives regularity near the singularity.

**Proposition 5.1.** Take $\alpha \in (0, k)$ and $n > 2k$, and let $v$ be a solution of (1-3) on $B_\rho(x_0) \subset B$, with $v > 0$ and $v \in \Gamma^1_k$. If
\[
\int_{B_\rho(x_0)} v^{(\alpha-2k)n/(2k)} \leq a
\]
for some small enough $a$ (not depending on $\rho$), then
\[(5-1) \quad \sup_{B_{\rho^2}(x_0)} |v^{-1}| \leq \frac{C}{\rho^{p/p}} \|v^{-1}\|_{L^p(B_{\rho^2}(x_0))}
\]
for all $p > (n-2k)k/(k+1)$. In particular, if
\[(5-2) \quad \int_{\varepsilon < |x| < 1} v^{(\alpha-2k)n/(2k)} < C < \infty
\]
for some constant $C$ independent of $\varepsilon$, the function $v$ is bounded near the origin.

**Proof.** The argument is similar to [González 2005a, Theorem 1.2] for the critical problem. Condition (5-2) is analogous to its volume smallness condition. \(\square\)

**Proof of Theorem 1.1.** Fix $x_0$ small enough and take $2R = |x_0|$. First, note that Hölder estimates with
\[
r = \frac{\delta - (k + 1)/k}{(2k - \alpha)n/(2k)} > 1 \quad \text{and} \quad 1 = \frac{1}{r} + \frac{1}{s}
\]
give, independently of $x_0$.

\begin{equation}
\int_{B_R(x_0)} v^{(\alpha-2k)n/(2k)} \leq \left( \int_{R \leq |x| \leq 3R} v^{(k+1)/k-\delta} \right)^{1/r} e^{\alpha n/s} \lesssim R^{(n-\delta-\alpha(k+1)/k)} 1/\gamma \leq R^0 < \infty.
\end{equation}

We cannot apply Proposition 5.1 directly to $v$. However, we could have started with the function $\tilde{v}(y) = A^{2k/(2k-\alpha)} v(y/A)$ that still satisfies the same equation $\sigma_k(\tilde{v}) = \tilde{v}^\alpha$, for some $A$ big enough and of the form $A = (\text{constant}) \int_{R \leq |x| \leq 3R} v^{(\alpha-2k)n/(2k)}$.

Since we are interested only in the local behavior near zero, we can assume that (5-1) gives an estimate for $v$, 

\[ \sup_{B_{R/4}(x_0)} |v^{-1}| \leq \frac{C}{R^{n/p}} \|v^{-1}\|_{L^p(B_{R}(x_0))} \]

for all $p > (n-2k)k/(k+1)$, and with $C$ depending on 

\[ \int_{R \leq |x| \leq 3R} v^{(\alpha-2k)n/(2k)}. \]

This estimate is uniformly bounded by a constant, independently of $R$, because of (5-3). It is also true that

\begin{equation}
\sup_{B_{R/2}(x_0)} |v^{-1}| \leq \frac{C}{|x_0|^{n/p}} \|v^{-1}\|_{L^p(|R \leq |x| \leq 3R|)}
\end{equation}

for all $p > (n-2k)k/(k+1)$. Set $p = \delta - \alpha (k+1)/k$; this choice is valid when $\alpha \in (0, k)$ and $n > 2k$. Use (4-12) again:

\[ \int_{R \leq |x| \leq 3R} v^{-p} \leq C|x_0|^{n-p/\gamma}, \]

and thus, from (5-4), we arrive at

\[ v^{-1}(x_0) \leq \frac{C}{|x_0|^{2k/(2k-\alpha)}}, \]

as desired. \qed

**Corollary 5.2** (Harnack). Under these hypotheses, there exists $M_0 > 0$ such that, for all $\rho > 0$ and $M \leq M_0$,

\begin{equation}
\sup_{\rho \leq |x| \leq \rho M} v^{-1} \leq C \inf_{\rho \leq |x| \leq \rho M} v^{-1},
\end{equation}

where $C$ is independent of $v$, $\rho$, and $M$. 

Proof. Once we get a supremum estimate (5-4) for a ball, standard elliptic theory yields the infimum estimate. If we write \( v^{-2} = u^{2/(n-2)} \), then \( u \) is a superharmonic function. To finish, use a covering argument for the annulus \( \{ \rho \leq |x| \leq \rho M \} \). □

Corollary 5.3. If \( v \) is a solution of (1-3), then either \( v^{-1} \) is bounded near the origin, or \( v^{-1}(x) \to \infty \) as \( x \to 0 \).

Proof. The argument follows the steps of [Gidas and Spruck 1981, Corollary 3.3], by using the second part of Proposition 5.1. □

6. Proof of Theorem 1.2

We have proved the estimate

\[
(6-1) \quad v^{-1}(x) \leq \frac{C}{|x|^{2/(2k-\alpha)}}
\]

Now we would like to get the opposite inequality. Suppose that

\[
\lim_{x \to 0} \inf |x|^{2/(2k-\alpha)} v^{-1}(x) = 0.
\]

From the Harnack estimate (5-5) follows that

\[
(6-2) \quad \lim_{x \to 0} |x|^{2/(2k-\alpha)} v^{-1}(x) = 0.
\]

We want to see that in this case the function \( v^{-1} \) is bounded near the origin and thus that the theorem follows. It suffices to establish (5-2).

Let’s review two results from [González 2005a]:

Proposition 6.1. Let \( v \) be a solution with \( v^{-1} \in \mathcal{C}^3(U) \), \( v > 0 \), \( v \in \Gamma^k \), and \( n > 2k \).

For all \( \varphi \in \mathcal{C}_0^\infty(U) \) and \( \theta \) a big positive integer,

\[
(6-3) \quad \int_U \sigma_k \varphi^\theta v^{-\gamma} \geq \sum_{s=1}^{k} c_{k-s}(\gamma) \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma} + E(\varphi),
\]

where

\[
(6-4) \quad E(\varphi) \lesssim \sum_{s=1}^{k} \left| \int_U \sum_{i,j} T_{ij}^{k-s} v_j \varphi_i |\nabla v|^{2(s-1)} \varphi^\theta v^{-1} v^1 \right|,
\]

and where the coefficients \( c_{k-s}(\gamma) \) are positive for all \( \gamma \) with

\[
(6-5) \quad \gamma > n - \frac{n - 2k}{k + 1}.
\]

Proposition 6.2. For all \( \varepsilon > 0 \), the error term (6-4) can be estimated by

\[
E(\varphi) \leq \varepsilon \sum_{s=1}^{k} \int_U \sigma_{k-s} |\nabla v|^{2s} \varphi^\theta v^{-\gamma} + C_\varepsilon \sum_{U_k} \int_U (\varphi^{\theta-\alpha_k} v^{2k-\gamma}),
\]
where the $U_k(\varphi)$'s are groups of derivatives of $\varphi$ of order $2k$, and $\alpha_k \in \mathbb{R}$ are constants depending on each of the $U_k$'s. These concepts are defined inductively in the following manner:

- For a fixed $s = 1, \ldots, k$, the starting point is
  
  $$U_s(\varphi)\varphi^{\alpha_s} = |\nabla \varphi|^{2s} \varphi^{-2s}.$$  

- For each integer $l = 0, 1, \ldots$ and $m = s + l$, and once given $U_m\varphi^{\alpha_m}$, the following step is of one of these three shapes:

  $$(6-6) \quad U_{m+1}\varphi^{-\alpha_{m+1}} = \begin{cases} U_{m}^{(m+1)/m} \varphi^{-\alpha_{m}(m+1)/m}, \\ |\nabla U_{m}|^{2(m+1)/2(m+1)-1} \varphi^{-\alpha_{m}2(m+1)-1}, \\ (|\nabla \varphi|^2 U_{m}) \varphi^{-\alpha_{m}-2}. \end{cases}$$  

- The ending point is when $m = s + l$ reaches $k$.

We will use (6-3) for a suitable cutoff function. Take $\varphi = \eta r$ with $\eta \in \mathcal{C}_0^\infty(B\setminus\{0\})$, such that

$$\eta = \begin{cases} 1 & \text{if } \varepsilon < |x| < R, \\ 0 & \text{if } |x| < \frac{1}{2}\varepsilon \text{ and } |x| > 2R, \end{cases}$$

and so that the derivatives have a good bound on $\frac{1}{2}\varepsilon < |x| < \varepsilon$ and $R < |x| < 2R$. The value of $\gamma$ will be chosen later. Rewrite (6-3) as

$$(6-7) \quad \int \sigma_k v^{-\gamma} \varphi^\theta \geq \sum_{s=1}^k \int \sigma_{k-s} |\nabla v|^{2s} v^{-\gamma} \varphi^\theta - \int T^{k-1}_{i,j} v_i \varphi_j \varphi^{\theta-1} v^{1-\gamma} + \tilde{E}(\varphi),$$

with

$$\tilde{E}(\varphi) \lesssim \sum_{s=2}^k \left| \int T^{k-s}_{i,j} v_i \varphi_j |\nabla v|^{2(s-1)} \varphi^{\theta-1} v^{1-\gamma} \right|,$$

since we will look more carefully at the term in $T^{k-1}$. Integration by parts gives

$$-\int \sum_{i,j} T^{k-1}_{i,j} v_i \varphi_j \varphi^{\theta-1} v^{1-\gamma} = -\frac{1}{1-\gamma} \int \sum_{i,j} T^{k-1}_{i,j} \partial_i (v^{2-\gamma}) \varphi_j \varphi^{\theta-1}$$

$$= \frac{1}{2-\gamma} \int \sum_{i,j} T^{k-1}_{i,j} \varphi_i \varphi^{\theta-1} v^{2-\gamma} + \frac{1}{2-\gamma} \int \sum_{i,j} \partial_i (T^{k-1}_{i,j}) \varphi_j \varphi^{\theta-1} v^{2-\gamma}$$

$$+ \frac{\theta-1}{2-\gamma} \int \sum_{i,j} T^{k-1}_{i,j} \varphi_i \varphi^{\theta-2} v^{2-\gamma}.$$
Substituting (2-5) into this, we get
\[-(n+2-\gamma)\int \sum_{i,j} T_{ij}^{k-1} v_i \varphi_j \varphi^{\theta-1} v^{1-\gamma} = \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi_j \varphi^{\theta-1} v^{2-\gamma} + (n-k+1) \int \sigma_{k-1} v_i \varphi_i \varphi^{\theta-1} v^{1-\gamma} + (\theta-1) \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi_j \varphi^{\theta-2} v^{2-\gamma}.\]

Now substitute this into (6-7):
\[
(6-8) \quad \int \sigma_k \varphi^\theta v^{-\gamma} \gtrsim \sum_{s=1}^k \int \sigma_{k-s} |\nabla v|^2 \varphi^\theta v^{-\gamma} + \frac{1}{n+2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi_j \varphi^{\theta-2} v^{2-\gamma}
+ \frac{\theta-1}{n+2-\gamma} \int \sum_{i,j} T_{ij}^{k-1} \varphi_i \varphi_j \varphi^{\theta-2} v^{2-\gamma}
- \frac{n-k+1}{n+2-\gamma} \int \sum_i \sigma_{k-1} v_i \varphi_i \varphi^{\theta-1} v^{1-\gamma} + \tilde{E}(\varphi).
\]  

Group all the error terms into
\[E(\varphi) \lesssim \sum_{s=1}^k \int \sigma_{k-s} |\nabla \varphi||\nabla v|\varphi^{\theta-1} v^{1-\gamma}.\]

Compute
\[
\varphi_i = \frac{x_i}{r} \eta + E_1(\varphi),
\]
\[
\varphi_{ij} = r^{-1} \left( -\frac{x_i x_j}{r^2} + \delta_{ij} \right) \eta + E_1(\varphi),
\]
\[
\sum_{i,j} T_{ij}^{k-1} \varphi_{ij} = r^{-1} \left( -\sum_{i,j} T_{ij}^{k-1} \frac{x_i x_j}{r^2} + (n-k+1) \sigma_{k-1} \right) \eta + E_1(\varphi).
\]

Since $T^{k-1}$ is positive definite and trace $T^{k-1} = (n-k+1)\sigma_{k-1}$, as long as we keep $1 < \theta$ we have
\[
\sum_{i,j} T_{ij}^{k-1} \left( \varphi_{ij} + (\theta-1) \varphi_i \varphi_j r^{-\theta} \right) \geq C(\theta) \sigma_{k-1} r^{-1} \eta^2 + E_1(\varphi),
\]
for some $C(\theta) > 0$. If we keep $\gamma < n+2$, we can conclude from (6-8) that
\[
(6-9) \quad E(\varphi) + E_1(\varphi) + \int \sigma_k \varphi^\theta v^{-\gamma}
\gtrsim \sum_{s=1}^k \int \sigma_{k-s} |\nabla v|^2 \varphi^\theta v^{-\gamma} + \int \sigma_{k-1} r^{-2} \varphi^\theta v^{2-\gamma}.
\]
We have not been very precise with the errors $E_1(\varphi)$; however, they are of a similar type to $E(\varphi)$ and can be treated in the same manner. Note that, in the positive cone,

$$\sigma_{k-1} \gtrsim \sigma_k^{(k-1)/k} = v^{\alpha(k-1)/k},$$

so, with (6-9) we have actually proved

$$(6-10) \quad E(\varphi) \gtrsim \int \left( v^{\alpha(k-1)/k + 2 - \gamma} r^{-2} - v^{-2} \right) \psi^\theta + \sum_{s=1}^k \int \sigma_{k-s} |\nabla v|^{2s} \psi^\theta v^{-\gamma}. $$

To handle $E(\varphi)$, we need to control the error terms that appear in Proposition 6.2. Using Lemma 6.3 below,

$$(6-11) \quad \int U_k(\varphi) \psi^\theta \alpha_k v^{2k-\gamma} \lesssim \int r^{-2k} \psi^\theta v^{2k-\gamma} + \frac{1}{\varepsilon^{2k}} \int_{|x| < \varepsilon} r^\theta v^{2k-\gamma} + \frac{1}{R^{2k}} \int_{|x| < 2R} r^\theta v^{2k-\gamma}$$

Looking one-by-one at the terms above, notice that

$$\frac{1}{\varepsilon^{2k}} \int_{|x| < \varepsilon} r^\theta v^{2k-\gamma} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

by using the previous estimate (6-1) and the definition of $\eta$, and as soon as

$$(6-12) \quad \gamma \gtrsim n - \alpha \left( \frac{n-2k}{2k} \right).$$

A similar argument gives

$$\frac{1}{R^{2k}} \int_{|x| < 2R} r^\theta v^{2k-\gamma} \leq C.$$

The other integral in (6-11) is bounded by

$$\int r^{-2k} \psi^\theta v^{2k-\gamma} \lesssim \int \left( v^{\alpha(k-1)/k + 2 - \gamma} r^{-2} \right) \left( v^{-\alpha(k-1)/k - 2 + 2k r^{-2k}} \right) \psi^\theta.$$

Our assumption (6-2) yields

$$v^{-\alpha(k-1)/k - 2 + 2k r^{-2k}} = o(1),$$

and thus from (6-10) we obtain

$$C \gtrsim \int \left( v^{\alpha(k-1)/k + 2 - \gamma} r^{-2} - v^{\alpha - \gamma} \right) \psi^\theta.$$

Again, because of (6-2), we have

$$r^2 v^{\alpha/2 - k} = o(1).$$

Theorem 1.1 gives
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\[ v^\alpha - \gamma \lesssim (v^{\alpha(k-1)/k + 2 - \gamma} r^{-2}) (r^2 v^{(\alpha-2k)/2}). \]

Comparing the orders, we quickly obtain

\[(6-13) \int v^{\alpha(k-1)/k} r^{-2} v^{2 - \gamma} \varphi^\theta < \infty. \]

This is precisely the term (5-2) that we need to estimate, because

\[(6-14) \int_{v \leq |x| \leq R} v^{(\alpha(k-1)/k - n + \alpha n/(2k) - 2 + \gamma)} \eta^\theta \]

after using Theorem 1.1 and choosing \( \theta \) and \( \gamma \) so that

\[(6-15) (-\alpha 1/ k - n + \alpha n/(2k) - 2 + \gamma) (\frac{2k}{2k - \alpha}) = -2 + \theta, \]

that is, picking

\[ \gamma = n - \alpha \left(\frac{n - 2k}{2k}\right) + \theta \left(1 - \frac{\alpha}{2k}\right). \]

This is an admissible value for \( \gamma \) because, when \( \alpha < 2k/(k+1) \), it can be chosen to satisfy (6-5), (6-12), \( \gamma < n + 2 \), and \( \theta > 1 \).

**Lemma 6.3.** For the cutoff \( \varphi = r \eta \) constructed in the previous proof,

\[ U_k(\varphi) \varphi^{\theta - a_k} \lesssim r^{-2k} \varphi^\theta + e^{-2k r^\theta} \chi_{\{r/2 \leq |x| < \epsilon\}} + R^{-2k} r^\theta \chi_{\{R \leq |x| < 2R\}} \]

**Proof:** The definition of the \( U_k \) was given in Proposition 6.2. We are just interested in the orders of \( r \) and \( \epsilon \). For fixed \( s = 1, \ldots, k \), the initial step is

\[ U_s(\varphi) \varphi^{\theta - 2s} = |\nabla \varphi|^{2s} \varphi^{\theta - 2s} \]

Next, assume that the result is true for \( m = s + l \):

\[ U_m(\varphi) \varphi^{\theta - m} \lesssim r^{-2m} \varphi^\theta + e^{-2m r^\theta} \eta^{2m}. \]

The proof for \( m + 1 \) follows easily from (6-6). \( \square \)

**References**


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