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ASYMPTOTIC ESTIMATION FOR A p -GINZBURG–LANDAU TYPE MINIMIZER IN HIGHER DIMENSIONS

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This paper is concerned with the asymptotic behavior of the minimizers u_ε of a p -Ginzburg–Landau type functional when $\varepsilon \rightarrow 0$. First the author discusses the location of zeros of u_ε qualitatively. Then the $W^{1,p}$ estimation of u_ε is set up. Finally, the author proves the $C^{1,\alpha}$ convergence of u_ε .

1. Introduction

Let $B_r = \{x \in \mathbb{R}^n; |x| < r\}$, where $n \geq 2$. Denote by u_ε the minimizer of the functional

$$E_\varepsilon(u) = \frac{1}{2} \int_{B_1} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{B_1 \setminus B_\varrho} (1 - |u|^2)^2 + \frac{1}{2\varepsilon^2} \int_{B_\varrho} |u|^2$$

in the function space $H_g^1(B_1, \mathbb{R}^n)$, where the mapping $g : \partial B_1 \rightarrow S^{n-1}$ is smooth and satisfies $\deg(g, \partial B_1) = d \neq 0$. The functional $E_\varepsilon(u)$ with $n = 2$ is related to the Ginzburg–Landau model of superconductivity with normal impurity inclusion such as superconducting–normal junctions [Chapman et al. 1995]. To represent the domains occupied by the superconducting materials and the normal conducting materials, we use $B_1 \setminus \bar{B}_\varrho$ and B_ϱ , respectively. The minimizer u_ε is the order parameter. In the physics literature, u_ε is called a Higgs field. The parameter ε , which has the dimension of length, depends on the material and its temperature. When the temperature is not too close to the critical temperature, ε is extremely small. The zeros of u_ε exist in B_1 since $d \neq 0$. They are known as the Ginzburg–Landau vortices which are of significance in the theory of superconductivity [Du et al. 1992; Tinkham 1975]. The asymptotic behavior of the minimizer u_ε was studied when both ε and ϱ converge to 0, and the vortex-pinning effect was discussed [Ding et al. 1998].

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Now, we consider the minimizer of

$$E_\varepsilon(u, B_1) = \frac{1}{p} \int_{B_1} |\nabla u|^p + \frac{1}{4\varepsilon^p} \int_{B_1 \setminus B_\varrho} (1 - |u|^2)^2 + \frac{1}{2\varepsilon^p} \int_{B_\varrho} |u|^2$$

with $p > 1$ and $p \neq n$ in the class $W = W_g^{1,p}(B_1, R^n)$. By means of the calculus of variations, we can see the existence of minimizers u_ε . As in [Ding et al. 1998], we are concerned with the asymptotic behavior of u_ε when ε and ϱ tend to 0. In this paper, we discuss the problem in the following cases:

$$\begin{aligned} \text{Case I:} \quad & \varrho \leq \varepsilon \quad \text{or} \quad \varrho = O(\varepsilon) \text{ as } \varepsilon \rightarrow 0; \\ \text{Case II:} \quad & \varrho > \varepsilon \quad \text{and} \quad \lim_{\varrho \rightarrow 0} \varepsilon/\varrho = 0. \end{aligned}$$

In Case I, by the same argument proving Theorem III.1 in [Bethuel et al. 1994], we can easily see that

$$(1-1) \quad E_\varepsilon(u_\varepsilon, B_1) \leq C(1 + \varepsilon^{n-p}).$$

However, in Case II, the proof of (1-1) seems to be difficult. In Section 2, we will establish the estimation for $E_\varepsilon(u_\varepsilon, B_1 \setminus B_{\gamma\varrho})$ with $\gamma > 1$; see Proposition 2.4. Based on these results, in Section 3 we describe the vortex-pinning effect, i.e., the location of the zeros of the minimizer.

Theorem 1.1. *Assume u_ε is a minimizer. Then there are finitely many points $a_1, a_2, \dots, a_N \in \bar{B}_1$, such that for any $\eta \in (0, 1/2)$, there is $h = h(\eta) > 0$ which is independent of $\varepsilon, \varrho \in (0, \varepsilon_0)$ with ε_0 sufficiently small, satisfying the following properties:*

In Case I,

$$(1-2) \quad \{x \in B_1; |u_\varepsilon(x)| < 1 - \eta\} \subset \bigcup_{i=1}^N B(a_i, h\varepsilon) \cup (B_{h\varepsilon} \cup B_\varrho).$$

If $h\varepsilon < \varrho \leq \varepsilon$, then

$$(1-3) \quad \{x \in B_\varrho; |u_\varepsilon(x)| \geq \eta\} \subset B_{h\varepsilon} \quad \text{and} \quad |u_\varepsilon(x)| < \eta, \forall x \in B_\varrho \setminus B_{h\varepsilon}.$$

In Case II with $p > n$, there exists $\gamma > 1$ such that

$$(1-4) \quad \{x \in B_1; |u_\varepsilon(x)| < 1 - \eta\} \subset \bigcup_{i=1}^N B(a_i, h\varepsilon) \cup B_{\gamma\varrho}.$$

In Case II with $n - t < p < n$, for t a constant in $(0, \min\{1/2, 4/n\})$, we have

$$(1-5) \quad \{x \in B_1; |u_\varepsilon(x)| < 1 - \eta\} \subset \bigcup_{i=1}^N B(a_i, h\varrho^{(n-1)/n} \varepsilon^{1/n}) \cup B_{\gamma\varrho}.$$

Remark 1. If the vortices (zeros of $|u_\varepsilon|$) concentrate in some region, we talk of the *pinning effect*. According to Theorem 1.1, the vortices converge to $\{0, a_1, \dots, a_N\}$ when both ϱ and ε tend to zero. When $h\varepsilon < \varrho \leq \varepsilon$, we investigate a fixed point $x_0 \in B_{h\varepsilon} \setminus \{0\}$ satisfying $|u_\varepsilon(x_0)| \geq \eta$. In this situation, the *superconductive state*

at x_0 appears. Letting ε go to zero, when ε becomes so small that $x_0 \in B_\varrho \setminus B_{h\varepsilon}$, the *normal conductive state* at x_0 may appear, since $|u_\varepsilon(x_0)| < \eta$. As ε becomes extremely small, $x_0 \in B_{1/2} \setminus B_\varrho$, so $|u_\varepsilon(x_0)| \geq 1 - \eta$. Again the superconductive state at x_0 appears. This shows the conductive state is complicated and may change near the origin when ε and ϱ tend to zero.

Next, we will set up the uniform estimation of $\|u_\varepsilon\|_{W^{1,p}}$. When $p > n$, the idea in [Ding et al. 1998] (coming from [Bethuel et al. 1994]) is not valid, since the coefficients C_2 and C_3 of ε^{n-p} in the upper bounds for $E_\varepsilon(u_\varepsilon, B_1)$ and $E_\varepsilon(u_\varepsilon, B_1 \setminus B_{\gamma\varrho})$, respectively, are not sufficiently accurate. (See Propositions 2.3 and 2.4.) The reason is that the conformal transformation of $\int |\nabla u_\varepsilon|^p dx$ is lost when $p \neq n$. Although $E_\varepsilon(u_\varepsilon, [B(a_i, R) \setminus B(a_i, h\varepsilon)] \cup [B_R \setminus B_{\gamma\varrho}])$ can be bounded below by $C_4(\varepsilon^{n-p} - 1)$, the constant C_4 may be smaller than C_2 and C_3 . Thus, it is impossible to get the uniform estimation of $E_\varepsilon(u_\varepsilon, K)$ as we do in the case $p = n$ [Bethuel et al. 1994; Ding et al. 1998; Han and Li 1996; Hong 1996], where K is an arbitrary compact subset of $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. In Section 4, we establish the uniform estimation by means of induction. However in the proof, there are few results linking the degrees of the zeros of u_ε and the singularities of the p -harmonic maps. Hence, the relation between d and N is still open.

Theorem 1.2. *Assume u_ε is a minimizer. Then $|u_\varepsilon| \leq 1$ a.e. on \bar{B}_1 . In addition, in Case I with $p \in (1, n)$, there exists a constant $C > 0$ which is independent of $\varepsilon, \varrho \in (0, \varepsilon_0)$, such that*

$$(1-6) \quad E_\varepsilon(u_\varepsilon, B_1) \leq C.$$

In Case I with $p > n$ or in Case II, for any compact $K \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$, there exists $C = C(K) > 0$ such that

$$(1-7) \quad E_\varepsilon(u_\varepsilon, K) \leq C.$$

Remark 2. Based on these results, we will set up the following convergences of the minimizer as ϱ and ε go to 0:

(1) In Case I with $p \in (1, n)$, obviously, $E_\varepsilon(u_\varepsilon, B_1) \leq E_\varepsilon(u_*, B_1)$, where u_* is a least map of the energy $\int_{B_1} |\nabla u|^p dx$ on $W_g^{1,p}(B_1, S^{n-1})$. In addition, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^p} \int_{B_\varrho} |u_*|^2 dx = C(\pi) \lim_{\varepsilon \rightarrow 0} \frac{\varrho^n}{\varepsilon^p} = 0.$$

Thus, by the weak lower semicontinuity of $\int |\nabla u|^p$, there is a subsequence u_{ε_k} of u_ε such that as $\varepsilon \rightarrow 0$, the subsequence u_{ε_k} converges strongly to u_p in $W^{1,p}(B_1)$, where u_p is a least map of the energy $\int_{B_1} |\nabla u|^p dx$ on $W_g^{1,p}(B_1, S^{n-1})$.

(2) In Case I with $p > n$ or in Case II, according to Theorem 1 in [Misawa 2001], we can conclude that for some subsequence u_{ε_k} , as $k \rightarrow \infty$, the subsequence u_{ε_k} converges weakly to u_p in $W^{1,p}(K)$, where u_p is a p-harmonic map on $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. Its proof is also similar to that of Theorem 1.2 in [Hong 1996, pp632-633].

(3) When $p > 2n - 2$, from [Lei and Wu 2000, §6] we can deduce that for some subsequence \tilde{u}_k of the regularized minimizer \tilde{u}_ε introduced in [Hong 1996], if k tends to ∞ , then $\tilde{u}_k \rightarrow u_p$ in $C^{1,\alpha}(K)$, $\alpha \in (0, 1)$, where u_p is a p-harmonic map on $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$.

Now, we shall loosen the constraint $p > 2n - 2$. The following theorem will be proved in Section 5.

Theorem 1.3. *Assume \tilde{u}_ε is a regularized minimizer and let K be any compact subset of $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. When $p \neq n$ and $p > n - t$, for t a constant in $(0, \min\{1/2, 4/n\})$, if $\varepsilon \rightarrow 0$, then there is a subsequence \tilde{u}_k of \tilde{u}_ε such that*

$$\tilde{u}_k \rightarrow u_p \text{ in } C^{1,\alpha}(K), \quad \alpha \in (0, 1),$$

where u_p is a p-harmonic map on $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$.

Remark 3. Via the uniform estimation in this paper, we prove the convergence of u_ε . The compactness only leads to the convergence for some subsequence. If the limit u_p is unique, the convergence can be verified for the whole sequence. However, the uniqueness of u_p is yet to be established.

When $p = n$, all the results above can still be deduced by analogous arguments in [Ding et al. 1998; Han and Li 1996; Hong 1996; Lei 2004].

2. Preliminaries

Proposition 2.1. *The minimizer $u_\varepsilon \in W$ satisfies*

(2-1)

$$\int_{B_1} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx - \frac{1}{\varepsilon^p} \int_{B_1 \setminus B_\varepsilon} u \phi (1 - |u|^2) \, dx + \frac{1}{\varepsilon^p} \int_{B_\varepsilon} u \phi |u|^2 \, dx = 0,$$

for all $\phi \in W^{1,p}(B_1, \mathbb{R}^n)$ where $\phi|_{\partial B_1} = 0$. Moreover, $|u_\varepsilon| \leq 1$ a.e. on \bar{B}_1 .

Proof. Using calculus of variations, set $u = u_\varepsilon$ in (2-1) and $\phi = u(|u|^2 - 1)_+$, where $(|u|^2 - 1)_+ = \min(k, \max(0, |u|^2 - 1))$, for k a positive constant. We then have

$$\begin{aligned} & \int_{B_1} |\nabla u|^p (|u|^2 - 1)_+ \, dx + 2 \int_{B_1} |\nabla u|^{p-2} (u \nabla u)^2 \, dx \\ & + \frac{1}{\varepsilon^p} \int_{B_1 \setminus B_\varepsilon} |u|^2 (|u|^2 - 1)_+^2 \, dx + \frac{1}{\varepsilon^p} \int_{B_\varepsilon} |u|^4 (|u|^2 - 1)_+ \, dx = 0, \end{aligned}$$

from which it follows that

$$\frac{1}{\varepsilon^p} \int_{B_1 \setminus B_\varrho} |u|^2 (|u|^2 - 1)_+^2 dx + \frac{1}{\varepsilon^p} \int_{B_\varrho} |u|^4 (|u|^2 - 1)_+ dx = 0.$$

Thus $|u| = 0$ or $(|u|^2 - 1)_+ = 0$ a.e. on B_1 , and hence $|u| \leq 1$ a.e. on \bar{B}_1 . \square

Proposition 2.2. *Assume that $u_\varepsilon \in W$ satisfies (2–1). Then for any $\rho > 0$, there exists a positive constant C_1 independent of ϱ and ε , such that for $x \in B(0, 1 - \rho\varepsilon)$,*

$$\|\nabla u_\varepsilon(x)\|_{L^\infty(B(x, \rho\varepsilon))} \leq C_1 \varepsilon^{-1}.$$

Proof. Let $y = x\varepsilon^{-1}$ in (2–1) and set $v(y) = u(x)$. Then for any $\phi \in W_0^{1,p}(B_\varepsilon, \mathbb{R}^n)$, we have

$$\begin{aligned} (2-2) \quad & \int_{B_{\varepsilon^{-1}}} |\nabla v|^{p-2} \nabla v \nabla \phi \, dy \\ &= \int_{B_{\varepsilon^{-1}} \setminus B(0, \varrho\varepsilon^{-1})} v(1 - |v|^2) \phi \, dy - \int_{B(0, \varrho\varepsilon^{-1})} v \phi |v|^2 \, dy. \end{aligned}$$

Taking $\phi = v\zeta^p$ for some $\zeta \in C_0^\infty(B_\varepsilon, \mathbb{R})$, we obtain the inequality

$$\begin{aligned} \int_{B_{\varepsilon^{-1}}} |\nabla v|^p \zeta^p \, dy &\leq p \int_{B_{\varepsilon^{-1}}} |\nabla v|^{p-1} \zeta^{p-1} |\nabla \zeta| |v| \, dy \\ &\quad + \int_{B_{\varepsilon^{-1}} \setminus B(0, \varrho\varepsilon^{-1})} |v|^2 (1 - |v|^2) \zeta^p \, dy + \int_{B(0, \varrho\varepsilon^{-1})} |v|^4 |\zeta^p \, dy. \end{aligned}$$

Choose $y \in B(0, \varepsilon^{-1} - \rho)$ such that $B(y, 4\rho) \subset B_{\varepsilon^{-1}}$. Taking $\zeta = 1$ in $B(y, 2\rho)$ and $\zeta = 0$ in $B_{\varepsilon^{-1}} \setminus B(y, 4\rho)$ satisfying $|\nabla \zeta| \leq C(\rho)$, we have

$$\int_{B(y, 4\rho)} |\nabla v|^p \zeta^p \leq C \int_{B(y, 4\rho)} |\nabla v|^{p-1} \zeta^{p-1} + C.$$

Using Hölder's inequality, we can derive that $\int_{B(y, 2\rho)} |\nabla v|^p \leq C$. Combining this with [1983, p. 244, lines 19–23] yields that

$$\|\nabla v\|_{L^\infty(B(y, \rho))}^p \leq C \int_{B(y, 2\rho)} (1 + |\nabla v|)^p \leq C.$$

Setting $x = y\varepsilon$ in this inequality completes the proof. \square

Proposition 2.3. *Let u_ε be a minimizer. Then there is a constant $C_2 > 0$ which is independent of $\varrho, \varepsilon \in (0, 1)$, such that*

$$E_\varepsilon(u_\varepsilon, B_1) \leq C_2(1 + \varepsilon^{n-p}) \quad \text{in Case I,}$$

$$E_\varepsilon(u_\varepsilon, B_1) \leq C_2(1 + \varepsilon^{n-p} + \varrho^{n-1} \varepsilon^{1-p}) \quad \text{in Case II.}$$

Proof. In Case I, let $y = x\varepsilon^{-1}$. Then

$$E_\varepsilon(u_\varepsilon, B_1) = \frac{1}{2} \int_{B_{\varepsilon^{-1}}} |\nabla u_\varepsilon|^p dy + \frac{1}{4} \int_{B_{\varepsilon^{-1}} \setminus B_{\rho\varepsilon^{-1}}} (1 - |u_\varepsilon|^2)^2 dy + \frac{1}{2} \int_{B_{\rho\varepsilon^{-1}}} |u_\varepsilon|^2 dy.$$

Clearly, there exists $u_1 \in W$ minimizing

$$F(u, B_1) = \frac{1}{2} \int_{B_1} |\nabla u|^p dy + \frac{1}{4} \int_{B_1} (1 - |u|^2)^2 dy + \frac{1}{2} \int_{B_1} |u|^2 dy.$$

Define

$$u_2 = \begin{cases} u_1, & \text{if } 0 < |y| < 1; \\ \frac{y}{|y|}, & \text{if } 1 \leq |y| \leq \varepsilon^{-1}. \end{cases}$$

Noticing that u_ε is a minimizer of $E_\varepsilon(u, B_1)$, we have

$$E_\varepsilon(u_\varepsilon, B_1) \leq E_\varepsilon(u_2, B_1) = F(u_1, B_1) + \frac{1}{2} \int_{B_{\varepsilon^{-1}} \setminus B_1} \left| \nabla \frac{y}{|y|} \right|^p dy \leq C_2(1 + \varepsilon^{n-p}).$$

In Case II, assume $\varrho < 1/4$. For any integer $1 \leq j \leq |d|$, take disjoint balls $\{B(x_i, R_0)\}$ for $1 \leq i \leq j$ in $B_1 \setminus B_{1/2}$, where R_0 is a sufficiently small constant. Set

$$w(x) = \begin{cases} v(x), & \text{if } x \in (B_1 \setminus B_{1/2}) \setminus \left(\bigcup_{i=1}^j B(x_i, R_0) \right); \\ (x - x_i)/|x - x_i|, & \text{if } x \in B(x_i, R_0) \setminus B(x_i, \varepsilon R_0), \quad 1 \leq i \leq j; \\ v_i(x), & \text{if } x \in B(x_i, \varepsilon R_0), \quad 1 \leq i \leq j; \\ (x/|x|)^{|d|-j}, & \text{if } x \in B_{1/2} \setminus B_{\varrho+\varepsilon}; \\ (|x| - \varrho/\varepsilon) (x/|x|)^{|d|-j}, & \text{if } x \in B_{\varrho+\varepsilon} \setminus B_\varrho; \\ 0, & \text{if } x \in B_\varrho, \end{cases}$$

where $(x/|x|)^m$, for m a positive integer, is the S^{n-1} -valued map given in n -dimensional ball coordinates by

$$(x/|x|)^m = (\cos m\theta_1, \sin m\theta_1 \cos m\theta_2, \dots, \sin m\theta_1 \cdots \sin m\theta_{n-2} \cos m\theta_{n-1}, \sin m\theta_1 \cdots \sin m\theta_{n-1}),$$

where $v \in W^{1,p}((B_1 \setminus B_{1/2}) \setminus \left(\bigcup_{i=1}^j B(x_i, R_0) \right), S^{n-1})$ satisfies

$$v|_{\partial B_1} = g, \quad v|_{\partial B_{1/2}} = (x/|x|)^{|d|-j}, \quad \text{and} \quad v|_{\partial B(x_i, R_0)} = x/|x|, \quad 1 \leq i \leq j,$$

and where v_i is a minimizer of $E_\varepsilon(u, B(x_i, \varepsilon R_0))$ in $W^{1,p}(B(x_i, \varepsilon R_0), R^n)$ satisfying

$$v_i|_{\partial B(x_i, \varepsilon R_0)} = (x - x_i)/|x - x_i|, \quad 1 \leq i \leq j.$$

By calculating, we have

$$\begin{aligned} E_\varepsilon(v, (B_1 \setminus B_{1/2}) \setminus (\bigcup_{i=1}^j B(x_i, R_0))) &\leq C; \\ E_\varepsilon(w, B(x_i, R_0) \setminus B(x_i, \varepsilon R_0)) &\leq C(1 + \varepsilon^{n-p}); \\ E_\varepsilon(v_i, B(x_i, \varepsilon R_0)) &\leq C(1 + \varepsilon^{n-p}). \end{aligned}$$

In addition,

$$\begin{aligned} E_\varepsilon(w, B_{1/2} \setminus B_{\varrho+\varepsilon}) &= \frac{1}{p} \int_{B_{1/2} \setminus B_{\varrho+\varepsilon}} \left| \nabla \left(\frac{x}{|x|} \right)^{|d|-j} \right|^p dx \\ &= \frac{(n-1)^{p/2}}{p} (|d|-j)^p |S^{n-1}| \int_{\varrho+\varepsilon}^{1/2} r^{n-p-1} dr \leq C(1 + \varepsilon^{n-p}). \end{aligned}$$

Since $0 \leq (r - \varrho)/\varepsilon \leq 1$ on $[\varrho, \varrho + \varepsilon]$, we obtain

$$\begin{aligned} E_\varepsilon(w, B_{\varrho+\varepsilon} \setminus B_\varrho) &= \frac{1}{p} \int_{B_{\varrho+\varepsilon} \setminus B_\varrho} \left(\left| \nabla \frac{|x| - \varrho}{\varepsilon} \right|^2 + \left(\frac{|x| - \varrho}{\varepsilon} \left| \nabla \left(\frac{x}{|x|} \right)^{|d|-j} \right|^2 \right)^{p/2} \right) dx \\ &\quad + \frac{1}{4\varepsilon^p} \int_{B_{\varrho+\varepsilon} \setminus B_\varrho} \left(1 - \left(\frac{|x| - \varrho}{\varepsilon} \right)^2 \right)^2 dx \\ &\leq \frac{C}{\varepsilon^p} \int_\varepsilon^{\varrho+\varepsilon} r^{n-1} dr \leq C\varepsilon^{-p} ((\varrho + \varepsilon)^n - \varepsilon^n) \leq C\varrho^{n-1} \varepsilon^{1-p}. \end{aligned}$$

Combining these estimates and noting that u_ε is a minimizer, we have

$$E_\varepsilon(u_\varepsilon, B_1) \leq E_\varepsilon(w, B_1) \leq C_2(\varepsilon^{n-p} + \varrho^{n-1} \varepsilon^{1-p}). \quad \square$$

Proposition 2.4. *In Case II, for any given $\gamma > 1$, there exists $C_3 > 0$ which is independent of $\varrho, \varepsilon \in (0, 1)$, such that*

$$E_\varepsilon(u_\varepsilon, B_1 \setminus B_{\gamma\varrho}) \leq C_3(1 + \varepsilon^{n-p}).$$

Proof. We prove the proposition by means of induction. Set

$$w_1 = \begin{cases} w, & \text{if } x \in B_1 \setminus B_{1/2}; \\ (x/|x|)^{|d|-j}, & \text{if } x \in B_{1/2} \setminus B_\varepsilon; \\ (|x|/\varepsilon) (x/|x|)^{|d|-j}, & \text{if } x \in B_\varepsilon. \end{cases}$$

For any $\gamma > 1$, there exists $\delta > 0$ such that $\gamma - \delta > 1$. According to Proposition 2.3, we have

$$(2-3) \quad E_\varepsilon(u_\varepsilon, B_1 \setminus B_{(\gamma-\delta+(k-1)n^{-1}\delta)\varrho}) \leq C(1 + \varepsilon^{n-p} + \varrho^{n-k} \varepsilon^{k-p})$$

with $k = 1$. Suppose (2-3) holds for $k = m$ with $m = 2, 3, \dots, n-1$. Then we shall verify it for $k = m+1$.

By the mean value theorem, there is $\sigma_m \in (\gamma - \delta + (m-1)n^{-1}\delta, \gamma - \delta + mn^{-1}\delta)$ such that

$$\begin{aligned} \int_{B_{\gamma\varrho} \setminus B_{(\gamma-\delta)\varrho}} (1 - |u_\varepsilon|^2)^2 dx &= \int_{(\gamma-\delta)\varrho}^{\gamma\varrho} \left(\int_{S^{n-1}} (1 - |u_\varepsilon(r, \zeta)|^2)^2 d\zeta \right) r^{n-1} dr \\ &= \int_{S^{n-1}} (1 - |u_\varepsilon(\sigma_m\varrho, \zeta)|^2)^2 d\zeta \cdot \int_{(\gamma-\delta)\varrho}^{\gamma\varrho} r^{n-1} dr \\ &= C^{-1} \varrho^n \int_{S^{n-1}} (1 - |u_\varepsilon(\sigma_m\varrho, \zeta)|^2)^2 d\zeta. \end{aligned}$$

This means that

$$(2-4) \quad \frac{1}{\varepsilon^p} \int_{S^{n-1}} (1 - |u_\varepsilon(\sigma_m\varrho, \zeta)|^2)^2 d\zeta = C\varrho^{-n} \varepsilon^{-p} \int_{B_{\gamma\varrho} \setminus B_{(\gamma-\delta)\varrho}} (1 - |u_\varepsilon|^2)^2 dx \\ \leq C\varrho^{-n} (1 + \varepsilon^{n-p} + \varrho^{n-m} \varepsilon^{m-p}),$$

by applying (2-3) with $k = m$. Define

$$\bar{u}(r) = \left(\frac{1}{|S^{n-1}|} \int_{S^{n-1}} |u(r, \zeta)|^2 d\zeta \right)^{1/2} \quad \text{for } r \in [0, 1].$$

Using Hölder's inequality, we easily get

$$(2-5) \quad (1 - |\bar{u}(r)|^2)^2 \leq \frac{1}{|S^{n-1}|} \int_{S^{n-1}} (1 - |u(r, \zeta)|^2)^2 d\zeta \quad \text{for } r \in [0, 1],$$

$$(2-6) \quad \left| \frac{d\bar{u}(r)}{dr} \right|^2 \leq \frac{1}{|S^{n-1}|} \int_{S^{n-1}} \left| \frac{\partial u(r, \zeta)}{\partial r} \right|^2 d\zeta \quad \text{a.e. } r \in [0, 1].$$

Let

$$W_m = \begin{cases} w_1, & \text{if } x \in B_1 \setminus B_{\sigma_m\varrho+\varepsilon}; \\ (|x| - \sigma_m\varrho/\varepsilon) (1 - \bar{u}(\sigma_m\varrho)) + \bar{u}(\sigma_m\varrho)w_1, & \text{if } x \in B_{\sigma_m\varrho+\varepsilon} \setminus B_{\sigma_m\varrho}; \\ \bar{u}w_1, & \text{if } x \in B_{\sigma_m\varrho}. \end{cases}$$

Obviously,

$$(2-7) \quad \int_{B_1} |\nabla w_1|^p + E_\varepsilon(w_1, B_1 \setminus B_{\sigma_m\varrho+\varepsilon}) \leq C(1 + \varepsilon^{n-p}).$$

From (2-4), (2-5), and $\varepsilon\varrho^{-1} \leq 1$, it follows that

$$\begin{aligned} J_m &:= \frac{1}{\varepsilon^p} \int_{\sigma_m\varrho}^{\sigma_m\varrho+\varepsilon} (1 - |\bar{u}(\sigma_m\varrho)|^2)^2 r^{n-1} dr \\ &\leq C\varrho^{-n} (1 + \varepsilon^{n-p} + \varrho^{n-m} \varepsilon^{m-p}) \int_{\sigma_m\varrho}^{\sigma_m\varrho+\varepsilon} r^{n-1} dr \\ &\leq C\varrho^{-n} (1 + \varepsilon^{n-p} + \varrho^{n-m} \varepsilon^{m-p}) \varepsilon\varrho^{n-1} \leq C(1 + \varepsilon^{n-p} + \varrho^{n-m-1} \varepsilon^{m+1-p}). \end{aligned}$$

So since $0 \leq (r - \sigma_m \varrho)/\varepsilon \leq 1$ for $r \in [\sigma_m \varrho, \sigma_m \varrho + \varepsilon]$, by (2–7) we have

$$\begin{aligned}
 (2-8) \quad E_\varepsilon(W_m, B_{\sigma_m \varrho + \varepsilon} \setminus B_{\sigma_m \varrho}) & \\
 & \leq C \int_{B_{\sigma_m \varrho + \varepsilon} \setminus B_{\sigma_m \varrho}} \left(|\nabla w_1|^p \left(|\bar{u}(\sigma_m \varrho)| + \frac{|x| - \sigma_m \varrho}{\varepsilon} (1 - |\bar{u}(\sigma_m \varrho)|) \right)^p \right. \\
 & \quad \left. + \left(\frac{1 - |\bar{u}(\sigma_m \varrho)|}{\varepsilon} \right)^p |\nabla |x||^p \right) dx \\
 & \quad + \frac{1}{4\varepsilon^p} \int_{B_{\sigma_m \varrho + \varepsilon} \setminus B_{\sigma_m \varrho}} \left(1 - (|\bar{u}(\sigma_m \varrho)| + \frac{|x| - \sigma_m \varrho}{\varepsilon} (1 - |\bar{u}(\sigma_m \varrho)|)) \right)^2 \\
 & \leq C \int_{B_{\sigma_m \varrho + \varepsilon} \setminus B_{\sigma_m \varrho}} |\nabla w_1|^p + C J_m \leq C(1 + \varepsilon^{n-p} + \varrho^{n-m-1} \varepsilon^{m+1-p}).
 \end{aligned}$$

Next, by the definition of w_1 and W_m and from (2–5) and (2–6), it follows that

$$(2-9) \quad E_\varepsilon(W_m, B_{\sigma_m \varrho} \setminus B_\varepsilon) \leq E_\varepsilon(u_\varepsilon, B_{\sigma_m \varrho} \setminus B_\varepsilon).$$

Finally, on B_ε ,

$$(2-10) \quad (|\nabla \bar{u}|^2 + |\bar{u}|^2 |\nabla w_1|^2)^{p/2} = (|\nabla \bar{u}|^2 + |\bar{u}|^2 \frac{|x|^2}{\varepsilon^2} \left| \nabla \left(\frac{x}{|x|} \right)^{|d|-j} \right|^2 + \frac{1}{\varepsilon^2} |\bar{u}|^2)^{p/2}.$$

Using the mean value theorem, we see that on B_ε ,

$$\begin{aligned}
 & \left(|\nabla \bar{u}|^2 + |\bar{u}|^2 \frac{|x|^2}{\varepsilon^2} \left| \nabla \left(\frac{x}{|x|} \right)^{|d|-j} \right|^2 + \frac{1}{\varepsilon^2} |\bar{u}|^2 \right)^{p/2} - (|\nabla \bar{u}|^2)^{p/2} \\
 & = \frac{p|\bar{u}|^2}{2} \left(\frac{1}{\varepsilon^2} + \frac{(|d|-j)^2}{\varepsilon^2} \right) \int_0^1 \left(s \left(|\nabla \bar{u}|^2 + |\bar{u}|^2 \frac{|x|^2}{\varepsilon^2} \left| \nabla \left(\frac{x}{|x|} \right)^{|d|-j} \right|^2 + \frac{1}{\varepsilon^2} |\bar{u}|^2 \right) \right. \\
 & \quad \left. + (1-s)(|\nabla \bar{u}|^2) \right)^{(p-2)/2} ds \\
 & = \frac{p|\bar{u}|^2}{2} \frac{1 + (|d|-j)^2}{\varepsilon^2} I.
 \end{aligned}$$

From Proposition 2.2 and (2–6), it follows that $I \leq C\varepsilon^{2-p}$. Substituting this into the preceding equality and combining with (2–10), we obtain

$$\begin{aligned}
 E_\varepsilon(\bar{u}w_1, B_\varepsilon) & = \frac{1}{p} \int_{B_\varepsilon} (|\nabla \bar{u}|^2 + |\bar{u}|^2 |\nabla w_1|^2)^{p/2} dx + \frac{1}{2\varepsilon^p} \int_{B_\varepsilon} w_1^2 |\bar{u}|^2 dx \\
 & \leq E_\varepsilon(\bar{u}, B_\varepsilon) + C\varepsilon^{2-p} \int_0^\varepsilon |\bar{u}|^2 \frac{1}{\varepsilon^2} r^{n-1} dr.
 \end{aligned}$$

By the definition of \bar{u} and (2–6), we have at last that

$$(2-11) \quad E_\varepsilon(\bar{u}w_1, B_\varepsilon) \leq E_\varepsilon(u_\varepsilon, B_\varepsilon) + C\varepsilon^{n-p}.$$

Since u_ε is a minimizer, it follows from (2-7)-(2-9) and (2-11) that

$$E_\varepsilon(u_\varepsilon, B_1) \leq E_\varepsilon(W_m, B_1) \leq C(1 + \varepsilon^{n-p} + \varrho^{n-m-1}\varepsilon^{m+1-p}) + E_\varepsilon(u_\varepsilon, B_{\sigma_m\varrho}).$$

Observing that $E_\varepsilon(u_\varepsilon, B_1 \setminus B_{(\gamma-\delta+mn^{-1}\delta)\varrho}) \leq E_\varepsilon(u_\varepsilon, B_1 \setminus B_{\sigma_m\varrho})$, we see that (2-3) holds with $k = m + 1$. Proposition 2.4 follows by taking $k = n$ in (2-3). \square

3. Location of zeros

At first, we will show that there is no zero of u_ε near the boundary ∂B_1 .

Theorem 3.1. *There is a constant $\rho > 0$, such that for $x \in B_1 \setminus B_{1-2\rho\varepsilon}$,*

$$|u_\varepsilon(x)| \geq 1/2.$$

Proof. Scaling $y = x\varepsilon^{-1}$ in (2-1) yields (2-2). According to the results of the C^α -estimation of v (see, for example, Theorem 1 and lines 19-21 on p. 104 of [Chen and DiBenedetto 1989]), there exist $C > 0$ and $\alpha \in (0, 1)$, such that for any $\rho \in (0, 1)$ and $y_0 \in \partial B_{\varepsilon^{-1}}$, we have $|v(y) - v(y_0)| \leq C|y - y_0|^\alpha$ for all $y \in B_{\varepsilon^{-1}} \cap B(y_0, 4\rho)$. Taking $\rho = 1/(8C)$, we obtain $|v(y)| \geq |v(y_0)| - C|y - y_0| \geq 1/2$. Letting $x = y\varepsilon$, easily implies the theorem. \square

Proposition 3.2. *Let u_ε be a minimizer of $E_\varepsilon(u, B_1)$. There exists a constant $C > 0$ which is independent of ϱ , $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small, such that*

$$(3-1) \quad \frac{1}{\varepsilon^n} \int_{B_1 \setminus B_\varrho} (1 - |u_\varepsilon|^2)^2 + \frac{1}{\varepsilon^n} \int_{B_\varrho} |u_\varepsilon|^2 \leq C \quad \text{in Case I,}$$

$$(3-2) \quad \frac{1}{\varepsilon\varrho^{n-1}} \int_{B_1 \setminus B_\varrho} (1 - |u_\varepsilon|^2)^2 + \frac{1}{\varepsilon\varrho^{n-1}} \int_{B_\varrho} |u_\varepsilon|^2 \leq C \quad \text{in Case II.}$$

Furthermore, in Case II with $p > n$, for any $\gamma > 1$, there is $C > 0$ independent of ϱ , $\varepsilon \in (0, \varepsilon_0)$ such that

$$(3-3) \quad \frac{1}{\varepsilon^n} \int_{B_1 \setminus B_{\gamma\varrho}} (1 - |u_\varepsilon|^2)^2 \leq C.$$

Proof. When $p > n$, (3-1)-(3-3) are corollaries of Proposition 2.3 and Proposition 2.4 by multiplying by ε^{p-n} or $\varepsilon^{p-1}\varrho^{1-n}$. When $1 < p < n$, the idea of the proof comes from [Struwe 1993]. Set $v[\varepsilon] = \inf\{E_\varepsilon(u, B_1); u \in W\}$. For fixed $u \in W$, the map $\varepsilon \rightarrow E_\varepsilon(u, B_1)$ is nonincreasing, and

$$-\frac{\partial}{\partial \varepsilon} E_\varepsilon(u, B_1) = \frac{p}{4\varepsilon^{p+1}} \int_{B_1 \setminus B_\varrho} (1 - |u_\varepsilon|^2)^2 + \frac{p}{2\varepsilon^{p+1}} \int_{B_\varrho} |u_\varepsilon|^2.$$

Noting $v[\varepsilon + \delta] \leq E_{\varepsilon+\delta}(u_\varepsilon, B_1) \leq E_\varepsilon(u_\varepsilon, B_1) = v[\varepsilon]$, we have

$$(3-4) \quad \frac{1}{4\varepsilon^{p+1}} \int_{B_1 \setminus B_\varrho} (1 - |u_\varepsilon|^2)^2 + \frac{1}{2\varepsilon^{p+1}} \int_{B_\varrho} |u_\varepsilon|^2 \\ = \lim_{\delta \rightarrow 0} \frac{E_\varepsilon(u_\varepsilon, B_1) - E_{\varepsilon+\delta}(u_\varepsilon, B_1)}{\delta} \leq \overline{\lim}_{\delta \rightarrow 0} \frac{v[\varepsilon] - v[\varepsilon + \delta]}{\delta} = -v'[\varepsilon].$$

We claim that there exists a constant $M > 0$ independent of ε and ϱ such that when both ϱ and ε tend to zero,

$$(3-5) \quad \begin{aligned} -\varepsilon^{p+1-n} v'[\varepsilon] &\leq M \quad \text{in Case I;} \\ -\varepsilon^p \varrho^{1-n} v'[\varepsilon] &\leq M \quad \text{in Case II.} \end{aligned}$$

Otherwise, we can find $\varepsilon_1 > 0$, such that if $\varepsilon, \varrho \in (0, \varepsilon_1)$, then $-v'[\varepsilon] > M\varepsilon^{n-p-1}$ in Case I, and $-v'[\varepsilon] > M\varepsilon^{-p}\varrho^{n-1}$ in Case II. Now, let $M = 2(n-p)(C_2+1)\varepsilon_1^{p-n}$ in Case I and $M = 2(C_2+1)(p-1)$ in Case II. Here, C_2 is the constant in Proposition 2.3. Integrating from ε to ε_1 , we obtain

$$\begin{aligned} v[\varepsilon] &\geq v[\varepsilon_1] - \int_\varepsilon^{\varepsilon_1} v'[\varepsilon] d\varepsilon > v[\varepsilon_1] + 2C_2 + 1 && \text{in Case I;} \\ v[\varepsilon] &\geq v[\varepsilon_1] - \int_\varepsilon^{\varepsilon_1} v'[\varepsilon] d\varepsilon > v[\varepsilon_1] + (2C_2 + 1)\varepsilon^{1-p}\varrho^{n-1} && \text{in Case II.} \end{aligned}$$

These contradict Proposition 2.3. Substituting (3-5) into (3-4), we can find a sufficiently small $\varepsilon_0 > 0$, such that (3-1) and (3-2) hold with $C = M + 1$. \square

Hereafter, we assume $\varepsilon, \varrho \in (0, \varepsilon_0)$. For any $\gamma > 1$, set

$$\begin{aligned} A &= B_{1-\rho\varepsilon} \setminus B_{\gamma\varrho} \quad \text{in Case II,} \\ A &= B_{1-\rho\varepsilon} \setminus B_\varrho \quad \text{in Case I.} \end{aligned}$$

Proposition 3.3. *Let u_ε be a minimizer of $E_\varepsilon(u, B_1)$. Then for any $\eta \in (0, 1/2)$, there exist positive constants λ, μ which are independent of ϱ, ε , such that*

(1) *In Case I or in Case II with $p > n$, if*

$$(3-6) \quad \frac{1}{\varepsilon^n} \int_{A \cap B(\cdot, 2l\varepsilon)} (1 - |u_\varepsilon|^2)^2 \leq \mu,$$

for $B(\cdot, 2l\varepsilon)$ a ball of radius $2l\varepsilon$ with $l \geq \lambda$, then $|u_\varepsilon(x)| \geq 1 - \eta$ for all $x \in A \cap B(\cdot, l\varepsilon)$.

(2) *In Case I, if*

$$(3-7) \quad \frac{1}{\varepsilon^n} \int_{B_\varrho \cap B(\cdot, 2l\varepsilon)} |u_\varepsilon|^2 \leq \mu,$$

then $|u_\varepsilon(x)| \leq \eta$ for all $x \in B_\varrho \cap B(\cdot, l\varepsilon)$.

Proof. Observe that there exists a constant $C_3 > 0$ which is independent of ϱ and ε , such that for $0 < r \leq 1$, if x is in B_1 , then $|B_1 \cap B(x, r)| \geq |A \cap B(x, r)| \geq C_3 r^n$. Let $\lambda = \eta/(2C_1)$ and $\mu = C_3 \eta^2 \lambda^n/4$.

Suppose that there is a point $x_0 \in A \cap B(\cdot, l\varepsilon)$ such that $|u_\varepsilon(x_0)| < 1 - \eta$. According to Proposition 2.2, we have

$$(3-8) \quad |u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C_1 \varepsilon^{-1} |x - x_0| = C_1 \lambda = \eta/2 \quad \text{for all } x \in B(x_0, \lambda\varepsilon),$$

and hence $(1 - |u_\varepsilon(x)|^2)^2 > \eta^2/4$ for all $x \in B(x_0, \lambda\varepsilon)$. Thus

$$\int_{B(x_0, \lambda\varepsilon) \cap A} (1 - |u_\varepsilon|^2)^2 > (\eta^2/4) |A \cap B(x_0, \lambda\varepsilon)| \geq C_3 \eta (\lambda\varepsilon)^n / 4 = \mu \varepsilon^n.$$

Since $x_0 \in B(\cdot, l\varepsilon) \cap A$ and $(B(x_0, \lambda\varepsilon) \cap A) \subset (B(\cdot, 2l\varepsilon) \cap A)$, it follows that

$$\int_{B(\cdot, 2l\varepsilon) \cap A} (1 - |u_\varepsilon|^2)^2 > \mu \varepsilon^n,$$

which contradicts (3-6). This proves (1), and the proof of (2) is analogous. \square

In Case II with $p \in (1, n)$, Proposition 2.2 is not sufficient to deduce Proposition 3.3. The reason is that in Case II, the estimation (3-2) is not accurate as (3-1), which forces us to investigate (3-8) on the larger ball $B(x_0, \lambda\varepsilon^{1/n} \varrho^{1-1/n})$. Proposition 2.2 is invalid since it only holds on the smaller ball $B(x_0, \lambda\varepsilon)$. To obtain Proposition 3.3, we instead use Proposition 3.4, though it only holds for p sufficiently close to the dimension n .

Proposition 3.4. *Assume u_ε is a minimizer of $E_\varepsilon(u, B_1)$. Then in Case II with $p \in (n-t, n)$ where $t \in (0, \min\{1/2, 4/n\})$, there exists a constant $C > 0$ such that for any $x, x_0 \in \bar{A}$,*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C |x - x_0|^\alpha \quad \text{for some } \alpha \in (0, 1 - n/(p+t)).$$

Proof. By the Reverse Hölder inequality (Proposition 3.5) and Proposition 2.4, we have $\|\nabla u_\varepsilon\|_{L^{p+t}(A)} \leq C \|\nabla u_\varepsilon\|_{L^p(A)} \leq C$ for some $t \in (0, \min\{1/2, 4/n\})$.

Since $|u_\varepsilon| \leq 1$ a.e. on B , we obtain $\|u_\varepsilon\|_{W^{1,p+t}(A)} \leq C$. When $p \in (n-t, n)$, by the embedding theorem we see that $|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C |x - x_0|^\alpha$ for any $x, x_0 \in \bar{A}$, for some $\alpha \in (0, 1 - n/(p+t))$. \square

Proposition 3.5 (Reverse Hölder inequality). *Assume $p > 1$ and u_ε is a minimizer of $E_\varepsilon(u, B_1)$. Then there exist constants $t \in (0, \min\{1/2, 4/n\})$, $R_0 \in (0, 1/2)$ and $C > 0$ which are independent of ε and ϱ such that for any $B(\cdot, R) \subset B_1$ with $2R < R_0$,*

$$\left(\int_{B(\cdot, R)} |\nabla u_\varepsilon|^q dx \right)^{1/q} \leq C \left(\int_{B(\cdot, 2R)} (|\nabla u_\varepsilon|^2 + 1)^{p/2} dx \right)^{1/p} \quad \text{for } q \in [p, p+2t).$$

The proof is completely analogous to that of Proposition 2.1 in [Lei 2004].

Proposition 3.6. *Assume u_ε is a minimizer. Then in Case II with $p \in (n - t, n)$ where t is the constant in Proposition 3.5, for any $\eta \in (0, 1/2)$, there exist positive constants λ, μ which are independent of ϱ, ε , such that if*

$$\frac{1}{\varepsilon \varrho^{n-1}} \int_{A \cap B(\cdot, 2l\varepsilon^{1/n} \varrho^{(n-1)/n})} (1 - |u_\varepsilon|^2)^2 \leq \mu,$$

where $B(\cdot, 2l\varepsilon^{1/n} \varrho^{(n-1)/n})$ is some ball of radius $2l\varepsilon^{1/n} \varrho^{(n-1)/n}$ with $l \geq \lambda$, then

$$|u_\varepsilon(x)| \geq 1 - \eta \quad \text{for all } x \in A \cap B(\cdot, l\varepsilon^{1/n} \varrho^{(n-1)/n}).$$

The proof is like that of Proposition 3.3; the only difference is that we apply Proposition 3.4. instead of Proposition 2.2.

To find the zeros of u_ε in Case I or in Case II with $p > n$, we may take (2–1) as a ruler to distinguish the ball of radius $\lambda\varepsilon$ which contains the zeros. Given $\gamma > 1$, let λ, μ be the same constants as in Proposition 3.3. If

$$\frac{1}{\varepsilon^n} \int_{B(x^\varepsilon, 2\lambda\varepsilon) \cap A} (1 - |u_\varepsilon|^2)^2 \leq \mu,$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called a good ball. Otherwise it is called a bad ball. Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying the following conditions:

- (i) $x_i^\varepsilon \in A$ for $i \in I$.
- (ii) $A \subset \bigcup_{i \in I} B(x_i^\varepsilon, \lambda\varepsilon)$.
- (iii) $B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset$ for $i \neq j$.

Set $J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon) \text{ is a bad ball}\}$.

Proposition 3.7. *There exists an integer N independent of ε exceeding the number $\text{Card } J_\varepsilon$ of bad balls.*

Proof. Since (iii) implies that every point in A can be covered by a finite number m of balls where m is independent of ε and ϱ , from (3–1) or (3–3) and the definition of bad ball, we have

$$\begin{aligned} \mu \varepsilon^n \text{Card } J_\varepsilon &\leq \sum_{i \in J_\varepsilon} \int_{B(x_i^\varepsilon, 2\lambda\varepsilon) \cap A} (1 - |u_\varepsilon|^2)^2 \\ &\leq m \int_{\bigcup_{i \in J_\varepsilon} B(x_i^\varepsilon, 2\lambda\varepsilon) \cap A} (1 - |u_\varepsilon|^2)^2 \leq m \int_{B_1 \setminus B_\varrho} (1 - |u_\varepsilon|^2)^2 \leq mC\varepsilon^n. \end{aligned}$$

Hence $\text{Card } J_\varepsilon \leq mC/\mu \leq N$. □

Proof of Theorem 1.1. Based on Proposition 3.7, by applying Theorem IV.1 of [Bethuel et al. 1994], we may modify the family of bad balls so that the new

family, denoted $\{B(x_i^\varepsilon, h\varepsilon); i \in J\}$, satisfies

$$\bigcup_{i \in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \bigcup_{i \in J} B(x_i^\varepsilon, h\varepsilon), \quad \text{Card } J \leq \text{Card } J_\varepsilon,$$

and

$$(3-9) \quad |x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, \quad i, j \in J, \quad i \neq j,$$

where h is a constant satisfying $\lambda \leq h = h(\eta) \leq \lambda 9^N = 9^N \eta / (2C_1)$. Choose $\eta > 0$ sufficiently small so that $h < 1$. Condition (3-9) implies that no two balls in the new family intersect. Thus the points x where $|u_\varepsilon(x)| \leq 1 - \eta$ are contained in these finite, disjoint bad balls $\{B(x_i, h\varepsilon)\}_{i=1}^N$ and $B_{h\varepsilon} \cup B_{\gamma Q}$. Combining this with Theorem 3.1, we obtain (1-2).

Similarly, (1-3) is obtained by applying (3-1) and Proposition 3.3(2); (1-4) is obtained by applying (3-3) and Proposition 3.3(1); lastly, (1-5) is obtained by applying (3-2) and Proposition 3.6. \square

For each $i = 1, 2, \dots, \text{Card } J$, there exists a sequence $\varepsilon_k \rightarrow 0$ such that the centers $x_i^{\varepsilon_k}$ approach either 0 or some $a_i \in \bar{B}_1$. There may be more than one such subsequence $x_i^{\varepsilon_k}$ converging to the same point. We denote by $0, a_1, a_2, \dots, a_N$ the distinct points in $\{0, a_i\}_{i=1}^{\text{Card } J}$.

From the discussion above, we also see that for any $\sigma > 0$,

$$(3-10) \quad |u_\varepsilon(x)| \geq 1/2 \quad \text{for all } x \in \bar{B}_1 \setminus \left(\bigcup_{j=1}^{\text{Card } J} B(a_j, \sigma) \cup B_\sigma \right).$$

4. Uniform estimation

Let u_ε be a minimizer of $E_\varepsilon(u, B_1)$. When $p \in (1, n)$, Propositions 2.3 and 2.4 imply (1-6) and (1-7), respectively. In this section we shall prove (1-7) when $p > n$.

Theorem 4.1. *Let $R > 0$ be small enough that $B(x, 2R) \Subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. Then there are constants $C > 0$ and $R_j = 2R - jR/([p] + 1)$ such that*

$$(4-1) \quad E_\varepsilon(u_\varepsilon, B_j) \leq C\varepsilon^{j-p}$$

for $j = n, n+1, \dots, [p]$, where $\varepsilon \in (0, \varepsilon_0)$ and $B_j = B(x, R_j)$.

For $j = n$, the inequality (4-1) is a corollary of Proposition 2.4. Suppose that (4-1) holds for all $j \leq m$. Then, in particular,

$$(4-2) \quad E_\varepsilon(u_\varepsilon, B_m) \leq C\varepsilon^{m-p}.$$

Suppose $m < [p]$. We want to prove (4-1) for $j = m+1$.

According to Proposition 2.1 and (3–10), we have $1/2 \leq |u_\varepsilon(x)| \leq 1$ for all $x \in B(x, 2R)$. As in the derivation of (2–4), by (4–2) and the mean value theorem, there is $r \in [R_{m+1/2}, R_m]$ such that

$$(4-3) \quad \int_{\partial B(x,r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p} \int_{\partial B(x,r)} (1 - |u_\varepsilon|^2)^2 d\xi \leq C\varepsilon^{m-p}.$$

Here ξ is the integration variable on $\partial B(x, r)$.

Proposition 4.2. *Denote $B(x, r)$ by B . If ρ_m is a minimizer of the functional*

$$E(\rho, B) = \frac{1}{p} \int_B (|\nabla \rho|^2 + 1)^{p/2} + \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2$$

on $W_{|u_\varepsilon|}^{1,p}(B, R^+ \cup \{0\})$, then $E(\rho_m, B) \leq C\varepsilon^{m-p+1}$.

Proof. Obviously, the minimizer ρ_m exists and satisfies

$$(4-4) \quad -\operatorname{div}(v^{(p-2)/2} \nabla \rho) = 1/\varepsilon^p(1 - \rho) \quad \text{on } B,$$

and

$$(4-5) \quad \rho|_{\partial B} = |u_\varepsilon|,$$

where $v = |\nabla \rho|^2 + 1$. Since $1/2 \leq |u_\varepsilon| \leq 1$, from the maximum principle it follows that

$$(4-6) \quad 1/2 \leq \rho_m \leq 1 \quad \text{on } \bar{B}.$$

Applying (4–2) we see easily that

$$(4-7) \quad E(\rho_m, B) \leq E(|u_\varepsilon|, B) \leq C E_\varepsilon(u_\varepsilon, B) \leq C\varepsilon^{m-p}.$$

Multiplying (4–4) by $(v \cdot \nabla \rho)$, where ρ denotes ρ_m , and integrating over B , we have

$$(4-8) \quad -\int_{\partial B} v^{(p-2)/2} (v \cdot \nabla \rho)^2 d\xi + \int_B v^{(p-2)/2} \nabla \rho \cdot \nabla (v \cdot \nabla \rho) \\ = \frac{1}{\varepsilon^p} \int_B (1 - \rho)(v \cdot \nabla \rho),$$

where ν denotes the unit outside norm vector on ∂B . Using (4–7) we obtain

$$(4-9) \quad \left| \int_B v^{(p-2)/2} \nabla \rho \cdot \nabla (v \cdot \nabla \rho) \right| \leq C\varepsilon^{m-p} + \frac{1}{p} \left| \int_B v \cdot \nabla (v^{p/2}) \right| \\ \leq C\varepsilon^{m-p} + \frac{1}{p} \int_{\partial B} v^{p/2} d\xi.$$

Combining (4-3), (4-5), and (4-7) we also have

$$\left| \frac{1}{\varepsilon^p} \int_B (1-\rho)(v \cdot \nabla \rho) \right| \leq \frac{1}{2\varepsilon^p} \left| \int_B (1-\rho)^2 \operatorname{div} v - \int_{\partial B} (1-\rho)^2 d\xi \right| \leq C\varepsilon^{m-p}.$$

Substituting this and (4-9) into (4-8) yields

$$(4-10) \quad \left| \int_{\partial B} v^{(p-2)/2} (v \cdot \nabla \rho)^2 d\xi \right| \leq C\varepsilon^{m-p} + \frac{1}{p} \int_{\partial B} v^{p/2} d\xi.$$

Applying (4-5), (4-3) and (4-10), we obtain for any $\delta \in (0, 1)$,

$$\begin{aligned} \int_{\partial B} v^{p/2} d\xi &= \int_{\partial B} v^{(p-2)/2} \left(1 + \sum_{i=1}^{n-1} (\tau_i \cdot \nabla \rho)^2 + (v \cdot \nabla \rho)^2 \right) d\xi \\ &\leq C(\delta)\varepsilon^{m-p} + (1/p + 2\delta) \int_{\partial B} v^{p/2} d\xi, \end{aligned}$$

where τ_i , $i = 1, 2, \dots, n-1$, denotes the unit tangent vector on ∂B and $\tau_i \perp \tau_j$ when $i \neq j$. Choosing $\delta > 0$ sufficiently small yields

$$(4-11) \quad \int_{\partial B} v^{p/2} d\xi \leq C\varepsilon^{m-p}.$$

Multiplying both sides of (4-4) by $(1-\rho)$ and integrating over B , we have

$$\int_B v^{(p-2)/2} |\nabla \rho|^2 + \frac{1}{\varepsilon^p} \int_B (1-\rho)^2 = - \int_{\partial B} v^{(p-2)/2} (v \cdot \nabla \rho) (1-\rho) d\xi.$$

Thus, applying Hölder's inequality, (4-3), (4-5), (4-6) and (4-11), we obtain

$$(4-12) \quad E(\rho_m, B) \leq C\varepsilon^{(m-p)(p-1)/p} \left| \int_{\partial B} (1-|u_\varepsilon|)^2 d\xi \right|^{1/p} \leq C\varepsilon^{m-p+1}. \quad \square$$

Remark 4. Comparing (4-12) with (4-7), we see that the exponent of ε in the upper bound of $E(\rho_m, B)$ is improved. We shall use ρ_m as a comparison function to improve the exponent of ε in the upper bound of $E_\varepsilon(u_\varepsilon, B)$.

Proposition 4.3. *Set $h = |u_\varepsilon|$. Then for any $\delta \in (0, 1/2)$, there is $C > 0$ such that*

$$\begin{aligned} \frac{1}{p} \int_B |\nabla h|^p + \frac{1}{4\varepsilon^p} \int_B (1-h^2)^2 &\leq C\varepsilon^{m-p+1} + \delta \int_B |\nabla u_\varepsilon|^p \\ &\quad + C \left(\int_{B(x, 2r)} |\nabla u_\varepsilon|^p + 1 \right) \left(\int_B (1-h^2)^2 \right)^{t/(p+t)}. \end{aligned}$$

Here t is the constant in Proposition 3.5.

Proof. Let $U_\varepsilon = \rho_m w$ on B and $U_\varepsilon = u_\varepsilon$ on $B_1 \setminus B$, where $w = u_\varepsilon/|u_\varepsilon|$. Since u_ε is a minimizer of $E_\varepsilon(u, G)$, we have

$$E_\varepsilon(u_\varepsilon, G) \leq E_\varepsilon(U_\varepsilon, B_1) = E_\varepsilon(\rho_m w, B) + E_\varepsilon(u_\varepsilon, B_1 \setminus B).$$

This means $E_\varepsilon(u_\varepsilon, B) \leq E_\varepsilon(\rho_m w, B)$. Noting that

$$\begin{aligned} & \int_B (|\nabla \rho_m|^2 + \rho_m^2 |\nabla w|^2)^{p/2} dx - \int_B (\rho_m^2 |\nabla w|^2)^{p/2} dx = \\ & \frac{p}{2} \int_B \int_0^1 ((|\nabla \rho_m|^2 + \rho_m^2 |\nabla w|^2)^{(p-2)/2} s + (\rho_m^2 |\nabla w|^2)^{(p-2)/2} (1-s)) ds |\nabla \rho_m|^2 dx \\ & \leq C \int_B (|\nabla \rho_m|^p + |\nabla \rho_m|^2 |\nabla w|^{p-2}) dx, \end{aligned}$$

and using Hölder's inequality, (4–6), and (4–12), we have, for any $\delta \in (0, 1)$,

$$\begin{aligned} E_\varepsilon(u_\varepsilon, B) & \leq E_\varepsilon(\rho_m w, B) \\ & \leq \frac{1}{p} \int_B (\rho_m^2 |\nabla w|^2)^{p/2} + C \int_B (|\nabla \rho_m|^p + |\nabla \rho_m|^2 |\nabla w|^{p-2}) + \frac{1}{4\varepsilon^p} \int_B (1 - \rho_m^2)^2 \\ & \leq \frac{1}{p} \int_B |\nabla w|^p + C\varepsilon^{m+1-p} + \delta \int_B |\nabla u_\varepsilon|^p. \end{aligned}$$

Combining this with Jensen's inequality we obtain

$$\begin{aligned} (4-13) \quad & \frac{1}{p} \int_B |\nabla h|^p + \frac{1}{p} \int_B (h^p - 1) |\nabla w|^p + \frac{1}{4\varepsilon^p} \int_B (1 - h^2)^2 \\ & \leq E_\varepsilon(u_\varepsilon, B) - \frac{1}{p} \int_B |\nabla w|^p \leq C\varepsilon^{m-p+1} + \delta \int_B |\nabla u_\varepsilon|^p. \end{aligned}$$

In view of (3–10) and Proposition 3.5, we get

$$\begin{aligned} (4-14) \quad & \frac{1}{p} \int_B (1 - h^p) |\nabla w_\varepsilon|^p \leq \frac{2^p}{p} \int_B (1 - h^p) h^p |\nabla w_\varepsilon|^p \\ & \leq C(R) \left(\int_{B(x, 2r)} |\nabla u_\varepsilon|^p + 1 \right) \left(\int_B (1 - h^2)^2 \right)^{t/(p+t)}. \end{aligned}$$

Substituting this into (4–13) yields

$$\begin{aligned} (4-15) \quad & \frac{1}{p} \int_B |\nabla h|^p + \frac{1}{4\varepsilon^p} \int_B (1 - h^2)^2 \leq C\varepsilon^{m-p+1} + \delta \int_B |\nabla u_\varepsilon|^p \\ & + C \left(\int_{B(x, 2r)} |\nabla u_\varepsilon|^p + 1 \right) \left(\int_B (1 - h^2)^2 \right)^{t/(p+t)}. \quad \square \end{aligned}$$

Proof of Theorem 4.1. Step 1. Using (3–10) we may write $w = u_\varepsilon / |u_\varepsilon|$ on $B(x, 3R)$. Substituting this into (2–1) yields that

$$\int_{B(x, 3R)} |\nabla u|^{p-2} (w \nabla h + h \nabla w) \nabla \psi = \frac{1}{\varepsilon^p} \int_{B(x, 3R)} h w \psi (1 - h^2)$$

or $\operatorname{div}(|\nabla u|^{p-2}(w\nabla h + h\nabla w)) + 1/\varepsilon^p h w(1 - h^2) = 0$ in the distribution sense. Taking $\psi = w\zeta$ where $\zeta \in W_0^{1,p}(B(x, 3R))$, and noting that $w\nabla w = \frac{1}{2}\nabla(|w|^2) = 0$, we obtain

$$(4-16) \quad \frac{1}{\varepsilon^p} \int_{B(x, 3R)} h(1 - h^2)\zeta = \int_{B(x, 3R)} |\nabla u|^{p-2}(\nabla h\nabla\zeta + h|\nabla w|^2\zeta).$$

In addition, we also have $\operatorname{div}(|\nabla u|^{p-2}(w\nabla h + h\nabla w)) \wedge w = 0$ in the distribution sense. Together with $|w| = 1$, this implies

$$\int_{B(x, 3R)} |\nabla u|^{p-2} h(w \wedge \nabla w) \nabla \zeta = 0.$$

Using this with Theorem 6.1 (which will be proved in Section 6), we can deduce that

$$(4-17) \quad \int_B |\nabla u|^{p-2} h^2 |\nabla w|^2 \leq C \left(\int_{B(x, 2r)} |\nabla u|^p \right)^{1-2/p}.$$

Applying (4-17) and Hölder's inequality we have, for any $\delta \in (0, 1)$,

$$(4-18) \quad \begin{aligned} \int_B |\nabla u|^p &= \int_B |\nabla u|^{p-2} (h^2 |\nabla w|^2 + |\nabla h|^2) \\ &\leq C \left(\int_{B(x, 2r)} |\nabla u|^p \right)^{1-2/p} + \delta \int_B |\nabla u|^p + C(\delta) \left(\int_B |\nabla h|^p \right). \end{aligned}$$

Substituting (4-15) into (4-18) and choosing $\delta > 0$ sufficiently small we see that

$$(4-19) \quad \begin{aligned} \int_B |\nabla u|^p &\leq C \left(\int_{B(x, 2r)} |\nabla u|^p \right)^{1-2/p} + C\varepsilon^{m-p+1} \\ &\quad + C \left(\int_{B(x, 2r)} |\nabla u_\varepsilon|^p + 1 \right) \left(\int_B (1 - h^2)^2 \right)^{t/(p+t)}. \end{aligned}$$

From (4-2) it follows that $\int_{B(x, 2r)} |\nabla u|^p \leq C\varepsilon^{m-p}$. Substituting this into (4-19) yields

$$(4-20) \quad \int_B |\nabla u|^p \leq C(\varepsilon^{m-p})^{1-2/p} + C\varepsilon^{m-p+1} + C\varepsilon^{m-p+mt/(p+t)} =: I_1 + I_2 + I_3.$$

Step 2. If $m \leq p/2$, then $m + 1 - p \leq (m - p)(1 - 2/p)$. Now, $I_1 \leq I_2$. Let k_0 be an integer such that $m + 1 \leq (1 + t/(p + t))^{k_0} m$.

Assume ζ is in $C_0^\infty(B(x, 2R), [0, 1])$ and satisfies $|\nabla\zeta| \leq C$ and $\zeta = 1$ on $B_{m+1/2}$. Taking the test function as $h\zeta(1-h)$ in (4–16), we have

$$\begin{aligned} \frac{1}{\varepsilon^p} \int_B h^2(1-h^2)\zeta(1-h) + \int_B |\nabla u|^{p-2} |\nabla h|^2 h \zeta \\ = \int_B |\nabla u|^{p-2} \nabla h \nabla \zeta h(1-h) + \int_B |\nabla u|^p \zeta(1-h) \leq C \int_B |\nabla u|^p. \end{aligned}$$

Noting that $\zeta = 1$ on $B_{m+1/2}$ and applying (4–20), we obtain

$$(4-21) \quad \int_{B_{m+1/2}} (1-h^2)^2 \leq C \varepsilon^{m(1+t/(p+t))} \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

On the other hand, as in the derivation of (4–13), for $B_{m+1/2}$ we rewrite Proposition 4.3 and still conclude that for any $\delta > 0$,

$$(4-22) \quad \frac{1}{p} \int_{B_{m+1/2}} |\nabla h|^p + \frac{1}{4\varepsilon^p} \int_{B_{m+1/2}} (1-h^2)^2 \\ \leq C \varepsilon^{m-p+1} + \frac{1}{p} \int_{B_{m+1/2}} (1-h^p) |\nabla w|^p + \delta \int_{B_{m+1/2}} |\nabla u_\varepsilon|^p.$$

To estimate the second term of the right-hand side of (4–22), we apply (4–21) to obtain

$$\frac{1}{p} \int_{B_{m+1/2}} (1-h^p) |\nabla w|^p \leq C \varepsilon^{(m+\frac{t}{p+t}m)\frac{t}{p+t}+m+\frac{t}{p+t}m-p} = C \varepsilon^{m(1+t/(p+t))^2-p}$$

by a similar derivation to (4–14). Substituting this into (4–22) yields

$$\frac{1}{p} \int_{B_{m+1/2}} |\nabla h|^p \leq C(\varepsilon^{m-p+1} + \varepsilon^{m(1+t/(p+t))^2-p}) + \delta \int_{B_{m+1/2}} |\nabla u_\varepsilon|^p.$$

Using this instead of (4–15) and choosing $\delta > 0$ sufficiently small we can improve (4–20) to

$$\int_{B_{m+1/2}} |\nabla u_\varepsilon|^p \leq C + C(\varepsilon^{m-p+1} + \varepsilon^{m(1+t/(p+t))^2-p}) \leq C \varepsilon^{m(1+t/(p+t))^2-p}.$$

We have improved the exponent $m(1+t/(p+t)) - p$ of ε to $m(1+t/(p+t))^2 - p$, though the integral domain B has shrunk to $B_{m+1/2}$. By induction, it can be derived in k_0 steps that

$$\int_{B_{m+1-1/2^{k_0-1}}} |\nabla u_\varepsilon|^p \leq C + C(\varepsilon^{m-p+1} + \varepsilon^{m(1+t/(p+t))^{k_0-p}}).$$

Noting the definition of k_0 , we obtain (4-2) for $j = m + 1$:

$$\int_{B_{m+1}} |\nabla u_\varepsilon|^p \leq \int_{B_{m+1-1/2^{k_0-1}}} |\nabla u_\varepsilon|^p \leq C(\varepsilon^{m-p+1} + 1).$$

Step 3. If $m > p/2$, then $(m-p)(1-2/p) < m+1-p$. Let $k \geq 3$ be an integer such that $(m-p)(1-2/p)^k \leq m+1-p < (m-p)(1-2/p)^{k+1}$. Now (4-20) becomes

$$\int_B |\nabla u|^p \leq C(\varepsilon^{m-p})^{1-2/p} + C\varepsilon^{m-p+mt/(p+t)}.$$

Proceeding as in Step 2, we improve the exponent $m-p+mt/(p+t)$ of ε to $(m-p)(1-2/p)$, since we can find $k_0 \in \mathbb{Z}$ such that $m(1+t/(p+t))^{k_0} - p$ is greater than $(m-p)(1-2/p)$. At the same time, the integral domain $B(x, r)$ shrinks. Namely, there is a constant $r_1 \in (R_{m+1}, r)$ such that

$$(4-23) \quad \int_{B(x, r_1)} |\nabla u_\varepsilon|^p \leq C\varepsilon^{(m-p)(1-2/p)}.$$

Hence as in the derivation of (4-21),

$$\int_{B(x, r)} (1-h^2)^2 \leq C\varepsilon^{(m-p)(1-2/p)+p}.$$

Substituting these into (4-19) we have

$$\begin{aligned} & \int_{B(x, r_1/2)} |\nabla u_\varepsilon|^p \\ & \leq C\varepsilon^{m+1-p} + C \left(\int_{B(x, r)} |\nabla u_\varepsilon|^p \right)^{1-\frac{2}{p}} + C \int_{B(x, r)} |\nabla u_\varepsilon|^p \left(\int_{B(x, r)} (1-h^2)^2 \right)^{\frac{t}{p+t}} \\ & \leq C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^2} + C\varepsilon^{(m-p)(1-2/p)+((m-p)(1-2/p)+p)t/(p+t)} \\ & \leq C\varepsilon^{(m-p)(1-2/p)^2} + C\varepsilon^{(m-p)(1-2/p)+((m-p)(1-2/p)+p)t/(p+t)}. \end{aligned}$$

Again by an argument analogous to Step 2, we improve the exponent of ε in the last term to $(m-p)(1-2/p)^2$. Namely, there is a constant $r_2 \in (R_{m+1}, r_1/2)$ such that

$$\int_{B(x, r_2)} |\nabla u_\varepsilon|^p \leq C\varepsilon^{(m-p)(1-2/p)^2}.$$

By induction, it follows that

$$\int_{B(x, r_{k-1})} |\nabla u_\varepsilon|^p \leq C\varepsilon^{(m-p)(1-2/p)^k}.$$

Combining this with (4–19), and noting the definition of k , we obtain

$$\begin{aligned} & \int_{B(x, r_{k-1}/2)} |\nabla u_\varepsilon|^p \\ & \leq C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^{k+1}} + C\varepsilon^{(m-p)(1-2/p)^k + ((m-p)(1-2/p)^k + p)(t/(p+t))} \\ & \leq C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)^k + ((m-p)(1-2/p)^k + p)(t/(p+t))}. \end{aligned}$$

By the same discussion as in Step 2, we may also improve the exponent of ε to $m+1-p$, and the integral domain shrinks. Namely, we have (4–2) with $j = m+1$:

$$\int_{B(x, r_{k-1}/2)} |\nabla u_\varepsilon|^p \leq C\varepsilon^{m+1-p}. \quad \square$$

Theorem 4.4 (Uniform estimation). *For any compact $K \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$, there exists a constant $C > 0$ independent of ε such that $E_\varepsilon(u_\varepsilon, K) \leq C$.*

Proof. We only prove the theorem for the ball $B(x, R)$ in $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. Theorem 4.1 shows that

$$(4-24) \quad E_\varepsilon(u_\varepsilon, B_{[p]}) \leq C\varepsilon^{[p]-p}.$$

The integral mean value theorem and (4–24) imply that there exists a constant $r \in [R_{[p]}, R_{[p]+1/2}]$ such that

$$(4-25) \quad \int_{\partial B(x, r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p} \int_{\partial B(x, r)} (1 - |u_\varepsilon|^2)^2 d\xi \leq C\varepsilon^{[p]-p}.$$

Consider the functional

$$E(\rho, B) = \frac{1}{p} \int_B (|\nabla \rho|^2 + 1)^{p/2} + \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2,$$

where $B = B(x, r)$. It is easy to see the existence of the minimizer $\rho_{[p]}$ of $E(\rho, B)$ on $W_{|u_\varepsilon|}^{1,p}(B, R^+ \cup \{0\})$. Similar to the proof of Proposition 4.2, from (4–24) and (4–25) we can deduce $E(\rho_{[p]}, B) \leq C\varepsilon^{[p]-p+1}$. Thus, for any $\delta \in (0, 1)$,

$$E_\varepsilon(u_\varepsilon, B) \leq E_\varepsilon(\rho_{[p]}w, B) \leq \frac{1}{p} \int_B |\nabla w|^p + C\varepsilon^{[p]+1-p} + \delta \int_B |\nabla u_\varepsilon|^p.$$

As in the derivation of (4–8), it follows that

$$(4-26) \quad \begin{aligned} \frac{1}{p} \int_B |\nabla h|^p + \frac{1}{4\varepsilon^p} \int_B (1 - h^2)^2 \\ \leq C\varepsilon^{[p]+1-p} + \delta \int_B |\nabla u_\varepsilon|^p + \frac{1}{p} \int_B (1 - h^p) |\nabla w|^p. \end{aligned}$$

To estimate the third term of the right-hand side, we shall do as in the proof of (4–14) and (4–15) and apply $(1/\varepsilon^p) \int_B (1 - h^2)^2 \leq C\varepsilon^{[p]-p}$, which is implied by

(4–24). As a result, there exists $t \in (0, 1/2)$ such that

$$\frac{1}{p} \int_B (1 - h^p) |\nabla w|^p \leq C \varepsilon^{[p] + [p]t / (p+t) - p}.$$

Substituting this into (4–26) yields

$$\frac{1}{p} \int_B |\nabla h|^p + \frac{1}{4\varepsilon^p} \int_B (1 - h^2)^2 \leq C(\varepsilon^{[p]+1-p} + \varepsilon^{[p] + [p]t / (p+t) - p}) + \delta \int_B |\nabla u_\varepsilon|^p.$$

Combining this with (4–18) and choosing δ sufficiently small, we obtain

$$\int_B |\nabla u_\varepsilon|^p \leq C \varepsilon^{[p]-p+1} + C \varepsilon^{[p]-p+tm/(p+t)} + C \varepsilon^{([p]-p)(1-2/p)} + C.$$

By a same argument of Steps 2 and 3, we may improve the exponents of ε in the second and the third terms of the right hand side to $[p] - p + 1$. Thus, for some shrinking domain $B_{[p]+1} \subset B$, there exists $C > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$, such that

$$\int_{B_{[p]+1}} |\nabla u_\varepsilon|^p \leq C + C \varepsilon^{[p]+1-p} \leq C. \quad \square$$

5. Convergence

There may be several minimizers of $E_\varepsilon(u, B_1)$. One of them, denoted by \tilde{u}_ε , can be obtained as the limit of a subsequence $u_\varepsilon^{\tau_k}$ of the minimizers u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u, G) = \frac{1}{p} \int_G (|\nabla u|^2 + \tau)^{p/2} dx + \frac{1}{4\varepsilon^p} \int_G (1 - |u|^2)^2 dx, \quad \text{for } \tau \in (0, 1)$$

in $W_g^{1,p}(B_1, R^n)$ as $\tau_k \rightarrow 0$, namely

$$(5-1) \quad \lim_{\tau_k \rightarrow 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon \text{ in } W^{1,p}(B_1, R^n).$$

We call \tilde{u}_ε the *regularized minimizer* of $E_\varepsilon(u, B_1)$. For the regularized minimizer we shall establish the $C^{1,\alpha}$ convergence when $p > n - t$ and $p \neq n$.

It is not difficult to see that the minimizer u_ε^τ of $E_\varepsilon^\tau(u, B_1)$ solves

$$(5-2) \quad -\operatorname{div}((|\nabla u|^2 + \tau)^{p-2} \nabla u) = \frac{1}{\varepsilon^p} u(1 - |u|^2) \quad \text{on } B_1$$

and satisfies $|u_\varepsilon^\tau| \leq 1$ on \bar{B}_1 . As (3–10) and Theorem 1.2 hold for u_ε^τ , the following results are also true: for any compact subset K of $B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$, there is $C > 0$ such that

$$(5-3) \quad |u_\varepsilon^\tau(x)| \geq 1/2 \quad \text{for all } x \in K$$

and

$$(5-4) \quad E_\varepsilon^\tau(u_\varepsilon^\tau, K) \leq C.$$

Proposition 5.1. *Assume $p > n - t$ and $p \neq n$, where t is the constant in Proposition 3.5. Then for any compact subset $K \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$ and arbitrary $l > 1$, there exists a constant $C > 0$ which is independent of ε, τ , such that*

$$(5-5) \quad \|\nabla u_\varepsilon^\tau\|_{L^l(K, \mathbb{R}^n)} \leq C = C(K, l).$$

Proof. Step 1. Write $v = |\nabla u|^2 + \tau$ in (5-2). Differentiating (5-2) with respect to x_j , we obtain

$$(5-6) \quad -(v^{(p-2)/2} u_{x_i})_{x_i x_j} = \frac{1}{\varepsilon^p} (u(1 - |u|^2))_{x_j}.$$

Take $R > 0$ so that $B = B(\cdot, 3R) \Subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. Let $\zeta \in C_0^\infty(B, [0, 1])$ be a function such that $\zeta = 1$ on $B(\cdot, R)$ and $\zeta = 0$ on $B(\cdot, 3R) \setminus B(\cdot, 2R)$, with $|\nabla \zeta| \leq C$ on $B(\cdot, 3R)$. Now integrate over $B(\cdot, 3R)$ the inner product of the both sides of (5-6) with $u_{x_j} v^b \zeta^2$ ($b \geq 0$) to obtain

$$\int_B (v^{(p-2)/2} u_{x_i})_{x_j} (\zeta^2 u_{x_j})_{x_i} = \frac{1}{\varepsilon^p} \int_B (1 - |u|^2) \zeta^2 (u_{x_j})^2 v^b - \frac{1}{2\varepsilon^p} \int_B \zeta^2 (|u|^2)_{x_j}^2 v^b.$$

Summing over $j = 1, 2, \dots, n$ and computing the term of the left hand side yields

$$(5-7) \quad \int_B \zeta^2 v^{(p+2b-2)/2} \sum_{j=1}^n |\nabla u_{x_j}|^2 + \frac{p+2b-2}{4} \int_B \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \\ + \frac{b(p-2)}{2} \int_B v^{(p+2b-6)/2} (\nabla u \cdot \nabla v)^2 \\ \leq \frac{1}{\varepsilon^p} \int_B (1 - |u|^2) \zeta^2 v^{b+1} + 2 \left| \sum_{j=1}^n \int_B (v^{(p-2)/2} \nabla u)_{x_j} u_{x_j} v^b \zeta \nabla \zeta \right|.$$

From (5-2) and (5-3), it follows that

$$(5-8) \quad \frac{1}{\varepsilon^p} (1 - |u|^2) = \frac{-u}{|u|^2} \operatorname{div}(v^{(p-2)/2} \nabla u).$$

Applying Young's inequality, we derive that for any $\delta \in (0, 1)$,

$$(5-9) \quad \frac{1}{\varepsilon^p} \int_B (1 - |u|^2) \zeta^2 v^{b+1} \leq C(\delta) \int_B \zeta^2 v^{(p+2b+2)/2} \\ + \delta \int_B \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 + \delta \int_B \zeta^2 \sum_{j=1}^n |\nabla u_{x_j}|^2 v^{(p+2b-2)/2},$$

where $\varepsilon, \tau \in (0, \varepsilon_0)$. Using the Young inequality again, for any $\delta \in (0, 1)$

$$(5-10) \quad \left| \sum_{j=1}^n \int_B (v^{(p-2)/2} \nabla u)_{x_j} u_{x_j} v^b \zeta \nabla \zeta \right| \\ \leq \delta \int_B v^{(p+2b-4)/2} |\nabla v|^2 \zeta^2 + C(\delta) \int_B v^{(p+2b)/2} |\nabla \zeta|^2.$$

Substituting (5-9)–(5-10) into (5-7) and choosing δ small enough yields

$$(5-11) \quad \int_B \zeta^2 v^{(p+2b-2)/2} \sum_{j=1}^n |\nabla u_{x_j}|^2 + \frac{p+2b-2}{4} \int_B \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \\ + \frac{b(p-2)}{2} \int_B v^{(p+2b-6)/2} (\nabla u \cdot \nabla v)^2 \\ \leq C \int_B \zeta^2 v^{(p+2b+2)/2} + C \int_B v^{(p+2b)/2} |\nabla \zeta|^2.$$

Step 2. When $p > 2$, all the terms of the left-hand side of (5-11) are nonnegative. When $1 < p < 2$, first observe that

$$v^{(p+2b-2)/2} \sum_{j=1}^n |\nabla u_{x_j}|^2 \geq \frac{1}{4} v^{(p+2b-4)/2} |\nabla v|^2.$$

Next, the third term of the left-hand side of (5-11) is not positive. However,

$$\frac{b(p-2)}{2} \int_B \zeta^2 v^{(p+2b-6)/2} (\nabla u \cdot \nabla v)^2 \geq \frac{b(p-2)}{2} \int_B \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2.$$

Hence, we can derive from (5-11) that

$$(5-12) \quad \int_B \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \leq C \int_B \zeta^2 v^{(p+2b+2)/2} + C \int_B v^{(p+2b)/2} |\nabla \zeta|^2.$$

To estimate $\int_B \zeta^2 v^{(p+2b+2)/2}$, we take $\phi = \zeta^{2/q} v^{(p+2b+2)/2q}$ in the interpolation inequality

$$(5-13) \quad \|\phi\|_{L^q} \leq C \|\nabla \phi\|_{L^\kappa}^\alpha \|\phi\|_{L^1}^{1-\alpha}, \quad q \in (1, n\kappa/(n-\kappa)),$$

where

$$\alpha = \left(1 - \frac{1}{q}\right) \left(1 - \frac{n-\kappa}{n\kappa}\right)^{-1} \in (0, 1).$$

Thus,

$$(5-14) \quad \int_B \zeta^2 v^{\frac{p+2b+2}{2}} \leq C \left(\int_B \zeta^{\frac{2}{q}} v^{\frac{p+2b+2}{2q}} \right)^{q(1-\alpha)} \times \left(\int_B \zeta^{\kappa(\frac{2}{q}-1)} |\nabla \zeta|^\kappa v^{\kappa \frac{p+2b+2}{2q}} + \frac{p+2b+2}{2q} \left(\int_B \zeta^2 v^{\frac{p+2b-4}{2}} |\nabla v|^2 \right)^{\frac{\kappa}{2}} \right)^{\frac{q\alpha}{\kappa}} \times \left(\int_B \zeta^{\frac{2\kappa}{2-\kappa}(\frac{2}{q}-1)} v^{\frac{2\kappa}{2-\kappa}(\frac{p+2b+2}{2q} - \frac{p+2b}{4})} \right)^{1-\frac{\kappa}{2}}.$$

Step 3. Since $p > n - t$ and $p \neq n$, we can choose κ such that $1 < \kappa < 2$ and $\kappa \in (2n(2-t)/(2(p+2b+2) - nt), 2n/(n+2))$. Using κ , fix q in the interval $(2t(p+2b+2)/(2(p+2b+t) - \kappa t), n\kappa/(n-\kappa)) \subset (1, 2)$. Thus, $q\alpha/2 < 1$ and

$$(5-15) \quad \frac{p+2b+2}{2q}, \frac{\kappa(p+2b+2)}{2q}, \frac{2\kappa}{2-\kappa} \left(\frac{p+2b+2}{2q} - \frac{p+2b}{4} \right) \leq \frac{p+2b+t}{2}.$$

Let $b = 0$. From Hölder's inequality, Proposition 3.5 and (5-4) it follows that

$$\int_B \zeta^{2/q} v^{(p+2)/2q} + \int_B \zeta^{\kappa(2/q-1)} |\nabla \zeta|^\kappa v^{\kappa(p+2)/(2q)} \leq C \left(\int_B v^{(p+2)/2} \right)^{(p+2)/((p+2)q)} \leq C \left(\int_B v^{p/2} \right)^{(p+2)/(pq)} \leq C.$$

Substituting this into (5-14), and again using Hölder's inequality, Proposition 3.5 and (5-4), we obtain that for any $\delta \in (0, 1)$,

$$\int_B \zeta^2 v^{(p+2)/2} \leq C + C \left(\int_B \zeta^2 v^{(p-4)/2} |\nabla v|^2 \right)^{q\alpha/2} \leq C(\delta) + \delta \int_B \zeta^2 v^{(p-4)/2} |\nabla v|^2,$$

since $q\alpha/2 < 1$. Substituting this into (5-12), we see that $\int_B \zeta^2 v^{(p-4)/2} |\nabla v|^2 \leq C$ or $\int_B \zeta^2 |\nabla w|^2 \leq C$, where $w = v^{p/4}$. Since (5-4) implies $\int_B \zeta^2 |w|^2 \leq C$, we have $\|\zeta w\|_{H^1(B,R)}^2 \leq C$, and thus the embedding inequality implies (5-5) when $n = 2$. If $n \geq 3$, the embedding inequality gives

$$(5-16) \quad \left(\int_B (\zeta w)^r \right)^{1/r} \leq C \|\zeta w\|_{H^1(B,R)} \leq C,$$

where $r \leq 2n/n - 2$. Now we set $G_i = B(x_0, R + R/2^i)$. Take ζ such that $\zeta = 1$ on G_1 and $\zeta = 0$ on $B(x_0, 3R) \setminus B(x_0, 2R)$. Noting that $p > n - t$ and $t < 4/n$, choose $r = 2 + 8/(np)$ in (5-16). Since $\zeta = 1$ on G_1 , we see that $\nabla u \in L^{s_1}(G_1)$

where $s_1 = p + 4/n$, and

$$(5-17) \quad \int_{G_1} |\nabla u|^{s_1} \leq C.$$

Step 4. To prove (5-5), we will choose $b > 0$ and proceed in the same way as in Steps 1, 2 and 3. However, Proposition 3.5 can not be applied, since it is only a result on the regularized functional $\int_{B_1} v^{p/2} dx$ and is not valid on $\int_{B_1} v^{s/2} dx$ for $s > p$. On the other hand, if we take $b \geq 2/n$ from now on, the inequalities $p > n - t$ and $t < 4/n$ imply that κ can be taken in $(2(2-t)n/(2p+4b+4-nt), 2n/(n+2))$ with $t = 0$. In view of this, suppose $t = 0$ in the following calculation when proceeding as in Step 3.

Write $w = v^{(p+2b)/4}$. Set $b = 2/n$ and take $\zeta = 1$ on G_2 and $\zeta = 0$ on $B_1 \setminus G_1$. Then from (5-17),

$$\int_{G_1} w^2 = \int_{G_1} v^{(p+4/n)/2} = \int_{G_1} |\nabla u|^{s_1} \leq C.$$

Noting (5-15) with $t = 0$, we use Hölder's inequality to estimate the terms of the right-hand side of (5-14). Combining with (5-12), we have, for any $\delta \in (0, 1)$,

$$\int_{G_1} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2 \leq C(\delta) \left(1 + \int_{G_1} w^2\right)^\lambda + \delta \int_{G_1} \zeta^2 v^{(p+2b-4)/2} |\nabla v|^2,$$

where $\lambda > 0$ only depends on n , p and b . Choosing δ sufficiently small, we obtain $\|\zeta w\|_{H^1(G_1)}^2 \leq C(1 + \int_{G_1} w^2)^\lambda \leq C$. Applying the embedding theorem to ζw and using that $\zeta = 1$ on G_2 , we obtain

$$(5-18) \quad \int_{G_2} |\nabla u|^{s_2} \leq C,$$

where $s_2 = s_1 + 4(n+2)/n^2 = p + 4/n + 4(n+2)/n^2 = p + 8/n + 8/n^2$.

Step 5. Reset b and ζ again. Applying (5-18) and following the same logic as Step 4, we can improve s_2 to $s_3 > s_2$. For any $l > 1$, proceeding inductively, we may at last find s_i for some i such that $s_i > l$ and

$$\int_{G_i} |\nabla u|^{s_i} \leq C,$$

where $G_i \subset B_R$. Thus (5-5) is proved. \square

We can extend Proposition 5.1 by means of Moser iteration.

Proposition 5.2. *Assume $p > n - t$ and $p \neq n$. Then for any compact subset $K \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$, there exists a constant $C = C(K) > 0$ independent of ε, τ , such that*

$$\|\nabla u_\varepsilon^\tau\|_{L^\infty(K, \mathbb{R}^n)} \leq C.$$

Proof. Given any $x_0 \in B(\cdot, 3R) \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$, let $r > 0$ be small such that $B(x_0, 2r) \subset B(\cdot, 2R)$. Denote $Q_m = B(x_0, r_m)$ where $r_m = r + r/2^m$. Choose $\zeta_m \in C_0^\infty(Q_m, R)$ such that $\zeta_m = 1$ on Q_{m+1} and $|\nabla \zeta_m| \leq Cr^{-1}2^m$, $m = 1, 2, \dots$. Integrate over Q_m the inner product of the both sides of (5–6) with $\zeta_m^2 v^b u_{x_j}$, $b \geq 1$. Then, as in the derivation of (5–12), we see that

$$(5-19) \quad \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \leq C \int_{Q_m} v^{(p+2b)/2} |\nabla \zeta_m|^2 + C \int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2}.$$

To estimate $\int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2}$, we take $\phi = \zeta_m^{2/q} v^{(p+2b+2)/2q}$ in the interpolation inequality (5–13). We then obtain

$$(5-20) \quad \int_{Q_m} \zeta_m^2 v^{(p+2b+2)/2} \leq C \left(\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/(2q)} \right)^{q(1-\alpha)} \\ \times \left(\left(\frac{2}{q} \right)^\kappa \int_{Q_m} \zeta_m^{\kappa(2/q-1)} |\nabla \zeta_m|^\kappa v^{\kappa(p+2b+2)/(2q)} \right. \\ \left. + \left(\frac{p+2b+2}{2q} \right)^\kappa \int_{Q_m} \zeta_m^{2\kappa/q} v^{\kappa((p+2b+2)/(2q)-1)} |\nabla v|^\kappa \right)^{q\alpha/\kappa}.$$

Now, we estimate the right-hand side. Choose $r \in (0, 1)$ sufficiently small such that $|Q_m| \leq 1$. Take $\kappa \in (2n/(p+4), 2n/(n+2)) \cap (1, 2)$; hence, q can be chosen in $((\kappa(p+4))/(p+2), n\kappa/(n-\kappa))$. This implies that (5–15) with $t = 0$ is also true since $b \geq 1$. By using Hölder's inequality, we have

$$\int_{Q_m} \zeta_m^{2/q} v^{(p+2b+2)/2q} \leq \left(\int_{Q_m} v^{(p+2b)/2} \right)^{(p+2b+2)/(q(p+2b))}, \\ \int_{Q_m} \zeta_m^{\kappa(2/q-1)} |\nabla \zeta_m|^\kappa v^{\kappa(p+2b+2)/(2q)} \leq \frac{2^m}{r^\kappa} \left(\int_{Q_m} v^{(p+2b)/2} \right)^{\kappa(p+2b+2)/(q(p+2b))},$$

and

$$\int_{Q_m} \zeta_m^{(2\kappa)/q} v^{\kappa((p+2b+2)/(2q)-1)} |\nabla v|^\kappa \\ \leq \left(\int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \right)^{\kappa/2} \left(\int_{Q_m} v^{(p+2b)/2} \right)^{\kappa((p+2b+2)/(q(p+2b))-1/2)}.$$

Combining these inequalities with (5–19) and (5–20) yields

$$(5-21) \quad I_1 \leq C \left(\left(\frac{2^m}{r} \right)^2 I_2 + \left(\frac{4^m}{r} \right)^{q\alpha} I_2^{1+2/(p+2b)} \right. \\ \left. + \left(\frac{p+2b+2}{2q} \right)^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/(p+2b)-q\alpha/2} \right),$$

where

$$I_1 = \int_{Q_m} \zeta_m^2 v^{(p+2b-4)/2} |\nabla v|^2 \quad \text{and} \quad I_2 = \int_{Q_m} v^{(p+2b)/2}.$$

Let $p + 2b = s^m$ and $w = v^{(p+2b)/4} = v^{s^m/4}$, with $s > 1$ to be determined later. Using the Young inequality to treat the last term on the right side of (5–21), we obtain

$$\begin{aligned} \left(\frac{s^m + 2}{2q}\right)^{q\alpha} I_1^{q\alpha/2} I_2^{1+2/(s^m)-q\alpha/2} \\ \leq \delta I_1 + C(\delta) \left(\frac{s^m + 2}{2q}\right)^{2q\alpha/(2-q\alpha)} I_2^{2(1+2/(s^m)-q\alpha/2)/(2-q\alpha)}. \end{aligned}$$

Substituting this into (5–21), we get

$$(5-22) \quad I_1 \leq C(\delta) \left(\left(\frac{2^m}{r}\right)^2 I_2 + \left(\frac{4^m}{r^t}\right)^{q\alpha} I_2^{1+2/s^m} \right. \\ \left. + \left(\frac{s^m + 2}{2q}\right)^{2q\alpha/(2-q\alpha)} I_2^{2(1+2/s^m-q\alpha/2)/(2-q\alpha)} \right).$$

By the embedding theorem, for any $s \in (1, n/(n-2)]$,

$$\begin{aligned} \int_{Q_m} (\zeta_m w)^{2s} \leq C(s) \left(\int_{Q_m} (\zeta_m w)^2 + \int_{Q_m} |\nabla(\zeta_m w)|^2 \right)^s \\ \leq C \left(\left(1 + \left(\frac{2^m}{r}\right)^2\right) I_2 + \left(\frac{s^m}{4}\right)^2 I_1 \right)^s. \end{aligned}$$

Combining this with (5–22) yields

$$(5-23) \quad \int_{Q_m} (\zeta_m w)^{2s} \leq C(s, r, q, \kappa) \left((1 + 4^m + s^{2m} 4^m) I_2 + s^{2m} 4^{q\alpha m} I_2^{1+2/s^m} \right. \\ \left. + s^{2m} s^{2q\alpha m/(2-q\alpha)} I_2^{(1+2/s^m-q\alpha/2)2/(2-q\alpha)} \right)^s.$$

If there is a subsequence of positive integers $\{m_i\}$ tending to infinity such that

$$I_2 = \int_{Q_{m_i}} v^{s^{m_i}/2} < 1,$$

then letting $m_i \rightarrow \infty$ immediately yields

$$(5-24) \quad \|v\|_{L^\infty(Q_\infty, R)} \leq C(r).$$

Otherwise, there must be a positive integer m_0 such that

$$I_2 = \int_{Q_m} v^{s^m/2} \geq 1 \quad \text{for } m \geq m_0.$$

Since $(1 + 2/s^m - q\alpha/2)(2/(2 - q\alpha)) = 1 + (2/s^m)(2/(2 - q\alpha)) > 1$, the exponent of the last term in (5–23) is higher than those of the other terms. Now we compare

the coefficients of the terms in (5–23). If we choose $s \in (1, \min\{n/(n-2), 2^{2-q\alpha}\})$, then 4^m and $s^{2q\alpha m/(2-q\alpha)}$ are less than $4^{(1+q\alpha)m}$. Thus,

$$\int_{Q_m} (\zeta_m w)^{2s} \leq C \left((s^2 4^{1+q\alpha})^m I_2^{1+(2/s^m)(2/(2-q\alpha))} \right)^s.$$

This means

$$\int_{Q_{m+1}} v^{s^{m+1}/2} \leq (C_0 C_1^m)^s \left(\int_{Q_m} v^{s^m/2} \right)^{(1+C_2/s^m)s},$$

where $C_1 = (s^2 4^{1+q\alpha})^s$, $C_2 = 4/(2-q\alpha)$, and C_0 is a positive constant. Using the iteration lemma [Lei 2004, Proposition 2.3] and Proposition 5.1, we also obtain the estimate (5–24), completing the proof of Proposition 5.2. \square

Proposition 5.3. *Assume $p > n - t$ and $p \neq n$. Suppose \tilde{u}_ε is a regularized minimizer. Then for any compact subset $K \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$, there exists a constant $C = C(K) > 0$ which is independent of ε such that*

$$(5-25) \quad \|(1/\varepsilon^p) (1 - |\tilde{u}_\varepsilon|^2)\|_{L^\infty(K)} \leq C.$$

Proof. Assume $B = B_R \Subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. Consider the inner product of the both sides of (5–2) with $u = u_\varepsilon^\tau$,

$$-\operatorname{div}(v^{(p-2)/2} \nabla u) u = (1/\varepsilon^p) |u|^2 (1 - |u|^2) = |u|^2 \psi,$$

where $\psi = \psi_\varepsilon^\tau = (1/\varepsilon^p) (1 - |u_\varepsilon^\tau|^2)$. Combining this and $\nabla \psi = -(2/\varepsilon^p) u \cdot \nabla u$ with the equality $-\operatorname{div}(v^{(p-2)/2} \nabla u) u = -\operatorname{div}(v^{(p-2)/2} u \cdot \nabla u) + v^{(p-2)/2} |\nabla u|^2$, and noting (5–3), we then obtain

$$(5-26) \quad (1/4) \psi \leq v^{(p-2)/2} |\nabla u|^2 + (\varepsilon^p/2) \operatorname{div}(v^{(p-2)/2} \nabla \psi) \quad \text{on } B.$$

At the point x_0 where ψ achieves its maximum on B , we have $\nabla \psi(x_0) = 0$ and $\Delta \psi(x_0) \leq 0$. Hence at x_0 ,

$$\operatorname{div}(v^{(p-2)/2} \nabla \psi) = v^{(p-2)/2} \Delta \psi + ((p-2)/2) v^{(p-4)/2} \nabla v \nabla \psi \leq 0.$$

Combining this with (5–26) and using Proposition 5.2, we derive that

$$(5-27) \quad \|(1/\varepsilon^p) (1 - |u_\varepsilon^\tau|^2)\|_{L^\infty(B)} \leq \psi(x_0) \leq C.$$

When $p > n$, from (5–1) it follows that

$$(5-28) \quad \lim_{\tau_k \rightarrow 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon \quad \text{in } C(\bar{B}_1).$$

When $p \in (n-t, n)$, Proposition 3.5 still holds for u_ε^τ . As in the proof of Proposition 3.4, combining (5–1) with the embedding theorem we deduce (5–28). Letting $\tau \rightarrow 0$ in (5–27) and using (5–28), we reach (5–25) by a finite-covering argument. \square

Proof of Theorem 1.3. According to Proposition 5.3, the right-hand side of the Euler–Lagrange equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = (1/\varepsilon^p)u(1 - |u|^2)$$

satisfied by \tilde{u} is bounded on every compact subset $K \subset B_1 \setminus \{0, a_1, a_2, \dots, a_N\}$. Thus Tolksdorf’s theorem [1983, p. 244, lines 19–23] yields that $\|\tilde{u}_\varepsilon\|_{C^{1,\beta}(K)} \leq C = C(K)$ for some $\beta \in (0, 1)$, where the constant C does not depend on ε . Letting $\varepsilon \rightarrow 0$, we find a subsequence \tilde{u}_k of \tilde{u}_ε and a map u_* such that $\tilde{u}_k \rightarrow u_*$ in $C^{1,\alpha}(K)$ for all $\alpha \in (0, \beta)$. In addition, Remark 2 implies $u_* = u_p$, completing the proof. \square

6. Proof of (4–17)

Theorem 6.1. *Assume $h = |u| \geq 1/2$ and let $w = u|u|^{-1}$. If $u \in W^{1,p}(B(x, 3R), R^n)$ satisfies*

$$(6-1) \quad \int_{B(x, 3R)} |\nabla u|^{p-2} h (w \wedge \nabla w) \nabla \zeta = 0 \quad \text{for all } \zeta \in W_0^{1,p}(B(x, 3R)),$$

then for any $\rho \in (0, 3R/2)$, there is $C > 0$ such that

$$\int_{B(x, \rho)} |\nabla u|^{p-2} h^2 |\nabla w|^2 \leq C \left(\int_{B(x, 2\rho)} |\nabla u|^p \right)^{1-2/p}.$$

Proof. Let $\{e_i\}_{i=1}^n$ be an orthogonal basis of R^n . Since $|w| = 1$ over $B(x, 3R)$, we have the formula in n -dimension ball coordinates

$$\begin{aligned} w &= \cos \theta_1 e_1 + \sin \theta_1 \cos \theta_2 e_2 + \sin \theta_1 \sin \theta_2 \cos \theta_3 e_3 + \dots \\ &\quad + \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1} e_{n-1} + \sin \theta_1 \dots \sin \theta_{n-2} \sin \theta_{n-1} e_n. \end{aligned}$$

As $h \geq 1/2$, there is no zero of u in $B(x, 3R)$. This implies $\deg(w, \partial\Omega) = 0$ for any $\Omega \subset B(x, 3R)$. Hence, $(\theta_1, \dots, \theta_{n-2}, \theta_{n-1}) \in [0, \pi] \times \dots \times [0, \pi] \times [0, 2\pi]$, and each θ_i is single-valued. Thus,

$$\begin{aligned} \nabla w &= -\sin \theta_1 \nabla \theta_1 e_1 + (\cos \theta_1 \cos \theta_2 \nabla \theta_1 - \sin \theta_1 \sin \theta_2 \nabla \theta_2) e_2 \\ &\quad + (\cos \theta_1 \sin \theta_2 \cos \theta_3 \nabla \theta_1 + \sin \theta_1 \cos \theta_2 \cos \theta_3 \nabla \theta_2 - \sin \theta_1 \sin \theta_2 \sin \theta_3 \nabla \theta_3) e_3 \\ &\quad + \dots + (\cos \theta_1 \sin \theta_2 \dots \sin \theta_{n-2} \cos \theta_{n-1} \nabla \theta_1 + \dots \\ &\quad \quad + \sin \theta_1 \dots \sin \theta_{n-3} \cos \theta_{n-2} \cos \theta_{n-1} \nabla \theta_{n-2} - \sin \theta_1 \dots \sin \theta_{n-1} \nabla \theta_{n-1}) e_{n-1} \\ &\quad + (\cos \theta_1 \sin \theta_2 \dots \sin \theta_{n-1} \nabla \theta_1 + \dots + \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1} \nabla \theta_{n-1}) e_n. \end{aligned}$$

Hence,

$$(6-2) \quad |\nabla w|^2 = |\nabla \theta_1|^2 + \sin^2 \theta_1 |\nabla \theta_2|^2 + \sin^2 \theta_1 \sin^2 \theta_2 |\nabla \theta_3|^2 + \dots \\ + \sin^2 \theta_1 \dots \sin^2 \theta_{n-2} |\nabla \theta_{n-1}|^2,$$

and there are $n(n-1)/2$ vectors in the formula

$$\begin{aligned}
w \wedge \nabla w = & \left((\cos \theta_2 \nabla \theta_1 - \cos \theta_1 \sin \theta_1 \sin \theta_2 \nabla \theta_2)(e_1 \wedge e_2) + \cdots \right. \\
& + \left(\prod_{i=1}^{n-1} \sin \theta_i \nabla \theta_1 + \cos \theta_1 \prod_{i=1}^{n-1} \sin \theta_i \nabla \theta_2 + \cdots \right. \\
& \quad \left. + \cos \theta_1 \prod_{i=1}^{n-2} \sin \theta_i \cos \theta_{n-1} \nabla \theta_{n-1} \right)(e_1 \wedge e_n) \\
& + \left((\sin^2 \theta_1 \cos \theta_3 \nabla \theta_2 - \sin^2 \theta_1 \sin \theta_2 \cos \theta_2 \sin \theta_3 \nabla \theta_3)(e_2 \wedge e_3) + \cdots \right. \\
& \quad + (\sin^2 \theta_1 \prod_{i=3}^{n-1} \sin \theta_i \nabla \theta_2 + \cdots \\
& \quad \left. + \sin^2 \theta_1 \cos \theta_2 \prod_{i=2}^{n-2} \sin \theta_i \cos \theta_{n-1} \nabla \theta_{n-1} \right)(e_2 \wedge e_n) + \cdots \\
& + \left(\prod_{i=1}^{n-3} \sin^2 \theta_i (\cos \theta_{n-1} \nabla \theta_{n-2} - \sin \theta_{n-2} \cos \theta_{n-2} \sin \theta_{n-1} \nabla \theta_{n-1})(e_{n-2} \wedge e_{n-1}) \right. \\
& \quad \left. + \prod_{i=1}^{n-3} \sin^2 \theta_i (\sin \theta_{n-1} \nabla \theta_{n-2} + \sin \theta_{n-2} \cos \theta_{n-2} \cos \theta_{n-1} \nabla \theta_{n-1})(e_{n-2} \wedge e_n) \right) \\
& + \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} \nabla \theta_{n-1} (e_{n-1} \wedge e_n).
\end{aligned}$$

The equality corresponding to $e_{n-1} \wedge e_n$ in the integral system (6–1) is

$$\int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i \nabla \theta_{n-1} \nabla \zeta = 0.$$

Letting $\zeta = \theta_{n-1} \xi^2$ where $\xi \in W_0^{1,p}(B(x, 3R))$, we obtain

$$\begin{aligned}
\int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i |\nabla \theta_{n-1}|^2 \xi^2 \\
\leq \left| \int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i (\xi \theta_{n-1}) \nabla \theta_{n-1} \nabla \xi \right|.
\end{aligned}$$

Using Hölder's inequality, we have, for any $\delta \in (0, 1)$,

$$\begin{aligned}
\int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i |\nabla \theta_{n-1}|^2 \xi^2 \\
\leq \delta \int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i |\nabla \theta_{n-1}|^2 \xi^2 \\
+ C(\delta) \int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i |\nabla \xi|^2 (\xi \theta_{n-1})^2.
\end{aligned}$$

Taking $\xi = 1$ over $B(x, \rho)$ and $\xi = 0$ over $B(x, 3R) \setminus B(x, 2\rho)$ and letting δ be sufficiently small, we get

$$\begin{aligned}
(6-3) \quad \int_{B(x, \rho)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-2} \sin^2 \theta_i |\nabla \theta_{n-1}|^2 & \leq C \int_{B(x, 2\rho)} |\nabla u|^{p-2} \\
& \leq C \left(\int_{B(x, 2\rho)} |\nabla u|^p \right)^{1-2/p}.
\end{aligned}$$

Next, we use the equalities corresponding to $e_{n-2} \wedge e_{n-1}$ and $e_{n-2} \wedge e_n$ in (6–1): the integrals over $B(x, 3R)$ of

$$|\nabla u|^{p-2} h^2 \prod_{i=1}^{n-3} \sin^2 \theta_i (\cos \theta_{n-1} \nabla \theta_{n-2} - \cos \theta_{n-2} \sin \theta_{n-2} \sin \theta_{n-1} \nabla \theta_{n-1}) \nabla \zeta$$

and

$$|\nabla u|^{p-2} h^2 \prod_{i=1}^{n-3} \sin^2 \theta_i (\sin \theta_{n-1} \nabla \theta_{n-2} + \cos \theta_{n-2} \sin \theta_{n-2} \cos \theta_{n-1} \nabla \theta_{n-1}) \nabla \zeta$$

both equal zero. Taking $\zeta = \theta_{n-2} \xi^2 \cos \theta_{n-1}$ and $\theta_{n-2} \xi^2 \sin \theta_{n-1}$ in these two integrals, respectively, we obtain

$$\int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-3} \sin^2 \theta_i (|\nabla \theta_{n-2}|^2 \xi^2 + \theta_{n-2} \nabla \theta_{n-2} \nabla \xi^2) = 0.$$

Similar to the derivation of (6–3), we have, for any $\rho \in (0, 3R/2)$,

$$(6-4) \quad \int_{B(x, \rho)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{n-3} \sin^2 \theta_i |\nabla \theta_{n-2}|^2 \leq C \left(\int_{B(x, 2\rho)} |\nabla u|^p \right)^{1-2/p}.$$

By means of induction, applying the equalities corresponding to $e_k \wedge e_{k+1}$, $e_k \wedge e_{k+2}$, \dots , $e_k \wedge e_n$ in (6–1), we find that

$$(6-5) \quad \int_{B(x, \rho)} |\nabla u|^{p-2} h^2 \prod_{i=1}^{k-1} \sin^2 \theta_i |\nabla \theta_k|^2 \leq C \left(\int_{B(x, 2\rho)} |\nabla u|^p \right)^{1-2/p},$$

for $k = 2, \dots, n-1$. At last, we can deduce

$$(6-6) \quad \int_{B(x, \rho)} |\nabla u|^{p-2} h^2 |\nabla \theta_1|^2 \leq C \left(\int_{B(x, 2\rho)} |\nabla u|^p \right)^{1-2/p}.$$

Combining the estimations (6–3)–(6–6) and using (6–2) completes the proof. \square

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