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GLOBAL PROPERTIES OF CONSTANT MEAN CURVATURE SURFACES IN $\mathbb{H}^2 \times \mathbb{R}$

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We discuss some aspects of the global behavior of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature H (known as H-surfaces). We prove a maximum principle at infinity for complete properly embedded H-surfaces with $H > 1/\sqrt{2}$, and show that the genus of a compact stable H-surface with $H > 1/\sqrt{2}$ is at most three.

1. Introduction

We discuss the global behavior of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature H, known as *H-surfaces*. Recently, work has been done on H-surfaces in product spaces $M^2 \times \mathbb{R}$ with M^2 a Riemannian surface. New examples were produced, as well as many theoretical results; see [Hauswirth 2003; Meeks and Rosenberg 2004; 2005; Nelli and Rosenberg 2002; Rosenberg 2002] for the case $H = 0$, and [Abresch and Rosenberg 2004; Hoffman et al. 2006; Sá Earp and Toubiana 2004] for the case $H \neq 0$.

We study the stability of H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and prove a maximum principle at infinity. Most of our results depend strongly on the following distance estimate:

Main Lemma. *Let M be a stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$. If $H > 1/\sqrt{3}$, then*

$$\text{dist}_M(p, \partial M) < \frac{2\pi}{\sqrt{3(3H^2 - 1)}} \qquad \text{for any } p \in M. \tag{1}$$

The analogue of this lemma for the Euclidean case was proved in [Ros and Rosenberg 2001].

The hypothesis $H > 1/\sqrt{3}$ seems to be necessary only for technical reasons. We believe that a similar estimate can be proved for $H > 1/2$. Whether H is smaller or greater than $1/2$ makes a profound difference in the behaviour of H-surfaces (comparable to whether H is smaller or greater than 1 in the case of ambient $\mathbb{H}^3$).

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For example, compact embedded H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ only exist for $H > 1/2$ (see [Hsiang and Hsiang 1989] and our Section 5).

The Lemma is used for proving the next two theorems:

Theorem A. *Let M_1 and M_2 be two disjoint H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that are complete, properly embedded, without boundary, and with $H > 1/\sqrt{2}$. The surface M_2 cannot lie in the mean convex side of M_1.*

By "mean convex side", we understand the following: Since M_1 is properly embedded, it separates $\mathbb{H}^2 \times \mathbb{R}$ into two connected components. The *mean convex side* of M_1 is the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus M_1$ towards which points the mean curvature vector of M_1.

Theorem B. *In $\mathbb{H}^2 \times \mathbb{R}$, there is no complete noncompact stable H-surface with $H > 1/\sqrt{3}$, either with compact boundary or without boundary.*

We also prove:

Theorem C. *Let M be a compact weakly stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$. If $H > 1/\sqrt{2}$, then the genus of M is at most three.*

In the last section we will give explicit examples of entire graphs with constant mean curvature $H \le 1/2$.

2. Proofs of the Main Lemma and Theorem B

We recall the definition of stability. Consider an immersion $x : M \to \mathbb{H}^2 \times \mathbb{R}$ of a 2-manifold M. Denote by dM the area form of M in the metric induced by the immersion x. A *variation* of x is a differentiable map $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{H}^2 \times \mathbb{R}$ such that $X_t = X(t, \cdot)$ is an immersion for each $t \in (-\varepsilon, \varepsilon)$, $X_0 = x$, and $X_t|_{\partial M} = x|_{\partial M}$.

Define the *area* and the *volume functions* $A, V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$A(t) = \int_M dM_t \qquad \text{and} \qquad V(t) = \int_{[0,t]\times M} X^* dv,$$

where dv is the volume element of the ambient space, X^* is the standard linear map on forms induced by X, and so $X^* dv$ is the induced (algebraic) volume form.

Let N be a unit vector field normal to M, and H the mean curvature function of M with respect to N. Writing $f = \left\langle \frac{\partial X}{dt}\Big|_{t=0}, N \right\rangle$, one has

$$\dot{A}(0) = -\int_M 2fH\, dM, \qquad \text{and} \qquad \dot{V}(0) = \int_M f\, dM.$$

See [Barbosa et al. 1988] for a proof.

Take M a surface with constant mean curvature H and consider the function $G : (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by $G(t) = A(t) + 2HV(t)$. Clearly, $\dot{G}(0) = 0$. Furthermore,

$$\ddot{G}(0)(f) = -\int_M \Big(f\Delta f + \big(\mathrm{Ric}(N) + |A|^2\big) f^2 \Big)\, dM, \tag{2}$$

where $|A|$ is the norm of the second fundamental form of M, while $\mathrm{Ric}(N)$ is the Ricci curvature of the ambient space in the direction of N. See again [Barbosa et al. 1988] for a proof.

We call $L = \Delta + |A|^2 + \mathrm{Ric}(N)$ *the stability operator* on M.

If M has constant mean curvature, we say that M is *stable* if $\ddot{G}(0)(f) \geq 0$ for all smooth functions f on M with compact support.

Our definition of stability is the strong notion of stability; it is what we use for obtaining certain diameter estimates (see Main Lemma). Usually, one also requires the test functions f to satisfy $\int_M f = 0$; that is, one only considers volume-preserving variations. The resulting notion of stability is called *weak stability*. For example, the geodesic spheres in space forms are weakly stable but not strongly stable. We use the weaker notion in Theorem C.

Remark 2.1. We prove that a vertical graph M in $\mathbb{H}^2 \times \mathbb{R}$ is stable. Assume that the mean curvature vector of M points above, and orient M by the mean curvature vector. It is enough to prove that, for any domain $D \subset M$ with compact closure, the first eigenvalue λ of L on D is positive. By contradiction, assume that $\lambda < 0$ and let f be a first eigenfunction. Then $Lf = -\lambda f$ and $f|_{\partial D} = 0$. We can assume that $f|_D > 0$. Let Φ_t be the variation of D such that $\big\langle \frac{d\Phi_t}{dt}\big|_{t=0}, N \big\rangle = f$, where N is the unit normal vector field that orients M. The first variation of the mean curvature for the normal variation fN is

$$\dot{H}(0)f = Lf = -\lambda f > 0.$$

Hence, for positive small t and at any interior point of the variation $\Phi_t(D)$, the mean curvature is greater than H. Translate D upward, such that $D \cap \Phi_t(D) = \varnothing$. Afterwards, translate D downward: at the first contact point between the translation of D and $\Phi_t(D)$, the mean curvature of D is smaller than the mean curvature of $\Phi_t(D)$, but D is above $\Phi_t(D)$. By the maximum principle, this is a contradiction.

Remark 2.2. Let $\{e_1, e_2\}$ be a principal frame for M, and λ_1, λ_2 the principal curvatures. We have

$$|A|^2 + \mathrm{Ric}(N) = 4H^2 - 2\lambda_1\lambda_2 - 1 - K_s,$$

where K_s is the sectional curvature of the ambient space for the plane in the direction (e_1, e_2). We used that the scalar curvature of the ambient space is -1. By the Gauss equation, $K_s + \lambda_1\lambda_2 = K$, the intrinsic Gauss curvature of M. Hence

$$|A|^2 + \mathrm{Ric}(N) = 4H^2 - \lambda_1\lambda_2 - 1 - K.$$

Then

(3) $$L-\Delta+K=4H^2-\lambda_1\lambda_2-1\geq 3H^2-1\geq 0 \qquad \text{for } H\geq 1/\sqrt{3},$$

(4) $$L-\Delta=4H^2-2\lambda_1\lambda_2-1-K_s\geq 2H^2-1\geq 0 \quad \text{for } H\geq 1/\sqrt{2}.$$

In order to obtain inequality (4), we used that $-1\leq K_s\leq 0$.

Main Lemma. *Let M be a stable H-surface in $\mathbb{H}^2\times\mathbb{R}$. If $H>1/\sqrt{3}$, then*

$$\operatorname{dist}_M(p,\partial M)<\frac{2\pi}{\sqrt{3(3H^2-1)}} \qquad \textit{for any } p\in M.$$

Proof. As M is stable, it was proved in [Fischer-Colbrie 1985] that there exists a function $u>0$ on M such that $Lu=0$ on M. Denote by ds^2 the metric on M induced by its immersion in $\mathbb{H}^2\times\mathbb{R}$, and let $d\tilde{s}^2=u^2ds^2$. Consider $p\in M$ and let $R>0$ be such that the ball B_R, centered at p and of ds-radius R, is contained in the interior of M. Let γ be a $d\tilde{s}$-minimizing geodesic in B_R joining p to ∂B_R. If a denotes the ds-length of γ, then $a\geq R$, and it is enough to prove that

$$a<\frac{2\pi}{\sqrt{3(3H^2-1)}}.$$

Let $\tilde{K}$ be the intrinsic Gauss curvature of M, and $\tilde{R}$ the length of γ, both in the $d\tilde{s}$ metric. As γ is minimizing, by the second variation formula one has that

(5) $$\int_0^{\tilde{R}}\left(\left(\frac{d\varphi}{d\tilde{s}}\right)^2-\tilde{K}\varphi^2\right)d\tilde{s}\geq 0$$

for any smooth function φ with $\varphi(0)=0$ and $\varphi(\tilde{R})=0$. From standard computations, we have

(6) $$\tilde{K}=\frac{1}{u^2}(K-\Delta\ln u) \qquad \text{and} \qquad \Delta\ln u=\frac{1}{u^2}\big(u\Delta u-|\nabla u|^2\big).$$

Set $c=3H^2-1>0$; then (3) implies that $L-\Delta+K\geq c$. As $Lu=0$, we have

(7) $$0\geq\Delta u+(c-K)u.$$

From (6) and (7),

(8) $$\tilde{K}=\frac{1}{u^2}(K-\Delta\ln u)\geq\frac{1}{u^2}\left(c+\frac{|\nabla u|^2}{u^2}\right)>0.$$

We replace (8) in (5), while letting φ denote, by abuse of notation, also the composition $\varphi\circ\tilde{s}$. Hence $\varphi(0)=\varphi(a)=0$, and

$$0<\int_0^a\left(c+\frac{|\nabla u|^2}{u^2}\right)\frac{\varphi^2}{u}ds\leq\int_0^a\tilde{K}\varphi^2u\,ds\leq\int_0^a\left(\frac{d\varphi}{d\tilde{s}}\right)^2u\,ds=\int_0^a\left(\frac{d\varphi}{ds}\right)^2\frac{ds}{u}.$$

To get rid of u from the denominator, we replace φ by $\varphi\sqrt{u}$ and, using dot notation for the derivative with respect to s, write

$$\int_0^a \Big(c+\frac{|\nabla u|^2}{u^2}\Big)\varphi^2\,ds \le \int_0^a \Big(\frac{\varphi^2\dot u^2}{4u^2}+\dot\varphi^2+\frac{\dot u}{u}\varphi\dot\varphi\Big)\,ds,$$

hence

$$\int_0^a \Big(-\frac{3\varphi^2\dot u^2}{4u^2}-c\varphi^2+\dot\varphi^2+\frac{\dot u}{u}\varphi\dot\varphi\Big)\,ds \ge 0.$$

Using the inequality $a^2+b^2\ge 2ab$ with $a=\frac{\sqrt{3}}{2}u^{-1}\dot u\varphi$ and $b=\frac{1}{\sqrt{3}}\dot\varphi$, we get

$$\frac{3\dot u^2\varphi^2}{4u^2}+\frac{\dot\varphi^2}{3}\ge u^{-1}\dot u\varphi\dot\varphi,$$

hence

$$\int_0^a \Big(\frac{4}{3}\dot\varphi^2-c\varphi^2\Big)\,ds \ge 0.$$

Integration by parts gives

$$\int_0^a \Big(\frac{4}{3}\ddot\varphi+c\varphi\Big)\varphi\,ds \le 0.$$

Choosing $\varphi=\sin(\pi s a^{-1})$ and $s\in[0,a]$, we have

$$\int_0^a \Big(c-\frac{4\pi^2}{3a^2}\Big)\sin^2(\pi s a^{-1})\,ds \le 0.$$

Finally,

$$c-\frac{4\pi^2}{3a^2}\le 0,$$

and this gives the desired inequality. □

Theorem B. *In $\mathbb{H}^2\times\mathbb{R}$ there is no complete noncompact stable surface of constant mean curvature $H>1/\sqrt{3}$, either with compact boundary or without boundary.*

Proof. As M is complete, one can find a sequence of points on M whose distances to any compact set diverge. This is in contradiction with (1). □

Remark 2.3. In the Main Lemma, suppose one replaces the stability hypothesis by the assumption that M has finite index (see [Fischer-Colbrie 1985] for the definition and for the proof of the next fact). For a compact subset K of M, there is a positive function u on $M\setminus K$ with $Lu=0$ on $M\setminus K$. The same argument as above shows that the distance from any point $p\in M\setminus K$ to $\partial(M\setminus K)$ is bounded by the same bound as in the lemma. Thus, if M is an H-surface of finite index in $\mathbb{H}^2\times\mathbb{R}$, with $H>1/\sqrt{3}$ and ∂M compact, then M is compact.

Denote by t the last coordinate in $\mathbb{H}^2\times\mathbb{R}$.

Corollary 2.4. *Let M be a compact H-surface embedded in $\mathbb{H}^2\times\mathbb{R}$, with $H > 1/\sqrt{3}$ and boundary in the horizontal geodesic plane $P = \{t=0\}$. The height h of M above P satisfies*

$$|h| < \frac{4\pi}{\sqrt{3(3H^2-1)}}. \tag{9}$$

Proof. For any two points p and q in M, one has $\mathrm{dist}_{\mathbb{H}^2\times\mathbb{R}}(p,q) \le \mathrm{dist}_M(p,q)$. Assume first that M is a vertical graph above the plane P. By Remark 2.1, such a graph is a stable surface. Hence, for any point $p \in M$ one has

$$|h(p)| \le \mathrm{dist}_{\mathbb{H}^2\times\mathbb{R}}(p,\partial M) \le \mathrm{dist}_M(p,\partial M) \le \frac{2\pi}{\sqrt{3(3H^2-1)}},$$

where the last step is inequality (1).

If M is not a graph, it is enough to prove the result for any component M' of M lying above the plane P. We apply to M' the Alexandrov reflection method with horizontal geodesic planes. One can do it exactly as in the Euclidean case (see [Ros and Rosenberg 1996]), because reflections with respect to horizontal geodesic planes are isometries of $\mathbb{H}^2\times\mathbb{R}$. By this method one obtains that, if the height of M' above the plane P is L, then the part of M' above height $\frac{1}{2}L$ is a vertical graph. So, by the first part of the proof,

$$L < \frac{4\pi}{\sqrt{3(3H^2-1)}}. \qquad \square$$

It is proved in [Hoffman et al. 2006] that

$$|h| \le \frac{4H}{2H^2-1} \qquad \text{for } H > 1/\sqrt{2}.$$

A straightforward computation shows that our estimate is sharper than the estimate in [Hoffman et al. 2006] in the range

$$\frac{1}{\sqrt{2}} \le h \le \left(\frac{4\pi^2-3+\sqrt{3(4\pi^2+3)}}{2(4\pi^2-9)}\right)^{1/2}.$$

On the other hand, if $H > 1/2$, the height of a compact rotational H-surface over its horizontal symmetry plane is

$$h_{\mathrm{rot}} = \frac{2H}{\sqrt{4H^2-1}}\tan^{-1}\frac{1}{\sqrt{4H^2-1}},$$

as computed in [Hsiang and Hsiang 1989]. Since

$$h_{\mathrm{rot}} < \frac{2\pi}{\sqrt{3(3H^2-1)}} \qquad \text{for } H > 1/\sqrt{3},$$

our estimate is not sharp.

3. Maximum principle at infinity

Theorem A. *Let M_1 and M_2 be two disjoint H-surfaces properly embedded in $\mathbb{H}^2 \times \mathbb{R}$, without boundary and with $H > 1/\sqrt{2}$. The surface M_2 cannot lie in the mean convex side of M_1.*

The analogue of this result in Euclidean space was proved in [Ros and Rosenberg 2001].

In order to prove Theorem A, we need to establish some notation and prove a lemma. Let M be an H-surface properly embedded in $\mathbb{H}^2 \times \mathbb{R}$. Take ε a positive number and, for each t with $|t| \leq \varepsilon$, consider the set

$$N_t = \left\{ q \in \mathbb{H}^2 \times \mathbb{R} \;\middle|\; \operatorname{dist}_{\mathbb{H}^2 \times \mathbb{R}}(q, M) = t \right\},$$

where $\operatorname{dist}_{\mathbb{H}^2 \times \mathbb{R}}$ is the signed distance function, chosen to be positive in the mean convex side of M. Given any domain of M with compact closure, if ε is sufficiently small, the N_t's foliate a neighborhood of this domain in $\mathbb{H}^2 \times \mathbb{R}$. For $t > 0$, the leaf N_t lies in the mean convex side of M; for $t < 0$, in the other side. Let Y be the unit vector field normal to the foliation N_t and orienting M in the direction of the mean curvature vector. Along M, the field Y points into the mean convex side of M. At any $x \in N_t$, let $H_t(x)$ denote the mean curvature of N_t with respect to Y.

Lemma 3.1. *If $H > 1/\sqrt{2}$ and ε is sufficiently small, then, for any $x \in N_t$ and $y \in N_s$,*

$$H_t(x) < H < H_s(y) \tag{10}$$

whenever $-\varepsilon < t < 0 < s < \varepsilon$. Moreover, $\operatorname{div} Y$ *is negative (here* div *is divergence in the ambient space).*

Proof. The first variation of the mean curvature for a normal variation fY gives

$$\left.\frac{dH_t}{dt}\right|_{t=0} f = \dot{H}(0) f = \Delta f + \big(|A|^2 + \operatorname{Ric}(Y)\big) f,$$

where $|A|$ is the norm of the second fundamental form of M and $\operatorname{Ric}(Y)$ is the Ricci curvature of $\mathbb{H}^2 \times \mathbb{R}$ in the direction of Y.

Choosing $f = 1$ and using (4),

$$\dot{H}(0) = |A|^2 + \operatorname{Ric}(Y) > 0 \qquad \text{for } H > 1/\sqrt{2},$$

and inequality (10) is proved.

Furthermore, at any point y of N_t one has

$$(\operatorname{div} Y)(y) = -2H_t(y).$$

Hence

$$(\operatorname{div} Y)(y) = -2H_s(y) < -2H < -2H_t(x) = (\operatorname{div} Y)(x) \tag{11}$$

for any $y \in N_s,\ x \in N_t,\ -\varepsilon < t < 0 < s < \varepsilon$. Therefore, div Y is negative. □

Proof of Theorem A. The first part of the proof explains the name "maximum principle at infinity". The proof is by contradiction: we assume that M_2 lies in the mean convex side of M_1.

We first prove that neither M_1 nor M_2 can be compact. If M_1 were compact, then the mean convex side of M_1 would be compact too, and M_2 would be properly embedded in a compact set; hence, M_2 would be compact. Moving M_1 towards M_2 by an isometry of the ambient space yields a first contact point, where the mean curvature vectors of M_1 and M_2 are equal. This gives a contradiction by the standard maximum principle; hence, M_1 cannot be compact. If M_2 were compact, then, by moving M_2 towards M_1 as before, one obtains a contradiction by the standard maximum principle. So, M_1 and M_2 are both noncompact. Further, when moving M_1 towards M_2 by an isometry of the ambient space, the first contact point cannot be finite, by the standard maximum principle.

We are left with the case in which the first contact point is at infinity. In this case, define W to be the closure of the component of $(\mathbb{H}^2 \times \mathbb{R}) \setminus (M_1 \cup M_2)$ with $\partial W = M_1 \cup M_2$. The boundary of W is not connected, and the mean curvature vector of M_1 points towards W.

Take $h > 1/2$ and let B_h be a compact rotational surface of constant mean curvature h. One knows that such surfaces exist for any $h > 1/2$, and that their mean convex sides are compact and exhaust the space as $h \to 1/2$; see [Hsiang and Hsiang 1989]. Let W_h be the intersection of W with the mean convex side of B_h.

Let S be a relatively compact domain in M_1 such that $\Gamma = \partial S$ is a smooth curve. For large h, the mean convex side of B_h contains the surface S. We will show that there exists a stable H-surface Σ in W_h with boundary Γ and homologous to S. Then, by taking the domain S in M_1 larger and larger, we will find points of Σ very far from its boundary Γ. This will give a contradiction, because the distance between a point of a stable H-surface and its boundary is bounded; see inequality (1).

We divide the proof of the existence of Σ in three steps.

Step 1: Define a functional F on integral mod 2 currents Q in W_h with $\partial Q = S \cup \Sigma$, and minimize it in W_h.

Step 2: If Q is a minimizer for F, then $\Sigma = \partial Q \setminus S$ is a stable H-surface.

Step 3: If Σ does not coincide with S, then Int Σ is contained in Int W_h.

Proof of Step 1. Consider the functional

$$F(Q) = A(\Sigma) + 2H\,V(Q)$$

defined on all integral mod 2 currents Q contained in W_h with both

(a) $\partial Q = \Sigma \cup S$, where Σ is a compact set of finite 2-mass; and

(b) $V(Q) \leq A(S)$, and $A(\Sigma) \leq A(S)$.

Here, $A(\Sigma)$ denotes the 2-mass of Σ, while $V(Q)$ denotes the 3-mass of Q.

If there exists no Q such that conditions (a) and (b) hold, then S is stable. We prove this by contradiction: Assume that, for any Q satisfying condition (a), one has either $V(Q) > A(S)$ or $A(\Sigma) > A(S)$. As $H \geq 1/\sqrt{2}$, it follows that in both cases $F(Q) > A(S)$. This means that there is no variation of S inside W_h that decreases the value of F. We claim that this implies that S is stable in the sense of Section 2.

Suppose by contradiction that this is not the case. Then, as in the proof of Remark 2.1, the first eigenvalue λ of the stability operator L on S is negative. We can assume that a first eigenfunction f is such that $f|_{\partial S} = 0$ and $f|_S > 0$. Then $Lf = -\lambda f > 0$ and $\ddot{G}(0)(f) < 0$. As $f|_S > 0$, the deformation arising from f is inside W_h, where the functionals G and F coincide. We found a deformation of F inside W_h that decreases F: a contradiction.

If one chooses the domain S on M_1 large enough to have interior points with distance from the boundary of S contradicting inequality (1), then S is not stable. For such an S, one can find integral currents Q such that conditions (a) and (b) are satisfied.

There exists a minimum Q of F in W_h. Furthermore, $\Sigma = \partial Q \setminus S$ is regular at interior points, that is, at points of $\Sigma \cap \operatorname{Int} W_h$, and Σ has mean curvature H at regular points. The existence follows from [Massari 1974, Theorem 1.1]; Massari states the result in Euclidean space, but, as his functional is the same as our F, the proofs are analogous.

The regularity follows from [Morgan 2003]. Actually, Morgan proves regularity for solutions of an isoperimetric problem, and his proof is local. In order to apply Morgan's result, we have to show that, in a small ball, a minimum of F and the solution of the isoperimetric problem coincide. Take $p \in \Sigma$ and let $B(p, \varepsilon) \subset \operatorname{Int}(W_h)$ be a small ball around p. Write $Q_\varepsilon = Q \cap B(p, \varepsilon)$ and $\Sigma_\varepsilon = \Sigma \cap B(p, \varepsilon)$.

We claim that Σ_ε is the solution of the isoperimetric problem for the volume $V(Q_\varepsilon)$ in the closed manifold $\bar{B}(p, \varepsilon)$, with fixed boundary $\partial \Sigma_\varepsilon$. If this is not the case, one can find a surface $\Sigma' \subset \bar{B}(p, \varepsilon)$, with $\partial \Sigma' = \partial \Sigma_\varepsilon$ but distinct from Σ_ε, with $A(\Sigma') < A(\Sigma_\varepsilon)$ and $V(Q') = V(Q_\varepsilon)$, where $Q' \subset \bar{B}(p, \varepsilon)$ is such that $\partial Q' = \Sigma' \cup (\partial Q_\varepsilon \setminus \Sigma_\varepsilon)$; see Figure 1. But then

$$F(Q') = A(\Sigma') + 2H\,V(Q') < A(\Sigma_\varepsilon) + 2H\,V(Q_\varepsilon) = F(Q_\varepsilon)$$

Hence Q would not be a minimum for F, a contradiction. □

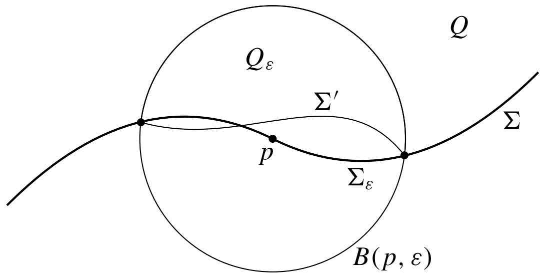

Figure 1

Proof of Step 2. We have to prove that, if Q is a minimum for F, then $\Sigma = \partial Q \setminus S$ is a stable surface in the sense of Section 2.

First, we argue that the mean curvature vector of Σ points outside Q. Suppose by contradiction that there is a point p of Σ such that the mean curvature vector at p points towards Q. Then one can find a compact subset K of the ambient space such that $p \in K \cap \Sigma$, ∂K is mean convex, and the mean curvature vectors at points of $\Sigma \cap K$ point towards Q. Let $\tilde{\Sigma}$ be the solution of the Plateau problem in $K \cap Q$ for the boundary $\partial(K \cap \Sigma)$. Denote by $\tilde{Q}$ the subset of $K \cap Q$ such that $\partial \tilde{Q} = (\Sigma \cap K) \cup \tilde{\Sigma}$. One has

$$F(Q \setminus \tilde{Q}) = V(Q \setminus \tilde{Q}) + 2H\big(A(\Sigma) - A(U \cap K) + A(\tilde{\Sigma})\big) < F(Q),$$

where the last inequality follows from $A(\tilde{\Sigma}) < A(\Sigma \cap K)$, as $\tilde{\Sigma}$ is area-minimizing. This is a contradiction with Q being a minimizer of F.

Now, suppose by contradiction that $\Sigma = \partial Q \setminus S$ is not a stable surface in the sense of Section 2. Then, as in the proof of Remark 2.1, the first eigenvalue λ for the stability operator L on Σ is negative. We can assume that a first eigenfunction f is such that $f|_{\partial\Sigma} = 0$ and $f|_{\Sigma} > 0$. Then $Lf = -\lambda f > 0$ and $\ddot{G}(0)(f) < 0$. As $f|_{\Sigma} > 0$, the deformation arising from f is inside W_h and outside Q, where the functionals G and F coincide. Hence, Σ cannot be a minimum for F in W_h: a contradiction. □

Proof of Step 3. We prove that, if Q is an integral current in W_h such that $\partial Q \setminus S$ touches ∂W_h, then there is an integral current Q' such that

$$S \subset \partial Q', \qquad \partial Q' \setminus S \subset \mathrm{Int}(W_h), \quad \text{and} \quad F(Q') < F(Q).$$

Hence, one can always choose a minimizing sequence Q_n for the functional F such that $\partial Q_n \setminus S$ stays inside $\mathrm{Int}(W_h)$, and there is a minimum Q of F such that $\Sigma = \partial Q \setminus S$ is a stable H-surface in $\mathrm{Int}(W_h)$ with boundary Γ and homologous to S.

We have four cases: $\Sigma = \partial Q \setminus S$ can touch ∂W_h along S, along $M_1 \setminus S$, along B_h, or along M_2.

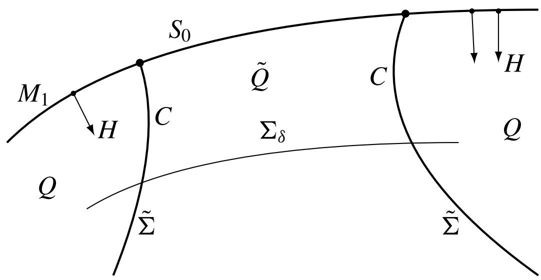

Figure 2

(i) *Case along* S. We can assume that Σ touches S in one connected component S_0; if this were not the case, we could repeat the same reasoning for each of the connected components. Foliate a neighbourhood of S_0 as in Lemma 3.1. Let $C \subset \partial Q$, $\tilde{\Sigma} \subset \partial Q$ and $\Sigma_\delta \subset N_\delta$ with $\delta < \varepsilon$ be such that $\partial Q = \tilde{\Sigma} \cup C \cup S_0$, as in Figure 2. Let $\tilde{Q}$ be the subset of W_h such that $\partial \tilde{Q} = \Sigma_\delta \cup C \cup S_0$. Set $Q' = Q \cup \tilde{Q}$, so that $\partial Q'$ does not touch S.

We prove that $F(Q') < F(Q)$. We have

$$F(Q') = A(\tilde{\Sigma}) + A(\Sigma_\delta) + 2H\,V(\tilde{Q}) + 2H\,V(Q),$$
$$F(Q) = A(\tilde{\Sigma}) + A(C) + A(S_0) + 2H\,V(Q).$$

After applying Stokes' theorem in $\tilde{Q}$ and using inequality (11), one gets

$$-2H\,V(\tilde{Q}) > \int_{\tilde{Q}} \operatorname{div} Y = A(\Sigma_\delta) + \int_C \langle Y, n_C\rangle - A(S_0),$$

where n_C is the unit normal vector to C pointing outwards $\tilde{Q}$. Then

$$A(\Sigma_\delta) + 2H\,V(\tilde{Q}) < A(S_0) + A(C).$$

This yields $F(Q') < F(Q)$ and thus excludes this case. (See [Ros and Rosenberg 2001] for another proof of this case.)

(ii) *Case along* $M_1 \setminus S$. Let S_0 be the subset of $M_1 \setminus S$ where Σ touches M_1. As in the proof of (i), we can assume that S_0 is connected, and foliate a neighbourhood of S_0 as in Lemma 3.1. Let $\tilde{Q} = Q \cap \{N_t \mid 0 < t < \delta\}$ with $\delta < \varepsilon$. Write $Q' = Q \setminus \tilde{Q}$, so that $\partial Q' \setminus S$ does not touch M_1.

We prove that $F(Q') < F(Q)$. Split $\partial \tilde{Q} = S_0 \cup \Sigma_\delta \cup C$, with $C \subset \partial Q$, $\Sigma_\delta \subset N_\delta$, and $\Sigma' = \partial Q \setminus (C \cup S_0)$, as in Figure 3. Applying Stokes' theorem in $\tilde{Q}$ and using inequality (11), one has

$$-2H\,V(\tilde{Q}) > \int_{\tilde{Q}} \operatorname{div} Y = -A(S_0) + A(\Sigma_\delta) + \int_C \langle Y, n_C\rangle,$$

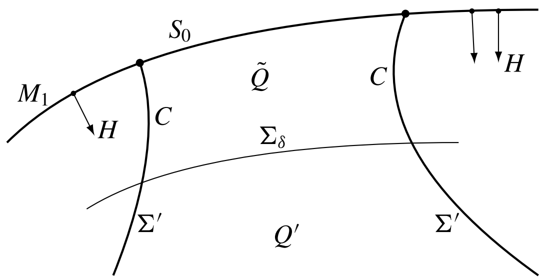

Figure 3

where n_C is the unit normal vector to C pointing outwards $\tilde{Q}$. Then

$$A(\Sigma_\delta) + 2H\,V(\tilde{Q}) < A(S_0) + A(C). \tag{12}$$

Furthermore,

$$F(Q) = 2H\,V(\tilde{Q}) + 2H\,V(Q') + A(\Sigma') + A(C) + A(S_0),$$
$$F(Q') = 2H\,V(Q') + A(\Sigma') + A(\Sigma_\delta).$$

From inequality (12) then follows that $F(Q') < F(Q)$.

(iii) *Case along B_h*. As Q lies in the mean convex side of B_h, a reasoning similar to that in (ii) excludes this case.

(iv) *Case along M_2*. The same proof as in (ii) yields that ∂Q cannot touch M_2, provided the mean curvature vector of M_2 points towards W_h.

If the mean curvature vector of M_2 points outwards W_h, we proceed as follows: Let S_0 be the subset of M_2 where Σ touches M_2; as before we can assume that S_0 is connected, and foliate a neighbourhood of S_0 as in Lemma 3.1. Let $\tilde{Q} = Q \cap \{N_t \mid -\delta < t < 0\}$ and $Q' = Q \setminus \tilde{Q}$; then Q' does not touch M_2.

We prove that $F(Q') < F(Q)$. Split $\partial\tilde{Q} = S_0 \cup \Sigma_{-\delta} \cup C$, with $C \subset \partial Q$ and $\Sigma_{-\delta} \subset N_{-\delta}$, and let $\Sigma' = \partial Q \setminus (C \cup S_0)$, as in Figure 4. Applying Stokes' theorem in $\tilde{Q}$ and using inequality (11), one has

$$-2H\,V(\tilde{Q}) < \int_{\tilde{Q}} \operatorname{div} Y = A(S_0) - A(\Sigma_{-\delta}) + \int_C \langle Y, n_C\rangle,$$

where n_C is the unit normal vector to C pointing outwards $\tilde{Q}$. Then

$$A(\Sigma_{-\delta}) < 2H\,V(\tilde{Q}) + A(S_0) + A(C). \tag{13}$$

Furthermore,

$$F(Q) = 2H\,V(\tilde{Q}) + 2H\,V(Q') + A(\Sigma') + A(C) + A(S_0),$$
$$F(Q') = 2H\,V(Q') + A(\Sigma') + A(\Sigma_{-\delta}).$$

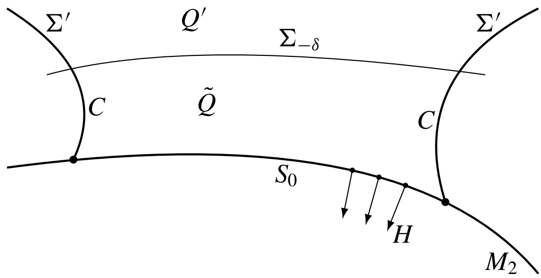

Figure 4

Then, from inequality (13) it follows that $F(Q') < F(Q)$, which excludes this last case. □

Now, we finish the proof of Theorem A.

Take $p_1 \in M_1$ and let S be the set of points of M_1 at distance from p_1 less than or equal to a fixed R. As M_1 is properly embedded, when R is large ∂S is outside any compact set of $\mathbb{H}^2 \times \mathbb{R}$, and hence S is not stable because of inequality (1). Let Σ be the stable H-surface in W_h with boundary Γ obtained by minimizing F. Take a point p_2 of M_2 and a path γ in Int W_h joining p_1 to p_2. One of the components of Σ intersects γ, since Σ is homologous to S and the intersection number of γ and S is one. As $\partial S = \partial \Sigma$, there exists a $p \in \gamma \cap \Sigma$ such that

$$\mathrm{dist}_\Sigma(p, \partial\Sigma) > \frac{2\pi}{\sqrt{3(3H^2-1)}}.$$

This contradicts inequality (1). □

4. Compact stable surfaces

Theorem C. *Let M be a compact weakly-stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$. If $H > 1/\sqrt{2}$, then the genus g of M is at most three.*

Proof. Let $\varphi : M \to \mathbb{S}^2$ be a meromorphic map satisfying

$$\deg \varphi \leq 1 + \left[\tfrac{1}{2}(g+1)\right] \qquad \text{and} \qquad \int_M \varphi = 0,$$

where $[x]$ means the integer part of x. For the existence of such a map φ, see for example [Ritoré and Ros 1992]. Applying the stability inequality of Section 2 to the three coordinates of φ and summing up, one has

$$0 \leq \int_M \left(|\nabla\varphi|^2 - (\mathrm{Ric}\, N + |A|^2)\right) dM.$$

Using that $|\nabla\varphi|^2 = 2\,\mathrm{Jac}(\varphi)$ and $\mathrm{Ric}(N) + |A|^2 = 4H^2 - 2K + K_s - 1$ (with the notation of Remark 2.2), one obtains

$$0 \le 8\pi \deg\varphi - \int_M \left(4H^2 + K_s - 1\right) dM + 2\int_M K\, dM.$$

Since $K_s \ge -1$, by the Gauss–Bonnet theorem one has

$$0 \le 8\pi \deg\varphi - \int_M \left(4H^2 - 2\right) dM + 8\pi(1-g).$$

Hence

(14) $$\frac{1}{2\pi}\int_M \left(H^2 - {}^1\!/_2\right) dM \le \left[{}^{(g+1)}\!/_2\right] + 2 - g.$$

Since $H > {}^1\!/\!\sqrt{2}$, this inequality gives $g \le 3$. □

Corollary 4.1. *Let M be a compact stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H > {}^1\!/\!\sqrt{2}$. Denote the genus of M by g.*

- *If $g = 0$ or 1, then $A(M) \le {}^{8\pi}\!/_{(2H^2-1)}$.*
- *If $g = 2$ or 3, then $A(M) \le {}^{4\pi}\!/_{(2H^2-1)}$.*

Proof. The estimates on the area of M follow immediately from inequality (14). □

Remark 4.2. In [Abresch and Rosenberg 2004] it is proved that any compact H-surface of genus zero is a rotational surface. We remark that such rotational compact H-surfaces exist only for $H > {}^1\!/_2$.

5. Surfaces of constant mean curvature $H \le {}^1\!/_2$

Theorem D. *For any $H \in (0, {}^1\!/_2]$ there exists a complete, vertical rotational graph R_H on $\mathbb{H}^2$ with constant mean curvature H.*

Proof. Consider the disk model for $\mathbb{H}^2$ and let $u : \mathbb{H}^2 \to \mathbb{R}$ be a C^2 function depending on $r = \sqrt{x_1^2 + x_2^2}$. Write $\dot u = du/dr$.

The graph of u is a rotational H-surface if and only if

(15) $$\ddot u + \frac{\tau^2 \dot u}{r} + \sqrt{F}\, r \dot u^3 = \frac{2H\tau^3}{F},$$

where $F = \left({}^{(1-r^2)}\!/_2\right)^2$ and $\tau = \sqrt{1 + F\dot u^2}$. A first integral of (15) is

$$\dot u = \frac{4Hr}{(1-r^2)\sqrt{1-4H^2r^2}}.$$

Let us describe the behaviour of such a solution. All values of r in $[0, 1)$ are allowed. When $H = {}^1\!/_2$, one can integrate explicitly and obtain

$$u(r) = \frac{2}{\sqrt{1-r^2}} + \text{constant}.$$

Hence, $\lim_{r\to 1} u(r) = \infty$. On the other hand, if $H \in (0, 1/2]$, then

$$u(r) > \int \frac{4Hr}{(1-r^2)} = -2H \ln(1-r^2).$$

Hence, $\lim_{r\to 1} u(r) = \infty$. □

Corollary 5.1. *For any simple closed curve Γ that is not homologically zero in the boundary of $\mathbb{H}^2 \times \mathbb{R}$, there exists no H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H \neq 0$ and asymptotic boundary Γ.*

Proof. If such a surface existed, then, by the maximum principle it would coincide with one of the rotational surfaces described above; this is a contradiction. □

Remark 5.2. The result of Corollary 5.1 is in contrast with the situation for $H = 0$; see [Nelli and Rosenberg 2002].

Corollary 5.3. *There is no compact H-surface with $H \in (0, 1/2]$ and embedded in $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. Assume by contradiction that a compact embedded H-surface Σ_0 exists for $H = H_0 \in (0, 1/2]$. Consider the rotational surface R_{H_0} from Theorem D. Comparing Σ_0 and R_{H_0}, by the maximum principle one obtains that they should coincide. This is a contradiction, as Σ_0 is compact, which is not the case with R_{H_0}. □

Remark 5.4. Corollary 5.3 is implicitly contained in [Hsiang and Hsiang 1989]. There, it is proved that any embedded compact surface of constant mean curvature must be rotational, and that the only embedded compact rotational surfaces have mean curvature greater than $1/2$. In the case of the sphere, Corollary 5.3 holds for immersed surfaces. This is contained in [Abresch and Rosenberg 2004], where the authors prove that any immersed constant-mean-curvature sphere is rotational.

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