Pacific Journal of Mathematics

Volume 226 No. 2 August 2006

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by

E. F. Beckenbach (1906–1982) F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840 is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

COMPLETE LOCALLY CONFORMALLY FLAT MANIFOLDS OF NEGATIVE CURVATURE

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We construct new examples of complete locally conformally flat manifolds of negative curvature by means of warped product and multiply warped product structures. Special attention is paid to those spaces with one-dimensional base, thus generalizing the Robertson–Walker spacetimes, and to those with higher-dimensional base of constant curvature.

1. Introduction

Locally conformally flat structures on Riemannian manifolds are natural generalizations of isothermal coordinate systems, which are available on Riemann surfaces. However, not every higher-dimensional Riemannian manifold admits a locally conformally flat structure, and it is difficult to provide a classification of those that do; this is still an open problem. Some partial results are known. A compact simply connected locally conformally flat manifold must be a Euclidean sphere [Kuiper 1949; Schoen and Yau 1988]. Locally symmetric manifolds which are locally conformally flat are either of constant sectional curvature or locally isometric to a product of two spaces of constant opposite sectional curvature [Lafontaine 1988; Yau 1973]. Complete locally conformally flat manifolds with nonnegative Ricci curvature have been studied by several authors; Zhu [1994] showed that their universal cover is in the conformal class of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{R} \times \mathbb{S}^{n-1}$, where $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ are spheres of constant sectional curvature. Such conformal equivalence can be specialized to isometric equivalence under some extra assumptions on the scalar curvature and the sign of the Ricci curvatures [Cheng 2001; Tani 1967] (see also [Carron and Herzlich 2004] and the references therein). In spite of the results on locally conformally flat manifolds of nonnegative curvature, to the best of our knowledge, there is a lack of information as concerns negative curvature. Henceforth, our purpose on this work is to construct new examples of complete locally conformally flat Riemannian manifolds with nonpositive curvature.

MSC2000: 53C15, 53C50.
Keywords: Locally conformally flat, warped product metric, Möbius equation, Obata equation.
Supported by projects BFM 2003-02949 and PGIDIT04PXIC20701PN (Spain).

Since their introduction by Bishop and O'Neill [1969], warped products have been a powerful tool for constructing manifolds of nonpositive curvature (see also [Bertola and Gouthier 2001]). Our aim, then, is to investigate the existence of locally conformally flat structures on manifolds equipped with a warped product structure, or more generally on multiply warped spaces, as being a natural generalization of warped products (see for example [Tojeiro 2004] and the references therein). Other generalizations of warped product structures, like twisted or multiply quasiwarped [Meumertzheim et al. 1999; Ponge and Reckziegel 1993; Tojeiro 2004] are not of interest for our purposes, since they reduce to warped and multiply warped spaces, respectively, if they are locally conformally flat [Brozos-Vázquez et al. 2005]. Another motivation for the consideration of locally conformally flat structures on manifolds equipped with a warped product metric comes from the fact that the Schouten tensor is Codazzi for any locally conformally flat manifold. Moreover, although the local structure of Codazzi tensors is not yet completely understood, the existence of such a tensor leads to warped product decompositions of the manifold in many cases [Bivens et al. 1981; Tojeiro 2004].

This paper is organized as follows. In Section 2 we recall basic facts on the curvature of warped and multiply warped spaces. Locally conformally flat multiply warped spaces are investigated in Section 3. Our approach relies on the fact that any multiply warped space is in the conformal class of a suitable product, a fact previously observed for warped product metrics [Lafontaine 1988], which has several implications on the geometry of the fibers and the base of the multiply warped space. A local description of locally conformally flat spaces with the underlying structure of a multiply warped product is then obtained from the fact that any warping function must define a global conformal transformation on the base which makes it of constant sectional curvature. Then the situation when the base has dimension 2 or higher reduces to the existence of nontrivial solutions of some Obata type equations on the base (sometimes called concircular transformations; see [Kühnel 1988; Tashiro 1965]) together with some compatibility conditions among the different warping functions. This analysis is carried out in Section 3A. Conditions become much weaker when the base is assumed to be one-dimensional, as shown in Section 3B, in accordance with Roberston–Walker type metrics, which are locally conformally flat independently of the warping function. Some global consequences are obtained in Section 4, where locally conformally flat warped product manifolds with complete base of constant curvature are classified, as well as multiply warped ones if the base is further assumed to be simply connected.

Applications of the results in Section 3 have already been found by R. Tojeiro in the study of conformal immersions into the Euclidean space [2006]. Moreover, multiply warped spaces with hyperbolic space as the base are of key interest, providing some new examples of complete locally conformally flat manifolds with nonpositive sectional curvature, and with nonpositive Ricci curvatures but no sign requirement on the sectional curvature.

2. Preliminaries

Let (B, g_B), (F_1, g_1), ..., (F_k, g_k) be Riemannian manifolds. The product manifold $M = B \times F_1 \times \cdots \times F_k$, equipped with the metric

$$g = g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k,$$

where $f_1, \ldots, f_k : B \to \mathbb{R}$ are positive functions, is called a *multiply warped product*. B is the *base*, $F_1, \ldots, F_k$ are the *fibers* and $f_1, \ldots, f_k$ are the *warping functions*. We will denote a multiply warped product manifold as above by $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$.

Remark 2.1. The general form of multiply warped products is slightly flexible, so we must adopt some criteria to identify multiply warped products with different form but which are essentially the same. They are:

C1. Warping functions are supposed to be nonconstant and any two warping functions which are multiples one to each other are written as the same function and the metric of the fiber is multiplied by the constant in order to do not modify the metric of the multiply warped product.

C2. Fibers with the same warping function are joined in one fiber.

Moreover, the possible order of the fibers is irrelevant for our purposes.

Next we fix some notation and criteria to be used in what follows. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. The Riemann curvature tensor R is the $(1, 3)$-tensor field on M defined by $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$, for all vector fields $X, Y, Z \in \mathfrak{X}(M)$. The Ricci tensor is the contraction of the curvature tensor given by $\rho(X, Y) = \text{trace}\{U \rightsquigarrow R(X, U)Y\}$ and the scalar curvature is obtained by contracting the Ricci tensor, $\tau = \text{trace}(\rho)$. For a vector field X on M the *divergence* of X is defined by $div\, X = \text{trace}\, \nabla X$. The *gradient* of a function $f : (M, g) \to \mathbb{R}$ is determined by $g(\nabla f, X) = X(f)$ and the *Laplacian* of f is defined by $\Delta f = div \nabla f$. Also, the linear map $h_f(X) = \nabla_X \nabla f$ is called the *Hessian tensor* of f on (M, g), and $H_f(X, Y) = g(h_f(X), Y)$ is called the *Hessian form* of f. Finally, note that $\Delta f = \text{trace}\, h_f$.

In order to study the properties of multiply warped products, we need some properties of their curvature tensor, obtained essentially in the same way as for warped products [Bishop and O'Neill 1969]. Therefore proofs are omitted. The nonzero components of the curvature tensor are

$$
\begin{aligned}
&R_{XY}Z = R^B_{XY}Z, && R_{V_iX}Y = \frac{1}{f_i}H_{f_i}(X,Y)V_i,\\
(1)\quad &R_{XU_i}V_i = \frac{\langle U_i, V_i\rangle}{f_i}\nabla_X\nabla f_i, && R_{U_jU_i}V_i = \frac{\langle U_i, V_i\rangle}{f_i f_j}\langle \nabla f_i, \nabla f_j\rangle U_j \ \text{ if } i\neq j,\\
&R_{U_iV_i}W_i = R^{F_i}_{U_iV_i}W_i - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2}\big(\langle U_i, W_i\rangle V_i - \langle V_i, W_i\rangle U_i\big),
\end{aligned}
$$

for all $X, Y, Z \in \mathfrak{L}(B)$ and $U_i, V_i, W_i \in \mathfrak{L}(F_i)$, where R^B and R^{F_i} denote the curvature tensor of (B, g_B) and (F_i, g_i), respectively. Here $H_{f_i}(X,Y)$ and ∇f_i denote the Hessian tensor and the gradient of the warping function f_i with respect to the Riemannian structure of (B, g_B). A straightforward calculation from (1) shows that the sectional curvature of M satisfies

$$
\begin{aligned}
&K_{XY} = K^B_{XY}, && K_{XU_i} = -\frac{H_{f_i}(X,X)}{f_i\|X\|^2},\\
(2)\quad &K_{U_iV_i} = \frac{1}{f_i^2}K^{F_i}_{U_iV_i} - \frac{\|\nabla f_i\|^2}{f_i^2}, && K_{U_iV_j} = -\frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} \ \text{ if } i \neq j,
\end{aligned}
$$

where K^B and K^{F_i} denote the sectional curvatures on the base B and the fiber F_i. Here the sectional curvature of a plane π is taken with the sign convention $K(\pi) = R(X, Y, X, Y)$, for any orthonormal base $\{X, Y\}$ of π.

3. Locally conformally flat multiply warped spaces

Recall that a Riemannian manifold (M, g) is *locally conformally flat* if every point in M admits a coordinate neighborhood $\mathfrak{U}$ which is conformal to Euclidean space $\mathbb{R}^n$; equivalently, if there is a diffeomorphism $\Phi : V \subset \mathbb{R}^n \to \mathfrak{U}$ such that $\Phi^* g = \Psi^2 g_{\mathbb{R}^s}$ for some positive function Ψ. Any surface is locally conformally flat, but not every higher-dimensional Riemannian manifold admits a locally conformally flat structure. Necessary and sufficient conditions for the existence of such a structure are the nullity of the Weyl tensor $W = R - C \odot g$ when $\dim M \geq 4$, and, in dimension three, the condition that the Schouten tensor

$$C = \frac{1}{n-2}\left(\rho - \frac{\tau}{2(n-1)}g\right)$$

be a Codazzi tensor. Here $\odot$ represents the Kulkarni–Nomizu product (see [Lafontaine 1988], for example). A nonflat locally decomposable Riemannian manifold is locally conformally flat if and only if it is locally equivalent to the product $N(c) \times \mathbb{R}$ of an interval and a space of constant sectional curvature, or to the product

$N_1(c) \times N_2(-c)$ of two spaces of opposite constant sectional curvature [Lafontaine 1988; Yau 1973].

Lemma 3.1. *Let $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a locally conformally flat multiply warped space.*

(i) *(B, g_B) is locally conformally flat.*

(ii) *(F_i, g_i) is a space of constant sectional curvature for all $i = 1, \ldots, k$, provided that* $\dim F_i \geq 2$.

Proof. For any $i = 1, \ldots, k$, write the multiply warped metric as

$$g = f_i^2 \left(\frac{1}{f_i^2} g_B \oplus \frac{f_1^2}{f_i^2} g_1 \oplus \cdots \oplus g_i \oplus \cdots \oplus \frac{f_k^2}{f_i^2} g_k \right).$$

Since f_i maps B to $\mathbb{R}^+$, this expression shows that g is in the conformal class of a suitable product metric tensor. Hence, the multiply warped metric is locally conformally flat if and only if so is the product metric of (F_i, g_i) and the multiply warped $\tilde{B} \times_{f_1/f_i} F_1 \times \cdots \times \widehat{F_i} \times \cdots \times_{f_k/f_i} F_k$ with base $\tilde{B} \equiv (B, f_i^{-2} g_B)$. This shows that either $\dim F_i = 1$ or otherwise it is of constant sectional curvature, and moreover that $\tilde{B} \times_{f_1/f_i} F_1 \times \cdots \times \widehat{F_i} \times \cdots \times_{f_k/f_i} F_k$ is of constant sectional curvature. Now the result is obtained by iterating this process. □

Remark 3.2. Note from the previous proof that if $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat, then so is $B \times_{f_1} F_1 \times \cdots \times_{f_{k-1}} F_{k-1}$.

3A. ***Locally conformally flat multiply warped spaces with base of dimension at least 2.*** Although the fibers of any locally conformally flat multiply warped space are of constant curvature, this necessary condition does not suffice for local conformal flatness since it strongly depends on the warping functions. In this section we obtain a local description of such warping functions. As a consequence, we will show the existence of some limitations on the number of fibers of a locally conformally flat multiply warped space and also on their geometries. Assuming that the base (B, g_B) is of constant sectional curvature, the necessary and sufficient conditions for local conformal flatness are as follows.

Theorem 3.3. *Let $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space with s-dimensional base B of constant sectional curvature, where $s \geq 2$. Then M is locally conformally flat if and only if the warping functions satisfy*

$$H_{f_i} = \frac{\Delta f_i}{s} g_B, \tag{3}$$

$$\frac{\Delta f_i}{f_i} + \frac{\Delta f_j}{f_j} = s \frac{\langle \nabla f_i, \nabla f_j \rangle}{f_i f_j} - s K^B \text{ if } i \neq j, \tag{4}$$

$$K^{F_i} = \|\nabla f_i\|^2 - \frac{2}{s} f_i \Delta f_i - f_i^2 K^B \text{ if } \dim F_i \geq 2, \tag{5}$$

where $i, j = 1, \dots, k$ and K^B and K^{F_i} denote the sectional curvatures of the base (B, g_B) and the fibers (F_i, g_i).

Proof. Condition (3) is equivalent to the constancy of the sectional curvature of the base of a locally conformally flat multiply warped space. Since (B, g_B) is locally conformally flat and $(B, f_i^{-2} g_B)$ is a space of constant sectional curvature by Lemma 3.1, we see that (B, g_B) is of constant sectional curvature if and only if the conformal deformation $g_B \mapsto f_i^{-2} g_B$ preserves the (unique) eigenspaces of the Ricci tensor, and this occurs if and only if f is a solution of the Möbius equation; this proves (3) (see [Kühnel 1988; Osgood and Stowe 1992]).

Next, consider the Weyl curvature tensor given by

$$W(X,Y,Z,T) = R(X,Y,Z,T) + \frac{\tau}{(n-1)(n-2)}\big(\langle X,Z\rangle\langle Y,T\rangle - \langle Y,Z\rangle\langle X,T\rangle\big) - \frac{1}{n-2}\big(\rho(X,Z)\langle Y,T\rangle - \rho(Y,Z)\langle X,T\rangle + \langle X,Z\rangle\rho(Y,T) - \langle Y,Z\rangle\rho(X,T)\big).$$

Also note from (1) that the nonzero components of the Ricci tensor of a multiply warped space $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ are given by

$$\begin{aligned} \rho(X,Y) &= \rho^B(X,Y) - \sum_i d_i \frac{H_{f_i}(X,Y)}{f_i}, \\ \rho(U_a,V_a) &= \rho^{F_a}(U_a,V_a) \\ &\quad -\langle U_a,V_a\rangle\left(\frac{\Delta f_a}{f_a} + (d_a-1)\frac{\langle\nabla f_a,\nabla f_a\rangle}{f_a^2} + \sum_{i\neq a} d_i \frac{\langle\nabla f_a,\nabla f_i\rangle}{f_a f_i}\right) \end{aligned} \tag{6}$$

for all $X, Y \in \mathfrak{L}(B)$ and $U_a, V_a \in \mathfrak{L}(F_a)$, where $d_i = \dim F_i$ and ρ^B and ρ^{F_i} denote the Ricci tensor of the base (B, g_B) and the fibers (F_i, g_i). The scalar curvature of M satisfies

$$\begin{aligned} \tau &= \tau^B + \sum_i \frac{1}{f_i^2}\tau^{F_i} \\ &\quad - 2\sum_i d_i \frac{\Delta f_i}{f_i} - \sum_i d_i(d_i-1)\frac{\langle\nabla f_i,\nabla f_i\rangle}{f_i^2} - \sum_i\sum_{j\neq i} d_i d_j \frac{\langle\nabla f_i,\nabla f_j\rangle}{f_i f_j}, \end{aligned} \tag{7}$$

where τ^B and τ^{F_i} denote the scalar curvatures of the base and the fibers.

Now, in order to show the necessity of (4) and (5), note that if M is locally conformally flat, then it follows from Remark 3.2 that the warped product space $B \times_{f_a} F_a$ is also locally conformally flat, for all $a = 1, \dots, k$, and thus its Weyl tensor vanishes. A straightforward calculation from (6) and (7) using that $H_{f_a} = (\Delta f_a / s)\, g_B$ shows that

$$W(X,Y,X,Y) = \frac{d_a(d_a-1)}{(s+d_a-1)(s+d_a-2)}\left(K^B + \frac{2}{s}\frac{\Delta f_a}{f_a} + \frac{K^{F_a}}{f_a^2} - \frac{\langle\nabla f_a,\nabla f_a\rangle}{f_a^2}\right)$$

for all orthogonal unit vectors $X, Y \in \mathfrak{L}(B)$, whence (5). We proceed in an analogous way to show the necessity of (4), just considering the multiply warped space $B \times_{f_a} F_a \times_{f_b} F_b$, which is also locally conformally flat for all $a \neq b \in \{1, \dots, k\}$. After some calculations from (6) and (7) and using the already proved Equation (5), we have

$$W(X, Y, X, Y) = \frac{2d_a d_b}{(s+d_a+d_b-1)(s+d_a+d_b-2)}\left(K^B + \frac{1}{s}\frac{\Delta f_a}{f_a} + \frac{1}{s}\frac{\Delta f_b}{f_b} - \frac{\langle \nabla f_a, \nabla f_b\rangle}{f_a f_b}\right)$$

for all orthogonal unit vectors $X, Y \in \mathfrak{L}(B)$, which proves (4).

Next we show that conditions (3)–(5) are indeed sufficient for M to be locally conformally flat. Note first that the a-priori nonzero components of the Weyl tensor in a local orthonormal frame $\{X, Y, \dots, U_1, V_1, \dots, U_a, V_a, \dots\}$ with $X, Y, \dots$ in $\mathfrak{L}(B)$ and $U_a, V_a, \dots$ in $\mathfrak{L}(F_a)$ are those given by $W(X, Y, X, Y)$, $W(X, U_a, X, U_a)$, $W(U_a, U_b, U_a, U_b)$ and $W(U_a, V_a, U_a, V_a)$. Now, a long but straightforward calculation from (6) and (7), using the equalities $H_{f_i} = (\Delta f_i / s)\, g_B$, shows that

$$\begin{aligned} W(X, Y, X, Y) &= \sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i}\right) \\ &\quad + \sum_i \sum_{j \neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j}\right), \end{aligned}$$

for all $X, Y \in \mathfrak{L}(B)$. Also, for $X \in \mathfrak{L}(B)$ and $U_a \in \mathfrak{L}(F_a)$, one has

$$\begin{aligned} W(X, U_a, X, U_a) &= \sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{s f_i}\right) \\ &\quad + \sum_i \sum_{j \neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle \nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{s f_i} + \frac{\Delta f_j}{s f_j}\right) \\ &\quad + \sum_{i \neq a} \frac{d_i}{n-2}\left(\frac{\langle \nabla f_a, \nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{s f_a} - \frac{\Delta f_i}{s f_i} - K^B\right) \\ &\quad + \frac{d_a - 1}{n-2}\left(\frac{\langle \nabla f_a, \nabla f_a\rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{s f_a} - K^B\right). \end{aligned}$$

Next, given $U_a \in \mathfrak{L}(F_a)$ and $U_b \in \mathfrak{L}(F_b)$, where $a \neq b$, we get

$$
\begin{aligned}
W(U_a, U_b, U_a, U_b) = {} & \sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{sf_i}\right) \\
& + \sum_i \sum_{j\neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{sf_i} + \frac{\Delta f_j}{sf_j}\right) \\
& + \sum_{i\neq a} \frac{d_i}{n-2}\left(\frac{\langle\nabla f_a, \nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{sf_a} - \frac{\Delta f_i}{sf_i} - K^B\right) \\
& + \sum_{i\neq b} \frac{d_i}{n-2}\left(\frac{\langle\nabla f_b, \nabla f_i\rangle}{f_b f_i} - \frac{\Delta f_b}{sf_b} - \frac{\Delta f_i}{sf_i} - K^B\right) \\
& + \frac{d_a-1}{n-2}\left(\frac{\langle\nabla f_a, \nabla f_a\rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{sf_a} - K^B\right) \\
& + \frac{d_b-1}{n-2}\left(\frac{\langle\nabla f_b, \nabla f_b\rangle}{f_b^2} - \frac{K^{F_b}}{f_b^2} - \frac{2\Delta f_b}{sf_b} - K^B\right) \\
& + \left(K^B - \frac{\langle\nabla f_a, \nabla f_b\rangle}{f_a f_b} + \frac{\Delta f_a}{sf_a} + \frac{\Delta f_b}{sf_b}\right),
\end{aligned}
$$

$$
\begin{aligned}
W(U_a, V_a, U_a, V_a) = {} & \sum_i \frac{d_i(d_i-1)}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i, \nabla f_i\rangle}{f_i^2} + \frac{K^{F_i}}{f_i^2} + \frac{2\Delta f_i}{sf_i}\right) \\
& + \sum_i \sum_{j\neq i} \frac{d_i d_j}{(n-1)(n-2)}\left(K^B - \frac{\langle\nabla f_i, \nabla f_j\rangle}{f_i f_j} + \frac{\Delta f_i}{sf_i} + \frac{\Delta f_j}{sf_j}\right) \\
& + \sum_{i\neq a} \frac{2d_i}{n-2}\left(\frac{\langle\nabla f_a, \nabla f_i\rangle}{f_a f_i} - \frac{\Delta f_a}{sf_a} - \frac{\Delta f_i}{sf_i} - K^B\right) \\
& + \frac{2(d_a-1)}{n-2}\left(\frac{\langle\nabla f_a, \nabla f_a\rangle}{f_a^2} - \frac{K^{F_a}}{f_a^2} - \frac{2\Delta f_a}{sf_a} - K^B\right) \\
& + \left(K^B - \frac{\langle\nabla f_a, \nabla f_a\rangle}{f_a^2} + \frac{K^{F_a}}{f_a^2} + \frac{2\Delta f_a}{sf_a}\right),
\end{aligned}
$$

for all $U_a, V_a \in \mathfrak{L}(F_a)$.

It follows from these expressions that the compatibility conditions (4) and (5) suffice to show the local conformal flatness of the multiply warped space M. □

Although Equations (3)–(5) characterize the warping functions of a locally conformally flat multiply warped space with base of constant curvature, they are difficult to deal with. However, they become simpler if the base is assumed to be locally Euclidean:

Theorem 3.4. *Let $M = \mathfrak{U}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ be a multiply warped space, where $\mathfrak{U}^s \subset \mathbb{R}^s$ with $s \geq 2$. Then M is locally conformally flat if and only if the warping functions satisfy*

$$f_i(\vec{\boldsymbol{x}}) = a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i \tag{8}$$

for all $\vec{\boldsymbol{x}} \in \mathfrak{U}^s$, where $a_i > 0$, $c_i \in \mathbb{R}$ and $\vec{\boldsymbol{b}}_i \in \mathbb{R}^s$. Moreover the warping functions are compatible in the sense that

$$\langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{b}}_j \rangle = 2(a_i c_j + a_j c_i), \qquad i \neq j \tag{9}$$

and the sectional curvature of each fiber of dim $F_i \geq 2$ *is given by*

$$K^{F_i} = \|\vec{\boldsymbol{b}}_i\|^2 - 4a_i c_i, \quad i, j = 1, \dots, k. \tag{10}$$

Proof. It follows from [Osgood and Stowe 1992] that the solutions of the Möbius equation in Euclidean space are given by $f_i(\vec{\boldsymbol{x}}) = a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i$ for some $a_i, c_i \in \mathbb{R}$ and $\vec{\boldsymbol{b}}_i \in \mathbb{R}^s$. The result follows by observing the equivalence between (4) and (5) in Theorem 3.3 and (9) and (10) in Theorem 3.4. □

Remark 3.5. We explain how the previous theorem can be extended for not necessarily flat locally conformally flat bases to get a local description of locally conformally flat multiply warped spaces. Since (B, g_B) is locally conformally flat, there exist local coordinates such that $g_B = \boldsymbol{\Psi}^2 g_{\mathfrak{U}^s}$. In such coordinates, the multiply warped metric satisfies

$$g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k = \boldsymbol{\Psi}^2 \left(g_{\mathfrak{U}^s} \oplus \left(\frac{f_1}{\boldsymbol{\Psi}}\right)^2 g_1 \oplus \cdots \oplus \left(\frac{f_k}{\boldsymbol{\Psi}}\right)^2 g_k \right).$$

Therefore the multiply warped product $g_B \oplus f_1^2 g_1 \oplus \cdots \oplus f_k^2 g_k$ is locally conformally flat if and only if $g_{\mathfrak{U}^s} \oplus (f_1/\boldsymbol{\Psi})^2 g_1 \oplus \cdots \oplus (f_k/\boldsymbol{\Psi})^2 g_k$ is. Hence the warping functions are determined locally by Theorem 3.3 up to a conformal factor $\boldsymbol{\Psi}$, since the warping functions, in local coordinates where $g_B = \boldsymbol{\Psi}^2 g_{\mathfrak{U}^s}$, are given by $f_i(\boldsymbol{x}) = \big(a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i\big)\boldsymbol{\Psi}$ for all $i = 1, \dots, k$.

Remark 3.6. Locally conformally flat multiply warped spaces can now be easily constructed as follows. Since any warping function of a locally conformally flat multiply warped space $M = \mathfrak{U}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is completely determined by scalars $a_i, c_i \in \mathbb{R}$ and vectors $\vec{\boldsymbol{b}}_i = (b_{i1}, \dots, b_{is}) \in \mathbb{R}^s$, consider the vectors $\vec{\xi}_i = (b_{i1}, \dots, b_{is}, a_i, c_i)$ in $\mathbb{R}^{s+2}$. Next, define a Lorentzian inner product in $\mathbb{R}^{s+2}$ by

$$\left(\begin{array}{ccc|cc} 1 & & & & \\ & \ddots^{\,s} & & & \\ & & 1 & & \\ \hline & & & 0 & -2 \\ & & & -2 & 0 \end{array}\right)$$

and note that equations (9) and (10) of Theorem 3.4 are interpreted in terms of the orthogonality $\vec{\xi}_i \perp \vec{\xi}_j$ (for all $i \neq j$) and $K^{F_i} = \|\vec{\xi}_i\|^2$ (whenever $\dim F_i \geq 2$), respectively. Thus Remark 3.5 has the following consequences:

(i) *A locally conformally flat space* $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ *has, at most,* $s+2$ *different fibers, where* $s = \dim B$.

(i) *For the sectional curvatures of the fibers* (F_i, g_i) *of a locally conformally flat multiply warped space, we have, whenever* $\dim F_i \geq 2$:

(ii.1) *At most* $\dim B + 1$ *fibers have positive curvature.*

(ii.2) *At most one fiber has nonpositive curvature.*

(iii) *For any locally conformally flat manifold* (B^s, g_B), *there exists* $s+2$ *locally defined warping functions* $f_i : \mathfrak{U} \subset B \to \mathbb{R}^+$ *and* (F_i, g_i) *spaces of constant curvature such that* $M = \mathfrak{U} \times_{f_1} F_1 \times \cdots \times_{f_{s+2}} F_{s+2}$ *is locally conformally flat.*

3B. ***Multiply warped spaces with one-dimensional base.*** Recall that a warped product $I \times_f F$ with one-dimensional base is locally conformally flat if and only if the fiber is a space of constant sectional curvature. Local conformal flatness is independent of the warping function f [Lafontaine 1988], in opposition to the case of higher-dimensional base just considered. In what remains of this section we look at the local structure of a locally conformally flat multiply warped space with one-dimensional base.

The characterization in the next theorem is essentially independent of the last warping function, as in the case of metrics of Robertson–Walker type.

Theorem 3.7. *Let* $M = I \times_{f_1} F_1 \cdots \times_{f_k} F_k$ *be a multiply warped space with one-dimensional base* I. *Then* M *is locally conformally flat if and only if, up to a reparametrization of* I, *one of the following conditions holds*:

(i) $M = I \times_f F$ *is a warped product with fiber* F *of constant sectional curvature* (*if* $\dim F \geq 2$) *and any* (*positive*) *warping function* f.

(ii) $M = I \times_{f_1} F_1 \times_{f_2} F_2$ *is a multiply warped product with two fibers of constant sectional curvature* (*if* $\dim F_i \geq 2$) *and warping functions*

$$f_1 = (\xi \circ f)\frac{1}{f'}, \qquad f_2 = \frac{1}{f'}$$

where f *is a strictly increasing function and* ξ *is a warping function making* $I \times_\xi F_1$ *of constant sectional curvature and* $(\xi \circ f) > 0$.

(iii) $M = I \times_{f_1} F_1 \times_{f_2} F_2 \times_{f_3} F_3$ *is a multiply warped product with three fibers of constant sectional curvature* (*if* $\dim F_i \geq 2$) *and warping functions*

$$f_1 = (\xi_1 \circ f)\frac{1}{f'}, \qquad f_2 = (\xi_2 \circ f)\frac{1}{f'}, \qquad f_3 = \frac{1}{f'}$$

where f is a strictly increasing function and ξ_i are warping functions making $I \times_{\xi_1} F_1 \times_{\xi_2} F_2$ of constant sectional curvature such that $\xi_i \circ f > 0$ for $i = 1, 2$.

Proof. This is a local consideration. Proceeding as in Lemma 3.1 it follows that M is locally conformally flat if and only if

$$\frac{1}{f_k^2} g_I \oplus \frac{f_1^2}{f_k^2} g_1 \oplus \cdots \oplus \frac{f_{k-1}^2}{f_k^2} g_{k-1}$$

is of constant sectional curvature opposite to K^{F_k} (if $\dim F_k \geq 2$), and hence $k \leq 3$ (see Remark 3.8). Now, since f_k is strictly positive, it defines a reparametrization on I by $\tau = \int 1/f_k$ to obtain a multiply warped metric $d\tau^2 \oplus \xi_1(\tau)^2 g_1 \oplus \cdots \oplus \xi_{k-1}(\tau)^2 g_{k-1}$ of constant sectional curvature, where the warping functions ξ_i are given in Remark 3.8. Hence $f_i(t) = \xi_i\big(\int 1/f_k\big) f_k(t)$ for $i = 1, \ldots, k-1$, and there are no constraints on the last warping function f_k. □

Remark 3.8. Observe from (2) that, if a multiply warped space M with one-dimensional base is of constant sectional curvature κ, then the warping functions satisfy $f_i'' + \kappa f_i = 0$ and $f_i'^2 + \kappa f_i^2 = K^{F_i}$, which is just an adjustment of the sectional curvatures of the fibers since $f_i'^2 + \kappa f_i^2$ is necessarily constant. Moreover, the necessary compatibility conditions among the different warping functions are given by $f_i' f_j' + \kappa f_i f_j = 0$, $(i \neq j)$, from where it follows that no more than two fibers are admissible. As a consequence, one obtains the following (see also [Mignemi and Schmidt 1998]):

(i) If $K^M = 0$, then $M = I \times_{\xi_1} F_1$ or $M = I \times_{\xi_1} F_1 \times_{\xi_2} F_2$, with warping functions given by $\xi_i(t) = a_i t + b_i$ and $K^{F_i} = a_i^2$ whenever $\dim F_i \geq 2$ for $i = 1, 2$. If the two fibers are different we have $a_1 a_2 = 0$.

(ii) If $K^M = c^2$, then $M = I \times_{\xi_1} F_1$ or $M = I \times_{\xi_1} F_1 \times_{\xi_2} F_2$, with warping functions given by $\xi_i(t) = a_i \sin ct + b_i \cos ct$ and $K^{F_i} = c^2(a_i^2 + b_i^2)$, whenever $\dim F_i \geq 2$ for $i = 1, 2$. If the two fibers are different we have $a_1 a_2 + b_1 b_2 = 0$.

(iii) If $K^M = -c^2$, then $M = I \times_{\xi_1} F_1$ or $M = I \times_{\xi_1} F_1 \times_{\xi_2} F_2$, with warping functions given by $\xi_i(t) = a_i \sinh ct + b_i \cosh ct$ and $K^{F_i} = c^2(a_i^2 - b_i^2)$, provided that $\dim F_i \geq 2$ for $i = 1, 2$. If the two fibers are different we have $a_1 a_2 - b_1 b_2 = 0$.

Remark 3.9. A generalization of the notion of warped product structures $B \times_f F$ to warped bundles has been developed in [Bishop and O'Neill 1969], where it is shown that those results which are local on B remain valid in the warped bundle framework. Therefore, previous results in this section can be generalized to warped bundles.

4. Some global considerations

The existence of nontrivial globally defined solutions of (3) on complete manifolds has significant geometrical consequences [Kühnel 1988]. They leads to:

Theorem 4.1. *Let $M = B \times_f F$ be a locally conformally flat warped product space with complete base (B, g_B) of constant curvature. Then one of the following occurs*:

(i) *B is isometric to the Euclidean space $\mathbb{R}^s$ and the warping function is given by $f(\vec{x}) = a\|\vec{x}\|^2 + \langle \vec{b}, \vec{x}\rangle + c$. Moreover $4ac - \|\vec{b}\|^2 > 0$, $a > 0$ and the fiber F is either one-dimensional or $K^F = \|\vec{b}\|^2 - 4ac < 0$.*

(ii) *B is isometric to a Euclidean sphere $\mathbb{S}^s$ and the warping function is given by*

$$f = -\frac{s-1}{\tau}\psi + \kappa,$$

where τ denotes the scalar curvature of $\mathbb{S}^s$, ψ is the restriction to the sphere of a function Ψ on $\mathbb{R}^{s+1}$ defined by $\Psi(\vec{x}) = \langle \vec{a}, \vec{x}\rangle$ for any $\vec{a} \in \mathbb{R}^{s+1}$, and κ is a constant greater than $(s-1)\|\vec{a}\|/\tau$. Moreover F is either one-dimensional or of constant negative curvature

$$K^F = \frac{(s-1)^2}{\tau^2}\|\vec{a}\|^2 - \kappa^2.$$

(iii) *B is isometric to a warped product $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N$, where N is a complete flat manifold and the warping function is given by $f(t) = \frac{\alpha}{\beta} e^{\beta t+\gamma} + c$ for some constants $\beta, c > 0$. Moreover F is either one-dimensional or $K^F = c^2\beta^2$.*

(iv) *B is isometric to the hyperbolic space $\mathbb{H}^s$ and the warping function is given by*

$$f(\vec{x}) = \frac{a\|\vec{x}\|^2 + \langle \vec{b}, \vec{x}\rangle + c}{x_s},$$

for some $\vec{b} \in \mathbb{R}^s$, where $a > 0$ and either $4ac - (b_1^2 + b_2^2 + \cdots + b_s^2) > 0$ or $4ac - (b_1^2 + b_2^2 + \cdots + b_{s-1}^2) \geq 0$ and $b_s \geq 0$. Moreover the fiber F is either one-dimensional or $K^F = \|\vec{b}\|^2 - 4ac$.

Proof. Since any warping function f defines a global conformal transformation that makes $(B, f^{-2}g_B)$ have constant curvature, it follows from [Kühnel 1988] that B is either a complete and simply connected space form or a warped product $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N$, where N is complete Ricci flat, and thus flat since B is necessarily locally conformally flat. Now the result will follow after a case by case consideration of the possible warping functions and the curvature of the induced metric $f^{-2}g_B$.

Next, observe that a solution of the Möbius equation in $\mathbb{R}^s$, $f(\vec{x}) = a\|\vec{x}\|^2 + \langle \vec{b}, \vec{x}\rangle + c$, is everywhere positive if and only if $4ac - \|\vec{b}\|^2 > 0$, $a > 0$, and (i) is obtained since the conformal metric $f^{-2}g_{\mathbb{R}^s}$ has constant curvature $4ac - \|\vec{b}\|^2 > 0$.

If $B \equiv \mathbb{S}^s$, it follows from [Brozos-Vázquez et al. 2005; Xu 1993] that any warping function is given by

$$f = -\frac{s-1}{\tau}\psi + \kappa,$$

where τ is as in the theorem's statement, ψ is a first eigenfunction of the Laplacian and κ is a constant making f positive. Hence ψ is the restriction to the sphere of a function Ψ on $\mathbb{R}^{s+1}$ defined by

$$\Psi(\vec{x}) = \langle \vec{a}, \vec{x}\rangle$$

for $0 \neq \vec{a} \in \mathbb{R}^{s+1}$, [Berger et al. 1971] and the sectional curvature of $(\mathbb{S}^s, f^{-2}g_{\mathbb{S}})$ is the constant $\kappa^2 - \big((s-1)^2/\tau^2\big)\|\vec{a}\|^2 > 0$, proving (ii).

In case (iii) the warping function f gives rise to a warped product decomposition of B as $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N$, where the warping function is of the form $f(t) = (\alpha/\beta)\, e^{\beta t+\gamma} + c$ for some positive constant c [Kühnel 1988]. This defines a global conformal transformation such that $(B, f^{-2}g_B)$ has constant curvature $-c^2\beta^2$; hence the result.

Finally, assume B to be hyperbolic space. We work in the half-space model, with domain $\{x_s < 0\}$ and metric obtained by a conformal deformation of the Euclidean metric: $(\mathbb{H}^s, x_s^{-2}g_{\mathbb{R}^s})$. The general form of the warping functions then arises from Remark 3.5. Next note that $a\|\vec{x}\|^2 + \langle \vec{b}, \vec{x}\rangle + c$ is positive in hyperbolic space if and only if $a > 0$ and the minimum of the paraboloid is positive $(4ac - \|\vec{b}\|^2 > 0)$ or occurs on the lower half-space (so $-b_s/(2a) \leq 0$) and the intersection of the paraboloid and the hyperplane $x_s = 0$ is positive, which gives

$$4ac - (b_1^2 + b_2^2 + \cdots + b_{s-1}^2) \geq 0.$$

Further note that the induced metric $f^{-2}g_B$ is of constant curvature $4ac - \|\vec{b}\|^2 > 0$ but it has no preferred sign in opposition to case (i). □

Theorem 4.2. *Let $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$, $k \geq 2$, be a locally conformally flat multiply warped product space with complete and simply connected base (B, g_B) of constant curvature. Then B is isometric to the hyperbolic space $\mathbb{H}^s$ and for each $k \leq s+2$ there exists locally conformally flat multiply warped spaces $M = \mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$.*

Proof. First of all, note that since B is assumed to be simply connected, the possibly warping functions reduce to cases (i), (ii) and (iv) in Theorem 4.1. Next, in order to show that a locally conformally flat multiply warped space whose base is the

Euclidean space or the sphere reduces to a warped product, an analysis of the curvature of the induced metric $(B, f^{-2}g_B)$ is needed. Assuming that the space $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat, so is $M_{ij} = B \times_{f_i} F_i \times_{f_j} F_j$, whose metric tensor can be expressed as

$$g_{M_{ij}} \equiv f_j^2 \left(\frac{1}{f_j^2} g_B \oplus \frac{1}{f_j^2} f_i^2 g_i \oplus g_j \right).$$

This shows that $M_{\hat{j}i} = B \times F_i$, equipped with the metric $(1/f_j^2)\, g_B \oplus (f_i^2/f_j^2)\, g_i$, has constant sectional curvature $K^{M_{\hat{j}i}}$. Since $M_{\hat{j}i}$ can be viewed as a warped product, it follows from (2) that

$$K^{M_{\hat{j}i}}(X \wedge U) = -\frac{f_j^3}{f_i} \widehat{H}_{f_i/f_j}(X, X)$$

for all unit vectors $X \in \mathfrak{L}(B)$, $U \in \mathfrak{L}(F_i)$, where $\widehat{H}_{f_i/f_j}$ denotes the Hessian of f_i/f_j with respect to the conformal metric $f_j^{-2} g_B$. Now, since

$$\widehat{H}_{f_i/f_j} = \frac{1}{f_j}\left(H_{f_i} - \frac{f_i}{f_j} H_{f_j} - \frac{1}{f_j} g_B(\nabla f_j, \nabla f_i) g_B + \frac{f_i}{f_j^2} g_B(\nabla f_j, \nabla f_j) g_B \right)$$

(see [García-Río and Kupeli 1999]), one gets

$$(11) \quad -K^{M_{\hat{j}i}} \frac{f_i}{f_j} g_B = f_j H_{f_i} - f_i H_{f_j} - g_B(\nabla f_j, \nabla f_i) g_B + \frac{f_i}{f_j} g_B(\nabla f_j, \nabla f_j) g_B.$$

Proceeding similarly, and expressing the metric tensor of $M_{ij} = B \times_{f_i} F_i \times_{f_j} F_j$ as

$$g_{M_{ij}} \equiv f_i^2 \left(\frac{1}{f_i^2} g_B \oplus \frac{1}{f_i^2} f_j^2 g_j \oplus g_i \right),$$

one also gets

$$(12) \quad -K^{M_{\hat{i}j}} \frac{f_j}{f_i} g_B = f_i H_{f_j} - f_j H_{f_i} - g_B(\nabla f_i, \nabla f_j) g_B + \frac{f_j}{f_i} g_B(\nabla f_i, \nabla f_i) g_B.$$

Now it follows from (11) and (12) that

$$(13) \qquad -K^{M_{\hat{j}i}} f_i^2 - K^{M_{\hat{i}j}} f_j^2 = \| f_j \nabla f_i - f_i \nabla f_j \|^2.$$

As an immediate application of this equality we have:

Proposition 4.3. *If $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a locally conformally flat multiply warped space, then the (constant) sectional curvature of $(B, f_i^{-2} g_B)$ cannot be nonnegative for two different warping functions.*

Proof. If $\| f_j \nabla f_i - f_i \nabla f_j \|^2 = 0$, then $\nabla \ln(f_i/f_j) = 0$, implying f_i is a multiple of f_j in opposition to Remark 2.1. This shows there exist no nontrivial locally conformally flat multiply warped metrics having Euclidean space or the sphere for base. □

Finally, in order to show the existence of complete locally conformally flat multiply warped products with base $\mathbb{H}^s$ and the maximum number of fibers, just consider the set of functions

$$\begin{aligned}
\bar{f}_1(\vec{\boldsymbol{x}}) &= \tfrac{1}{4}(s+4)\|\vec{\boldsymbol{x}}\|^2 + x_1 + \cdots + x_{s-1} + (s+2)x_s + s + 1,\\
\bar{f}_2(\vec{\boldsymbol{x}}) &= \tfrac{1}{4}(s+4)\|\vec{\boldsymbol{x}}\|^2 + x_1 + \cdots + x_{s-1} + s x_s + s - 1,\\
\bar{f}_3(\vec{\boldsymbol{x}}) &= \|\vec{\boldsymbol{x}}\|^2 + 3x_s + 2,\\
\bar{f}_4(\vec{\boldsymbol{x}}) &= \|\vec{\boldsymbol{x}}\|^2 + x_{s-1} + 2x_s + 2,\\
\bar{f}_5(\vec{\boldsymbol{x}}) &= \|\vec{\boldsymbol{x}}\|^2 + x_{s-2} + 2x_s + 2,\\
&\vdots\\
\bar{f}_{s+2}(\vec{\boldsymbol{x}}) &= \|\vec{\boldsymbol{x}}\|^2 + x_1 + 2x_s + 2.
\end{aligned}$$

These functions are positive in hyperbolic space and satisfy the compatibility conditions in Theorem 3.4. Hence, proceeding as in Remark 3.5, one sees that $f_i(\vec{\boldsymbol{x}}) = \bar{f}_i(\vec{\boldsymbol{x}})/x_s$ are positive warping functions on $\mathbb{H}^s$ that define a locally conformally flat multiply warped space for either one- or higher-dimensional fibers of suitable constant curvature as in Remark 3.6. This completes the proof of the theorem. □

Remark 4.4. If $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is locally conformally flat with compact base B, then $k = 1$. Indeed, let f_i, f_j be two distinct warping functions. Proceeding as in Lemma 3.1, we conclude that $(B, f_i^{-2} g_B)$ and $(B, f_i^{-2} g_B)$ are of constant sectional curvature. Since f_i/f_j is not constant it follows that $(B, f_i^{-2} g_B)$ and $(B, f_i^{-2} g_B)$ are conformal metrics of constant curvature, and thus Euclidean spheres [Kühnel 1988], from which the result follows.

Examples of complete locally conformally flat manifolds of nonpositive curvature. Proceeding as in [Bishop and O'Neill 1969], note that a multiply warped manifold $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is complete if and only if the base and all the fibers are so. In such a case, the sectional curvature is nonpositive if and only if the following conditions are satisfied:

(a) the sectional curvatures of the base and the fibers are nonpositive: $K^B \leq 0$ and $K^{F_i} \leq 0$.

(b) The warping functions are convex, i.e., H_{f_i} is positive semidefinite.

(c) $\langle \nabla f_i, \nabla f_j \rangle \geq 0$ for all $i \neq j$.

Condition (a) can be omitted whenever the base and the corresponding fiber are one-dimensional.

A complete locally conformally flat multiply warped space with simply connected base of constant curvature is of nonpositive sectional curvature if and only if one of the following conditions holds:

(i) $B \equiv \mathbb{R}^s$, and then $\mathbb{R}^s \times_f F$ is of nonpositive sectional curvature for any warping function f as in Theorem 4.1.

(ii) $B \equiv \mathbb{H}^s$, and then $\mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is of nonpositive sectional curvature if and only if the warping functions

$$f_i(\vec{x}) = \frac{a_i \|\vec{x}\|^2 + \langle \vec{b}_i, \vec{x} \rangle + c_i}{x_s}$$

satisfy

$$f_i \geq 2b_{is} \quad \text{if } \dim F_i \geq 2 \qquad \text{and} \qquad 1 \geq \frac{b_{is}}{f_i} + \frac{b_{js}}{f_j} \quad \text{for all } i \neq j.$$

The simplest examples illustrating this situation are as follows.

(a) Let M be the product manifold $M = \mathbb{H}^2 \times F_1^d \times F_2$ equipped with the multiply warped metric tensor defined by the warping functions

$$f_1(\vec{x}) = \kappa \frac{\frac{1}{2}\|\vec{x}\|^2 + x_2 + 1}{x_2}, \qquad f_2(\vec{x}) = \frac{\frac{1}{4}\|\vec{x}\|^2 + x_2 + \frac{1}{2}}{x_2},$$

where F_2 is one-dimensional and F_1 is either one-dimensional or of negative sectional curvature $K^{F_1} = -\kappa^2$.

(b) Let M be the product manifold $M = \mathbb{H}^2 \times {F_1}^d \times F_2$ equipped with the multiply warped metric tensor defined by the warping functions

$$f_1(\vec{x}) = \frac{\frac{1}{4}\|\vec{x}\|^2 + x_1 + 1}{x_2}, \qquad f_2(\vec{x}) = \frac{\frac{1}{2}\|\vec{x}\|^2 + 2x_1 + x_2 + 2}{x_2}.$$

where F_2 is one-dimensional and F_1 is either one-dimensional or flat.

Further, if $M = B \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a locally conformally flat multiply warped space with base of constant sectional curvature, then it follows from (3)–(5) that M has at most $(k+1)$-different Ricci curvatures given by

$$\begin{aligned} \lambda_B &= (s-1)K^B - \frac{1}{s}\sum_i d_i \frac{\Delta f_i}{f_i}, \\ \lambda_{F_a} &= (s-1)K^B - \frac{1}{s}\sum_i d_i \frac{\Delta f_i}{f_i} - (n-2)\left(K^B + \frac{1}{s}\frac{\Delta f_a}{f_a}\right). \end{aligned} \tag{14}$$

A straightforward calculation shows that examples (a) have exactly three different Ricci curvatures, but only two occur in case (b).

Remark 4.5. Observe that the base and the fibers of a multiply warped product play completely different roles. For instance, if M is a warped product with compact base and nonpositive sectional curvature, then it follows from (2) that the warping function satisfies $H_f \geq 0$, and thus f is constant, which shows that M must be a direct product. In opposition, one can easily construct examples of locally conformally flat multiply warped spaces of nonpositive sectional curvature with compact fibers. In addition to examples (b) above, those metrics in Theorem 4.1(iii) can also be viewed as multiply warped metrics with one-dimensional base. A straightforward calculation shows that $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N \times_{\frac{\alpha}{\beta} e^{\beta t+\gamma}+c} F$ is of nonpositive sectional curvature if and only if F is one-dimensional. Further note that both N and F can be chosen to be compact. Further, $\mathbb{R} \times_{\alpha e^{\beta t+\gamma}} N \times_{(\alpha/\beta)\, e^{\beta t+\gamma}+c} F$ has three distinct Ricci curvatures and therefore is not isometric to example (b), where only two distinct Ricci curvatures occur.

Remark 4.6. As an immediate application of (14), a locally conformally flat multiply warped space $M = \mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ $(s \geq 2)$ has nonpositive Ricci curvature if and only if the warping functions

$$f_i(\vec{\boldsymbol{x}}) = \frac{a_i \|\vec{\boldsymbol{x}}\|^2 + \langle \vec{\boldsymbol{b}}_i, \vec{\boldsymbol{x}} \rangle + c_i}{x_s}$$

satisfy

$$\sum_i d_i \frac{b_{is}}{f_i} \leq n-1 \quad \text{for all } i = 1, \ldots, k,$$

$$(n-2)\frac{b_{is}}{f_i} + \sum_j d_j \frac{b_{js}}{f_j} \leq n-1 \quad \text{for all } i \neq j \in \{1, \ldots, k\}.$$

The simplest examples of complete locally conformally flat manifolds with nonpositive Ricci curvature consist of

$$\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2 \times_{f_3} \mathbb{S}^2 \times_{f_4} \mathbb{H}^2$$

with warping functions

$$f_1(\vec{\boldsymbol{x}}) = \frac{\frac{3}{2}\|\vec{\boldsymbol{x}}\|^2 + x_1 + 4x_2 + 3}{x_2}, f_2(\vec{\boldsymbol{x}}) = \frac{\|\vec{\boldsymbol{x}}\|^2 + 3x_2 + 2}{x_2},$$

$$f_3(\vec{\boldsymbol{x}}) = \frac{\frac{1}{2}\|\vec{\boldsymbol{x}}\|^2 + x_1 + 2x_2 + 2}{x_2}, f_4(\vec{\boldsymbol{x}}) = \frac{\|\vec{\boldsymbol{x}}\|^2 + x_1 + 2x_2 + 1}{x_2}.$$

The same conclusions hold for the multiply warped spaces $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2 \times_{f_3} \mathbb{S}^2$, $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2 \times_{f_2} \mathbb{S}^2$ and $\mathbb{H}^2 \times_{f_1} \mathbb{S}^2$. Also note from (13) that if $\mathbb{H}^s \times_{f_1} F_1 \times \cdots \times_{f_k} F_k$ is a locally conformally flat space of nonpositive sectional curvature, there is at most one fiber F_a with $\dim F_a \geq 2$, which must necessarily be of nonpositive sectional curvature. This shows that none of the examples above has nonpositive sectional curvature.

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Received November 22, 2004. Revised February 15, 2005.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

ANALYTIC STABILITY OF THE CR CROSS-CAP

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For $m < n$, any real analytic m-submanifold of complex n-space with a nondegenerate CR singularity is shown to be locally equivalent, under a holomorphic coordinate change, to a fixed real algebraic variety defined by linear and quadratic polynomials. The situation is analogous to Whitney's stability theorem for cross-cap singularities of smooth maps. The complex analyticity of the normalizing transformation is proved using a rapid convergence argument.

1. Introduction

For $m \leq n$, if a real m-manifold M is embedded in $\mathbb{C}^n$, then for each point $\boldsymbol{x}$ on M there are two possibilities: the tangent m-plane at $\boldsymbol{x}$ may contain a complex line, so M is said to be *CR singular* at $\boldsymbol{x}$, or it may not, so M is said to be *totally real* at $\boldsymbol{x}$. This article will consider the local extrinsic geometry of a real analytically embedded M near a CR singular point, in the case when the CR singularity satisfies some natural nondegeneracy properties and $\frac{2}{3}(n+1) \leq m < n$ (so $(m, n) = (4, 5)$ is the case of lowest dimension). The main result is an algebraizability property: there exists a holomorphic coordinate change in a neighborhood of $\boldsymbol{x}$ so that M is real algebraic in the new coordinate system. In fact, M will be biholomorphically equivalent to a fixed normal form variety, so that, unlike the well-known $m = n$ case, nondegenerate CR singularities have no continuous invariants under biholomorphisms.

The analysis of normal forms near CR singular points is part of the program of studying the local equivalence problem for real m-submanifolds of $\mathbb{C}^n$, as described in [Baouendi et al. 2000]. Normal forms for CR singular real n-manifolds in $\mathbb{C}^n$, where $m = n \geq 2$, have been the subject of much study; see, for example, [Bishop 1965; Moser 1985; Moser and Webster 1983; Webster 1985]. Real surfaces in $\mathbb{C}^n$ ($m = 2$, $n \geq 3$) have been considered in [Harris 1981; 1983; Coffman 2004], and real threefolds in $\mathbb{C}^4$ in [Coffman 2006]. A formal normal form for a CR singular real 4-manifold in $\mathbb{C}^5$ was found in [Beloshapka 1997] and [Coffman 1997] — it was shown that there exists a transformation (not unique) defined by formal power

MSC2000: 32V40, 32S05.
Keywords: normal form, CR singularity, real submanifold.

series, taking M to the normal form. The new result here is the existence of a normalizing transformation defined by series that are convergent in a neighborhood of the singularity.

2. Topological considerations

We briefly recall some topological properties of CR singularities. We could consider real submanifolds of any complex manifold, but since the main result on the normalization is about the local geometry, we can begin by assuming M is a smoothly immersed real m-manifold in $\mathbb{C}^n$.

The most basic invariant of a CR singularity at a point $\boldsymbol{x} \in M$ is the number $\boldsymbol{j}(\boldsymbol{x}) = \dim_{\mathbb{C}} T_{\boldsymbol{x}} \cap J_{\boldsymbol{x}} T_{\boldsymbol{x}}$, where $T_{\boldsymbol{x}}$ is the real tangent space of M at $\boldsymbol{x}$ and $J_{\boldsymbol{x}}$ is the complex structure operator corresponding to scalar multiplication by i on the tangent space of the ambient complex manifold. The number $\boldsymbol{j}(\boldsymbol{x})$ is the dimension of the largest complex subspace tangent to M at $\boldsymbol{x}$, so $0 \leq \boldsymbol{j}(\boldsymbol{x}) \leq m/2$.

One way to keep track of $\boldsymbol{j}(\boldsymbol{x})$ is the following construction. For $m \leq n$, let G be the grassmannian variety of real m-subspaces in $\mathbb{C}^n \cong \mathbb{R}^{2n}$; see [Garrity 2000; Coffman 1997]. The real m-subspaces T such that $\dim_{\mathbb{C}} T \cap iT \geq j$ form a subvariety $\mathfrak{D}_j$ of real codimension $2j(n-m+j)$ in G. The occurrence of complex tangents of an immersion corresponds to the intersection of $\mathfrak{D}_j$ with the image of the Gauss map $M \to G : \boldsymbol{x} \mapsto T_{\boldsymbol{x}}$, and the immersion could be called "generic" if the Gauss map meets each stratum $\mathfrak{D}_j \setminus \mathfrak{D}_{j+1}$ transversely. So, generic immersions of M in $\mathbb{C}^n$ are totally real outside a subset of M of codimension $2(n-m+1)$, and if $m < \frac{2}{3}(n+1)$, a generic immersion of M is totally real everywhere. This resembles the bounds in Whitney's embedding and immersion theorems [1944a; 1944b]. In the range $\frac{2}{3}(n+1) \leq m \leq n$, CR singularities are topologically stable — small smooth perturbations of a generic immersion with a CR singular point will still have a CR singular point. For compact real submanifolds of complex manifolds, there are topological obstructions to the property of being totally real at every point, and the CR singularities can be enumerated by characteristic class formulas. See [Domrin 1995a; 1995b; Coffman 1997] and references therein on this topic.

The case addressed by this paper is $\frac{2}{3}(n+1) \leq m < n$, and $\boldsymbol{j}(\boldsymbol{x}) = 1$; that is, only points $\boldsymbol{x}$ where exactly one complex line is tangent at $\boldsymbol{x}$ will be considered, and only in dimension cases where the CR singularity is stable under smooth perturbations of the immersion. As mentioned in the Introduction, the $m = n$ case has a qualitatively different local geometry than the $m < n$ case and is not considered here. The cases $(m, n) = (2, 3)$ or $(3, 4)$, considered in [Coffman 2004; 2006], fall outside the topological stability range. The case $(m, n) = (4, 5)$, considered in [Beloshapka 1997; Domrin 1995a; Coffman 1997; 2002], has the lowest dimensions in the range, and the generic singularity is isolated (codimension 4 in M).

3. The quadratic normal form

Let the ambient complex space be $\mathbb{C}^n$, with coordinates $(z_1, \dots, z_n)$. The real and imaginary parts of the coordinate functions are labeled $z_j = x_j + iy_j$ for $j = 1, \dots, n$. Let M be a real analytic m-dimensional submanifold embedded in $\mathbb{C}^n$, with $m < n$, and let $\boldsymbol{x}$ be a point on M at which M is tangent to a complex line but not to any complex 2-plane — in terms of the previous section, $\boldsymbol{j}(\boldsymbol{x}) = 1$, which we regard as a nondegeneracy assumption, since for M in general position, the points where $\boldsymbol{j}(\boldsymbol{x}) > 1$ form a subset of higher codimension.

By a translation that moves $\boldsymbol{x}$ to the origin $\vec{0}$, and then a complex linear transformation of $\mathbb{C}^n$, the tangent space $T = T_{\vec{0}}$ of M can be assumed to be the one spanned by $(x_1, y_1, x_2, \dots, x_{m-1})$, and thus to contain the z_1-axis. Then there is some neighborhood Δ of the origin in $\mathbb{C}^n$ so that the defining equations of M in Δ are in the form of a graph over a neighborhood of the origin in T:

$$\begin{aligned} y_s &= H_s(z_1, \bar{z}_1, x_2, \dots, x_{m-1}) \\ z_u &= h_u(z_1, \bar{z}_1, x_2, \dots, x_{m-1}), \end{aligned} \tag{1}$$

where H_s, for $s = 2, \dots, m-1$, is a real-valued real analytic function, and h_u, for $u = m, \dots, n$, is a complex-valued real analytic function, with H_s and h_u defined in a neighborhood of the origin in T, and vanishing to second order at $(x_1, y_1, x_2, \dots, x_{m-1}) = (0, \dots, 0)$. The expression "$x_2, \dots, x_{m-1}$" is abbreviated as just x. So, the defining functions are of the following form:

$$\begin{aligned} H_s(z_1, \bar{z}_1, x) = \alpha_s z_1^2 + \beta_s z_1 \bar{z}_1 + \gamma_s \bar{z}_1^2 + \sum \delta_s^{s_1} z_1 x_{s_1} + \sum \epsilon_s^{s_1} \bar{z}_1 x_{s_1} + \sum \theta_s^{s_1 s_2} x_{s_1} x_{s_2} \\ + E_s(z_1, \bar{z}_1, x), \\ h_u(z_1, \bar{z}_1, x) = \alpha_u z_1^2 + \beta_u z_1 \bar{z}_1 + \gamma_u \bar{z}_1^2 + \sum \delta_u^{s_1} z_1 x_{s_1} + \sum \epsilon_u^{s_1} \bar{z}_1 x_{s_1} + \sum \theta_u^{s_1 s_2} x_{s_1} x_{s_2} \\ + e_u(z_1, \bar{z}_1, x), \end{aligned}$$

with E_s, e_u having terms of degree three or higher. Each of these functions can be expressed as the restriction to $\{(z_1, \zeta, x) \in \mathbb{C}^m : \zeta = \bar{z}_1, x = \bar{x}\}$ of an m-variable series with complex coefficients:

$$\begin{aligned} H_s(z_1, \zeta, x) = \alpha_s z_1^2 + \beta_s z_1 \zeta + \gamma_s \zeta^2 + \sum \delta_s^{s_1} z_1 x_{s_1} + \sum \epsilon_s^{s_1} \zeta x_{s_1} \\ + \sum \theta_s^{s_1 s_2} x_{s_1} x_{s_2} + \sum_{a+b+I \geq 3} E_s^{abI} z_1^a \zeta^b x^I \\ h_u(z_1, \zeta, x) = \alpha_u z_1^2 + \beta_u z_1 \zeta + \gamma_u \zeta^2 + \sum \delta_u^{s_1} z_1 x_{s_1} + \sum \epsilon_u^{s_1} \zeta x_{s_1} \\ + \sum \theta_u^{s_1 s_2} x_{s_1} x_{s_2} + \sum_{a+b+I \geq 3} e_u^{abI} z_1^a \zeta^b x^I, \end{aligned}$$

where x^I abbreviates $x_2^{i_2} x_3^{i_3} \cdots x_{m-1}^{i_{m-1}}$ and $a+b+I$ abbreviates $a+b+i_2+\ldots+i_{m-1}$. Each of the series in (z_1, ζ, x) converges on some set of the form

$$\{(z_1, \zeta, x) : |z_1| < r,\ |\zeta| < r,\ |x_s| < r\},$$

with $r > 0$, to a complex analytic function, with $\gamma_s = \bar{\alpha}_s$, $\epsilon_s^{s_1} = \overline{\delta_2^{s_1}}$, etc., so that $H_s(z_1, \bar{z}_1, x)$ and $E_s(z_1, \bar{z}_1, x)$ are real-valued.

Definition 3.1. A (formal) monomial $Cz_1^a \zeta^b x^I$ (with complex coefficient C) has *degree* $a + b + I$. A (convergent or formal) power series in m variables, say $e(z_1, \zeta, x) = \sum e^{abI} z_1^a \zeta^b x^I$, is said to have *degree* d if $e^{abI} = 0$ for all (a, b, I) such that $a+b+I < d$. Sometimes a series of degree d will be abbreviated $O(d)$. An ordered k-tuple of series $(e_1, \ldots, e_k)$ has degree d if all its components have degree d.

Definition 3.2. Similarly for n variables, a monomial $Cz_1^{a_1} \cdots z_n^{a_n}$ has degree $a_1 + \cdots + a_n$, but we will also work with the *weight* $a_1 + \cdots + a_{m-1} + 2a_m + \cdots + 2a_n$. A series $p(\vec{z}) = \sum p^{a_1 \ldots a_n} z_1^{a_1} \ldots z_n^{a_n}$ has *weight* W if $p^{a_1 \ldots a_n} = 0$ when $a_1 + \cdots + a_{m-1} + 2a_m + \cdots + 2a_n < W$.

We consider two coordinate systems for a neighborhood of the origin in $\mathbb{C}^n$: the previously mentioned $\vec{z} = (z_1, \ldots, z_n)$, and a new system $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n)$, with $\tilde{z}_j = \tilde{x}_j + \tilde{y}_j$. The two systems are related by the change of coordinates

$$\tilde{z} = \vec{z} + \vec{p}(\vec{z}), \tag{2}$$

where $\vec{p}(\vec{z}) = (p_1(\vec{z}), \ldots, p_n(\vec{z}))$ and each component p_j is a holomorphic function of $z_1, \ldots, z_n$ whose series expansion has weight 2, and for $j \geq m$, also degree 2. Such a transformation of $\mathbb{C}^n$ has invertible linear part, so it is invertible on some neighborhood of $\vec{0}$. In the calculations of this section, we will neglect considering the size of that neighborhood, and consider only points close enough to the origin, but the size of the domain of $\vec{p}$ will be important information in later sections.

The goal of this section is to establish some nondegeneracy conditions on the defining equations (1), by using complex linear transformations and nonlinear transformations of the form (2) to put the quadratic terms of (1) into a normal form. Similar calculations have already been done in the case $m = n$ and the cases $(m, n) = (2, 3)$, $(3, 4)$, and $(4, 5)$, in [Bishop 1965; Coffman 2004; 2006; Beloshapka 1997], respectively, so we will skip some of the computational details.

As the first special case of a transformation of the form (2) to be used, let $p_1 = 0$ and let $p_2, \ldots, p_n$ be homogeneous quadratic polynomials in $z_1, \ldots, z_{m-1}$. Using such a transformation, the quadratic terms in h_u that are products of z_1 and x only, without a $\bar{z}_1$ factor, can be eliminated in the new coordinate system, or their coefficients $(\alpha_u,\ \delta_u^{s_1},\ \theta_u^{s_1 s_2})$ can be altered to attain any complex values, by a suitable choice of $\vec{p}$. This transformation may also change some higher-degree terms but does not alter the coefficients β_u, γ_u, $\epsilon_u^{s_1}$. Similarly, the quadratic terms without $\bar{z}_1$ in each H_s can also be eliminated by a transformation $\tilde{z} = \vec{z} + \vec{p}$, which simultaneously eliminates their conjugates (using $\gamma_s = \bar{\alpha}_s$), leaving only the mixed term $\beta_s z_1 \bar{z}_1$.

The result of this preliminary normalization is that for any CR singular submanifold M of the form (1), there exists a quadratic coordinate transformation of the form (2) with $p_1 = 0$, so that M has the following general normal form. In a local coordinate system $\vec{z}$ in some neighborhood of the CR singularity, the defining equations of M are of the form (1), with

$$\begin{aligned} y_s &= H_s(z_1, \bar{z}_1, x) = \beta_s z_1 \bar{z}_1 + O(3), \\ z_u &= h_u(z_1, \bar{z}_1, x) = \beta_u z_1 \bar{z}_1 + \gamma_u \bar{z}_1^2 + \sum \epsilon_u^{s_1} \bar{z}_1 x_{s_1} + O(3). \end{aligned}$$

At this point we consider which invertible complex linear transformations of $\mathbb{C}^n$ fix the tangent plane T with coordinates (z_1, x). The matrix representation of such a transformation must be of the form $\tilde{z} = \boldsymbol{A} z$, where

$$\boldsymbol{A} = \begin{pmatrix} a_1 & a_2 \dots a_{m-1} & a_m \dots a_n \\ 0 & \boldsymbol{R} & * \\ 0 & 0 & \boldsymbol{C} \end{pmatrix}.$$

The entries $a_1, \dots, a_n$ are complex, with $a_1 \neq 0$, the $(m-2) \times (m-2)$ block $\boldsymbol{R}$ has real entries and a nonzero determinant, and the $(n-m+1) \times (n-m+1)$ block $\boldsymbol{C}$ has complex entries and a nonzero determinant.

The *first nondegeneracy condition* is that the $(n-m+1) \times 2$ block of coefficients β_u, γ_u in the functions h_u satisfies

$$\text{rank} \begin{pmatrix} \beta_m & \gamma_m \\ \vdots & \vdots \\ \beta_n & \gamma_n \end{pmatrix} = 2. \tag{3}$$

In particular, this requires $m < n$. In this nondegenerate case, there is a linear transformation of $\mathbb{C}^n$ which uses the block $\boldsymbol{C}$ in the complex matrix above to put these coefficients into a row echelon form:

$$\begin{aligned} z_t &= h_t(z_1, \bar{z}_1, x) = \sum \epsilon_t^{s_1} \bar{z}_1 x_{s_1} + O(3), \\ z_{n-1} &= h_{n-1}(z_1, \bar{z}_1, x) = \bar{z}_1^2 + \sum \epsilon_{n-1}^{s_1} \bar{z}_1 x_{s_1} + O(3), \\ z_n &= h_n(z_1, \bar{z}_1, x) = z_1 \bar{z}_1 + \sum \epsilon_n^{s_1} \bar{z}_1 x_{s_1} + O(3), \end{aligned}$$

for $t = m \dots, n-2$, or there are no h_t expressions if $m = n-1$.

As a consequence of the first nondegeneracy condition on the functions h_u, the functions H_s can also be simplified. There is a linear transformation with components $\tilde{z}_s = z_s - i\beta_s z_n$ (where for $s = 2, \dots, m-1$, the complex coefficients $-i\beta_s$ are entries from the $*$ block of the matrix $\boldsymbol{A}$) that eliminates the $\beta_s z_1 \bar{z}_1$ terms from each function H_s. This may introduce more terms of the form $z_1 x_{s_1}$ or $\bar{z}_1 x_{s_1}$ in the H_s functions, which can be eliminated by $\tilde{z}_s = z_s + p_s$ quadratic transformations as done previously, without reintroducing any $z_1\bar{z}_1$ terms.

A linear transformation of the form $\tilde{z}_1 = z_1 + \sum a_s z_s$, using the block $a_2, \dots, a_{m-1}$ from the matrix $\boldsymbol{A}$, can eliminate the terms of the form $\bar{z}_1 x_{s_1}$ from either the h_{n-1} quantity or the h_n quantity, but generally not both at once — we make the choice to eliminate the terms $\sum \epsilon_{n-1}^{s_1} \bar{z}_1 x_{s_1}$ from h_{n-1}. This may introduce more terms of the form $z_1 x_{s_1}$ or $x_{s_1} x_{s_2}$ in the other h_u functions, which can be eliminated by $\tilde{z}_u = z_u + p_u$ quadratic transformations as done previously.

The real and imaginary parts of the coefficients $\epsilon_u^{s_1}$, for $u = m, \dots, n-2$ and $u = n$, on the terms $\bar{z}_1 x_{s_1}$, $s_1 = 2, \dots, m-1$, form a real $2(n-m) \times (m-2)$ matrix, in this expression where the left-hand side is a column $(n-m)$-vector:

$$(4) \quad \Big(\sum \epsilon_u^{s_1} \bar{z}_1 x_{s_1}\Big)_{u=m,\dots,n-2,n}$$
$$= \begin{pmatrix} 1 & i & \cdots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & 1 & i \end{pmatrix} \begin{pmatrix} \operatorname{Re}\epsilon_m^2 & \operatorname{Re}\epsilon_m^3 & \dots & \operatorname{Re}\epsilon_m^{m-1} \\ \operatorname{Im}\epsilon_m^2 & \operatorname{Im}\epsilon_m^3 & \dots & \operatorname{Im}\epsilon_m^{m-1} \\ \vdots & & & \vdots \\ \operatorname{Re}\epsilon_{n-2}^2 & \operatorname{Re}\epsilon_{n-2}^3 & \dots & \operatorname{Re}\epsilon_{n-2}^{m-1} \\ \operatorname{Im}\epsilon_{n-2}^2 & \operatorname{Im}\epsilon_{n-2}^3 & \dots & \operatorname{Im}\epsilon_{n-2}^{m-1} \\ \operatorname{Re}\epsilon_n^2 & \operatorname{Re}\epsilon_n^3 & \dots & \operatorname{Re}\epsilon_n^{m-1} \\ \operatorname{Im}\epsilon_n^2 & \operatorname{Im}\epsilon_n^3 & \dots & \operatorname{Im}\epsilon_n^{m-1} \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \bar{z}_1.$$

The *second nondegeneracy condition* is that this real matrix has rank $2(n-m)$. It follows that the number of x_s directions, $m-2$, must be greater than or equal to the number $2(n-m)$, and this is equivalent to $m \geq \frac{2}{3}(n+1)$, exactly the lower bound of the dimensions of topological stability, as discussed in Section 2.

When the second nondegeneracy condition holds, the real $\boldsymbol{R}$ block of the matrix $\boldsymbol{A}$ can transform the x_s variables to put the real matrix above into echelon form, transforming the real and imaginary parts of the $\epsilon_u^{s_1}$ coefficients, without altering the $\bar{z}_1^2$ and $z_1\bar{z}_1$ terms. We get the following quadratic normal form for a nondegenerate CR singularity:

$$(5) \quad \begin{aligned} y_s &= H_s(z_1, \bar{z}_1, x) = E_s(z_1, \bar{z}_1, x) = O(3), \\ z_t &= h_t(z_1, \bar{z}_1, x) = \bar{z}_1 x_{2(t-m+2)} + i\bar{z}_1 x_{2(t-m+2)+1} + e_t(z_1, \bar{z}_1, x), \\ z_{n-1} &= h_{n-1}(z_1, \bar{z}_1, x) = \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x), \\ z_n &= h_n(z_1, \bar{z}_1, x) = z_1\bar{z}_1 + \bar{z}_1 x_2 + i\bar{z}_1 x_3 + e_n(z_1, \bar{z}_1, x), \end{aligned}$$

with $s = 2, \dots, m-1$, $t = m, \dots, n-2$, or, again, there are no h_t expressions if $m = n-1$. If $m > \frac{2}{3}(n+1)$, then the $x_2, \dots, x_{2n-2m+1}$ variables appear in the quadratic part of the normal form but the variables $x_{2n-2m+2}, \dots, x_{m-1}$ do not. In fact, near the origin, the locus of CR singularities (with $\boldsymbol{j}(\boldsymbol{x}) = 1$) is a codimension $2(n-m+1)$ submanifold of M whose tangent space at the origin is the real subspace with coordinates $x_{2n-2m+2}, \dots, x_{m-1}$.

Having stated these two nondegeneracy conditions, we are now ready to state the main result:

Proposition 3.3. *Given $\frac{2}{3}(n+1) \le m < n$, let M be a real analytic m-submanifold of $\mathbb{C}^n$ with a CR singularity at $\boldsymbol{x}$, with $\boldsymbol{j}(\boldsymbol{x}) = 1$. If its local defining equations* (*of the form* (1)) *satisfy both nondegeneracy conditions* (*the full rank of the coefficient matrices* (3), (4)) *so that they can be put into the form* (5), *then there exists a holomorphic coordinate change $\tilde{z} = \vec{z} + \vec{p}$ as in* (2), *in a neighborhood of $\vec{0} \in \mathbb{C}^n$, transforming the equations* (5) *into the real algebraic normal form*

$$
\begin{aligned}
\tilde{y}_s &= 0 && \text{for } s = 2 \dots, m-1, \\
\tilde{z}_t &= \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) && \text{for } t = m \dots, n-2, \\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2, \\
\tilde{z}_n &= \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3).
\end{aligned}
\tag{6}
$$

The real algebraic variety defined by (6) is denoted $\widetilde{M}^{m,n}$, or more briefly $\widetilde{M}$. The example $\widetilde{M}^{4,5}$ is exactly the normal form of [Beloshapka 1997]. The proposition states that any real analytic M satisfying only $\boldsymbol{j}(\boldsymbol{x}) = 1$ at a point and both quadratic nondegeneracy conditions is locally biholomorphically equivalent to the real algebraic model. This is the "analytic stability" mentioned in the title, and it is apparently analogous to stability theorems in the singularity theory of smooth maps, where any sufficiently nondegenerate singularity is equivalent under a change of coordinates to a unique polynomial model. The equations for $\widetilde{M}$ resemble the normal forms for smooth maps with cross-cap (or "S_1") singularities, as in [Whitney 1958; Haefliger 1961; Golubitsky and Guillemin 1973, §VII.4], and $\widetilde{M}$ and the images of the singular maps also have similar structures as a cartesian product when the singularity is not isolated. The main difference between Whitney's normal forms and (6) is that the quantities in (6) are not monomials, and cannot be simultaneously transformed into monomials by holomorphic coordinate transformations in the nondegenerate case. More will be said about the analogies with singularity theory in Section 8.

In the case $m = \frac{2}{3}(n+1)$ when the singularity is isolated, some of the topological invariants mentioned in Section 2 depend on an orientation of M, so it may be useful to consider normalizing transformations that fix a given orientation of the tangent plane T. This corresponds to the real block $\boldsymbol{R}$ of matrix $\boldsymbol{A}$ having a positive determinant, and the last equation of the normal form (5) falls into two cases: $z_n = \bar{z}_1(z_1 + x_2 \pm i x_3)$. The two normal forms are equivalent under the biholomorphic transformation $\tilde{z}_3 = -z_3$, but this reverses the orientation of T. In the remaining sections we will not be concerned with the orientation.

4. A functional equation

To show the existence of a normalizing transformation, we will set up a system of nonlinear functional equations, so that any solution $\vec{p}$ of the system will define a normalizing transformation $\tilde{z} = \vec{z} + \vec{p}$ as in (2). In addition to finding a formal power series solution, we will also have to show that the solution is convergent in some neighborhood of the origin. The method of proof is the rapid convergence technique, as used in [Moser 1985] and [Coffman 2004]. Rather than trying to solve the system of equations directly, we first find an approximate solution by solving a related system of linear equations. Iteration of this process gives a sequence of approximations that approach an exact solution. The issue of the domain of convergence of the exact solution was not addressed by [Beloshapka 1997], and was left open in [Coffman 1997]. In this latter paper, each approximate solution in the sequence was constructed only on a domain a fraction of the size of the previous one in the sequence — when the domains shrink to a point, the limit is an exact formal series solution, but no conclusion can be drawn about its analyticity. The new step here, which is crucial for the method of [Moser 1985] to be applicable, is the construction of a sequence of approximate solutions whose domains shrink slowly enough so their diameters are bounded below by a positive constant.

Starting with the quadratic part of the defining equations in normal form (5), we consider the effect of a coordinate change (2). As previously mentioned, the $\tilde{z} = \vec{z} + \vec{p}$ transformation is (at least formally) invertible near $\vec{0}$, and it may be useful to think of $\tilde{z} = \vec{z} + \vec{p}$ as having identity linear part, although there could be linear terms with weight 2, for example, $\tilde{z}_1 = z_1 + a_n z_n$.

In terms of $\tilde{z}$ and $\vec{z}$, consider the system of equations

$$
\begin{aligned}
(7)\qquad & 0 = \operatorname{Im}(\tilde{z}_s) = \operatorname{Im}(z_s + p_s(\vec{z})),\\
& 0 = \tilde{z}_t - (\bar{\tilde{z}}_1 \tilde{x}_{2(t-m+2)} + i \bar{\tilde{z}}_1 \tilde{x}_{2(t-m+2)+1}),\\
& \quad = z_t + p_t(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}\operatorname{Re}(z_{2(t-m+2)} + p_{2(t-m+2)}(\vec{z}))\\
& \qquad\qquad - i\,\overline{(z_1 + p_1(\vec{z}))}\operatorname{Re}(z_{2(t-m+2)+1} + p_{2(t-m+2)+1}(\vec{z})),\\
& 0 = \tilde{z}_{n-1} - \bar{\tilde{z}}_1^2 = z_{n-1} + p_{n-1}(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}^2,\\
& 0 = \tilde{z}_n - \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i \tilde{x}_3)\\
& \quad = z_n + p_n(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}\big(z_1 + p_1(\vec{z}) + \operatorname{Re}(z_2 + p_2(\vec{z})) + i \operatorname{Re}(z_3 + p_3(\vec{z}))\big).
\end{aligned}
$$

In order to get (6) to be the defining equations for M in the $\tilde{z}$ coordinates, the preceding equalities must hold for points $\vec{z}$ on M and near $\vec{0}$. So, we can replace the $\vec{z} = (z_1, \dots, z_n)$ expressions in (7) by the defining functions (5):

$$\vec{z} = \big(z_1, x_2 + iH_2(z_1, \bar{z}_1, x), \dots, h_n(z_1, \bar{z}_1, x)\big), \tag{8}$$

to get a system of equations where the right-hand side functions depend only on $z_1, \bar{z}_1, x$:

$$\begin{aligned} 0 &= \operatorname{Im}(x_s + iH_s + p_s(\vec{z})) = E_s(z_1, \bar{z}_1, x) + \operatorname{Im} p_s(\vec{z}), \\ 0 &= e_t(z_1, \bar{z}_1, x) + p_t(\vec{z}) - \overline{p_1(\vec{z})}(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) \\ &\qquad - \bar{z}_1\big(\operatorname{Re} p_{2(t-m+2)}(\vec{z}) + i \operatorname{Re} p_{2(t-m+2)+1}(\vec{z})\big) \\ &\qquad - \overline{p_1(\vec{z})}\big(\operatorname{Re} p_{2(t-m+2)}(\vec{z}) + i \operatorname{Re} p_{2(t-m+2)+1}(\vec{z})\big), \\ 0 &= e_{n-1}(z_1, \bar{z}_1, x) + p_{n-1}(\vec{z}) - 2\bar{z}_1\overline{p_1(\vec{z})} - \overline{p_1(\vec{z})}^2 \\ 0 &= e_n(z_1, \bar{z}_1, x) + p_n(\vec{z}) - \bar{z}_1\big(p_1(\vec{z}) + \operatorname{Re} p_2(\vec{z}) + i \operatorname{Re} p_3(\vec{z})\big) \\ &\qquad - \overline{p_1(\vec{z})}(z_1 + x_2 + ix_3) - \overline{p_1(\vec{z})}\big(p_1(\vec{z}) + \operatorname{Re} p_2(\vec{z}) + i \operatorname{Re} p_3(\vec{z})\big). \end{aligned} \tag{9}$$

The components of $\vec{e} = (E_2, \dots, E_{m-1}, e_m, \dots, e_n)$ appear in two ways — as terms in each equation of (9), and also in the $\vec{z}$ input (8) for each $p_j(\vec{z})$ in (9), $j = 1, \dots, n$. So, given $\vec{e}$, if we happen to have an exact solution $\vec{p}$ of the system of functional equations above, the conclusion of Proposition 3.3 holds and we are done. However, (9) is a nonlinear system in the unknown quantity $\vec{p}$, where in addition to the composition with the given defining functions (8), there are products of the components p_j and their complex conjugates.

As a first step in solving for $\vec{p}$ in terms of $\vec{e}$, consider the system of simpler equations:

$$\begin{aligned} 0 &= E_s(z_1, \bar{z}_1, x) + \operatorname{Im} p_s(\vec{z}), \\ 0 &= e_t(z_1, \bar{z}_1, x) + p_t(\vec{z}) - \overline{p_1(\vec{z})}(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) \\ &\qquad - \bar{z}_1\big(\operatorname{Re} p_{2(t-m+2)}(\vec{z}) + i \operatorname{Re} p_{2(t-m+2)+1}(\vec{z})\big), \\ 0 &= e_{n-1}(z_1, \bar{z}_1, x) + p_{n-1}(\vec{z}) - 2\bar{z}_1\overline{p_1(\vec{z})}, \\ 0 &= e_n(z_1, \bar{z}_1, x) + p_n(\vec{z}) \\ &\qquad - \bar{z}_1\big(p_1(\vec{z}) + \operatorname{Re} p_2(\vec{z}) + i \operatorname{Re} p_3(\vec{z})\big) - \overline{p_1(\vec{z})}(z_1 + x_2 + ix_3), \end{aligned} \tag{10}$$

where the $\vec{z}$ input for each p_j is

$$\vec{z} = \big(z_1, x_2, \dots, x_{m-1}, \bar{z}_1(x_4 + ix_5), \dots, \bar{z}_1^2, \bar{z}_1(z_1 + x_2 + ix_3)\big). \tag{11}$$

This simplifies $p_j(\vec{z})$ by considering only the linear and quadratic parts of the input (8). Also, the products of p_j are dropped, so that these are (real) linear equations.

To see how the new equations are related to the original system, suppose $\vec{e}$ has degree $d \geq 3$, and that $\vec{p}$ is a solution of (10)–(11) so that $p_1, \ldots, p_{2n-2m+1}$ have weight $\geq d-1$, and $p_{2n-2m+2}, \ldots, p_n$ have weight $\geq d$. Evaluating the right-hand side of (9) with this solution for $\vec{p}$ evidently results in expressions of degree $\geq 2d-2$. Converting these expressions in $z_1, \bar{z}_1, x$ to $\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}$ and equating them to the $\tilde{z}$ expressions in (7) gives the higher-order terms of the new defining equations for M in the $\tilde{z}$ coordinate system. (It will be shown later (Theorem 6.5) that in fact for $\vec{z} \in M$ close enough to $\vec{0}$, $z_1, \bar{z}_1, x$ are real analytic functions of $\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}$.) So, while a solution $\vec{p}$ of the linearized equations is just an approximation to the solution of the original system, using such a $\vec{p}$ to define a coordinate transformation does have the effect of nearly doubling the order of vanishing of the $\vec{e}$ quantity.

5. A solution of the linear equation

The goal of this section is to construct a solution $\vec{p}$ of the system of linear equations (10)–(11), given the higher-order terms of the defining equations, $\vec{e}$. Considering $\vec{p}$ and $\vec{e}$ as formal power series, such a solution exists but is not unique — this fact, together with the approximate doubling of the degree mentioned in the previous section and iteration of the linearization procedure, is enough to show the (already known, as mentioned previously) formal equivalence of M and $\widetilde{M}$. The solution $\vec{p}$ constructed here will be an n-tuple of series in $\vec{z} = (z_1, \ldots, z_n)$ with the following properties: the size of the domain of convergence of $\vec{p}$ is comparable in a certain sense to the size of the domain of $\vec{e}$, and also a suitable norm of $\vec{p}$ is bounded in terms of a suitable norm of $\vec{e}$.

Notation 5.1. For $\boldsymbol{r} = (r_1, \ldots, r_N) \in \mathbb{R}^N$, with all $r_j > 0$, define a polydisc in $\mathbb{C}^N$ by

$$\mathbb{D}_{\boldsymbol{r}} = \{(z_1, \ldots, z_N) : |z_j| < r_j\}.$$

As special cases, let

$$D_r = \mathbb{D}_{(r,r,\ldots,r)} \subseteq \mathbb{C}^m \quad \text{and} \quad \Delta_r = \mathbb{D}_{(r,\ldots,r,2r^2,\ldots,2r^2,r^2,3r^2)} \subseteq \mathbb{C}^n,$$

where there are $m-1$ radius lengths r and $n-m-1$ radius lengths $2r^2$, in the $z_m, \ldots, z_{n-2}$ coordinate directions.

The initial assumption on the defining equations is that

$$\vec{e}(z_1, \bar{z}_1, x) = (E_2, \ldots, E_{m-1}, e_m, \ldots, e_n)$$

is real analytic, so there is some $r > 0$ so that each component of $\vec{e}$ is the restriction to $\{\zeta = \bar{z}_1, x = \bar{x}\}$ of a multivariable power series in (z_1, ζ, x) with center $(0, 0, \ldots, 0)$ and complex coefficients which converges on a complex polydisc $D_r \subseteq \mathbb{C}^m$ (or, equivalently, a complex analytic function on D_r).

Notation 5.2. For a complex-valued function $e(z_1, \zeta, x)$ of m complex variables, which is defined on some set containing the polydisc D_r, define the norm

$$|e|_r = \sup_{(z_1,\zeta,x)\in D_r} |e(z_1, \zeta, x)|.$$

For an $(n-1)$-tuple $\vec{e} = (E_2, \dots, e_n)$, define

$$|\vec{e}|_r = |E_2|_r + \cdots + |e_n|_r.$$

For a complex-valued function $p(z_1, \dots, z_n)$ of n complex variables, which is defined on some set containing the polydisc Δ_r, define the norm

$$\|p\|_r = \sup_{\vec{z}\in\Delta_r} |p(\vec{z})|.$$

With this notation, we can further assume $r > 0$ is small enough so that $|\vec{e}(z_1, \zeta, x)|_r$ is finite. Given $\vec{e}$ with degree ≥ 3, the eventual goal is to find some $\tilde{r}$, $0 < \tilde{r} \leq r$, and a holomorphic map $\vec{p} : \Delta_{\tilde{r}} \to \mathbb{C}^n$, so that the transformation $\tilde{z} = \vec{z} + \vec{p}(\vec{z})$ is a biholomorphism with domain $\Delta_{\tilde{r}}$ taking M to $\widetilde{M}$. That is, if $\vec{z} \in M \cap \Delta_{\tilde{r}}$, then $\tilde{z}$ satisfies (6). However, in this section we are only looking for $\vec{p}$ that is a solution of (10)–(11).

Some steps of the proof of Theorem 5.6 below will decompose series into subseries and their complex conjugates, where these preliminary lemmas on the $|e|_r$ norm will be useful.

Lemma 5.3. *Given $0 < R < r$ and complex coefficients a_{jkI}, b_{jkI}, if*

$$\Bigl|\sum a_{jkI} z_1^j \zeta^k x^I\Bigr|_r \leq K$$

and for complex x with $|x_s| < r$, $j, k = 0, 1, 2, 3, \dots$,

$$\Bigl|\sum_I b_{jkI} x^I\Bigr| \leq \Bigl|\sum_I a_{jkI} x^I\Bigr|,$$

then

$$\Bigl|\sum b_{jkI} z_1^j \zeta^k x^I\Bigr|_R \leq \frac{Kr^2}{(r-R)^2}.$$

Proof. For $(z_1, \zeta, x) \in D_r$, these series are absolutely convergent and equal:

$$\sum a_{jkI} z_1^j \zeta^k x^I = \sum_{j=0}^{\infty}\biggl(\sum_{k=0}^{\infty}\Bigl(\sum_I a_{jkI} x^I\Bigr)\zeta^k\biggr) z_1^j.$$

Using Cauchy's estimate [Ahlfors 1979] twice, we obtain

$$\biggl|\sum_{k=0}^{\infty}\Bigl(\sum_I a_{jkI} x^I\Bigr)\zeta^k\biggr| \leq \frac{K}{r^j} \quad \text{and} \quad \Bigl|\sum_I a_{jkI} x^I\Bigr| \leq \frac{K}{r^k r^j}.$$

For $(z_1, \zeta, x) \in D_r$, the series $\sum b_{jkI} z_1^j \zeta^k x^I$ is absolutely convergent, and for $(z_1, \zeta, x) \in D_R$:

$$\begin{aligned}
\left|\sum b_{jkI} z_1^j \zeta^k x^I\right| &= \left|\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_I b_{jkI} x^I\right)\zeta^k\right) z_1^j\right| \\
&\leq \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\left|\sum_I b_{jkI} x^I\right| |\zeta|^k\right)|z_1|^j \\
&\leq \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\left|\sum_I a_{jkI} x^I\right| |\zeta|^k\right)|z_1|^j \\
&\leq \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{K}{r^k r^j}|\zeta|^k\right)|z_1|^j = \sum_{j,k} K\left(\frac{|\zeta|}{r}\right)^k\left(\frac{|z_1|}{r}\right)^j \\
&= K\frac{1}{1-|\zeta|/r}\,\frac{1}{1-|z_1|/r} = \frac{Kr^2}{(r-|\zeta|)(r-|z_1|)} < \frac{Kr^2}{(r-R)^2}.
\end{aligned}$$

□

In the applications of the lemma, for each pair (j, k), the coefficients b_{jkI} will either be zero for all I or equal to a_{jkI} for all I, so the estimate in the hypothesis is satisfied.

Notation 5.4. On the complex vector space of formal power series, define the real structure operator

$$e = \sum e^{abI} z_1^a \zeta^b x^I \mapsto e' = \sum \overline{e^{abI}} \zeta^a z_1^b x^I. \tag{12}$$

Lemma 5.5. *For $r > 0$, the restriction of the map* (12) *to the subspace* $\{e : |e|_r < \infty\}$ *is an isometry.*

Proof. The equality of norms uses a change of variables that does not change the radius length r.

$$\begin{aligned}
|e'|_r &= \sup_{(z_1,\zeta,x)\in D_r}\left|\sum \overline{e^{abI}} \zeta^a z_1^b x^I\right| = \sup_{(\zeta',z_1',x')=(\bar z_1,\bar\zeta,\bar x)\in D_r}\left|\sum \overline{e^{abI}}\, \overline{z_1'}^a\, \overline{\zeta'}^b\, \overline{x'}^I\right| \\
&= \sup_{(\zeta',z_1',x')\in D_r}\left|\overline{\sum e^{abI}(z_1')^a(\zeta')^b(x')^I}\right| = \sup_{(z_1',\zeta',x')\in D_r}\left|\sum e^{abI}(z_1')^a(\zeta')^b(x')^I\right| \\
&= |e|_r.
\end{aligned}$$

□

Of course, this map is a representation of complex conjugation: given a series $e(z_1, \bar z_1, x)$ for real x, which "complexifies" to $e = e(z_1, \zeta, x)$ for $(z_1, \zeta, x) \in D_r$ for the purposes of finding its norm as in Notation 5.2, expanding $\overline{e(z_1, \bar z_1, x)}$ as a series in $(z_1, \bar z_1, x)$ and then complexifying gives $e' = e'(z_1, \zeta, x)$.

In an attempt to simplify the notation by avoiding an excess of indices in an already intricate calculation, the following theorem will focus on one particular dimension pair (m, n). In order to represent the most general behavior, we want $m < n-1$, so there is a z_t equation in (5), and also $m > \frac{2}{3}(n+1)$, so there is a variable x_{m-1} that does not appear in the quadratic part of the defining equations. The smallest pair where both conditions occur is $m = 7$, $n = 9$, so we will be considering a real 7-manifold in $\mathbb{C}^9$, where the coordinates of the tangent plane are $z_1, x_2, \dots, x_6$, and the CR singular locus in M near $\vec{0}$ is a real curve tangent to the x_6 axis at the origin.

Theorem 5.6. *Given $r > 0$ and $\vec{e}(z_1, \zeta, x)$ convergent on D_r with $|\vec{e}|_r < \infty$ and degree $d \geq 3$, there exists $\vec{p}$ that is convergent on Δ_r and satisfies these properties*:

(a) *$\vec{p}$ solves the following case of the system of equations* (10)–(11):

$$(13)\qquad \begin{aligned} 0 &= E_s(z_1, \bar{z}_1, x) + \operatorname{Im} p_s(\vec{z}) \qquad \text{for } s = 2, \dots, 6,\\ 0 &= e_7(z_1, \bar{z}_1, x) + p_7(\vec{z}) - \overline{p_1(\vec{z})}(x_4 + i x_5) - \bar{z}_1(\operatorname{Re} p_4(\vec{z}) + i \operatorname{Re} p_5(\vec{z})),\\ 0 &= e_8(z_1, \bar{z}_1, x) + p_8(\vec{z}) - 2\bar{z}_1\overline{p_1(\vec{z})},\\ 0 &= e_9(z_1, \bar{z}_1, x) + p_9(\vec{z})\\ &\qquad - \bar{z}_1(p_1(\vec{z}) + \operatorname{Re} p_2(\vec{z}) + i \operatorname{Re} p_3(\vec{z})) - \overline{(p_1(\vec{z}))}(z_1 + x_2 + i x_3), \end{aligned}$$

where

$$(14)\qquad \vec{z} = \big(z_1, x_2, x_3, x_4, x_5, x_6, \bar{z}_1(x_4 + i x_5), \bar{z}_1^2, \bar{z}_1(z_1 + x_2 + i x_3)\big).$$

(b) $\|p_1\|_r \leq 3|e_8|_r/(2r)$, $\|p_8\|_r \leq 4|e_8|_r$ *and*, *for any* $0 < R < r$,

$$\|p_2\|_R \leq \frac{3|e_9|_r + 18|e_8|_r}{R} + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_2|_r + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_3|_r,$$

$$\|p_3\|_R \leq \frac{3|e_9|_r + 18|e_8|_r}{R} + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_2|_r + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_3|_r,$$

$$\|p_4\|_R \leq \frac{3|e_7|_r + 9|e_8|_r}{R} + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_4|_r + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_5|_r,$$

$$\|p_5\|_R \leq \frac{3|e_7|_r + 9|e_8|_r}{R} + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_4|_r + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_5|_r,$$

$$\|p_6\|_R \leq \frac{20r^2}{(r-R)^2}|E_6|_r,$$

$$\|p_7\|_R \leq 4|e_7|_r + 12|e_8|_r + 8R\Big(\frac{r^2}{(r-R)^2} + 1\Big)(|E_4|_r + |E_5|_r),$$

$$\|p_9\|_R \leq 4|e_9|_r + 24|e_8|_r + 8R\Big(\frac{r^2}{(r-R)^2} + 1\Big)(|E_2|_r + |E_3|_r).$$

Proof. First, notice that if $\vec{p}(z_1, \dots, z_9)$ is a formal series solution of (13)–(14), it does not follow that $\vec{p}$ is convergent at any point (other than the origin). For example, with any component p_j, the series expressions $p_j(\vec{z})$ and

$$p_j(\vec{z}) + ((z_1 + z_2 + iz_3)^2 z_8 - z_9^2) \cdot \mathrm{Q}(\vec{z}) \tag{15}$$

are formally the same when restricted to $\vec{z}$ as in (14), for any (possibly divergent) series Q. So, if one formal solution $\vec{p}$ exists, then there exist infinitely many divergent solutions. There may also exist formal series solutions that are convergent only on some neighborhood of the origin much smaller than that claimed in the theorem.

Continuing with the abbreviation $x = x_2, x_3, x_4, x_5, x_6$, and also using $z = z_2, z_3, z_4, z_5, z_6$, the following choice of normalization will simplify the construction of the solution $\vec{p}$ satisfying the claimed convergence and bounds:

$$p_1(\vec{z}) = p_1(z_1, z, z_8), \qquad p_j(\vec{z}) = p_j^E(z_1, z, z_8) + z_9 p_j^O(z_1, z, z_8),$$

for $j = 2, \dots, 9$. Note that $\vec{p}$ does not depend on z_7, and the first component p_1 does not depend on z_9. We may make the further assumption that p_1 is an even function of z_1: $p_1(z_1, z, z_8) = p_1(-z_1, z, z_8)$. The remaining components, p_j, have some terms not depending on z_9, labeled p_j^E, and other terms which have exactly one linear factor of z_9. The p_j^E and p_j^O terminology corresponds to even and odd powers of $\bar{z}_1$ which appear after the substitution of (14) into $\vec{p}$. The choice that $\vec{p}$ has at most linear terms in $z_9 = \bar{z}_1(z_1 + x_2 + ix_3)$ is made to avoid high powers of the nonmonomial quantity $\bar{z}_1(z_1 + x_2 + ix_3)$, since as in [Coffman 1997], any multinomial coefficients in the series expansion of $\vec{p}(\vec{z})$ could be large enough to affect the size of the domain of convergence.

We begin with the e_8 equation of the system (13). If the series expansion of e_8 had only even powers of $\bar{z}_1$, then it would be a very simple matter to compare the coefficients of $e_8(z_1, \bar{z}_1, x)$ and $p_8^E(z_1, x, \bar{z}_1^2)$, and get a solution of the equation with $p_8^O = p_1 = 0$. The odd powers of $\bar{z}_1$ in e_8 make the p_8^O and p_1 quantities necessary to solve the equation. The consideration of the terms of the components of the given quantity $\vec{e}$ which are even or odd in $\bar{z}_1$ was part of the analysis of [Beloshapka 1997] and [Coffman 1997] of the formal normal form problem, and even/odd decompositions also appeared in analogous calculations in [Whitney 1943]. However, to deal with the nonmonomial property of the quadratic normal form, there will be some rearrangements of the terms in the series which were not required in Whitney's work. First, decompose e_8 into even and odd parts e_{8A}, e_{8B}, e_{8C}, and then apply an add-and-subtract trick to e_{8C}, as follows:

$$e_8 = \sum e_8^{abI} z_1^a \bar{z}_1^b x^I = e_{8A} + e_{8B} + e_{8C},$$

$$e_{8A} = \textstyle\sum_{b \text{ even}} e_8^{abI} z_1^a \bar{z}_1^b x^I,$$

$$e_{8B} = \textstyle\sum_{a \text{ even}, b \text{ odd}} e_8^{abI} z_1^a \bar{z}_1^b x^I,$$
$$e_{8C} = \textstyle\sum_{a, b \text{ odd}} e_8^{abI} z_1^a \bar{z}_1^b x^I = e_{8D} + e_{8E},$$
$$e_{8D} = \bar{z}_1(z_1 + x_2 + ix_3) \textstyle\sum_{a, b \text{ odd}} e_8^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I,$$
$$e_{8E} = -(x_2 + ix_3) \textstyle\sum_{a, b \text{ odd}} e_8^{abI} z_1^{a-1} \bar{z}_1^b x^I.$$

Let $f_8 = e_{8B} + e_{8E}$, so $f_8(z_1, \bar{z}_1, x)$ is even in z_1 and odd in $\bar{z}_1$. Then, combining $e_8 = e_{8A} + e_{8D} + f_8$ with the normalization for p_1 and p_8 in the e_8 equation from (13), a straightforward (by construction) comparison of coefficients yields

$$\begin{aligned} 0 &= e_{8A}(z_1, \bar{z}_1, x) + p_8^E(z_1, x, \bar{z}_1^2), \\ 0 &= e_{8D}(z_1, \bar{z}_1, x) + \bar{z}_1(z_1 + x_2 + ix_3) p_8^O(z_1, x, \bar{z}_1^2), \\ 0 &= f_8(z_1, \bar{z}_1, x) - 2\bar{z}_1 \overline{p_1(z_1, x, \bar{z}_1^2)}. \end{aligned}$$

If $p_8^E(z_1, z, z_8) = \sum p_8^{acI} z_1^a z_8^c z^I$, then the coefficient p_8^{acI} must be equal to $-e_8^{a,2c,I}$, and we get an estimate for the norm of p_8^E on the polydisc $\Delta_r \subseteq \mathbb{C}^9$:

$$\begin{aligned} \|p_8^E\|_r &= \sup_{\vec{z} \in \Delta_r} |p_8^E(\vec{z})| = \sup_{\substack{|z_1|<r,\, |x_s|<r, \\ |\zeta^2|<r^2}} \left|p_8^E(z_1, x, \zeta^2)\right| = \sup_{(z_1,\zeta,x) \in D_r} \left|-e_{8A}(z_1, \zeta, x)\right| \\ &= |e_{8A}|_r = \left|\tfrac{1}{2}(e_8(z_1, \zeta, x) + e_8(z_1, -\zeta, x))\right|_r \leq |e_8|_r. \end{aligned}$$

By using the averaging formula to extract the even part of e_8, we can just apply the triangle inequality to get the estimate for the subseries instead of Lemma 5.3. There is a similar estimate for the other component p_8^O, but this time the Schwarz Lemma [Ahlfors 1979] is used in two steps:

$$\begin{aligned} \|z_9 p_8^O\|_r &\leq \|z_9\|_r \|p_8^O\|_r = 3r^2 \sup_{\substack{|z_1|<r,\, |x_s|<r, \\ |\zeta^2|<r^2}} |p_8^O(z_1, x, \zeta^2)| \\ &= 3r^2 \sup_{(z_1,\zeta,x) \in D_r} \left| \frac{-e_{8D}(z_1, \zeta, x)}{\zeta(z_1 + x_2 + ix_3)} \right| = 3r^2 \sup_{(z_1,\zeta,x) \in D_r^*} \left| \frac{-e_{8C}(z_1, \zeta, x)}{z_1 \zeta} \right| \\ &\leq 3r^2 \sup_{(z_1,\zeta,x) \in D_r^*} \frac{(|z_1|/r)\, \sup_{|z_1|<r} |e_{8C}|}{|z_1||\zeta|} \\ &\leq 3r \sup_{(z_1,\zeta,x) \in D_r^*} \frac{\sup_{|z_1|<r} \left|(|\zeta|/r)\, \sup_{|\zeta|<r} |e_{8C}|\right|}{|\zeta|} = 3|e_{8C}|_r \\ &= \tfrac{3}{4}\left|e_8(z_1, \zeta, x) - e_8(z_1, -\zeta, x) - e_8(-z_1, \zeta, x) + e_8(-z_1, -\zeta, x)\right|_r \\ &\leq 3|e_8|_r. \end{aligned}$$

In some of these steps, we restricted to the open subset

$$D_r^* = D_r \setminus \big(\{z_1 = 0\} \cup \{\zeta = 0\}\big),$$

which avoids division by 0 but, by the maximum principle, does not affect the supremum.

From $f_8 = e_{8B} + e_{8E}$ and the Schwarz Lemma,

$$\begin{aligned}|f_8|_r &\leq |e_{8B}|_r + |e_{8E}|_r \\ &= \tfrac{1}{4}\big|e_8(z_1,\zeta,x) - e_8(z_1,-\zeta,x) + e_8(-z_1,\zeta,x) - e_8(-z_1,-\zeta,x)\big|_r \\ &\qquad + \left|-\frac{x_2+ix_3}{z_1}e_{8C}\right|_r \\ &\leq |e_8|_r + |x_2+ix_3|_r \cdot \frac{1}{r}|e_{8C}|_r \leq 3|e_8|_r.\end{aligned}$$

Solving for p_1 involves complex conjugation, so we take care to work out a few steps. By comparing the coefficients of $\overline{f_8}$ and p_1, we see that if $p_1(z_1, z, z_8) = \sum_{\alpha \text{ even}} p_1^{\alpha\beta I} z_1^\alpha z_8^\beta z^I$, then $p_1^{\alpha\beta I} = \frac{1}{2}\overline{f_8^{2\beta,\alpha+1,I}}$. Using the Schwarz Lemma and Lemma 5.5, we obtain

$$\begin{aligned}\|p_1\|_r &= \sup_{(z_1,\zeta,x)\in D_r}\Big|\sum_{\alpha \text{ even}} p_1^{\alpha\beta I} z_1^\alpha \zeta^{2\beta} x^I\Big| = \sup_{(z_1,\zeta,x)\in D_r}\Big|\sum_{\alpha \text{ even}} \tfrac{1}{2}\overline{f_8^{2\beta,\alpha+1,I}} z_1^\alpha \zeta^{2\beta} x^I\Big| \\ &= \sup_{(z_1,\zeta,x)\in D_r}\left|\sum_{a \text{ even}, b \text{ odd}} \frac{\overline{f_8^{abI}}\zeta^a z_1^b x^I}{2z_1}\right| = \left|\frac{f_8'(z_1,\zeta,x)}{2z_1}\right|_r \\ &\leq \frac{1}{2r}|f_8'|_r = \frac{1}{2r}|f_8|_r \leq \frac{3}{2r}|e_8|_r.\end{aligned}$$

By construction, p_1 has weight $d-1$ and p_8 has weight d.

Moving next to the E_6 equation of (13), split the real valued series E_6 into subseries, some real and some in complex conjugate pairs:

$$\begin{aligned}E_6 &= e_{6A} + \overline{e_{6A}} + E_{6B} + e_{6C} + \overline{e_{6C}} + e_{6D} + \overline{e_{6D}} + E_{6E}, \\ e_{6A} &= \textstyle\sum_{a>b,\, b \text{ even}} E_6^{abI} z_1^a \bar{z}_1^b x^I, \\ E_{6B} &= \textstyle\sum_{a \text{ even}} E_6^{aaI} z_1^a \bar{z}_1^a x^I, \\ e_{6C} &= \textstyle\sum_{a>b,\, a \text{ even},\, b \text{ odd}} E_6^{abI} z_1^a \bar{z}_1^b x^I, \\ e_{6D} &= \textstyle\sum_{a>b,\, a,b \text{ odd}} E_6^{abI} z_1^a \bar{z}_1^b x^I, \\ E_{6E} &= \textstyle\sum_{a \text{ odd}} E_6^{aaI} z_1^a \bar{z}_1^a x^I.\end{aligned}$$

By Lemma 5.3, we have

$$|e_{6A}|_R \leq \frac{r^2}{(r-R)^2}|E_6|_r,$$

and all the other subseries have the same bound. We rearrange two of these subseries to be able to compare coefficients with p_6:

$$\begin{aligned}
e_{6D} &= e_{6F} + e_{6G},\\
e_{6F} &= (z_1 + x_2 + ix_3)\bar{z}_1 \textstyle\sum_{a > b,\, a,\, b \text{ odd}} E_6^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I,\\
e_{6G} &= -(x_2 + ix_3) \textstyle\sum_{a > b,\, a,\, b \text{ odd}} E_6^{abI} z_1^{a-1} \bar{z}_1^{b} x^I,\\
E_{6E} &= e_{6H} + \overline{e_{6H}} + e_{6I} + \overline{e_{6I}},\\
e_{6H} &= \tfrac{1}{2}(z_1 + x_2 + ix_3)\bar{z}_1 \textstyle\sum_{a \text{ odd}} E_6^{aaI} z_1^{a-1} \bar{z}_1^{a-1} x^I,\\
e_{6I} &= -\tfrac{1}{2}(x_2 + ix_3) \textstyle\sum_{a \text{ odd}} E_6^{aaI} z_1^{a-1} \bar{z}_1^{a} x^I,
\end{aligned}$$

and collect some of these subseries back together:

$$\begin{aligned}
f_{6A}(z_1, \bar{z}_1, x) &= e_{6A} + \overline{e_{6I}} = \textstyle\sum_{a > b,\, b \text{ even}} f_{6A}^{abI} z_1^a \bar{z}_1^b x^I,\\
f_{6C}(z_1, \bar{z}_1, x) &= \overline{e_{6C}} + \overline{e_{6G}} = \textstyle\sum_{a < b,\, a \text{ odd},\, b \text{ even}} f_{6C}^{abI} z_1^a \bar{z}_1^b x^I,
\end{aligned}$$

so

$$E_6 = f_{6A} + \overline{f_{6A}} + E_{6B} + f_{6C} + \overline{f_{6C}} + e_{6F} + \overline{e_{6F}} + e_{6H} + \overline{e_{6H}}.$$

The unknown p_6 can also be expressed as a sum of subseries:

$$\begin{aligned}
p_6 &= p_6^E(z_1, z, z_8) + z_9 p_6^O(z_1, z, z_8),\\
p_6^E &= p_{6A} + p_{6B} + p_{6C},\\
p_{6A} &= \textstyle\sum_{\alpha > 2\gamma} p_{6A}^{\alpha\gamma I} z_1^\alpha z^I z_8^\gamma,\\
p_{6B} &= \textstyle\sum p_{6B}^{\gamma I} z_1^{2\gamma} z^I z_8^\gamma,\\
p_{6C} &= \textstyle\sum_{\alpha < 2\gamma,\, \alpha \text{ odd}} p_{6C}^{\alpha\gamma I} z_1^\alpha z^I z_8^\gamma,\\
p_6^O &= p_{6D} + p_{6E},\\
p_{6D} &= \textstyle\sum_{\alpha > 2\gamma,\, \alpha \text{ even}} p_{6D}^{\alpha\gamma I} z_1^\alpha z^I z_8^\gamma,\\
p_{6E} &= \textstyle\sum p_{6E}^{\gamma I} z_1^{2\gamma} z^I z_8^\gamma.
\end{aligned}$$

Comparing coefficients, the equation $0 = E_6 - \frac{1}{2}i(p_6 - \overline{p_6})$ from (13) turns into these five equations and their complex conjugates:

$$\begin{aligned}
0 &= f_{6A} - \tfrac{1}{2}ip_{6A},\\
0 &= \tfrac{1}{2}E_{6B} - \tfrac{1}{2}ip_{6B},\\
0 &= f_{6C} - \tfrac{1}{2}ip_{6C},\\
0 &= e_{6F} - \tfrac{1}{2}i(z_1 + x_2 + ix_3)\bar{z}_1 p_{6D},\\
0 &= e_{6H} - \tfrac{1}{2}i(z_1 + x_2 + ix_3)\bar{z}_1 p_{6E}.
\end{aligned}$$

Solving for each component of p_6 gives a weight d quantity, and using Lemma 5.5, the Schwarz Lemma, and the previously mentioned estimates for the subseries of E_6, we get these estimates:

$$\begin{aligned}
\|p_{6A}\|_R &= |-2if_{6A}|_R = 2|e_{6A} + e'_{6I}|_R \le 2(|e_{6A}|_R + |e_{6I}|_R),\\
&\le 2\left(|e_{6A}|_R + \left|\frac{-(x_2+ix_3)E_{6E}}{2z_1}\right|_R\right),\\
&\le 2\left(\frac{r^2|E_6|_r}{(r-R)^2} + \frac{2R}{2}\cdot\frac{1}{R}\cdot\frac{r^2|E_6|_r}{(r-R)^2}\right) = \frac{4r^2}{(r-R)^2}|E_6|_r,
\end{aligned}$$

$$\|p_{6B}\|_R = \left|-2i\cdot\tfrac{1}{2}E_{6B}\right|_R \le \frac{r^2}{(r-R)^2}|E_6|_r,$$

$$\begin{aligned}
\|p_{6C}\|_R &= |-2if_{6C}|_R = 2|e'_{6C} + e'_{6G}|_R,\\
&\le 2\left(|e_{6C}|_R + \left|\frac{-(x_2+ix_3)e_{6D}}{z_1}\right|_R\right),\\
&\le 2\left(\frac{r^2|E_6|_r}{(r-R)^2} + 2R\cdot\frac{1}{R}\cdot\frac{r^2|E_6|_r}{(r-R)^2}\right) = \frac{6r^2}{(r-R)^2}|E_6|_r,
\end{aligned}$$

$$\begin{aligned}
\|z_9p_{6D}\|_R &\le \|z_9\|_R\|p_{6D}\|_R = 3R^2\left|-2i\frac{e_{6F}}{(z_1+x_2+ix_3)\zeta}\right|_R = 6R^2\left|\frac{e_{6D}}{z_1\zeta}\right|_R,\\
&\le 6|e_{6D}|_R \le \frac{6r^2}{(r-R)^2}|E_6|_r,
\end{aligned}$$

$$\|z_9p_{6E}\|_R \le 3R^2\left|-2i\frac{e_{6H}}{(z_1+x_2+ix_3)\zeta}\right|_R = 6R^2\left|\frac{e_{6E}}{2z_1\zeta}\right|_R \le \frac{3r^2}{(r-R)^2}|E_6|_r.$$

Finding p_2, p_3, p_4, p_5 is a bit trickier since each appears in more than one equation of (13). We will simultaneously solve for p_4, p_5, p_7, using a more involved comparison of coefficients, and similarly but independently, also p_2, p_3, p_9.

To find p_4, p_5, and p_7, we consider the E_4, E_5, e_7 equations of (13), and use the previously found solution for p_1 to get the system with the unknowns on the left-hand side and the known $O(d)$ quantities on the right-hand side:

$$\operatorname{Im} p_4 = -E_4, \tag{16}$$

$$\operatorname{Im} p_5 = -E_5, \tag{17}$$

$$p_7 - \bar{z}_1 \operatorname{Re} p_4 - i\bar{z}_1 \operatorname{Re} p_5 = -e_7 + (x_4 + i x_5)\overline{p_1} \tag{18}$$

Starting with the right-hand side of (16), the following decomposition of E_4 is different from that of E_6:

$$\begin{aligned}
E_4 &= E_{4A} + e_{4B} + \overline{e_{4B}} + e_{4C} + \overline{e_{4C}} + E_{4D},\\
E_{4A} &= \textstyle\sum_{a,\,b \text{ even}} E_4^{abI} z_1^a \bar{z}_1^b x^I,\\
e_{4B} &= \textstyle\sum_{a\,>\,b,\,a \text{ odd},\,b \text{ even}} E_4^{abI} z_1^a \bar{z}_1^b x^I,\\
e_{4C} &= \textstyle\sum_{a\,>\,b,\,a \text{ even},\,b \text{ odd}} E_4^{abI} z_1^a \bar{z}_1^b x^I,\\
E_{4D} &= \textstyle\sum_{a,\,b \text{ odd}} E_4^{abI} z_1^a \bar{z}_1^b x^I = e_{4E} + e_{4F},\\
e_{4E} &= (z_1 + x_2 + i x_3)\bar{z}_1 \textstyle\sum_{a,\,b \text{ odd}} E_4^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I,\\
e_{4F} &= -(x_2 + i x_3) \textstyle\sum_{a,\,b \text{ odd}} E_4^{abI} z_1^{a-1} \bar{z}_1^b x^I.
\end{aligned}$$

The E_{4A} piece is simply an even part, so $|E_{4A}|_r \le |E_4|_r$, and similarly for the odd part, $|E_{4D}|_r \le |E_4|_r$. The other two subseries satisfy the estimate from Lemma 5.3:

$$|e_{4B}|_R \le \frac{r^2}{(r-R)^2}|E_4|_r \quad \text{and} \quad |e_{4C}|_R \le \frac{r^2}{(r-R)^2}|E_4|_r.$$

We regroup some of these subseries:

$$\begin{aligned}
m_4(z_1, \bar{z}_1, x) &= \overline{e_{4B}} + e_{4C} + e_{4F} = \textstyle\sum_{a \text{ even},\,b \text{ odd}} m_4^{abI} z_1^a \bar{z}_1^b x^I,\\
f_4(z_1, \bar{z}_1, x) &= E_{4A} + e_{4B} + \overline{e_{4C}} + \overline{m_4} = \textstyle\sum_{b \text{ even}} f_4^{abI} z_1^a \bar{z}_1^b x^I.
\end{aligned}$$

Hence,

$$E_4 = f_4 + e_{4E} + m_4 - \overline{m_4}.$$

Similarly, $E_5 = f_5 + e_{5E} + m_5 - \overline{m_5}$, where f_5 is even in $\bar{z}_1$ and m_5 is even in z_1 and odd in $\bar{z}_1$.

The estimates follow from Lemma 5.5 and the Schwarz Lemma:

$$\begin{aligned}|m_4|_R &= |e'_{4B}+e_{4C}+e_{4F}|_R \le |e_{4B}|_R+|e_{4C}|_R+|e_{4F}|_R\\ &\le \frac{r^2}{(r-R)^2}|E_4|_r+\frac{r^2}{(r-R)^2}|E_4|_r+|x_2+ix_3|_R\left|\frac{E_{4D}}{z_1}\right|_R\\ &\le \left(\frac{2r^2}{(r-R)^2}+2\right)|E_4|_r,\\ |f_4|_R &= |E_{4A}+2e_{4B}+2e'_{4C}+e'_{4F}|_R \le \left(\frac{4r^2}{(r-R)^2}+3\right)|E_4|_r,\end{aligned}$$

and similarly for m_5 and f_5.

From the right-hand side of (18), let $f_7(z_1,\bar z_1,x)=-e_7+(x_4+ix_5)\overline{p_1(z_1,x,\bar z_1^2)}$, so

$$|f_7|_r \le |-e_7|_r+\left|(x_4+ix_5)\frac{f_8}{2\zeta}\right|_r \le |e_7|_r+3|e_8|_r. \tag{19}$$

It splits into even and odd parts, $f_7=f_{7A}+f_{7B}+f_{7C}$, with

$$\begin{aligned}f_{7A} &= \textstyle\sum_{b\text{ even}} f_{7A}^{abI} z_1^a\bar z_1^b x^I\\ f_{7B} &= \textstyle\sum_{a,\,b\text{ odd}} f_{7B}^{abI} z_1^a\bar z_1^b x^I,\\ f_{7C} &= \textstyle\sum_{a\text{ even},\,b\text{ odd}} f_{7C}^{abI} z_1^a\bar z_1^b x^I,\end{aligned}$$

with $|f_{7A}|_r \le |f_7|_r$ and the same bound for f_{7B}, f_{7C}. Let

$$\begin{aligned}g_{7A} &= f_{7A}+i\bar z_1 m_4-\bar z_1 m_5 = \textstyle\sum_{b\text{ even}} g_{7A}^{abI} z_1^a\bar z_1^b x^I,\\ g_{7B} &= f_{7B}-i\bar z_1\overline{m_4}+\bar z_1\overline{m_5} = \textstyle\sum_{a,\,b\text{ odd}} g_{7B}^{abI} z_1^a\bar z_1^b x^I = g_{7C}+g_{7D},\\ g_{7C} &= (z_1+x_2+ix_3)\bar z_1 \textstyle\sum_{a,\,b\text{ odd}} g_{7B}^{abI} z_1^{a-1}\bar z_1^{b-1} x^I,\\ g_{7D} &= -(x_2+ix_3)\textstyle\sum_{a,\,b\text{ odd}} g_{7B}^{abI} z_1^{a-1}\bar z_1^{b} x^I.\end{aligned}$$

Then

$$f_7 = g_{7A}+g_{7C}+g_{7D}+f_{7C}-\bar z_1(im_4-i\overline{m_4}-m_5+\overline{m_5})$$

is in a form that compares to the left-hand side of (18) to give

$$p_7^E = g_{7A}, \tag{20}$$

$$(z_1+x_2+ix_3)\bar z_1 p_7^O = g_{7C}, \tag{21}$$

$$-\bar z_1(\text{Re } p_4+i\text{ Re } p_5) = g_{7D}+f_{7C}-\bar z_1(im_4-i\overline{m_4}-m_5+\overline{m_5}). \tag{22}$$

Equations (20) and (21) determine p_7, with the estimates

$$\begin{aligned}\|p_7^E\|_R &= |g_{7A}|_R = |f_{7A} + i\zeta m_4 - \zeta m_5|_R \\ &\le |f_7|_r + R\left(\frac{2r^2}{(r-R)^2}+2\right)|E_4|_r + R\left(\frac{2r^2}{(r-R)^2}+2\right)|E_5|_r\end{aligned}$$

$$\begin{aligned}\|z_9 p_7^O\|_R &\le 3R^2\left|\frac{g_{7C}}{(z_1+x_2+ix_3)\zeta}\right|_R = 3R^2\left|\frac{g_{7B}}{z_1\zeta}\right|_R \le 3|f_{7B} - i\zeta m_4' + \zeta m_5'|_R \\ &\le 3|f_7|_r + 3R\left(\frac{2r^2}{(r-R)^2}+2\right)|E_4|_r + 3R\left(\frac{2r^2}{(r-R)^2}+2\right)|E_5|_r.\end{aligned}$$

Dividing (22) by $-\bar{z}_1$, then considering the real and imaginary parts and recalling (16) and (17), we get the system

$$\begin{aligned}&\operatorname{Re} p_4 = \operatorname{Re}\frac{g_{7D}+f_{7C}}{-\bar{z}_1} + im_4 - i\overline{m_4}, &&\operatorname{Re} p_5 = \operatorname{Im}\frac{g_{7D}+f_{7C}}{-\bar{z}_1} + im_5 - i\overline{m_5}, \\ &\operatorname{Im} p_4 = -E_4 = -f_4 - e_{4E} - m_4 + \overline{m_4}, &&\operatorname{Im} p_5 = -E_5 = -f_5 - e_{5E} - m_5 + \overline{m_5}.\end{aligned}$$

It is at this point that the second nondegeneracy condition — the full rank of the coefficient matrix (4) — is used: if the quadratic term $\epsilon_7^5\bar{z}_1x_5$ in h_7 had coefficient 0 instead of i, then $\operatorname{Re} p_5$ would not appear in the e_7 equality of (13), and (18) could not be solved this way.

Recombining the real and imaginary parts of p_4, p_5, there is (by construction) a convenient cancellation:

$$\begin{aligned}p_4 &= \operatorname{Re} p_4 + i\operatorname{Im} p_4 = \operatorname{Re}\frac{g_{7D}+f_{7C}}{-\bar{z}_1} - if_4 - ie_{4E}, \\ p_5 &= \operatorname{Re} p_5 + i\operatorname{Im} p_5 = \operatorname{Im}\frac{g_{7D}+f_{7C}}{-\bar{z}_1} - if_5 - ie_{5E}.\end{aligned}$$

These equations were set up so that e_{4E} and e_{5E} are the only terms on the right-hand side with odd powers of $\bar{z}_1$, so $p_4 = p_4^E + z_9p_4^O$ and $p_5 = p_5^E + z_9p_5^O$ are each determined by a comparison of coefficients, and by construction, p_4 and p_5 have weight $d-1$ and satisfy the estimates

$$\begin{aligned}\|p_4^E\|_R &= \left|\frac{g_{7D}}{-2\zeta} + \frac{f_{7C}}{-2\zeta} + \frac{g_{7D}'}{-2z_1} + \frac{f_{7C}'}{-2z_1} - if_4\right|_R \\ &\le \frac{1}{R}|g_{7D}|_R + \frac{1}{R}|f_{7C}|_R + |f_4|_R \\ &\le \frac{1}{R}\left|-(x_2+ix_3)\frac{g_{7B}}{z_1}\right|_R + \frac{1}{R}|f_7|_R + |f_4|_R \\ &\le \frac{3|e_7|_r + 9|e_8|_r}{R} + \left(\frac{8r^2}{(r-R)^2}+7\right)|E_4|_r + \left(\frac{4r^2}{(r-R)^2}+4\right)|E_5|_r\end{aligned}$$

$$\|p_5^E\|_R = \left|\frac{g_{7D}}{-2i\zeta} + \frac{f_{7C}}{-2i\zeta} - \frac{g'_{7D}}{-2iz_1} - \frac{f'_{7C}}{-2iz_1} - if_5\right|_R$$
$$\le \frac{3|e_7|_r + 9|e_8|_r}{R} + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_4|_r + \left(\frac{8r^2}{(r-R)^2} + 7\right)|E_5|_r,$$
$$\|z_9 p_4^O\|_R \le \|z_9 p_4^O\|_r \le 3r^2\left|\frac{-ie_{4E}}{(z_1+x_2+ix_3)\zeta}\right|_r = 3r^2\left|\frac{E_{4D}}{z_1\zeta}\right|_r \le 3|E_4|_r$$
$$\|z_9 p_5^O\|_R \le \|z_9 p_5^O\|_r \le 3r^2\left|\frac{-ie_{5E}}{(z_1+x_2+ix_3)\zeta}\right|_r = 3r^2\left|\frac{E_{5D}}{z_1\zeta}\right|_r \le 3|E_5|_r.$$

The method of finding p_2, p_3, p_9 can be copied from the solution of p_4, p_5, p_7. In the place of (16), (17), (18), the system to be solved is

$$\begin{aligned} \operatorname{Im} p_2 &= -E_2, \\ \operatorname{Im} p_3 &= -E_3, \\ p_9 - \bar{z}_1 \operatorname{Re} p_2 - i\bar{z}_1 \operatorname{Re} p_3 &= -e_9 + (z_1+x_2+ix_3)\overline{p_1} + \bar{z}_1 p_1, \end{aligned} \tag{23}$$

and the right-hand side of the third equation can be abbreviated f_9, in analogy with f_7. The estimate (19) changes to

$$|f_9|_r \le |-e_9|_r + \left|(z_1+x_2+ix_3)\frac{f_8}{2\zeta}\right|_r + \left|\zeta\frac{f'_8}{2z_1}\right|_r \le |e_9|_r + 6|e_8|_r.$$

Both the construction of the solution and the estimates proceed by only changing the subscripts from 4, 5, 7 to 2, 3, 9, and adjusting the estimate for f_9 to get the claimed results — the second nondegeneracy condition on the quadratic part of h_9 is used here also in the same way. □

Corollary 5.7. *Given* $\frac{2}{3}(n+1) \le m < n$, $r > 0$, *and* $\vec{e}(z_1, \zeta, x)$ *convergent on* D_r *with* $|\vec{e}|_r < \infty$ *and degree* $d \ge 3$, *there exists* $\vec{p}$ *that is convergent on* Δ_r, *solves the system of equations* (10)–(11), *satisfies*

$$\|p_1\|_r \le \frac{3}{2r}|e_{n-1}|_r \quad \textit{and} \quad \|p_{n-1}\|_r \le 4|e_{n-1}|_r,$$

and, for any $0 < R < r$, *satisfies*

$$\|p_2\|_R \le \frac{3|e_n|_r + 18|e_{n-1}|_r}{R} + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_2|_r + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_3|_r,$$
$$\|p_3\|_R \le \frac{3|e_n|_r + 18|e_{n-1}|_r}{R} + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_2|_r + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_3|_r,$$
$$\|p_n\|_R \le 4|e_n|_r + 24|e_{n-1}|_r + 8R\Big(\frac{r^2}{(r-R)^2} + 1\Big)(|E_2|_r + |E_3|_r).$$

Further, if $m > \frac{2}{3}(n+1)$, *then*

$$\|p_s\|_R \leq \frac{20r^2}{(r-R)^2}|E_s|_r \quad \text{for } s = 2n-2m+2, \ldots, m-1,$$

and if $m < n-1$, *then for* $t = m, \ldots, n-2$,

$$\|p_{2(t-m+2)}\|_R \leq \frac{3|e_t|_r + 9|e_{n-1}|_r}{R} + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_{2(t-m+2)}|_r + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_{2(t-m+2)+1}|_r,$$

$$\|p_{2(t-m+2)+1}\|_R \leq \frac{3|e_t|_r + 9|e_{n-1}|_r}{R} + \Big(\frac{4r^2}{(r-R)^2} + 4\Big)|E_{2(t-m+2)}|_r + \Big(\frac{8r^2}{(r-R)^2} + 10\Big)|E_{2(t-m+2)+1}|_r,$$

$$\|p_t\|_R \leq 4|e_t|_r + 12|e_{n-1}|_r + 8R\Big(\frac{r^2}{(r-R)^2} + 1\Big)(|E_{2(t-m+2)}|_r + |E_{2(t-m+2)+1}|_r).$$

Proof. The method of solution from the proof of Theorem 5.6 groups the system of equations into smaller subsystems that can be solved sequentially, so the generalization from $(7, 9)$ to (m, n) can be accomplished by a straightforward relabeling of subscripts (described below), resulting in similar estimates as claimed. The nondegeneracy conditions remain essential for any (m, n).

The solution claimed by the corollary can be chosen to have the following form, where now z abbreviates $z_2, \ldots, z_{m-1}$ and again p_1 is even in z_1:

$$p_1(\vec{z}) = p_1(z_1, z, z_{n-1})$$
$$p_j(\vec{z}) = p_j^E(z_1, z, z_{n-1}) + z_n p_j^O(z_1, z, z_{n-1}),$$

for $j = 2, \ldots, n$. For $m < n-1$, this $\vec{p}$ does not depend on $z_m, \ldots, z_{n-2}$.

The e_{n-1} equation from (10) determines p_{n-1} and p_1, exactly as in the solution of the e_8 equation, replacing the subscript 8 with $n-1$ in the first part of the preceding proof. The subscript 1 does not change.

If $m > \frac{2}{3}(n+1)$, then each of the $3m-2n-2$ individual E_s equations, $s = 2n-2m+2, \ldots, m-1$, independently determines p_s, in analogy with the solution for p_6 in terms of E_6 in the second part of the preceding proof. If $m = \frac{2}{3}(n+1)$ (the case of an isolated singularity), there are no equations analogous to the proof's E_6 equation.

The subsystem of three equations determining p_2, p_3, p_n in terms of E_2, E_3, e_n, and p_1, can be solved in analogy with the above E_2, E_3, e_9 group of equations (23), only the subscript 9 needs to change to n. If $m = n-1$, then those three equations are the only remaining ones in the system.

If $m < n-1$, then there are $n-m-1$ more subsystems of three equations, to be solved for $p_{2(t-m+2)}$, $p_{2(t-m+2)+1}$, p_t, $t = m, \dots, n-2$, in terms of $E_{2(t-m+2)}$, $E_{2(t-m+2)+1}$, e_t, and p_1, in analogy with equations (16)–(18). Solving each of these subsystems depends only on having solved for p_1, and not any other equations in the system (10). □

It is not yet claimed that using the solution $\vec{p}$ of Theorem 5.6 or Corollary 5.7 in (2) defines a local biholomorphism; this will be shown later (Theorem 6.4), under certain conditions on $\vec{e}$ and r. The most important property so far of the solution $\vec{p}$ is that the norms of its components can be estimated on Δ_R for R less than, but arbitrarily close to, r.

Corollary 5.8. *Given $\frac{2}{3}(n+1) \le m < n$, there is a constant $c_1 > 0$ (depending only on m, n) such that, for any $\vec{p}, \vec{e}$ as in Corollary 5.7 and any radius lengths ρ, r with $\frac{1}{2} < \rho < r \le 1$, we have*

$$\max_{j=1,\dots,n} \left\{ \|p_j\|_\rho \right\} \le \frac{c_1 |\vec{e}|_r}{(r-\rho)^2} \quad \textit{and} \quad \max_{j=1,\dots,n} \left\{ \sum_{k=1}^{n} \left\| \frac{dp_k}{dz_j} \right\|_\rho \right\} \le \frac{c_1 |\vec{e}|_r}{(r-\rho)^3}.$$

Proof. Let $R = \frac{1}{2}(\rho + r)$. The bound on each p_j follows from $\|p_j\|_\rho \le \|p_j\|_R$ and the bounds from the previous corollary, using $\frac{1}{2} < R < r \le 1$, and $16 < 1/(r-R)^2 = 4/(r-\rho)^2$. The bounds for the derivatives of p_k follow from this consequence of Cauchy's estimate (for which see [Ahlfors 1979]): *If $0 < R_2 < R_1$ and $f(w)$ is holomorphic and bounded by K for $|w| < R_1$, then df/dw is bounded by $K/(R_1 - R_2)$ for $|w| < R_2$.*

This fact can be applied with $K = \|p_k\|_R$ and $R_1 - R_2 = R - \rho = \frac{1}{2}(r-\rho)$ for the $z_1, \dots, z_{m-1}$ derivatives, $R_1 - R_2 = R^2 - \rho^2 > R - \rho = \frac{1}{2}(r-\rho)$ for the z_{n-1} derivatives, and $R_1 - R_2 = 3R^2 - 3\rho^2 > \frac{3}{2}(r-\rho)$ for the z_n derivatives. The $z_m, \dots, z_{n-2}$ derivatives are zero by construction. □

The lower bound $r > \frac{1}{2}$ was important for the previous corollary, but it is not a significant *a priori* restriction on the manifold M. By a real rescaling $\vec{z} \mapsto (a_1 z_1, \dots, a_1 z_{m-1}, {a_1}^2 z_m, \dots, {a_1}^2 z_n)$, with $a_1 > 0$, equations (5) can be assumed to define M for $|z_1| < 1$, $|x_s| < 1$; and for any $\eta > 0$, there is a rescaling making $|\vec{e}|_1$ less than η.

6. The new defining equations and some estimates

To get a solution of the nonlinear equation (9) by iterating the solution of the linear equation, the rapid convergence technique will apply, closely following the methods used in [Moser 1985] on a different CR singularity problem. Each step along the way to a proof of Proposition 3.3 is stated as a theorem.

Substituting the linear equation's normalized solution $\vec{p}$ from Corollary 5.7 into $E_2, \ldots, e_n$ in the right-hand side of the nonlinear equation (9) gives a quantity $\vec{q}$ depending on z_1, $\bar{z}_1$, x. Let

$$\vec{z} = (z_1, x, \ldots, \bar{z}_1(x_{2(t-m+2)} + i x_{2(t-m+2)+1}), \ldots, \bar{z}_1^2, \bar{z}_1(z_1 + x_2 + i x_3)), \tag{24}$$

as in (11), let

$$\begin{aligned}\vec{z} + \vec{e} = \big(z_1, x_2 + i E_2, \ldots, x_{m-1} + i E_{m-1}, \ldots,&\\ \bar{z}_1(x_{2(t-m+2)} + i x_{2(t-m+2)+1}) + e_t, \ldots,&\\ \bar{z}_1^2 + e_{n-1}, \bar{z}_1(z_1 + x_2 + i x_3) + e_n\big),&\end{aligned} \tag{25}$$

as in (8), and then define

$$\vec{q}(z_1, \bar{z}_1, x) = (Q_2, \ldots, Q_{m-1}, q_m, \ldots, q_n)$$

by

$$\begin{aligned} Q_s &= \operatorname{Im}\big(p_s(\vec{z}+\vec{e}) - p_s(\vec{z})\big), \\ q_t &= p_t(\vec{z}+\vec{e}) - p_t(\vec{z}) - (x_{2(t-m+2)} + i x_{2(t-m+2)+1})\overline{(p_1(\vec{z}+\vec{e}) - p_1(\vec{z}))} \\ &\quad - \bar{z}_1 \cdot \operatorname{Re}\big(p_{2(t-m+2)}(\vec{z}+\vec{e}) - p_{2(t-m+2)}(\vec{z})\big) \\ &\quad - i\bar{z}_1 \operatorname{Re}\big(p_{2(t-m+2)+1}(\vec{z}+\vec{e}) - p_{2(t-m+2)+1}(\vec{z})\big) \\ &\quad - \overline{p_1(\vec{z}+\vec{e})}\big(\operatorname{Re} p_{2(t-m+2)}(\vec{z}+\vec{e}) + i \operatorname{Re} p_{2(t-m+2)+1}(\vec{z}+\vec{e})\big), \\ q_{n-1} &= p_{n-1}(\vec{z}+\vec{e}) - p_{n-1}(\vec{z}) - 2\bar{z}_1\overline{(p_1(\vec{z}+\vec{e}) - p_1(\vec{z}))} - \overline{(p_1(\vec{z}+\vec{e}))}^2, \\ q_n &= p_n(\vec{z}+\vec{e}) - p_n(\vec{z}) - \bar{z}_1(p_1(\vec{z}+\vec{e}) - p_1(\vec{z})) \\ &\quad - \bar{z}_1(\operatorname{Re}\big(p_2(\vec{z}+\vec{e}) - p_2(\vec{z})\big) + i \operatorname{Re}\big(p_3(\vec{z}+\vec{e}) - p_3(\vec{z})\big)) \\ &\quad - (z_1 + x_2 + i x_3)\overline{(p_1(\vec{z}+\vec{e}) - p_1(\vec{z}))} \\ &\quad - \overline{(p_1(\vec{z}+\vec{e}))}(p_1(\vec{z}+\vec{e}) + \operatorname{Re} p_2(\vec{z}+\vec{e}) + i \operatorname{Re} p_3(\vec{z}+\vec{e})). \end{aligned} \tag{26}$$

To outline the role of $\vec{q}$ in the argument, the next step (Theorem 6.2) will suppose that $\vec{p}(z_1, \ldots, z_n)$ is complex analytic on Δ_ρ and $|\vec{e}|_\sigma$ is small enough that $\vec{z} \in \Delta_\sigma$ implies $\vec{z} + \vec{e} \in \Delta_\rho$; hence $\vec{q}$ is a real analytic function for $(z_1, \bar{z}_1, x) \in D_\sigma$. If $\vec{q}(z_1, \bar{z}_1, x)$ happens to be identically zero, the manifold M has been brought to normal form by the functions $\vec{p}$. Otherwise, the degree of $\vec{q}$ is at least $2d-2$ by the construction of the solution $\vec{p}$, and defining $\vec{q}(z_1, \zeta, x)$ by Equations (26), with ζ formally substituted for $\bar{z}_1$ and allowing complex x, the norm $|\vec{q}|_\sigma$ can be bounded in terms of the norm of $\vec{e}$. Then later, in the proof of Theorem 6.6, converting $\vec{q}(z_1, \bar{z}_1, x)$ into an expression in $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$ and equating it to the $\tilde{z}$ polynomial expression in (7) gives the defining equations of M in the $\tilde{z}$ coordinate system.

The case $\tilde{N} = N$ of the next result is [Coffman 2004, Lemma 4.1].

Lemma 6.1. *Let $f = (f_1, \dots, f_{\tilde{N}}) : \mathbb{D}_{\boldsymbol{r}} \to \mathbb{C}^{\tilde{N}}$ be a holomorphic map with*

$$\max_{j=1,\dots,N} \left\{ \sum_{k=1}^{\tilde{N}} \sup_{\vec{z}\in\mathbb{D}_{\boldsymbol{r}}} \left| \frac{df_k}{dz_j}(\vec{z}) \right| \right\} \le K.$$

Then, for $\vec{z}, \vec{z}\,' \in \mathbb{D}_{\boldsymbol{r}}$,

$$\sum_{k=1}^{\tilde{N}} |f_k(\vec{z}\,') - f_k(\vec{z})| \le K \sum_{j=1}^{N} |z'_j - z_j|.$$

Theorem 6.2. *There are some constants $c_2 > 0$ and $\delta_1 > 0$ (depending on m, n) such that if $\frac{1}{2} < \sigma < r \le 1$, and $\vec{e}$ is as in Corollary 5.7, with $|\vec{e}|_r \le \delta_1(r - \sigma)$, then*

$$|\vec{q}|_\sigma \le \frac{c_2 |\vec{e}|_r^2}{(r-\sigma)^3}.$$

Proof. Let $\rho = \frac{1}{2}(r + \sigma)$. Note that if $\delta_1 \le \frac{1}{2}$, the formal series for $\vec{q}$ is convergent on D_σ, since then, for $(z_1, \zeta, x) \in D_r$, $|x_s + iE_s| < \sigma + \delta_1(r - \sigma) \le \sigma + (\rho - \sigma) = \rho$, $|\zeta(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) + e_t| < 2\sigma^2 + (\rho - \sigma) < 2\sigma^2 + (\rho - \sigma)(2(\rho + \sigma)) = 2\rho^2$, and similarly $|\zeta^2 + e_{n-1}| < \rho^2$ and $|(z_1 + x_2 + ix_3)\zeta + e_n| < 3\rho^2$, so $\vec{z} + \vec{e} \in \Delta_\rho$, which is contained in the domain of $\vec{p}$ by Corollary 5.7. The case $N = n$, $\tilde{N} = 1$, $\mathbb{D}_{\boldsymbol{r}} = \Delta_\rho$ of Lemma 6.1 applies to $p_k : \Delta_\rho \to \mathbb{C}$, with

$$\max_{j=1,\dots,n} \left\{ \left\| \frac{dp_k}{dz_j} \right\|_\rho \right\} \le K = \frac{c_1 |\vec{e}|_r}{(r-\rho)^3},$$

by Corollary 5.8, and $\vec{z}\,' = \vec{z} + \vec{e} \in \Delta_\rho$, so the conclusion is

$$|p_k(\vec{z} + \vec{e}) - p_k(\vec{z})| \le K(|E_2|_r + \cdots + |e_n|_r) = \frac{c_1 |\vec{e}|_r}{(r-\rho)^3} |\vec{e}|_r = \frac{8c_1 |\vec{e}|_r^2}{(r-\sigma)^3}.$$

This provides bounds for the differences that appear in (26), and the remaining terms are the products, where we can use $\frac{1}{2} < \sigma < \rho < r \le 1$, the bound of Theorem 5.6 on the p_1 factor, and the bounds of Corollary 5.8 on the other factors. For example, for the q_t equation of (26), in a case where $t = m < n - 1$, part of the expression is the product

$$\sup_{D_\sigma} \left| (p_1(\vec{z} + \vec{e}))' \frac{p_4(\vec{z} + \vec{e}) + (p_4(\vec{z} + \vec{e}))' + ip_5(\vec{z} + \vec{e}) + i(p_5(\vec{z} + \vec{e}))'}{2} \right|$$

$$\le \|p_1\|_\rho \big(\|p_4\|_\rho + \|p_5\|_\rho \big) \le \frac{3}{2r} |e_{n-1}|_r \frac{2c_1 |\vec{e}|_r}{(r-\rho)^2} < \frac{6c_1 |\vec{e}|_r^2}{(r-\rho)^2} < \frac{12c_1 |\vec{e}|_r^2}{(r-\sigma)^3}. \qquad \square$$

The following lemma on inverse functions will be used twice, in the construction of the new coordinate system and the new defining equations; a proof by a standard iteration procedure is sketched in [Coffman 2004].

Lemma 6.3. *Suppose* $0 < R_{2,k} < R_{1,k}$ *for* $k = 1, \dots, N$, *so that*

$$\mathbb{D}^2 = \mathbb{D}_{(R_{2,1},\dots,R_{2,N})} \subseteq \mathbb{D}^1 = \mathbb{D}_{(R_{1,1},\dots,R_{1,N})}.$$

Let $f(\vec{z}) = (f_1(z_1, \dots, z_N), \dots, f_N(z_1, \dots, z_N))$ *be holomorphic on* $\mathbb{D}^1$, *with*

$$\max_{j=1,\dots,N} \left\{ \sum_{k=1}^{N} \sup_{\vec{z}\in\mathbb{D}^1} \left| \frac{df_k}{dz_j}(\vec{z}) \right| \right\} \leq K < 1$$

and

$$\sum_{k=1}^{N} \sup_{\vec{z}\in\mathbb{D}^2} |f_k(\vec{z})| \leq (1-K) \min_{k=1,\dots,N} \{R_{1,k} - R_{2,k}\}.$$

Given $\vec{w} \in \mathbb{D}^2$, *there exists a unique solution* $\vec{z} \in \mathbb{D}^1$ *of the equation*

$$\vec{w} = \vec{z} + f(\vec{z}),$$

and this solution satisfies

$$\sum_{k=1}^{N} |z_k - w_k| \leq \frac{1}{1-K} \sum_{k=1}^{N} |f_k(\vec{w})|.$$

Theorem 6.4. *There is some constant* $\delta_2 > 0$ (*depending on* m, n) *so that for any radius lengths* $\frac{1}{2} < \sigma < r \leq 1$, *and* $\vec{e}$, $\vec{p}$ *as in Corollary 5.7, with* $|\vec{e}|_r \leq \delta_2(r-\sigma)^3$ *and* $\rho = \frac{1}{2}(r+\sigma)$, *the transformation*

$$\Psi : \vec{z} = (z_1, \dots, z_n) \mapsto \tilde{z} = \bigl(z_1 + p_1(\vec{z}), \dots, z_n + p_n(\vec{z})\bigr)$$

has a holomorphic inverse $\psi(\tilde{z}) = \vec{z}$ *such that* $\tilde{z} \in \Delta_\sigma$ *implies* $\psi(\tilde{z}) \in \Delta_\rho$.

Proof. By Corollary 5.8,

$$\max_{j=1,\dots,n} \left\{ \sum_{k=1}^{n} \left\| \frac{dp_k}{dz_j} \right\|_\rho \right\} \leq \frac{c_1|\vec{e}|_r}{(r-\rho)^3} \leq \frac{c_1\delta_2(r-\sigma)^3}{(r-\rho)^3} = 8\delta_2 c_1 \leq \tfrac{1}{2} = K,$$

if $\delta_2 \leq 1/(16c_1)$. Also by Corollary 5.8,

$$\sum_{k=1}^{n} \|p_k\|_\sigma \leq \frac{nc_1|\vec{e}|_r}{(r-\sigma)^2} \leq nc_1\delta_2(r-\sigma) \leq (1-K)(\rho-\sigma)$$

if $\delta_2 \leq 1/(4nc_1)$. The hypotheses of Lemma 6.3 are satisfied with $\Delta_\sigma \subseteq \Delta_\rho$, and $R_{1,k} - R_{2,k} \geq \rho - \sigma$, so given $\tilde{z} \in \Delta_\sigma$, there exists a unique $\vec{z} \in \Delta_\rho$ such that $\tilde{z} = \bigl(z_1 + p_1(\vec{z}), \dots, z_n + p_n(\vec{z})\bigr)$. This defines ψ so that $\Psi \circ \psi$ is the identity map on Δ_σ. □

For $(z_1, \zeta, x) \in D_{R_1} \subseteq \mathbb{C}^m$, define $z^c \in \mathbb{C}^n$ by

$$\begin{aligned} z^c = (z_1,\ & x_2 + iE_2(z_1, \zeta, x), \dots, x_{m-1} + iE_{m-1}(z_1, \zeta, x), \dots, \\ & \zeta(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) + e_t(z_1, \zeta, x), \dots, \\ & \zeta^2 + e_{n-1}(z_1, \zeta, x), \zeta(z_1 + x_2 + ix_3) + e_n(z_1, \zeta, x)), \end{aligned}$$

and define a map $\tau : D_{R_1} \to \mathbb{C}^m$ by

$$\begin{aligned} \tau(z_1, \zeta, x) &= \big(\tau_1(z_1, \zeta, x), \dots, \tau_m(z_1, \zeta, x)\big) \\ &= \big(z_1 + p_1(z^c), \zeta + (p_1(z^c))', x_2 + \tfrac{1}{2}(p_2(z^c) + (p_2(z^c))'), \dots, \\ &\qquad x_{m-1} + \tfrac{1}{2}(p_{m-1}(z^c) + (p_{m-1}(z^c))')\big). \end{aligned}$$

Theorem 6.5. *There is some constant $\delta_3 > 0$ (depending on m, n) so that for any radius lengths $\frac{1}{2} < r' < r \le 1$, with $\sigma = r' + \frac{1}{3}(r - r')$, and any $\vec{e}$, $\vec{p}$ as in Corollary 5.7, with $|\vec{e}|_r \le \delta_3(r - r')^3$, the transformation $\tau : (z_1, \zeta, x) \mapsto (\tilde{z}_1, \tilde{\zeta}, \tilde{x})$ has a holomorphic inverse $\phi(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) = (z_1, \zeta, x)$ such that if $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) \in D_{r'}$, then $\phi(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) \in D_\sigma$.*

Proof. Let $\rho = r' + \frac{2}{3}(r - r')$, so $\sigma - r' = \rho - \sigma = r - \rho = \frac{1}{3}(r - r') < \frac{1}{6}$, and let $\bar{r} = \frac{1}{2}(r + r')$, so $\frac{1}{2} < r' < \sigma < \bar{r} < \rho < r \le 1$. If $(z_1, \zeta, x) \in D_{\bar{r}}$, and $\delta_3 \le \frac{2}{3}$, then $|E_2(z_1, \zeta, x)| \le \delta_3(r - r')^3 = 216\delta_3(\rho - \bar{r})^3 < (216/12^2)\,\delta_3(\rho - \bar{r}) \le \rho - \bar{r}$, and similarly $|e_{n-1}(z_1, \zeta, x)| < \rho^2 - \bar{r}^2$, etc., so $z^c \in \Delta_\rho$, and $\vec{p}(z^c)$ and τ are well-defined and holomorphic on $D_{\bar{r}}$. Using Cauchy's estimate as in Corollary 5.8, for $(z_1, \zeta, x) \in D_\sigma$,

$$\left|\frac{d}{dz_1} p_2(z^c)\right| \le \frac{|p_2(z^c)|_{\bar{r}}}{\bar{r} - \sigma} \le \frac{\|p_2\|_\rho}{\frac{1}{2}(\rho - \sigma)} \le \frac{2c_1|\vec{e}|_r}{(\rho - \sigma)(r - \rho)^2} = \frac{54c_1|\vec{e}|_r}{(r - r')^3}.$$

Similarly, the derivative of each term, $p_1(z^c)$, $p_s(z^c)$, $(p_1(z^c))'$, $(p_s(z^c))'$, with respect to each variable z_1, ζ, x_s, is bounded by a comparable quantity, so there is some constant $c_3 > 0$ (depending on m, n) so that

$$\begin{aligned} \max_{j=2,\dots,m-1} \Bigg\{ & \left|\frac{dp_1(z^c)}{dz_1}\right|_\sigma + \left|\frac{d((p_1(z^c))')}{dz_1}\right|_\sigma + \sum_{s=2}^{m-1} \left|\frac{d(\frac{1}{2}(p_s(z^c) + (p_s(z^c))'))}{dz_1}\right|_\sigma, \\ & \left|\frac{dp_1(z^c)}{d\zeta}\right|_\sigma + \left|\frac{d((p_1(z^c))')}{d\zeta}\right|_\sigma + \sum_{s=2}^{m-1} \left|\frac{d(\frac{1}{2}(p_s(z^c) + (p_s(z^c))'))}{d\zeta}\right|_\sigma, \\ & \left|\frac{dp_1(z^c)}{dx_j}\right|_\sigma + \left|\frac{d((p_1(z^c))')}{dx_j}\right|_\sigma + \sum_{s=2}^{m-1} \left|\frac{d(\frac{1}{2}(p_s(z^c) + (p_s(z^c))'))}{dx_j}\right|_\sigma \Bigg\} \\ & \le c_3|\vec{e}|_r/(r - r')^3 \le c_3\delta_3 \le \tfrac{1}{2} \end{aligned}$$

if $\delta_3 \le 1/(2c_3)$. It also follows from Corollary 5.8 that

$$
\begin{aligned}
|p_1(z^c)|_{r'} + |(p_1(z^c))'|_{r'} + \sum_{s=2}^{m-1} \left|\tfrac{1}{2}(p_s(z^c) + (p_s(z^c))')\right|_{r'} \\
\leq 2\|p_1\|_\rho + \sum_{s=2}^{m-1} \|p_s\|_\rho \leq \frac{mc_1|\vec{e}|_r}{(r-\rho)^2} \leq \frac{mc_1\delta_3(r-r')^3}{(r-\rho)^2} \\
= 9mc_1\delta_3(r-r') \leq \tfrac{1}{2}(\sigma - r')
\end{aligned}
$$

if $\delta_3 \leq 1/(54mc_1)$. So, by Lemma 6.3, given $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) \in D_{r'}$, there exists a unique $(z_1, \zeta, x) \in D_\sigma$ such that $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) = \tau(z_1, \zeta, x)$. □

By inspection of the form of τ, if $(z_1, \zeta, x) \in D_\sigma$ and $\tau(z_1, \zeta, x) = (\tilde{z}_1, \tilde{\zeta}, \tilde{x})$, then $\tau(\bar{\zeta}, \bar{z}_1, \bar{x}) = (\bar{\tilde{\zeta}}, \bar{\tilde{z}}_1, \bar{\tilde{x}})$. If, further, $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) = (\bar{\tilde{\zeta}}, \bar{\tilde{z}}_1, \bar{\tilde{x}}) \in D_{r'}$, then $(z_1, \zeta, x) = (\bar{\zeta}, \bar{z}_1, \bar{x})$ by uniqueness of the inverse. In particular, if $|\tilde{z}_1| < r'$ and for $s = 2, \ldots, m-1$, $\tilde{x}_s$ is real and $|\tilde{x}_s| < r'$, then $\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$ is of the form $(z_1, \bar{z}_1, x)$ for some z_1 with $|z_1| < \sigma$ and x real with $|x_s| < \sigma$. Such (z_1, x) is unique, given $(\tilde{z}_1, \tilde{x})$: suppose there were (z_1^0, x^0) with $|z_1^0| < \sigma$, $|x_s^0| < \sigma$, x^0 real, such that

$$
\begin{aligned}
\tilde{z}_1 &= \tau_1(z_1^0, \overline{z_1^0}, x^0), \\
\tilde{x}_s &= \tau_{s+1}(z_1^0, \overline{z_1^0}, x^0) \quad \text{for } s = 2, \ldots, m-1.
\end{aligned}
$$

Then the second component $\tau_2(z_1^0, \overline{z_1^0}, x^0)$ can be calculated to have some value $\tilde{\zeta}$, so $\tau(z_1^0, \overline{z_1^0}, x^0) = (\tilde{z}_1, \tilde{\zeta}, \tilde{x})$. By the formula for τ, $\tilde{\zeta} = \bar{\tilde{z}}_1$, so $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) \in D_{r'}$ and $(z_1^0, \overline{z_1^0}, x^0) = \phi(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) = \phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) = (z_1, \bar{z}_1, x)$, so we can conclude from the uniqueness of Lemma 6.3 that $z_1^0 = z_1$ and $x^0 = x$.

Theorem 6.6. *There exist constants $c_4 > 0$ and $\delta_4 > 0$ (depending on m, n) such that for any $\frac{1}{2} < r' < r \leq 1$ (with σ, ρ as in the previous theorem), and any $\vec{e}$ as in Corollary 5.7 with $|\vec{e}|_r \leq \delta_4(r-r')^3$, there exist a holomorphic map*

$$
\Psi : \Delta_\rho \to \mathbb{C}^n, \quad (z_1, \ldots, z_n) \mapsto (\tilde{z}_1, \ldots, \tilde{z}_n),
$$

with a holomorphic inverse $\psi : \Delta_\sigma \to \Delta_\rho$, and a holomorphic map $\tilde{e} = (\tilde{E}_2, \ldots, \tilde{e}_n)$ from $D_{r'}$ to $\mathbb{C}^{n-1}$, such that the defining equations for M are

$$
\begin{aligned}
\tilde{y}_s &= \tilde{E}_s(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}), \\
\tilde{z}_t &= \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + \tilde{e}_t(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}), \\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}), \\
\tilde{z}_n &= \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}),
\end{aligned}
$$

for $|\tilde{z}_1| < r'$, $|\tilde{x}_s| < r'$. The degree of $\tilde{e}$ is at least $2d-2$, and

$$
|\tilde{e}|_{r'} \leq \frac{c_4|\vec{e}|_r^2}{(r-r')^3}.
$$

Proof. Initially, choose $\delta_4 \le \min\{\frac{8}{3}\delta_1, \frac{8}{27}\delta_2, \delta_3\}$, so that Theorems 6.2, 6.4, 6.5 apply, and define Ψ, ψ, $\vec{q}$, and ϕ in terms of the given $\vec{e}$ and the functions $\vec{p}$ constructed in Corollary 5.7. Define $\tilde{e}$ to be the composite of holomorphic maps $\vec{q}\circ\phi : D_{r'} \to \mathbb{C}^{n-1}$, so that by Theorem 6.2,

$$|\tilde{e}|_{r'} \le |\vec{q}|_\sigma \le \frac{c_2|\vec{e}|_r^2}{(r-\sigma)^3} = \frac{c_2|\vec{e}|_r^2}{(\frac{2}{3}(r-r'))^3}.$$

Since $\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$ has no constant terms, and $\vec{q}$ has degree $\ge 2d-2$ by construction, $\tilde{e}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$ also has degree at least $2d-2$.

Given $\tilde{z}_1$, $\tilde{x}$ such that $|\tilde{z}_1| < r'$, and $\tilde{x}$ is real with $|\tilde{x}_s| < r'$, define quantities $\tilde{z}_2, \dots, \tilde{z}_n$ by

$$\begin{aligned}
\tilde{z}_s &= \tilde{x}_s + i\tilde{E}_s(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}),\\
\tilde{z}_t &= \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + \tilde{e}_t(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})\\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}),\\
\tilde{z}_n &= \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}),
\end{aligned} \tag{27}$$

and define $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_n)$. The claim of the theorem is that $\psi(\tilde{z}) \in M$.

If ${\delta_4}^2 \le 32/(81c_2)$, then

$$|\tilde{e}|_{r'} \le \frac{c_2(\delta_4(r-r')^3)^2}{(r-\sigma)^3} = c_2(\delta_4)^2\frac{3^6}{2^3}(\sigma-r')^3 \le c_2(\delta_4)^2\frac{3^6}{2^3 6^2}(\sigma-r') \le \sigma-r',$$

so $\tilde{z} \in \Delta_\sigma$, the domain of ψ.

By Theorem 6.5, there exists a unique (z_1, x) (the first and last components of $(z_1, \bar{z}_1, x) = \phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$) such that $|z_1| < \sigma$, x is real with $|x_s| < \sigma$, and

$$\begin{aligned}
\tilde{z}_1 &= z_1 + p_1(z_1, x_2 + iE_2(z_1, \bar{z}_1, x), \dots, \bar{z}_1(z_1 + x_2 + ix_3) + e_n(z_1, \bar{z}_1, x)),\\
\tilde{x}_s &= x_s + \operatorname{Re}\big(p_s(z_1, x_2 + iE_2(z_1, \bar{z}_1, x), \dots, \bar{z}_1(z_1 + x_2 + ix_3) + e_n(z_1, \bar{z}_1, x))\big).
\end{aligned}$$

Then define quantities $z_2, \dots, z_n$ by

$$\begin{aligned}
z_s &= x_s + iE_s(z_1, \bar{z}_1, x),\\
z_t &= \bar{z}_1(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) + e_t(z_1, \bar{z}_1, x),\\
z_{n-1} &= \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x),\\
z_n &= \bar{z}_1(z_1 + x_2 + ix_3) + e_n(z_1, \bar{z}_1, x),
\end{aligned}$$

and define, as in (11) and (24), $\vec{z} = (z_1, x, \dots, \bar{z}_1(z_1 + x_2 + ix_3))$ and $\vec{z} + \vec{e} = (z_1, z_2, \dots, z_n)$ as in (8) and (25). Since $|z_1| < \sigma < r$ and $|x_s| < \sigma < r$, $\vec{z} + \vec{e} \in M$, and if $\delta_4 \le \frac{4}{3}$, then

$$|\vec{e}|_\sigma \le |\vec{e}|_r \le \delta_4(r-r')^3 = \delta_4 \cdot 27(\rho-\sigma)^3 < \delta_4 \frac{27}{6^2}(\rho-\sigma) \le (\rho-\sigma),$$

so $\vec{z}+\vec{e} \in \Delta_\rho$, which is contained in the domain of $\vec{p}$.

Next, by the construction of $\vec{q}$, $\tilde{e}$, and $\tilde{z}$, we see that $\Psi(\vec{z}+\vec{e})$ equals

$$\begin{aligned}
&\big(z_1+p_1(\vec{z}+\vec{e}),\dots,z_n+p_n(\vec{z}+\vec{e})\big)\\
&= \Big(\tilde{z}_1,\dots,\tilde{x}_s+iE_s(z_1,\bar{z}_1,x)+i\operatorname{Im} p_s(\vec{z}+\vec{e}),\dots,\\
&\qquad \overline{\tilde{z}_1-p_1(\vec{z}+\vec{e})}\big(\tilde{x}_{2(t-m+2)}-\operatorname{Re} p_{2(t-m+2)}(\vec{z}+\vec{e})\big)\\
&\qquad\qquad +i\overline{\tilde{z}_1-p_1(\vec{z}+\vec{e})}\big(\tilde{x}_{2(t-m+2)+1}-\operatorname{Re} p_{2(t-m+2)+1}(\vec{z}+\vec{e})\big)\\
&\qquad\qquad +e_t(z_1,\bar{z}_1,x)+p_t(\vec{z}+\vec{e}),\dots,\\
&\qquad \overline{\tilde{z}_1-p_1(\vec{z}+\vec{e})}^2+e_{n-1}(z_1,\bar{z}_1,x)+p_{n-1}(\vec{z}+\vec{e}),\\
&\qquad \overline{\tilde{z}_1-p_1(\vec{z}+\vec{e})}\big(\tilde{z}_1-p_1(\vec{z}+\vec{e})+\tilde{x}_2-\operatorname{Re} p_2(\vec{z}+\vec{e})+i\tilde{x}_3-i\operatorname{Re} p_3(\vec{z}+\vec{e})\big)\\
&\qquad\qquad\qquad +e_n(z_1,\bar{z}_1,x)+p_n(\vec{z}+\vec{e})\Big)\\
&= \big(\tilde{z}_1,\dots,\tilde{x}_s+iQ_s(z_1,\bar{z}_1,x),\dots,\\
&\qquad\qquad \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)}+i\tilde{x}_{2(t-m+2)+1})+q_t(z_1,\bar{z}_1,x),\dots,\\
&\qquad\qquad\qquad \bar{\tilde{z}}_1^2+q_{n-1}(z_1,\bar{z}_1,x),\bar{\tilde{z}}_1(\tilde{z}_1+\tilde{x}_2+i\tilde{x}_3)+q_n(z_1,\bar{z}_1,x)\big)\\
&= \big(\tilde{z}_1,\dots,\tilde{x}_s+iQ_s(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x})),\dots,\\
&\qquad\qquad \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)}+i\tilde{x}_{2(t-m+2)+1})+q_t(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x})),\dots,\\
&\qquad\qquad\qquad \bar{\tilde{z}}_1^2+q_{n-1}(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x})),\bar{\tilde{z}}_1(\tilde{z}_1+\tilde{x}_2+i\tilde{x}_3)+q_n(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}))\big)\\
&= \big(\tilde{z}_1,\dots,\tilde{x}_s+i\tilde{E}_s(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}),\dots,\\
&\qquad\qquad \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)}+i\tilde{x}_{2(t-m+2)+1})+\tilde{e}_t(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}),\dots,\\
&\qquad\qquad\qquad \bar{\tilde{z}}_1^2+\tilde{e}_{n-1}(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}),\bar{\tilde{z}}_1(\tilde{z}_1+\tilde{x}_2+i\tilde{x}_3)+\tilde{e}_n(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x})\big)\\
&= \tilde{z}.
\end{aligned}$$

Here we have used the fact that $\vec{p}$ is a solution of (10)–(11). By the uniqueness of Theorem 6.4, $\psi(\tilde{z}) = \vec{z}+\vec{e}$ lies in M. □

7. Composition of approximate solutions

The previous theorem's quadratic estimate on the size of $\tilde{e}$ in terms of $\vec{e}$ allows for the rapid convergence of a sequence of approximations. A couple of technical lemmas will be needed to measure the behavior of composite mappings. Theorem 7.7, which is the last step in proving Proposition 3.3, uses these lemmas and the estimates of the previous section to prove convergence of a sequence of transformations, following the ideas of [Moser 1985].

Notation 7.1. For $R_1 > 0$ and a $n \times n$ matrix of complex-valued functions $F = (F_{kj}(\vec{z}))$ on Δ_{R_1}, define

$$|||F|||_{R_1} = \max_{j=1,\dots,n} \left\{ \sum_{k=1}^{n} \sup_{\vec{z} \in \Delta_{R_1}} |F_{kj}(\vec{z})| \right\}.$$

This "maximum column sum" norm has already appeared, in Corollary 5.8 and Lemmas 6.1 and 6.3, in the case where $F = \mathrm{D}f = \mathrm{D}_{\vec{z}} f$, the Jacobian matrix of some map $f : \Delta_{R_1} \to \mathbb{C}^n$ at $\vec{z} \in \Delta_{R_1}$. The 3×3 case of the following lemma was proved in [Coffman 2004].

Lemma 7.2. *If* $|||A|||_{R_1} < 1$, *then* $\mathbb{1} + A$ *is invertible* (*where* $\mathbb{1}$ *is the* $n \times n$ *identity matrix*), *and*

$$|||(\mathbb{1} + A)^{-1}|||_{R_1} \le \frac{1}{1 - |||A|||_{R_1}}.$$

We need an elementary fact from the calculus of one real variable:

Lemma 7.3. *If* μ_k *is a sequence such that* $0 \le \mu_k < 1$ *and* $\sum_{k=0}^{\infty} \mu_k$ *is a convergent series*, *then the sequence of partial products*

$$\prod_{k=0}^{N} \frac{1}{1 - \mu_k}$$

is bounded above by some positive limit.

Notation 7.4. For $\nu = 0, 1, 2, \dots$, define a sequence $\left\{1, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \dots\right\}$ by the formula

$$r_\nu = \frac{1}{2}\left(1 + \frac{1}{\nu + 1}\right).$$

Note that $\frac{1}{2} < r_\nu \le 1$, and the sequence is decreasing, with

$$r_\nu - r_{\nu+1} = \frac{1}{2(\nu+1)(\nu+2)} \quad \text{and} \quad \frac{r_{\nu+1} - r_{\nu+2}}{r_\nu - r_{\nu+1}} = \frac{\nu+1}{\nu+3} \ge \frac{1}{3}.$$

Notation 7.5. Define $\sigma_\nu = r_{\nu+1} + \frac{1}{3}(r_\nu - r_{\nu+1})$, $\rho_\nu = r_{\nu+1} + \frac{2}{3}(r_\nu - r_{\nu+1})$, as in Theorem 6.5.

Recall that given $\eta > 0$, there is some scaling transformation so that $M \cap \Delta_1$ is defined by (5), with $\vec{e}$ holomorphic on D_1, degree $d \ge 3$, and $|\vec{e}|_1 \le \eta$.

Notation 7.6. Set $\vec{e}_0 = \vec{e}$ (so $|\vec{e}_0|_{r_0} = |\vec{e}|_1 \le \eta$), and inductively define the formal series $\vec{e}_{\nu+1}(z_1, \zeta, x)$ in terms of $\vec{e}_\nu(z_1, \zeta, x)$, by the $\vec{e} \mapsto \tilde{e}$ procedure of Theorem 6.6, with $r = r_\nu$, $r' = r_{\nu+1}$. Each $\vec{e}_\nu$ defines, as in the previous theorems, functions $\vec{p}_\nu$, $\vec{q}_\nu$, Ψ_ν, ψ_ν, ϕ_ν, and the degree of $\vec{e}_\nu$ is denoted d_ν.

Also recall that the degree $d_{\nu+1}$ of $\vec{e}_{\nu+1}$ is at least $2d_\nu - 2$; it can be checked that this, together with $d_0 = d \geq 3$, implies $d_\nu \geq 2^\nu + 2$.

The plan is to show that the bound for $\vec{e}_\nu$ in the hypothesis of Theorem 6.6 holds for all ν, to get a sequence of transformations $\psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_\nu}$, so that the composition

$$\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_0}$$

is well-defined, $\vec{e}_\nu$ is holomorphic on D_{r_ν}, and $\lim_{\nu\to\infty} |\vec{e}_\nu|_{r_\nu} = 0$.

Theorem 7.7. *There exists $\eta > 0$ (depending on m, n) so that if $\vec{e}_0$ and M are as described above, then there exists a holomorphic transformation $\psi : \Delta_{1/2} \to \mathbb{C}^n$, with a holomorphic inverse Ψ, and such that if $\tilde{z} \in \widetilde{M} \cap \Delta_{1/2}$, then $\psi(\tilde{z}) \in M$.*

Proof. Set $\delta_5 = \min\{\delta_4, 1/(27c_4)\}$ and choose $0 < \eta < \min\{\delta_5/64, 1/(1728c_1)\}$. It will be shown that $|\vec{e}_\nu|_{r_\nu} \leq \delta_5(r_\nu - r_{\nu+1})^3$ implies $|\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \delta_5(r_{\nu+1} - r_{\nu+2})^3$. By Theorem 6.6, $|\vec{e}_\nu|_{r_\nu} \leq \delta_4(r_\nu - r_{\nu+1})^3$ and $|\vec{e}_\nu|_{r_\nu} \leq (r_\nu - r_{\nu+1})^3/(27c_4)$ imply

$$|\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \frac{c_4|\vec{e}_\nu|^2_{r_\nu}}{(r_\nu - r_{\nu+1})^3} \leq \frac{1}{27}|\vec{e}_\nu|_{r_\nu};$$

this already suggests a geometric decrease in the sequence of norms. Then, using the properties of the sequence r_ν,

$$\tfrac{1}{27}|\vec{e}_\nu|_{r_\nu} \leq \tfrac{1}{27}\delta_5(r_\nu - r_{\nu+1})^3 \leq \delta_5(r_{\nu+1} - r_{\nu+2})^3,$$

which proves the claimed implication. Using this as an inductive step, and starting the induction with $|\vec{e}_0|_{r_0} \leq \eta < \frac{1}{64}\delta_5 = \delta_5(r_0 - r_1)^3$, the hypothesis of Theorem 6.6 is satisfied for all ν. The first of three conclusions from Theorem 6.6 is that $\vec{e}_\nu$ is holomorphic on D_{r_ν}, with degree $d_\nu \geq 2^\nu + 2$, and $|\vec{e}_\nu|_{r_\nu} \leq 27^{-\nu}\eta$. Secondly, $\psi_0 \circ \ldots \circ \psi_\nu$ is a well-defined holomorphic map $\Delta_{\sigma_\nu} \to \Delta_{\rho_0}$, and $\Psi_\nu \circ \ldots \circ \Psi_0$ is well-defined and holomorphic on the image $(\psi_0 \circ \ldots \circ \psi_\nu)(\Delta_{\sigma_\nu})$, so that

$$\Psi_\nu \circ \ldots \circ \Psi_0 \circ \psi_0 \circ \ldots \circ \psi_\nu$$

is the identity on Δ_{σ_ν}. The third conclusion is that if $|\tilde{z}_1| < r_{\nu+1}$ and $|\tilde{x}_s| < r_{\nu+1}$, and $\tilde{z}$ is defined as in (27) with $\tilde{e} = \vec{e}_{\nu+1}$, then $(\psi_0 \circ \ldots \circ \psi_\nu)(\tilde{z}) \in M$. For any $\vec{z} = (z_1, \ldots, z_n) \in \Delta_{1/2}$, the sequence (depending on ν) $(\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu)(\vec{z})$ is contained in $\Delta_{\rho_0} = \Delta_{11/12}$. The following argument, beginning with several applications of Lemma 6.1, shows this sequence is a Cauchy sequence, and converges to some value $\psi(\vec{z})$.

We have

$$
\begin{aligned}
(28)\quad & \sum_{k=1}^{n} \left|(\psi_0 \circ \ldots \circ \psi_{\nu+1})_k(\vec{z}) - (\psi_0 \circ \ldots \circ \psi_\nu)_k(\vec{z})\right| \\
&= \sum_{k=1}^{n} \left|(\psi_0)_k((\psi_1 \circ \ldots \circ \psi_{\nu+1})(\vec{z})) - (\psi_0)_k((\psi_1 \circ \ldots \circ \psi_\nu)(\vec{z}))\right| \\
&\leq |||\mathrm{D}\psi_0|||_{\rho_1} \cdot \sum_{j=1}^{n} \left|(\psi_1 \circ \ldots \circ \psi_{\nu+1})_j(\vec{z}) - (\psi_1 \circ \ldots \circ \psi_\nu)_j(\vec{z})\right| \\
&\leq \left(\prod_{\ell=0}^{\nu} |||\mathrm{D}\psi_\ell|||_{\rho_{\ell+1}}\right) \cdot \sum_{j=1}^{n} \left|(\psi_{\nu+1})_j(\vec{z}) - z_j\right|.
\end{aligned}
$$

By the estimate from Lemma 6.3, with $f = \vec{p}_{\nu+1}$ and $K = \frac{1}{2}$ from the proof of Theorem 6.4, and then using the bound for $\vec{p}$ from Corollary 5.8,

$$
\begin{aligned}
\sum_{j=1}^{n} |(\psi_{\nu+1})_j(\vec{z}) - z_j| &\leq \frac{1}{1-\frac{1}{2}} \sum_{j=1}^{n} |(\vec{p}_{\nu+1})_j(\vec{z})| \leq 2 \sum_{j=1}^{n} \|(\vec{p}_{\nu+1})_j\|_{1/2} \\
&\leq 2 \sum_{j=1}^{n} \|(\vec{p}_{\nu+1})_j\|_{\rho_{\nu+1}} \leq 2n \frac{c_1 |\vec{e}_{\nu+1}|_{r_{\nu+1}}}{(r_{\nu+1} - \rho_{\nu+1})^2} \\
&= 18n \frac{c_1 |\vec{e}_{\nu+1}|_{r_{\nu+1}}}{(r_{\nu+1} - r_{\nu+2})^2} = 72nc_1(\nu+2)^2(\nu+3)^2 |\vec{e}_{\nu+1}|_{r_{\nu+1}} \\
&\leq \frac{72nc_1(\nu+2)^2(\nu+3)^2\eta}{27^{\nu+1}}. \qquad \square
\end{aligned}
$$

It follows from $\mathrm{D}_{\vec{z}}\psi_\ell = (\mathbb{1} + \mathrm{D}_{\psi_\ell(\vec{z})}\vec{p}_\ell)^{-1}$ and Lemma 7.2 that

$$
|||\mathrm{D}\psi_\ell|||_{\rho_{\ell+1}} = |||(\mathbb{1} + \mathrm{D}_{\psi_\ell(\vec{z})}\vec{p}_\ell)^{-1}|||_{\rho_{\ell+1}} \leq |||(\mathbb{1} + \mathrm{D}\vec{p}_\ell)^{-1}|||_{\rho_\ell} \leq \frac{1}{1 - |||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell}}.
$$

Then, by Lemma 7.3, the product from (28) is bounded above by some constant $c_5 > 0$, since by Corollary 5.8,

$$
\begin{aligned}
\sum_{\ell=0}^{\infty} |||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell} &\leq \sum_{\ell=0}^{\infty} \frac{c_1 |\vec{e}_\ell|_{r_\ell}}{(r_\ell - \rho_\ell)^3} = \sum_{\ell=0}^{\infty} \frac{27c_1 |\vec{e}_\ell|_{r_\ell}}{(r_\ell - r_{\ell+1})^3} \\
&= \sum_{\ell=0}^{\infty} 216(\ell+1)^3(\ell+2)^3 c_1 |\vec{e}_\ell|_{r_\ell} \leq \sum_{\ell=0}^{\infty} \frac{216(\ell+1)^3(\ell+2)^3 c_1 \eta}{27^\ell},
\end{aligned}
$$

a convergent infinite series with terms less than 1.

The inequality

$$\sum_{k=1}^{n} \left|(\psi_0 \circ \ldots \circ \psi_{\nu+1})_k(\vec{z}) - (\psi_0 \circ \ldots \circ \psi_\nu)_k(\vec{z})\right| \leq \frac{72nc_1c_5(\nu+2)^2(\nu+3)^2\eta}{27^{\nu+1}}$$

is enough to show that the sequence of composite functions converges pointwise and uniformly to a function ψ on $\Delta_{1/2}$.

Remark. Although some details remain to be checked, it seems plausible that a similar rapid convergence argument could be used to prove an analogous analytic stability property for a nondegenerate CR singularity of a real 3-manifold in $\mathbb{C}^4$, as conjectured in [Coffman 2006].

8. Analogy with singularity theory

To continue with the theme of analogies between the normal form result and the properties of Whitney's cross-cap singularity, we briefly consider the notion of complexification. If the defining equations of a real m-submanifold M in $\mathbb{C}^n$ with a CR singularity at $\vec{0}$ are given as a graph over the tangent space as in (1), then M can also be considered as the image of a real analytic parametrization

$$\pi : \mathbb{R}^m \to \mathbb{R}^{2n}, \quad (z, x) \mapsto (z, x, H_s(z, x), h_u(z, x)).$$

Then the spaces $\mathbb{R}^m$, $\mathbb{R}^{2n}$ can be embedded as totally real subspaces of $\mathbb{C}^m$, $\mathbb{C}^{2n}$, and there is a complex analytic map $\pi_c : \mathbb{C}^m \to \mathbb{C}^{2n}$ which restricts to π on the totally real subspaces. In the following examples, composing with a projection $P : \mathbb{C}^{2n} \to \mathbb{C}^n$ gives a holomorphic map $P \circ \pi_c$ which restricts to π on the totally real $\mathbb{R}^m$ subspace, and its image is a complex subvariety of $\mathbb{C}^n$ containing M. Even though π_c is an embedding, the composite $P \circ \pi_c$ can be singular, and the image of its critical point set contains the CR singular locus of M. For details and more examples of this construction, see [Webster 1985; Coffman 2002; Coffman 2003], and to be more precise, these maps should be considered only in some neighborhood of the origin in the domain and target.

Example 8.1. In the case $m = n = 2$ [Bishop 1965], the local defining equation of a real surface with a nondegenerate CR singularity in $\mathbb{C}^2$ can be normalized to $z_2 = \beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1 + O(3)$, where the coefficient $\beta \geq 0$ is a biholomorphic invariant. Considering the real embedding's quadratic part,

$$\pi : (z_1, \bar{z}_1) \mapsto \left(z_1,\ \bar{z}_1,\ z_2 = \beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1,\ \bar{z}_2 = \overline{\beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1}\right)$$

is a real analytic map from the totally real subspace $\{(z_1, w_1) : w_1 = \bar{z}_1\}$ of $\mathbb{C}^2$ to the totally real subspace $\{(z_1, w_1, z_2, w_2) : w_1 = \bar{z}_1, w_2 = \bar{z}_2\}$ of $\mathbb{C}^4$, which extends to a complex analytic embedding

$$\pi_c : (z_1, w_1) \mapsto \big(z_1, w_1, \beta(z_1^2 + w_1^2) + z_1 w_1, \beta(w_1^2 + z_1^2) + w_1 z_1\big).$$

Then composing with the projection $P : \mathbb{C}^4 \to \mathbb{C}^2$ that forgets the w_1, w_2 variables in the target gives a map $P \circ \pi_c : (z, w) \mapsto (z, \beta(z^2 + w^2) + zw)$. For $\beta > 0$, this is a ramified two-to-one map onto $\mathbb{C}^2$ [Moser and Webster 1983; Webster 1985], and is analogous to Whitney's fold singularity $(x, y) \mapsto (x, y^2)$.

Example 8.2. An example of a cubic normal form for a CR singular surface in $\mathbb{C}^2$ in the $\beta = 0$ case is $z_2 = z_1\bar{z}_1 + \bar{z}_1^3$ [Moser 1985]. The map $P \circ \pi_c : (z_1, w_1) \mapsto (z_1, z_1 w_1 + w_1^3)$ is analogous to Whitney's cusp, $(x, y) \mapsto (x, xy + y^3)$.

Example 8.3. An example of a surface M in $\mathbb{C}^3$ with a topologically unstable CR singularity, considered in [Coffman 2004], has real equations $z_2 = \bar{z}_1^2$, $z_3 = z_1\bar{z}_1$, which complexify to $P \circ \pi_c : (z_1, w_1) \mapsto (z_1, w_1^2, z_1 w_1)$, exactly Whitney's normal form for the parametrization of the cross-cap singularity. The image of $P \circ \pi_c$ in $\mathbb{C}^3$ is $\{z_1^2 z_2 - z_3^2 = 0\}$, a singular complex hypersurface (Whitney's "umbrella" surface), and the smallest complex variety containing M.

Example 8.4. For the normal form variety $\widetilde{M}^{4,5}$, a parametrization $\mathbb{C}^4 \to \mathbb{C}^{10} \to \mathbb{C}^5$ of the complexification is

$$(z_1, w_1, z_2, z_3) \mapsto (z_1, z_2, z_3, w_1^2, w_1(z_1 + z_2 + iz_3)).$$

The real manifold $\widetilde{M}^{4,5}$ is the image of the restriction of this map to the totally real subspace $\{w_1 = \bar{z}_1, z_2 = \bar{z}_2, z_3 = \bar{z}_3\}$ in the domain. The holomorphic map $\mathbb{C}^4 \to \mathbb{C}^5$ parametrizes a singular complex hypersurface $\mathcal{H}$, which is the product of Whitney's cross-cap surface and a complex 2-plane, and the image $\{(z_1 + z_2 + iz_3)^2 z_4 - z_5^2 = 0\}$ is the smallest complex variety in $\mathbb{C}^5$ containing $\widetilde{M}^{4,5}$; a similar expression appeared in (15). The geometry of $\widetilde{M}^{4,5} \subseteq \mathcal{H}$ is considered in [Coffman 2003, §8], but with a different expression for the quadratic normal form.

Example 8.5. In general, the real variety $\widetilde{M}^{m,n}$ is contained in a singular subvariety of complex dimension m in $\mathbb{C}^n$, the defining ideal of which contains, for example, $(z_1 + z_2 + iz_3)^2 z_{n-1} - z_n^2$. As a consequence of Proposition 3.3, any real analytic M is not a local uniqueness set for holomorphic functions in a neighborhood of a nondegenerate CR singularity; compare [Harris 1983].

For surfaces in $\mathbb{C}^2$, the two-to-one nature of the complexification $\mathbb{C}^2 \to \mathbb{C}^2$ as in Example 8.1 was used in [Moser and Webster 1983] to solve a normal form problem in the $0 < \beta < \frac{1}{2}$ case. Their methods are different from that of this paper; for example in the $(m, n) = (4, 5)$ case, the map $\mathbb{C}^4 \to \mathbb{C}^5$ from Example 8.4 is generally one-to-one, the two-to-one locus being contained in a complex subvariety in the domain as shown in [Coffman 2003].

Normal forms for the complexifications that look more like Whitney's monomial normal forms would be possible using a larger group, where the z and w variables could be transformed independently. Under the subgroup used to normalize the CR singularity, one expects equivalence classes of maps to be smaller, and continuous parameters ("moduli") to appear sooner (for more and for lower-order terms). However, invariants which distinguish maps under the larger group will still distinguish them under the smaller group. One may speculate that invariants of the complexification, such as the intrinsic derivative, the Boardman sequence, Jacobian extensions, etc., could provide a coarse but general beginning to the development of a CR singularity theory analogous to the singularity theory of maps [Golubitsky and Guillemin 1973; Porteous 1971].

Acknowledgments

The author thanks the referee(s) for comments on this, and also on an earlier version, which circulated as a preprint entitled *Formal stability of the CR cross-cap*, and which was based on [Coffman 1997]. The question of convergence answered by Proposition 3.3 was posed to the author by S. Webster.

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Received December 5, 2004.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

QUOTIENTS OF MILNOR K-RINGS, ORDERINGS, AND VALUATIONS

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We define and study the Milnor K-ring of a field F modulo a subgroup of the multiplicative group of F. We compute it in several arithmetical situations, and study the reflection of orderings and valuations in this ring.

Introduction

Let F be a field and let $F^\times$ be its multiplicative group. The Milnor K-ring $K_*^M(F)$ of F is the tensor (graded) algebra of the $\mathbb{Z}$-module $F^\times$ modulo the homogeneous ideal generated by all elements $a_1 \otimes \cdots \otimes a_r$, where $1 = a_i + a_j$ for some $1 \le i < j \le r$ [Milnor 1970]. Alongside with $K_*^M(F)$, the quotients $K_*^M(F)/m = K_*^M(F)/mK_*^M(F)$, for m a positive integer, also play an important role in many arithmetical questions. In this paper we study a natural generalization of these two functors. Specifically, we consider a subgroup S of $F^\times$ and define the graded ring $K_*^M(F)/S$ to be the quotient of the tensor algebra over $F^\times/S$ modulo the homogeneous ideal generated by all elements $a_1S\otimes\cdots\otimes a_rS$, where $1 \in a_iS + a_jS$ for some $1 \le i < j \le r$. The graded rings $K_*^M(F)$ and $K_*^M(F)/m$ then correspond to $S = \{1\}$ and $S = (F^\times)^m$, respectively.

The ring-theoretic structure of $K_*^M(F)/S$ reflects many of the main arithmetical properties of F, especially those related to orderings and valuations. We illustrate this by computing it in the following situations:

(1) $F^\times/S$ is a finite cyclic group. Here, if F has no orderings containing S then $K_*^M(F)/S$ is trivial in degrees > 1. Otherwise $K_*^M(F)/S$ coincides in degrees > 1 with the tensor algebra over $\{\pm 1\}$ (Theorem 4.1). This includes as a special case the computation of the Milnor K-ring of finite fields, which goes back to Steinberg and Milnor [Milnor 1970, Example 1.5].

(2) There is a (Krull) valuation v on F whose 1-units are contained in S. We show that under a mild assumption, $K_*^M(F)/S$ is then obtained from the corresponding K-ring of the residue field and from $v(F^\times)/v(S)$ by means of a

MSC2000: primary 19F99; secondary 12J10, 12J15, 12E30.
Keywords: generalized Milnor K-rings, orderings, valuations.
The research has been supported by the Israel Science Foundation grant No. 8008/02–1.

natural algebraic construction analogous to the construction of a polynomial ring over a given ring (Section 5).

(3) $F^\times/S$ is finitely generated, and is generated by the 1-units of a rank-1 valuation v such that S is open in the v-topology on F. We then prove that $K_*^M(F)/S$ is trivial in degrees > 1 (Theorem 6.2).

(4) $F^\times/S$ is finite, and there is a rank-1 valuation v on F with mixed characteristics $(0, p)$ such that $S = (F^\times)^p(1 + p^2\mathfrak{m}_v)$, where $\mathfrak{m}_v$ is the valuation ideal (when v is Henselian the latter condition just means that $S = (F^\times)^p$). We show that then $K_*^M(F)/S$ is either the Milnor K-ring of a finite extension of $\mathbb{Q}_p$, or else it is trivial in degrees > 1 and $v(F^\times)$ is p-divisible (Theorem 7.4). The proof is based on the vanishing theorem of (3) above.

These results are mostly of a *local* nature. In a forthcoming paper we compute the functor $K_*^M(F)/S$ in *global* situations, where S is related to a family of orderings and valuations.

Studying Milnor's K-theory modulo a subgroup S by means of the functor $K_*^M(F)/S$ resembles the reduced theory of quadratic forms: there one studies quadratic forms modulo a preordering T on F via the *reduced* Witt ring functor $W_T(F)$, rather than the classical Witt ring; see [Lam 1983; Becker and Köpping 1977] for details.

Furthermore, the celebrated Bloch–Kato–Milnor conjecture predicts that $K_*^M(F)$ is isomorphic to the Galois cohomology of the absolute Galois group G_F of F with respect to twisted cyclotomic actions [Kahn 1997]. Similarly, when p is a prime number and F contains a primitive p-th root of unity, $K_*^M(F)/p$ is related to the Galois cohomology ring of the maximal pro-p Galois group $G_F(p)$ of F with its trivial action on $\mathbb{Z}/p$. From this viewpoint, the generalized functor $K_*^M(F)/S$ serves in some sense as an analog of the Galois cohomology of an arbitrary relative Galois group $\mathrm{Gal}(E/F)$ of F.

1. κ-Structures

In this section we define a convenient target category for the generalized Milnor K-ring functor. Recall that the Milnor K-ring of a field F is a graded ring with degree 1 component $F^\times$. Furthermore, the element -1 of $F^\times$ plays a special role in this ring. The idea is therefore to consider graded rings with a distinguished element in degree 1, which will play the role of -1. The resulting category of "κ-structures" is a slight modification of the "κ-algebras" defined in [Bass and Tate 1973], in the sense that we require in addition that the degree-0 component of the graded ring is $\mathbb{Z}$. This turns out to be useful for defining natural constructions in the resulting category, such as extensions (see below) and direct products. Other formal categories that were studied in the context of quadratic form theory (like abstract

Witt rings, quaternionic structures, or abstract spaces of orderings [Marshall 1980; 1996]) are specializations of the category of κ-structures in a natural way.

Denote the tensor algebra of an abelian group Γ by $\mathrm{Tens}(\Gamma)$. We let

$$\kappa = \bigoplus_{r=0}^{\infty} \kappa_r = \mathrm{Tens}(\{\pm 1\}),$$

and denote the nontrivial element of $\kappa_1 \cong \mathbb{Z}/2$ by ε. Thus $\kappa_0 = \mathbb{Z}$ and for all $r \geq 1$, $\kappa_r = \{0, \varepsilon^r\} \cong \mathbb{Z}/2$.

Definition 1.1. A *κ-structure* consists of a graded ring $A = \bigoplus_{r=0}^{\infty} A_r$ and a graded ring homomorphism $\kappa \to A$ such that

(i) $A_0 = \mathbb{Z}$, and the homomorphism $\kappa \to A$ is the identity in degree 0,

(ii) A_1 generates A as a ring, and

(iii) the image ε_A of ε in A satisfies $a^2 = \varepsilon_A a = a\varepsilon_A$ for all $a \in A_1$.

For every $a, b \in A_1$ we have $ab + ba = (a+b)^2 - a^2 - b^2 = 0$, by (iii). Thus A is anticommutative. A *morphism* $A \to B$ of κ-structures is a graded ring homomorphism which commutes with the structural homomorphisms $\kappa \to A$ and $\kappa \to B$.

The category of κ-structures has direct products. Namely, the direct product $\prod_{i\in I} A_i$ of κ-structures A_i for $i \in I$, is defined by $(\prod_{i\in I} A_i)_0 = \mathbb{Z}$ and $(\prod_{i\in I} A_i)_r = \prod_{i\in I} (A_i)_r$ for $r \geq 1$, with the natural multiplicative structure. The homomorphism $\kappa_r \to \prod_{i\in I} (A_i)_r$ is given by $\varepsilon \mapsto (\varepsilon_{A_i})_{i\in I}$.

Recall that the tensor product in the category of *graded rings* is defined by

$$A \otimes_{\mathbb{Z}} B = \bigoplus_{r=0}^{\infty} \Big(\bigoplus_{i+j=r} A_i \otimes_{\mathbb{Z}} B_j \Big),$$

with the product given by

$$(a \otimes b)(a' \otimes b') = (-1)^{i'j} aa' \otimes bb'$$

for $a \in A_i$, $a' \in A_{i'}$, $b \in B_j$, $b' \in B_{j'}$. Given κ-structures A, B, we define their *tensor product* in the category of κ-structures to be $A \otimes_\kappa B = (A \otimes_{\mathbb{Z}} B)/I$, where I is the homogeneous ideal generated by $\varepsilon_A \otimes 1_B - 1_A \otimes \varepsilon_B$. The homomorphism $\kappa \to A \otimes_\kappa B$ is given by $\varepsilon \mapsto \varepsilon_A \otimes 1_B + I = 1_A \otimes \varepsilon_B + I$. Since A and B are anticommutative, so is $A \otimes_{\mathbb{Z}} B$. Further, given $a \in A_1$ and $b \in B_1$ we have $(a \otimes 1_B)^2 = (\varepsilon_A \otimes 1_B)(a \otimes 1_B)$ and $(1_A \otimes b)^2 = (1_A \otimes \varepsilon_B)(1_A \otimes b)$, so by the anticommutativity,

$$(a \otimes 1_B + 1_A \otimes b)^2 + I = (\varepsilon_A \otimes 1_B)(a \otimes 1_B + 1_A \otimes b) + I$$

in $(A\otimes_\kappa B)_2$. This implies the first equality in (iii) for $A\otimes_\kappa B$. The second is proved similarly, showing that $A\otimes_\kappa B$ is a κ-structure. There are canonical morphisms $\iota: A\to A\otimes_\kappa B$ and $\iota': B\to A\otimes_\kappa B$ with respect to which $A\otimes_\kappa B$ is the coproduct of A and B in the category of κ-structures (in the sense of, e.g., [Lang 1984, I §7]). One has $A\cong A\otimes_\kappa \kappa$ and $B\cong \kappa\otimes_\kappa B$ via these morphisms.

Next we construct free objects in this category. Let Γ be an abelian group. We define $\kappa[\Gamma]$ to be the quotient of $\mathrm{Tens}(\kappa_1\oplus\Gamma)$ by the homogeneous ideal generated by all elements $\varepsilon\otimes\gamma-\gamma\otimes\gamma$, where $\gamma\in\Gamma$. Replacing γ by $\varepsilon+\gamma$ one sees that this ideal also contains $\gamma\otimes\varepsilon-\gamma\otimes\gamma$. The obvious embedding $\kappa_1\hookrightarrow\kappa_1\oplus\Gamma$ induces a graded ring homomorphism $\kappa\to\kappa[\Gamma]$. Then $\kappa[\Gamma]$ is a κ-structure satisfying the following universal property (which follows from the universal property of the tensor algebra):

For every κ-structure B and an abelian group homomorphism $\theta:\Gamma\to B_1$ there exists a unique morphism $\kappa[\Gamma]\to B$ extending θ.

Given a κ-structure A, we call $A[\Gamma]=A\otimes_\kappa\kappa[\Gamma]$ the *extension* of A by Γ. When $A=\kappa$ it coincides with our previous notation. This extends Serre's construction mentioned in [Milnor 1970, p. 323]. We identify $(A[\Gamma])_1$ with $A_1\oplus\Gamma$, and we let $\iota: A\to A[\Gamma]$ be the canonical morphism.

Lemma 1.2. *Let $\varphi: A\to B$ be a morphism of κ-structures and let $\theta:\Gamma\to B_1$ be a homomorphism of abelian groups. There exists a unique morphism $A[\Gamma]\to B$ extending θ which commutes with φ and ι.*

Proof. The universal property of $\kappa[\Gamma]$ yields a unique morphism $\kappa[\Gamma]\to B$ extending θ. Now use the fact that the tensor product is a coproduct. □

Corollary 1.3. *For a κ-structure A and abelian groups Γ_1,Γ_2 one has*

$$(A[\Gamma_1])[\Gamma_2]\cong A[\Gamma_1\oplus\Gamma_2].$$

Example 1.4. Let A be a κ-structure and let Γ be a cyclic group with generator γ. For every $i\geq 1$, we have $\gamma^i=\varepsilon_A^{i-1}\gamma$ in $A[\Gamma]$, by (iii) of Definition 1.1. It follows that $(A[\Gamma])_r=A_r\oplus(A_{r-1}\otimes_{\mathbb{Z}}\Gamma)$ for $r\geq 1$.

2. The functor $K_*^M(F)/S$

Let F be a field and let S be a subgroup of $F^\times$. For $r\geq 0$, let

$$(F^\times/S)^{\otimes r}=(F^\times/S)\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}(F^\times/S)\quad (r \text{ times}).$$

Let $\mathrm{St}_{F,r}(S)$ be the subgroup of $(F^\times/S)^{\otimes r}$ generated by all elements $a_1S\otimes\cdots\otimes a_rS$ such that $1\in a_iS+a_jS$ for some $i\neq j$. Generalizing standard terminology, we call such elements *Steinberg elements*. Let

$$K_r^M(F)/S=(F^\times/S)^{\otimes r}/\mathrm{St}_{F,r}(S).$$

In particular, $K_0^M(F)/S = \mathbb{Z}$ and $K_1^M(F)/S = F^\times/S$. For $t \geq 0$, one has

$$\mathrm{St}_{F,r}(S) \otimes_{\mathbb{Z}} (F^\times/S)^{\otimes t} \subseteq \mathrm{St}_{F,r+t}(S), \quad (F^\times/S)^{\otimes t} \otimes_{\mathbb{Z}} \mathrm{St}_{F,r}(S) \subseteq \mathrm{St}_{F,r+t}(S).$$

Therefore

$$K_*^M(F)/S = \bigoplus_{r=0}^{\infty} K_r^M(F)/S$$

is a graded ring respect to the multiplication induced by the tensor product. We call it the *Milnor K-ring of F modulo S*. Given $a_1, \dots, a_r \in F^\times$, we denote the image of $a_1 S \otimes \cdots \otimes a_r S$ in $K_r^M(F)/S$ by $\{a_1, \dots, a_r\}_S$.

When $S = \{1\}$ we obtain the classical Milnor K-ring $K_*^M(F) = \bigoplus_{r=0}^{\infty} K_r^M(F)$ of F as in [Milnor 1970]. In this case we write as usual $\{a_1, \dots, a_n\}$ for $\{a_1, \dots, a_n\}_S$. In general, we have graded ring homomorphisms $\mathrm{Tens}(F^\times/S) \to K_*^M(F)/S$ and $K_*^M(F) \to K_*^M(F)/S$.

Define a graded ring homomorphism $\kappa \to K_*^M(F)/S$ by setting $\varepsilon \mapsto -S \in F^\times/S$. Since the identities $\{a, a\}_S = \{-1, a\}_S = \{a, -1\}_S$ of part (iii) of Definition 1.1 are well known to hold when $S = \{1\}$ [Milnor 1970, §1], they also hold in $K_*^M(F)/S$. Hence $K_*^M(F)/S$ is a κ-structure.

Proposition 2.1. *For positive integers m, r and for $S = (F^\times)^m$ we have*

$$K_r^M(F)/S = K_r^M(F)/m.$$

Proof. There is an obvious graded ring homomorphism $\varphi : K_r^M(F)/m \to K_r^M(F)/S$ commuting with the canonical projections from $(F^\times)^{\otimes r}$. Conversely, suppose $a_1, \dots, a_r \in F^\times$ and $a_1 S \otimes \cdots \otimes a_r S \in \mathrm{St}_{F,r}(S)$, that is, $1 = a_i \alpha^m + a_j \beta^m$ for some $i < j$ and $\alpha, \beta \in F^\times$. Then

$$\{a_1, \dots, a_r\} \in \{a_1, \dots, a_i\alpha^m, \dots, a_j\beta^m, \dots, a_r\} + mK_r^M(F) = mK_r^M(F).$$

We obtain a projection $\psi : K_r^M(F)/S \to K_r^M(F)/m$ which also commutes with the projections from $(F^\times)^{\otimes r}$. Thus φ and ψ are inverse maps, hence isomorphisms. □

We consider the class of all pairs (F, S) where F is a field and $S \leq F^\times$ as a category, in which morphisms $(F, S) \to (F_1, S_1)$ are pairs of compatible embeddings $F \hookrightarrow F_1$, $S \hookrightarrow S_1$. For such a pair and for $r \geq 0$, we have a group homomorphism $(F^\times/S)^{\otimes r} \to (F_1^\times/S_1)^{\otimes r}$ mapping $\mathrm{St}_{F,r}(S)$ to $\mathrm{St}_{F_1,r}(S_1)$. It therefore induces a κ-structure morphism $\mathrm{Res} : K_*^M(F)/S \to K_*^M(F_1)/S_1$, which we call the *restriction* morphism. The map $(F, S) \mapsto K_*^M(F)/S$ is thus a covariant functor from the category of pairs (F, S) to the category of κ-structures.

A topology on a field F is called a *ring topology* if the addition and multiplication maps $F \times F \to F$ are continuous. We will need:

Proposition 2.2. *Let $\mathcal{T}$ be a ring topology on a field F_1 and let F be a subfield of F_1 which is $\mathcal{T}$-dense in F_1. Let S be a subgroup of $F^\times$ and let S_1 be a $\mathcal{T}$-open subgroup of $F_1^\times$ containing S. Then* $\mathrm{Res} : K_*^M(F)/S \to K_*^M(F_1)/S_1$ *is an epimorphism. When $S = F \cap S_1$, it is an isomorphism.*

Proof. For every $a \in F_1^\times$, we have $F \cap aS_1 \neq \varnothing$ by the density assumption. Hence the natural homomorphism $F^\times/S \to F_1^\times/S_1$ is surjective. Consequently, so is $\mathrm{Res} : K_*^M(F)/S \to K_*^M(F_1)/S_1$.

Suppose that $S = F \cap S_1$. For each r the induced map $(F^\times/S)^{\otimes r} \to (F_1^\times/S_1)^{\otimes r}$ is an isomorphism. Therefore the injectivity of Res would follow by a snake lemma argument once we show that the induced map $\mathrm{St}_{F,r}(S) \to \mathrm{St}_{F_1,r}(S_1)$ is surjective. To this end, take a generator $a_1S_1 \otimes \cdots \otimes a_rS_1 \in \mathrm{St}_{F_1,r}(S_1)$, where $a_1, \ldots, a_r \in F_1^\times$ and $1 \in a_iS_1 + a_jS_1$ for some distinct i, j. By continuity, there exist nonempty $\mathcal{T}$-open subsets $V, W \subseteq S_1$ such that $a_iV + a_jW \subseteq S_1$. Using the density assumption we find $x_1, \ldots, x_r \in F$ with $x_i \in a_iV$, $x_j \in a_jW$, and $x_l \in a_lS_1$ for all $l \neq i, j$. Then $x_i + x_j \in S_1 \cap F = S$, so $x_1S \otimes \cdots \otimes x_rS \in \mathrm{St}_{F,r}(S)$. Furthermore, $x_1S \otimes \cdots \otimes x_rS$ maps to $a_1S_1 \otimes \cdots \otimes a_rS_1$ under the homomorphism above, as required. □

3. Orderings

Let again F be a field, and let S be a subgroup of $F^\times$. Following standard terminology (see, e.g., [Neukirch et al. 2000, p. 191]), we call the map $\mathrm{Bock}_{F,S} : F^\times/S \to K_2^M(F)/S$, $\{x\}_S \mapsto \{x\}_S^2 = \{x, -1\}_S$, the *Bockstein operator* of the subgroup S of F. It is clearly a group homomorphism.

Lemma 3.1. *If* $\mathrm{Bock}_{F,S}$ *is injective then S is additively closed.*

Proof. It suffices to show that $1 + S \subseteq S$. To this end take $s \in S$. Then

$$\mathrm{Bock}_{F,S}(\{1+s\}_S) = \{1+s, -1\}_S = \{1+s, -s\}_S = 0.$$

By injectivity, $\{1+s\}_S = 0$, so $1 + s \in S$. □

By an *ordering* on F, we mean an additively closed subgroup P of $F^\times$ such that $F^\times = P \cup -P$. Recall that a ring is reduced if it has no nonzero nilpotent elements. The following fact is a variant of [Bass and Tate 1973, I, Theorem (3.1)].

Proposition 3.2. *The following conditions are equivalent*:

(a) $K_*^M(F)/S \cong \kappa$ *as κ-structures*;

(b) $F^\times = S \cup -S$ *and $K_*^M(F)/S$ is reduced*;

(c) $F^\times = S \cup -S$ *and* $\{-1, -1\}_S \neq 0$;

(d) *S is an ordering on F.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c): Immediate.
(c) $\Rightarrow$ (d): By Lemma 3.1 and the assumptions, S is additively closed. The rest is clear.
(d) $\Rightarrow$ (a): We first show that $\mathrm{St}_{F,r}(S)$ is trivial for all $r \geq 2$. Take $a_1, \dots, a_r \in F^\times$ with $1 \in a_i S + a_j S$ for some distinct $1 \leq i, j \leq r$. If a_i, a_j were both in $-S$ then we would get $-1 \in S + S \subseteq S$, a contradiction. Hence at least one of a_i, a_j must be in S. It follows that $a_1 S \otimes \cdots \otimes a_r S = 1$ in $(F^\times/S)^{\otimes r}$, as claimed.

Consequently, $K_*^M(F)/S = \mathrm{Tens}(F^\times/S) \cong \mathrm{Tens}(\{\pm 1\}) = \kappa$ as graded rings. Further, this is a κ-structure isomorphism. □

A *preordering* on F is an additively closed subgroup S of $F^\times$ containing $(F^\times)^2$ but not -1. Preorderings can be characterized K-theoretically as follows.

Proposition 3.3. *Suppose that* $(F^\times)^2 \leq S < F^\times$. *The following conditions are equivalent*:

(a) *S is a preordering on F*;

(b) $\mathrm{Bock}_{F,S}$ *is injective.*

Proof. (a) $\Rightarrow$ (b): Let $x \in F^\times$ satisfy $\{x\}_S^2 = 0$ and let P be an ordering on F containing S. Then $\{x\}_P^2 = 0$, and since $K_*^M(F)/P \cong \kappa$ is reduced (Proposition 3.2), $\{x\}_P = 0$, i.e., $x \in P$. As a preordering, S is the intersection of all the orderings P containing it [Lam 1983, Theorem 1.6]. Consequently, $x \in S$, as desired.
(b) $\Rightarrow$ (a): In light of Lemma 3.1, S is additively closed. By assumption, there exists $x \in F^\times \setminus S$. By injectivity, $\{x, -1\}_S \neq 0$. Hence $-1 \notin S$, so S is a preordering. □

4. The cyclic case

Using the K-theoretic analysis of orderings obtained in the previous section, we can now completely describe $K_*^M(F)/S$ when $F^\times/S$ is a finite cyclic group.

Theorem 4.1. *Let F be a field and let S be a subgroup of $F^\times$ such that $F^\times/S$ is finite and cyclic. Then one of following holds*:

(a) $K_r^M(F)/S = 0$ *for all* $r \geq 2$;

(b) $(F^\times : S) = 2m$ *with m odd, and there exists a unique ordering P on F containing S. Furthermore*, $\mathrm{Res} : K_*^M(F)/S \to K_*^M(F)/P$ $(\cong \kappa)$ *is an isomorphism in all degrees* $r \geq 2$.

Proof. Let $p_1^{d_1} \cdots p_n^{d_n}$ be the primary decomposition of $(F^\times : S)$. For each $1 \leq i \leq n$, choose $a_i \in F^\times$ such that the coset $\{a_i\}_S$ generates the p_i-primary part of $F^\times/S$. Let $a = a_1 \cdots a_n$. Then the coset $\{a\}_S$ generates $F^\times/S$, and one has $\{a, a\}_S = \{a, -1\}_S = \sum_{i=1}^n \{a_i, -1\}_S$.

Assume that (a) does not hold, i.e., $K_r^M(F)/S \neq 0$ for some $r \geq 2$. Since the canonical map $(F^\times/S)^r \to K_r^M(F)/S$ is multilinear, $\{a, \dots, a\}_S$ generates $K_r^M(F)/S$. Hence $\{a, \dots, a\}_S \neq 0$, and therefore $\{a, a\}_S \neq 0$. It follows that $\{a_i, -1\}_S \neq 0$ for some $1 \leq i \leq n$. We obtain that the orders of $\{a_i, -1\}_S$ and of $\{-1\}_S$ are precisely 2. Furthermore, $p_i^{d_i}\{a_i, -1\}_S = 0$, so we must have $p_i = 2$. Therefore $2^{d_i-1}\{a_i\}_S = \{-1\}_S$, and we get

$$2^{d_i-1}\{a_i, -1\}_S = 2^{d_i-1}\{a_i, a_i\}_S = \{a_i, -1\}_S \neq 0.$$

This implies that $d_i = 1$. Consequently, $(F^\times : S) = 2m$, with m odd.

Let P be the unique subgroup of $F^\times$ of index 2 which contains S. Then P/S is cyclic of order m, and is generated by $\{a^2\}_S$. Since $\{-1\}_S$ has order 2 in $F^\times/S$, it is not in P/S. Therefore $F^\times = P \cup -P$.

Next we claim that $1+P \subseteq P$. Indeed, suppose that $x \in P$. In particular, $x \neq -1$. Take s, t with $-x \in a^s S$ and $1+x \in a^t S$. Then

$$0 = \{-x, 1+x\}_S = \{a^s, a^t\}_S = st\{a, a\}_S.$$

Now $-x \notin P$, so s is odd. But $\{a, a\}_S = \{a, -1\}_S$ has order 2. It follows that t must be even, i.e., $1+x \in P$. Therefore P is additively closed, hence an ordering.

Finally, for every r, the functorial map $K_r^M(F)/S \to K_r^M(F)/P$ is clearly surjective. When $r \geq 2$, the group $K_r^M(F)/S$ is generated by $\{a, a, \dots, a\}_S = \{a, -1, \dots, -1\}_S$, so it has order at most 2. By Proposition 3.2, $K_r^M(F)/P$ has order 2. Consequently, the above map is an isomorphism, and (b) holds.

For the uniqueness part of (b), assume that $S \leq P' < F^\times$ is another ordering on F. Then $4|(F^\times : P \cap P')|(F^\times : S) = 2m$, contrary to the fact that m is odd. □

Corollary 4.2. *Let S be a subgroup of $F^\times$ with $F^\times/S$ cyclic of prime power order. Then either $K_r^M(F)/S = 0$ for all $r \geq 2$, or S is an ordering (hence $K_*^M(F)/S \cong \kappa$).*

As mentioned in the introduction, Theorem 4.1 generalizes the well-known fact that $K_2^M(F) = 0$ for a finite field F [Milnor 1970, Example 1.5; Fesenko and Vostokov 1993, IX, Proposition 1.3]. Indeed, $F^\times$ is cyclic [Lang 1984, VII §5, Theorem 11] and since char $F > 0$, there are no orderings on F.

5. *S*-compatible valuations

Recall that a (Krull) *valuation* on a field F is a group homomorphism v from $F^\times$ into an ordered abelian group $(\Gamma, \leq)$ such that $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in F$ with $x \neq -y$. One defines $v(0)$ to be a formal value $+\infty$ which is strictly larger than every value in Γ. Let O_v be the valuation ring of v, and $\mathfrak{m}_v$ its maximal ideal. Thus $x \in F$ lies in O_v (respectively $\mathfrak{m}_v$) if and only if $v(x) \geq 0$ (respectively $v(x) > 0$). Let $O_v^\times$ be the unit group of O_v, let $G_v = 1 + \mathfrak{m}_v$ be the group of principal units of v, let $\bar{F}_v = O_v/\mathfrak{m}_v$ be the residue field of v, and let $\pi_v : O_v \to \bar{F}_v,\ a \mapsto \bar{a},$ be the canonical projection.

Let S a subgroup of $F^\times$. Its push-down $\bar{S}_v = \pi_v(S \cap O_v^\times)$ under v is a subgroup of $\bar{F}_v^\times$. The maps v and π_v induce short exact sequences of abelian groups

$$1 \to S \cap O_v^\times \to S \xrightarrow{v} v(S) \to 0$$

and

$$1 \to S \cap G_v \to S \cap O_v^\times \xrightarrow{\pi_v} \bar{S}_v \to 1.$$

In particular, this holds for $S = F^\times$. The snake lemma therefore gives rise to canonical exact sequences

$$1 \to O_v^\times/(S \cap O_v^\times) \to F^\times/S \xrightarrow{v^*} v(F^\times)/v(S) \to 0 \tag{5–1}$$

and

$$1 \to G_v/(S \cap G_v) \to O_v^\times/(S \cap O_v^\times) \xrightarrow{\pi_v^*} \bar{F}_v^\times/\bar{S}_v \to 1. \tag{5–2}$$

Following [Arason et al. 1987], we say that the valuation v is *S-compatible* if $G_v \le S$. (When $S = (F^\times)^p$ for p prime and char $\bar{F}_v \neq p$, this is a weak form of Hensel's lemma; see [Wadsworth 1983, Proposition 1.2].) Then the sequences (5–1) and (5–2) combine to a single canonical short exact sequence

$$1 \to \bar{F}_v^\times/\bar{S}_v \xrightarrow{\eta} F^\times/S \xrightarrow{v^*} v(F^\times)/v(S) \to 0, \tag{5–3}$$

where for $a \in O_v^\times$ with residue $\bar{a}$ we set $\eta(\{\bar{a}\}_{\bar{S}_v}) = \{a\}_S$.

We will be interested in situations where (5–1) splits. For example, this is so in the following cases:

1. $v(F^\times) \cong \mathbb{Z}$ and $S = \{1\}$. Then a section of v^* corresponds to a choice of a uniformizer for v.
2. $(F^\times)^p \le S$ for some prime number p. In fact, then $F^\times/S$ and $v(F^\times)/v(S)$ are free $(\mathbb{Z}/p)$-modules.
3. $(F^\times)^q \le S \le (F^\times)^q O_v^\times$, where $q = p^s$ is a prime power. Indeed, the group $v(F^\times)$ is torsion-free, hence a flat $\mathbb{Z}$-module. Thus $v(F^\times)/v(S) = v(F^\times)/q$ is a flat $\mathbb{Z}/q$-module. Since $\mathbb{Z}/q$ is a nilpotent local ring, it is a consequence of the Nakayama lemma [Matsumura 1980, 3.G] that $v(F^\times)/q$ is a free $\mathbb{Z}/q$-module.

We now obtain a connection between valuations and extensions of κ-structures, in the sense of Section 1.

Theorem 5.1. *Let F be a field and let S be a subgroup of $F^\times$. Every section of* (5–1) *canonically induces an epimorphism of κ-structures*

$$K_*^M(F)/S \longrightarrow (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)].$$

This morphism is injective if and only if v is S-compatible.

Proof. Let $\theta : v(F^\times)/v(S) \to F^\times/S$ be a section of v^*. Take $S \le \Delta \le F^\times$ with $\Delta/S = \mathrm{Im}(\theta)$. Then $F^\times/S = (SO_v^\times/S) \times (\Delta/S)$. Thus every $x \in F^\times$ can be written as $x = ab$ with $a \in O_v^\times$ and $b \in \Delta$. We set $\bar{a} = \pi_v(a)$ and write $[v(b)]_S$ for the coset of $v(b)$ in $v(F^\times)/v(S)$. We obtain a well-defined group epimorphism

$$(5\text{–}4) \qquad F^\times/S \to (\bar{F}_v^\times/\bar{S}_v) \oplus (v(F^\times)/v(S)), \quad \{x\}_S \mapsto \{\bar{a}\}_{\bar{S}_v} + [v(b)]_S.$$

This abelian group epimorphism uniquely extends to a graded ring epimorphism

$$\lambda : \mathrm{Tens}(F^\times/S) \to (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)].$$

We claim that λ is trivial on $\mathrm{St}_{F,r}(S)$ for all r. It suffices to show that when $x, y \in F^\times$ and $1 \in xS + yS$, we have $\lambda(\{x\}_S \otimes \{y\}_S) = 0$. We may assume that $1 = x + y$. Write $x = ab$ and $y = cd$, with $a, c \in O_v^\times$ and $b, d \in \Delta$. Then

$$\begin{aligned}\lambda(xS \otimes yS) &= \big(\{\bar{a}\}_{\bar{S}_v} + [v(b)]_S\big) \cdot \big(\{\bar{c}\}_{\bar{S}_v} + [v(d)]_S\big) \\ &= \{\bar{a}, \bar{c}\}_{\bar{S}_v} + \big(\{\bar{a}\}_{\bar{S}_v} \cdot [v(d)]_S - \{\bar{c}\}_{\bar{S}_v} \cdot [v(b)]_S\big) + [v(b)]_S \cdot [v(d)]_S.\end{aligned}$$

To show that this expression vanishes, we distinguish between four cases:

Case I: $x \in G_v$. Here we can take $a = x$ and $b = 1$. Then $\{\bar{a}\}_{\bar{S}_v} = 0$ and $[v(b)]_S = 0$, so the assertion is clear.

Case II: $x \in \mathfrak{m}_v$. Then $y \in G_v$, so we can take $c = y$ and $d = 1$. Hence $\{\bar{c}\}_{\bar{S}_v} = 0$ and $[v(d)]_S = 0$, and we are done again.

Case III: $x \in O_v^\times \setminus G_v$. Then $y = 1 - x \in O_v^\times$, so we can take $a = x$, $b = 1$, $c = y$, and $d = 1$. Hence $\lambda(xS \otimes yS) = \{\bar{x}, \overline{1-x}\}_{\bar{S}_v} = 0$ once again.

Case IV: $x^{-1} \in \mathfrak{m}_v$. For any a, b as above, $y = a(x^{-1}-1) \cdot b$, with $a(x^{-1}-1) \in O_v^\times$. Thus we may take $c = a(x^{-1} - 1)$ and $d = b$. Then $\{\bar{c}\}_{\bar{S}_v} = \{-\bar{a}\}_{\bar{S}_v}$. Further, $\{\bar{a}\}_{\bar{S}_v} - \{-\bar{a}\}_{\bar{S}_v} = \{-\bar{1}\}_{\bar{S}_v}$ and $\{\bar{a}, -\bar{a}\}_{\bar{S}_v} = 0$. It follows that

$$\lambda(xS \otimes yS) = \{-\bar{1}\}_{\bar{S}_v} \cdot [v(b)]_S + [v(b)]_S \cdot [v(b)]_S = 0,$$

using property (iii) of Definition 1.1.

This proves the claim. Consequently, λ induces an epimorphism of κ-structures

$$\bar{\lambda} : K_*^M(F)/S \longrightarrow (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)],$$

as desired.

For the second assertion of the theorem, suppose that v is S-compatible. Then (5–3) is exact. The abelian group monomorphism η of (5–3) induces a morphism $\mathrm{Tens}(\bar{F}_v^\times/\bar{S}_v) \to \mathrm{Tens}(F^\times/S)$ of graded rings. Since $G_v \le S$, it maps $\mathrm{St}_{\bar{F}_v,r}(\bar{S}_v)$ into $\mathrm{St}_{F,r}(S)$ for every $r \ge 1$. Hence it induces a κ-structure morphism

$K_*^M(\bar{F}_v)/\bar{S}_v \to K_*^M(F)/S$. By the universal property of extensions (Lemma 1.2), there exists a unique κ-structure morphism $\bar{\nu}$ which extends the section θ and for which the following diagram commutes (where ι is the canonical morphism as in Section 1):

$$\begin{array}{ccc} K_*^M(\bar{F}_v)/\bar{S}_v & \xrightarrow{\iota} & (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)] \\ & \searrow & \downarrow{\scriptstyle \bar{\nu}} \\ & & K_*^M(F)/S. \end{array}$$

In degree 1, $\bar{\nu}$ coincides with the isomorphism $\eta \oplus \theta$. Hence it is surjective in all degrees. By construction, $\bar{\lambda}$ is given in degree 1 by the map (5–4). It follows that $\bar{\lambda} \circ \bar{\nu} = \mathrm{id}$ in degree 1, and therefore in all degrees. This proves that $\bar{\nu}$ is injective. Therefore both $\bar{\nu}$ and $\bar{\lambda}$ are isomorphisms.

Conversely, suppose that $\bar{\lambda}$ is an isomorphism. Its definition in degree 1 shows that it maps $G_v S/S$ trivially. Hence $G_v \le S$, as required. □

Remark 5.2. When v is a discrete valuation and $S = \{1\}$, the first part of Theorem 5.1 is due to Bass and Tate [1973, I, Proposition 4.3]. They also prove its second part when (F, v) is a complete, discretely valued field with positive residue characteristic prime to m and when $S = (F^\times)^m$ [Bass and Tate 1973, I, Corollary 4.7]. Note that in the latter case, v is S-compatible by Hensel's lemma. Wadsworth [1983, §2] proves Theorem 5.1 for any valued field (F, v) when $S = (F^\times)^q G_v$ and q is a prime power.

Remark 5.3. The epimorphism of Theorem 5.1 is functorial in the following sense: suppose (F_1, v_1) is a valued field extension of (F, v), and suppose that $S \le F^\times$, $S_1 \le F_1^\times$, and $S \le S_1$. Further assume there exist homomorphic sections θ and θ_1 of the projections

$$v^* : F^\times/S \to v(F^\times)/v(S), \quad v_1^* : F_1^\times/S_1 \to v_1(F_1^\times)/v_1(S_1)$$

induced by v and v_1, respectively. Moreover, suppose that the following square commutes:

$$\begin{array}{ccc} v(F^\times)/v(S) & \xrightarrow{\theta} & F^\times/S \\ \downarrow & & \downarrow \\ v_1(F_1^\times)/v_1(S_1) & \xrightarrow{\theta_1} & F_1^\times/S_1. \end{array}$$

Then the epimorphisms given in Theorem 5.1 and the restriction morphisms induce a square:

$$\begin{array}{ccc}
K_*^M(F)/S & \longrightarrow & (K_*^M(\bar{F}_v)/\bar{S}_v)[v(F^\times)/v(S)] \\
\downarrow & & \downarrow \\
K_*^M(F_1)/S_1 & \longrightarrow & (K_*^M((\bar{F}_1)_{v_1})/(\bar{S}_1)_{v_1})[v_1(F_1^\times)/v_1(S_1)].
\end{array}$$

This square commutes in degree 1, hence in all degrees.

Remark 5.4. There are partial converses to Theorem 5.1. Namely, if $S = (F^\times)^p$ for a prime number p and if $K_*^M(F)/S$ is an extension of some κ-structure by $(\mathbb{Z}/p)^d$, then apart from some well-understood exceptional cases, F is equipped with an S-compatible valuation v with $v(F^\times)/pv(F^\times) \cong (\mathbb{Z}/p)^d$. Indeed, this follows from the results of [Jacob 1981; Ware 1981; Arason et al. 1987; Hwang and Jacob 1995]; see [Efrat 1999] for a K-theoretic formulation of this line of results.

6. A vanishing theorem

Recall that a valuation v on F induces a ring topology $\mathcal{T}_v$ on F, with basis consisting of all sets $a + bO_v$, where $a, b \in F$ and $b \neq 0$. For $0 < \gamma \in v(F^\times)$ the set

$$W_\gamma = \{x \in F^\times \mid v(1-x) \geq \gamma\}$$

is a $\mathcal{T}_v$-open subgroup of $G_v = 1 + \mathfrak{m}_v$.

Lemma 6.1. *Let v be a valuation on the field F. Let S be a subgroup of $F^\times$ such that $G_v/(S \cap G_v)$ is a finitely generated group. Then there exists $0 < \gamma \in v(F^\times)$ such that*

(i) $SG_v = SW_\gamma$, *and*

(ii) *if* char $\bar{F}_v = p$ *then* $1 + pO_v \leq W_\gamma$.

Proof. We choose $a_1, \dots, a_n \in \mathfrak{m}_v$ such that the cosets of $1 - a_i$, $i = 1, \dots, n$, generate $G_v/(S \cap G_v)$. Hence $(1-a_i)S$, $i = 1, \dots, n$, generate SG_v/S. Take any $0 < \gamma \leq \min\{v(a_1), \dots, v(a_n)\}$. Then $1 - a_i \in W_\gamma$, $i = 1, \dots, n$. Combined with $W_\gamma \leq G_v$, this shows that $SW_\gamma/S = SG_v/S$. When char $\bar{F}_v = p$ we take

$$\gamma = \min\{v(p), v(a_1), \dots, v(a_n)\}. \qquad \square$$

One says that the valuation v on F has *rank* 1 (or that it is *Archimedean*) if $v(F^\times)$ embeds in $\mathbb{R}$ as an ordered abelian group. Equivalently, for every $0 <$

$\alpha, \gamma \in v(F^\times)$ there exists a positive integer s such that $\alpha < s\gamma$ [Bourbaki 1972, VI §4.5, Proposition 8].

Theorem 6.2. *Let v be a valuation of rank 1 on the field F. Let S be a $\mathcal{T}_v$-open subgroup of $F^\times$ such that $F^\times/S$ is finitely generated and $F^\times = SG_v$. Then $K_r^M(F)/S = 0$ for all $r \geq 2$.*

Proof. It suffices to show that $aS \otimes bS \in \mathrm{St}_{F,2}(S)$ for $a, b \in G_v$. Suppose that this is not the case. In particular, $a, b \notin S$. Lemma 6.1 yields $0 < \gamma \in v(F^\times)$ such that $F^\times = SG_v = SW_\gamma$.

We define inductively a sequence $c_1, c_2, \ldots \in G_v$ such that for each i,

$$1 - c_i \in (1-b)(1-W_\gamma)^{i-1}, \quad aS \otimes bc_i^{-1}S \in \mathrm{St}_{F,2}(S).$$

We can take $c_1 = b$. Next suppose that c_i has already been constructed. Since $aS \otimes bS \notin \mathrm{St}_{F,2}(S)$ we have $c_i \neq 1$. Choose $y_i \in S$ such that $a/(1-c_i^{-1}) \in y_i W_\gamma$. As $a \notin S$ and $y_i \in S$, we may define $c_{i+1} = c_i(1 - y_i^{-1}a)$. Since $c_i \in G_v$ we have $y_i^{-1}a \in (1-c_i^{-1})W_\gamma \subseteq \mathfrak{m}_v$. Hence $c_{i+1} \in G_v$. Now

$$\frac{1-c_{i+1}}{1-c_i} = 1 - \frac{y_i^{-1}a}{1-c_i^{-1}} \in 1 - W_\gamma,$$

so by the induction hypothesis, $1 - c_{i+1} \in (1-b)(1-W_\gamma)^i$. Furthermore,

$$\begin{aligned} aS \otimes bc_{i+1}^{-1}S &= aS \otimes bc_i^{-1}S - aS \otimes (1 - y_i^{-1}a)S \\ &= aS \otimes bc_i^{-1}S - y_i^{-1}aS \otimes (1 - y_i^{-1}a)S \in \mathrm{St}_{F,2}(S). \end{aligned}$$

This completes the inductive construction.

Since v has rank 1, the sets $(1-W_\gamma)^s$, $s = 1, 2, 3, \ldots$, form a local basis for $\mathcal{T}_v$ at 0. As $b \neq 1$, the set $(1-b)^{-1}(1-S)$ is a $\mathcal{T}_v$-open neighborhood of 0. Hence there exists a positive integer t such that $(1-W_\gamma)^t \subseteq (1-b)^{-1}(1-S)$. Then $1 - c_{t+1} \in (1-b)(1-W_\gamma)^t \subseteq 1 - S$, and so $c_{t+1} \in S$. We conclude that $aS \otimes bS = aS \otimes bc_{t+1}^{-1}S \in \mathrm{St}_{F,2}(S)$, a contradiction. □

7. Wild valuations of rank 1

In this section we study $K_*^M(F)$ when F is a field of characteristic 0 equipped with a valuation v such that $\mathrm{char}\,\bar{F}_v = p > 0$. First we assume that v is a discrete valuation. Thus $\mathfrak{m}_v = aO_v$ for some $a \in \mathfrak{m}_v$. For $i \geq 1$, the map $1 + \mathfrak{m}_v^i \to \bar{F}_v$, $1 + a^i b \mapsto \pi_v(b)$, is a group homomorphism with kernel $1 + \mathfrak{m}_v^{i+1}$.

Lemma 7.1. *Let $(E, u)/(F, v)$ be an extension of discrete valued fields with the same value group and residue field. Then*:

(a) $(1 + \mathfrak{m}_u^i)/(1 + \mathfrak{m}_v^i) \cong E^\times/F^\times$ *canonically for all $i \geq 1$.*

(b) *For every $\mathcal{T}_u$-open subgroup S of $E^\times$, one has $E^\times = F^\times S$.*

Proof. (a) For $i = 1$ this follows from the exact sequences (5–1) and (5–2), with F, S replaced by E, $F^\times$, respectively. For $1 \le i$, the preceding remark gives a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 1+\mathfrak{m}_v^{i+1} & \longrightarrow & 1+\mathfrak{m}_v^{i} & \longrightarrow & \bar{F}_v & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\Vert & & \\
0 & \longrightarrow & 1+\mathfrak{m}_u^{i+1} & \longrightarrow & 1+\mathfrak{m}_u^{i} & \longrightarrow & \bar{E}_u & \longrightarrow & 1.
\end{array}$$

The snake lemma gives rise to a canonical isomorphism

$$(1+\mathfrak{m}_u^{i+1})/(1+\mathfrak{m}_v^{i+1}) \xrightarrow{\sim} (1+\mathfrak{m}_u^{i})/(1+\mathfrak{m}_v^{i}),$$

so we are done by induction.

(b) Since u is discrete, the subgroups $1+\mathfrak{m}_u^i$, $i = 1, 2, 3, \dots$, form a local basis for $\mathcal{T}_u$ at 1. Hence there exists i with $1+\mathfrak{m}_u^i \le S$. By (a), $E^\times = F^\times(1+\mathfrak{m}_u^i)$, so $E^\times = F^\times S$. □

Now let p be a prime number and let $q = p^d$ be a p-power with $d \ge 1$.

Proposition 7.2. *Let v be a discrete valuation on a field F such that* $\mathrm{char}\, F = 0$ *and* $\mathrm{char}\, \bar{F}_v = p$. *Let (E, u) be the completion of (F, v) and let $S = (F^\times)^q(1+q^2\mathfrak{m}_v)$. Then* $\mathrm{Res} : K_*^M(F)/S \to K_*^M(E)/q$ *is an isomorphism.*

Proof. By the Hensel–Rychlik lemma [Fesenko and Vostokov 1993, II (1.3), Corollary 2], $1 + q^2\mathfrak{m}_u \le (E^\times)^q$. In particular, $(E^\times)^q$ is $\mathcal{T}_u$-open in E. By Lemma 7.1(b), $E^\times = F^\times(1+q\mathfrak{m}_u)$. Hence $(E^\times)^q = (F^\times)^q(1+q^2\mathfrak{m}_u)$. It follows that $F \cap (E^\times)^q = (F^\times)^q(1+q^2\mathfrak{m}_v) = S$.

Since F is $\mathcal{T}_u$-dense in E, the assertion now follows from Proposition 2.2. □

Note that here the field E is a complete, discrete valued field of characteristic 0 and finite residue field of characteristic p. Therefore it is a finite extension of $\mathbb{Q}_p$. For a detailed analysis of the Milnor K-ring of such fields, refer to [Fesenko and Vostokov 1993, Chapter IX].

The following theorem extends arguments of Pop, which are implicit in the proof of [Pop 1988, Korollar 2.7]. In Theorem 7.4 below we use it in conjunction with Theorem 6.2 to compute the functor $K_*^M(F)/S$ in another mixed characteristic situation.

Theorem 7.3. *Let v be a valuation of rank* 1 *on a field F such that* $\mathrm{char}\, F = 0$ *and* $\mathrm{char}\, \bar{F}_v = p$. *Suppose that $F^\times/(F^\times)^p(1+p\mathfrak{m}_v)$ is finite. Then either*

(a) *$v(F^\times)$ is discrete and $\bar{F}_v$ is finite, or*

(b) *$v(F^\times)$ is p-divisible and $\bar{F}_v$ is perfect.*

Proof. Let $S=(F^\times)^p(1+p\mathfrak{m}_v)$. We break the argument into five parts.

Part I: $\bar F_v$ is perfect. Indeed, $\bar S_v=(\bar F_v^\times)^p$. By the exact sequences (5–1) and (5–2), $\bar F_v^\times/(\bar F_v^\times)^p$ is finite. Since char $\bar F_v=p$, this quotient must be trivial [Efrat 2003, Corollary 1.6], as desired.

Part II: $S\cap G_v=G_v^p(1+p\mathfrak{m}_v)$. To show this, consider the commutative diagram of exponentiations by p:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_v & \longrightarrow & O_v^\times & \longrightarrow & \bar F_v^\times & \longrightarrow & 1\\
 & & \downarrow p & & \downarrow p & & \downarrow p & & \\
1 & \longrightarrow & G_v & \longrightarrow & O_v^\times & \longrightarrow & \bar F_v^\times & \longrightarrow & 1.
\end{array}$$

Since char $\bar F_v=p$, the right vertical map is injective. The snake lemma implies that $(O_v^\times)^p\cap G_v=G_v^p$. Hence also $(F^\times)^p\cap G_v=G_v^p$. Since $1+p\mathfrak{m}_v\le G_v$ we obtain

$$S\cap G_v=((F^\times)^p\cap G_v)(1+p\mathfrak{m}_v)=G_v^p(1+p\mathfrak{m}_v).$$

Part III: $S\cap G_v\subseteq(1-S)(1+p\mathfrak{m}_v)$. Indeed, recall that $p\,|\binom{p}{i}$ for $i=1,\dots,p-1$. Hence for every $a\in\mathfrak{m}_v$ we have

$$(1-a)^p\in 1-a^p+p\mathfrak{m}_v=(1-a^p)(1+p\mathfrak{m}_v)\subseteq(1-S)(1+p\mathfrak{m}_v).$$

Thus $G_v^p\subseteq(1-S)(1+p\mathfrak{m}_v)$. Now use Part II.

Part IV: $v(F^\times)$ is either discrete or p-divisible. In view of the structure of the ordered group $\mathbb{R}$, it suffices to find $0<\gamma\in v(F^\times)$ such that for every $b\in F$ with $0<v(b)<\gamma$, one has $v(b)\in pv(F^\times)$. Since $F^\times/S$ is finite, the sequences (5–1) and (5–2) imply that $G_v/(S\cap G_v)$ is also finite. Hence we may take γ as in Lemma 6.1. By property (i) of W_γ and since $W_\gamma\le G_v$, we have $1-b\in G_v=(SW_\gamma)\cap G_v=(S\cap G_v)W_\gamma$. It therefore follows from part III and from property (ii) of W_γ that $1-b\in(1-S)W_\gamma$. So choose $s\in S$ with $1-b\in(1-s)W_\gamma$. As $W_\gamma\le G_v$ we get $1-s\in G_v$. Hence

$$v(b-s)=v\left(\frac{b-s}{1-s}\right)=v\left(1-\frac{1-b}{1-s}\right)\ge\gamma.$$

Since $v(b)<\gamma$, necessarily $v(b)=v(s)\in v(S)=pv(F^\times)$, as desired.

Part V: When $v(F^\times)$ is discrete, $\bar F_v$ is finite. Indeed, as we have observed, in this case

$$G_v/(1+\mathfrak{m}_v^2)=(1+\mathfrak{m}_v)/(1+\mathfrak{m}_v^2)\cong\bar F_v.$$

Using again that $p \mid \binom{p}{i}$ for $1 \le i \le p-1$, we get

$$G_v^p(1+p\mathfrak{m}_v) \le 1+\mathfrak{m}_v^2.$$

In light of Part II, this gives rise to a group epimorphism $G_v/(S\cap G_v) \to \bar{F}_v$. We have already noted that $G_v/(S\cap G_v)$ is finite. Consequently so is $\bar{F}_v$. □

Theorem 7.4. *Let v be a valuation of rank* 1 *on a field F such that* char $F=0$ *and* char $\bar{F}_v = p$. *Let $S=(F^\times)^q(1+q^2\mathfrak{m}_v)$ and suppose that $(F^\times : S)<\infty$. Then one of the following holds*:

(a) *$v(F^\times)$ is discrete, $\bar{F}_v$ is finite, and $K_*^M(F)/S \cong K_*^M(E)/q$ for the completion E of F with respect to v;*

(b) *$v(F^\times)$ is p-divisible and $K_r^M(F)/S=0$ for all $r\ge 2$.*

Proof. We have $v(S)=qv(F^\times)$ and $\bar{S}_v=(\bar{F}_v^\times)^q$. Since

$$(F^\times)^q(1+q^2\mathfrak{m}_v) \le (F^\times)^p(1+p\mathfrak{m}_v),$$

the finiteness assumption implies that $(F^\times : (F^\times)^p(1+p\mathfrak{m}_v))<\infty$. By Theorem 7.3, one of the following cases occurs:

Case (i): $v(F^\times)$ is discrete and $\bar{F}_v$ is finite. Then we apply Proposition 7.2.

Case (ii): $v(F^\times)$ is p-divisible and $\bar{F}_v$ is perfect. Then $v(S)=v(F^\times)$ and $\bar{S}_v = \bar{F}_v^\times$. The exact sequences (5–1) and (5–2) therefore show that $F^\times = SG_v$. Since S is $\mathcal{T}_v$-open in F, Theorem 6.2 implies that $K_r^M(F)/S=0$ for $r\ge 2$. □

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Received August 29, 2004. Revised January 14, 2005.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

UPPER BOUNDS FOR THE NUMBER OF LIMIT CYCLES THROUGH LINEAR DIFFERENTIAL EQUATIONS

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Consider the differential equation $\dot{x} = y$, $\dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + y^3$ in the plane. We prove that if a certain solution of an associated linear ordinary differential equation does not change sign, there is an upper bound for the number of limit cycles of the system. The main ingredient of the proof is the Bendixson–Dulac criterion for ℓ-connected sets. Some concrete examples are developed.

1. Main results

Although second order ordinary differential equations of the form $\ddot{x} = f(x, \dot{x})$ are some of the easiest autonomous planar differential equations, most problems concerning the study of the number of periodic solutions remain open. For instance, even if we consider the *Kukles system* $\dot{x} = y$, $\dot{y} = f_3(x, y)$, where f_3 is a polynomial of degree at most 3, the maximum number of limit cycles that it can have is still unknown.

This paper deals with the problem of finding methods to establish upper bounds for the number of limit cycles of planar differential equations of the form

$$\dot{x} = y, \quad \dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + y^3, \tag{1–1}$$

where the functions h_i are smooth enough.

The proof of our main result is based on the use of the generalized Bendixson–Dulac criterion for ℓ-connected sets. Recall that an open subset U of $\mathbb{R}^2$ is said to be *ℓ-connected* if its fundamental group $\pi_1(U)$ is the free group in ℓ generators. This method has already been used with similar goals by several authors; see for instance [Cherkas 1997; Lloyd 1979; Yamato 1979; Cherkas and Grin' 1997; 1998; Gasull and Giacomini 2002]. The novelty of our approach is that we are able to reduce the computation of an upper bound for the number of limit cycles of the

MSC2000: 34C07, 34C05, 34A30, 37C27.
Keywords: ordinary differential equation, limit cycle, Bendixson–Dulac criterion, linear ordinary differential equation.
The first author is partially supported by the DGES grant number BFM2002-04236 and CONACIT grant number 2001SGR-00173.

differential system (1–1) to the study of the positiveness of a certain function, which has the important property of satisfying a linear ordinary differential equation.

More precisely, our main result is this:

Theorem A. *Let X be the vector field associated with the differential equation*

$$\dot{x} = y, \quad \dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + y^3, \tag{1–2}$$

and fix a positive integer number n. Then there exists a constructive procedure, detailed in Lemmas 2.2 and 2.4, to associate with X two functions $f_n(x, y)$ and $M_n(x)$, such that

(i) $\operatorname{div}\big(|f_n(x, y)|^{-3/n} X(x, y)\big) = -\frac{3}{n} \operatorname{sgn}\big(f_n(x, y)\big)|f_n(x, y))|^{-1-3/n} M_n(x)$, *and*

(ii) *the function $y = M_n(x)$ is defined for all $x \in \mathbb{R}$ and is a solution of a linear ordinary differential equation of the form*

$$s_{n,n+1}(x)\, y^{(n+1)}(x) + s_{n,n}(x)\, y^{(n)}(x) + \cdots + s_{n,1}(x)\, y'(x) + s_{n,0}(x)\, y(x) = 0.$$

Assume furthermore that $M_n(x)$ does not change sign and vanishes only at finitely many points. Then:

(iii) *The limit cycles of* (1–2) *do not cut the curves $\{f_n(x, y) = 0\}$.*

(iv) *The number of limit cycles of* (1–2) *contained in an ℓ-connected component U of $\mathbb{R}^2 \setminus \{f_n(x, y) = 0\}$ is at most ℓ. All these limit cycles are hyperbolic and their stability is given by the sign of $M_n(x)$ and the sign of $f_n(x, y)$ in the region occupied by the limit cycle.*

From the proof of the theorem it is easy to observe that, with small modifications, it can also be applied to systems for which the second equation is $\dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3(x)y^3$.

In Section 3 we study a simple example, the van der Pol equation, to show how the method works.

In Section 4 we study the number of limit cycles of the system

$$\dot{x} = y, \quad \dot{y} = -x^3 + dxy^2 + y^3.$$

For this system the expected upper bound is of one limit cycle, but as far as we know this is still an open question. The authors have studied this problem by using several existing methods in the literature but no progress has been possible. For this reason we have selected this problem to test the effectiveness of the new method proposed. A motivation for its study is also given at the beginning of Section 4. The results obtained are detailed in Section 5.

In these two examples we see that we can reduce the study of the number of limit cycles of a planar polynomial system to the study of a linear ordinary differential equation. Although the study of this last equation is not easy and requires special tricks for each concrete application, it provides a new way for trying to control the number of limit cycles for special classes of planar polynomial systems. Also, for the main example developed in Section 4, we can see that the final step goes to a one-variable nonlinear equation. To end this introduction we would like to stress this last scheme:

$$\begin{matrix}\text{Planar ordinary} \\ \text{nonlinear differential equation}\end{matrix} \longrightarrow \begin{matrix}\text{Linear ordinary} \\ \text{differential equation}\end{matrix} \longrightarrow \begin{matrix}\text{Nonlinear} \\ \text{equation}\end{matrix}$$

2. Preliminary results and proof of Theorem A

First we recall the *generalized Bendixson–Dulac criterion*. For various proofs, see [Lloyd 1979; Yamato 1979; Gasull and Giacomini 2002].

Proposition 2.1 (Generalized Bendixson–Dulac Criterion). *Consider a* $\mathscr{C}^1$ *differential system*

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

and set $X = (P, Q)$. *Let* U *be an open* ℓ*-connected subset of* $\mathbb{R}^2$ *with a smooth boundary. Assume that*

$$\operatorname{div}(X) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

does not change sign on U *and vanishes only on a null measure Lebesgue set. Then the system can have at most* ℓ *periodic orbits contained in* U. *Each such orbit is hyperbolic and its stability is given by the sign of* $\operatorname{div}(X)$.

We now turn to preliminary computations needed to prove Theorem A.

Lemma 2.2. *Consider the system* (1–2),

$$\begin{aligned}\dot{x} &= y =: P(x, y),\\ \dot{y} &= h_0(x) + h_1(x)y + h_2(x)y^2 + y^3 =: Q(x, y),\end{aligned}$$

and fix a positive integer number n. *There is a constructive procedure to find* $n+1$ *functions* $r_{n,i}(x)$, $i = 0, \ldots, n$, *satisfying the following condition*:

Let $y(x) = g_n(x)$ *be any solution of the order-*$(n{+}1)$ *linear ordinary differential equation*

$$y^{(n+1)}(x) + r_{n,n}(x)\, y^{(n)}(x) + \cdots + r_{n,1}(x)\, y'(x) + r_{n,0}(x)\, y(x) = 0, \tag{2–1}$$

and let $g_{n,i}(x)$, *where* $i = 0, \ldots, n-1$, *be defined in terms of* $h_0(x)$, $h_1(x)$, $h_2(x)$, $g_n(x)$, *their derivatives, and* $g_{n,n}(x) := g_n(x)$. *Then, setting*

$$f_n(x, y) := g_{n,0}(x) + g_{n,1}(x)y + g_{n,2}(x)y^2 + \cdots + g_{n,n}(x)y^n,$$

the expression

$$M_n := \langle \nabla f_n, (P, Q)\rangle - \frac{n}{3} f_n \operatorname{div}(P, Q)$$

is a function of only the x-variable.

Proof. Instead of dealing with a general n and for the sake of clarity, we present the details of the proof just for the case $n = 2$. The general case can be handled in the same way. Also, for the sake of brevity and during this proof, when a function of x appears that we do not want to specify, we will write $*$.

Take $f_2(x, y) = g_{2,0}(x) + g_{2,1}(x)y + g_{2,2}(x)y^2 := g_0(x) + g_1(x)y + g_2(x)y^2$. Then,

$$\begin{aligned} M_2(x, y) &= \big\langle \nabla f_2, (P, Q)\big\rangle - \tfrac{2}{3}\,\mathrm{div}(P, Q) f_2 \\ &= \big(g_2'(x) + \tfrac{2}{3}\, g_2(x)h_2(x) - g_1(x)\big)y^3 \\ &\quad + \big(g_1'(x) + \tfrac{4}{3}\, g_2(x)h_1(x) - \tfrac{1}{3}\, g_1(x)h_2(x) - 2\, g_0(x)\big)y^2 \\ &\quad + \big(g_0'(x) + \tfrac{1}{3}\, g_1(x)h_1(x) - \tfrac{4}{3}\, h_2(x)g_0(x) + 2\, g_2(x)h_0(x)\big)y \\ &\quad + \big(g_1(x)h_0(x) - \tfrac{2}{3}\, h_1(x)g_0(x)\big). \end{aligned}$$

By choosing

$$\begin{aligned} g_0(x) &= \tfrac{1}{2}\big(g_1'(x) + \tfrac{4}{3}\, g_2(x)h_1(x) - \tfrac{1}{3}\, g_1(x)h_2(x)\big), \\ g_1(x) &= g_2'(x) + \tfrac{2}{3}\, g_2(x)h_2(x), \end{aligned} \tag{2–2}$$

we ensure that the coefficients of y^2 and y^3 in M_2 vanish. Observe that $g_1(x) = g_2'(x) + *\, g_2(x)$ and that $g_0(x) = g_2''(x)/2 + *\, g_2'(x) + *\, g_2(x)$. If we substitute these equalities into the coefficient of y in the expression for M_2, we obtain $g_2'''(x)/2 + *\, g_2''(x) + *\, g_2'(x) + *\, g_2(x)$. By imposing that this last expression be identically zero, we get the linear ordinary differential equation (2–1) given in the statement of the lemma. Hence for these values of the functions g_i, where $i = 0, 1, 2$, the expression of M_2 is the function of one variable

$$M_2(x) = g_1(x)h_0(x) - \tfrac{2}{3}\, h_1(x)g_0(x), \tag{2–3}$$

as we wanted to prove. □

Remark 2.3. (i) From the proof of Lemma 2.2 it is easy to observe that if all the functions h_i appearing in system (1–2) are polynomials, then all the functions $r_{n,i}$ are polynomials as well.

(ii) If in system (1–2) instead of considering $\dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + y^3$, we take $\dot{y} = h_0(x) + h_1(x)y + h_2(x)y^2 + h_3(x)y^3$, then a similar result can be proved. The main difference is that the function h_3 and its powers appear in the denominators of the expressions of $r_{n,i}$. Hence, all the computations make sense on only the strips where $h_3(x)$ does not vanish.

Lemma 2.4. *Let $M_n(x)$ be the one-variable function described in Lemma 2.2. Then there exists an order-$(n+1)$ linear ordinary differential equation*

$$s_{n,n+1}(x)\,y^{(n+1)}(x)+s_{n,n}(x)\,y^{(n)}(x)+\cdots+s_{n,1}(x)\,y'(x)+s_{n,0}(x)\,y(x)=0, \tag{2–4}$$

such that $y = M_n(x)$ is one of its solutions. Here the functions $s_{n,i}$, for $i = 0, \ldots, n+1$, can be explicitly obtained from all the functions appearing in Lemma 2.2.

Proof. As for Lemma 2.2, we detail the proof just for the case $n = 2$. We continue denoting a generic smooth function of the variable x by $*$. From (2–3) and (2–2), we have $M := M_2 = *\,g_2''+*\,g_2'+*\,g_2$ and from the proof of Lemma 2.2 we obtain $g_2'''(x)+*\,g_2''(x)+*\,g_2'(x)+*\,g_2(x)=0$. Hence, if we differentiate the first equality three times and the second one twice, we get the linear system

$$\begin{pmatrix} 0 & 0 & 1 & * & * & * & 0 \\ 0 & 1 & * & * & * & * & 0 \\ 1 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * & M \\ 0 & 0 & * & * & * & * & M' \\ 0 & * & * & * & * & * & M'' \\ * & * & * & * & * & * & M''' \end{pmatrix} \begin{pmatrix} g_2^{V} \\ g_2^{IV} \\ g_2''' \\ g_2'' \\ g_2' \\ g_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since for all x, this system has the nonzero solution given by the function $g_2(x)$ and its derivatives, the determinant of the matrix is identically zero. By developing it from its last column, we get the linear ordinary differential equation satisfied by M given in the statement of the lemma. □

Remark 2.5. A main difference between the linear differential equations of order $n+1$ satisfied by g_n and M_n and those given in (2–1) and (2–4), respectively, is that the coefficient of the highest order derivative is identically 1 in the first case, and a function of x in the second case. Hence, one could think that M_n is not defined for all x, but this is not the case because it is also given by (2–3). In other words, M_n is a solution of the linear equation (2–4) defined for all real x.

Proof of Theorem A. (i) From Lemma 2.2 we can construct a function $f_n(x, y)$ such that

$$M_n := \langle \nabla f_n, X\rangle - \frac{n}{3}\,\mathrm{div}(P, Q)\,f_n$$

depends just on x, as we wanted to prove.

(ii) By Lemma 2.4, the function $y = M_n(x)$ satisfies the linear ordinary differential equation (2–4). Furthermore, by Remark 2.5 we know that it is defined for all $x \in \mathbb{R}$.

Assume from now on that $M_n(x)$ does not change sign and vanishes only on a finite set of points.

(iii) Since

$$M_n|_{f_n=0} = \langle \nabla f_n, X\rangle|_{f_n=0},$$

from the control on the sign of M_n, the periodic orbits of (1–2) never cut the curves $\{f_n = 0\}$, because the flow associated with X crosses each one of them either inwards or outwards.

(iv) Instead of considering the vector field X, we take the new one $|f_n|^{-3/n}X$. From the previous paragraph, we know that none of the limit cycles of X intersect $\{f_n = 0\}$. Hence each limit cycle is contained in a connected component U of $\mathbb{R}^2 \setminus \{f_n = 0\}$. Note that

$$\begin{aligned}\operatorname{div}\big(|f_n|^{-3/n}X\big) &= \big\langle \nabla(|f_n|^{-3/n}), X\big\rangle + |f_n|^{-3/n}\operatorname{div}(X)\\ &= -\tfrac{3}{n}\operatorname{sgn}(f_n)|f_n|^{-1-3/n}\langle\nabla f_n, X\rangle + |f_n|^{-3/n}\operatorname{div}(X)\\ &= -\tfrac{3}{n}\operatorname{sgn}(f_n)|f_n|^{-1-3/n}\big(\langle\nabla f_n, X\rangle - \tfrac{1}{3}nf_n\operatorname{div}(X)\big)\\ &= -\tfrac{3}{n}\operatorname{sgn}(f_n)|f_n|^{-1-3/n}M_n.\end{aligned}$$

Therefore, $\operatorname{div}(|f_n|^{-3/n}X)$ does not change sign on U. By using the generalized Bendixson–Dulac criterion (Proposition 2.1), the theorem follows. □

3. A first example: the van der Pol equation

The uniqueness of the limit cycle of the van der Pol equation can be proved by several different methods. We have chosen this simple example to illustrate our approach. Recall that the van der Pol equation is

$$\dot{x} = y - \varepsilon\big(\tfrac{1}{3}\,x^3 - x\big), \quad \dot{y} = -x.$$

It can be transformed into the form (1–2) by interchanging x and y and then changing y to $-y$. This gives

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon\big(\tfrac{1}{3}\,y^3 - y\big). \tag{3–1}$$

To prove the uniqueness of the limit cycle of this system by our method, we will apply Theorem A with $n = 2$. Notice that since $f_2(x, y) = g_0(x) + g_1(x)y + g_2(x)y^2$, for this value of n all the connected components of $\mathbb{R}^2\setminus\{f_2 = 0\}$ are either simply connected or 1-connected. Furthermore, there is at most one 1-connected component that surrounds the origin.

With the notation introduced in Theorem A, we have

$$\begin{aligned} g_0(x) &= \tfrac{3}{2}(3g_2''(x) - \tfrac{4}{3}g_2(x)), \quad g_1(x) = -3g_2'(x),\\ M_2(x) &= \tfrac{1}{3}(4g_2(x) + 9xg_2'(x) - 9g_2''(x)),\end{aligned} \tag{3–2}$$

where we have taken $\varepsilon = 1$ to simplify the calculations. The function $g_2(x)$ is any solution of the third order linear differential equation

$$9y'''(x) - 6y'(x) - 4xy(x) = 0. \tag{3–3}$$

We have to choose a suitable solution g_2 such that its associated M_2 does not change sign. From the classical theory of linear differential equations, (see for instance [Ince 1927; Wasow 1965]), the solutions of this equation are analytic and entire. We can write them as

$$g_2(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with $a_3 = \frac{1}{9}a_1$ and

$$a_n = \frac{4a_{n-4} + 6(n-2)a_{n-2}}{9n(n-1)(n-2)}, \quad \text{for } n \geq 4. \tag{3–4}$$

For facilitating the control of the sign of $M_2(x)$ we take the even solution of (3–3), defined by the conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

With these initial conditions, it is clear from (3–4) that all the nonzero coefficients a_n are positive. Hence, g_2 as well as all of its derivatives, are positive for positive x. Furthermore, from (3–2) we see that M_2 is an even function with $M_2(0) = 4/3$. Hence, it suffices to study the sign of $M_2(x)$ for $x > 0$. Let us prove that the m-th derivative of $M_2(x)$ is positive for $x > 0$ and any $m \geq 3$. By taking the third derivative of $M_2(x)$ from (3–2), and by using (3–3), we obtain

$$M_2'''(x) = \tfrac{2}{3}\Big(\tfrac{68}{9}xg_2(x) + \big(2x^2 + \tfrac{13}{3}\big)g_2'(x) + xg_2''(x)\Big).$$

This equality and the properties of $g_2(x)$ and its derivatives imply that $M_2'''(x) > 0$ for all positive x. Furthermore, by taking more derivatives of this expression, and using the equality $g_2'''(x) = \frac{1}{9}\big(6g_2'(x) + 4xg_2(x)\big)$ at each step, we obtain only positive coefficients during all of the computations. Hence, our assertion follows about the derivatives of M_2. By using Taylor's Formula and some straightforward computations, we get

$$M_2(x) = \tfrac{4}{3} - \tfrac{2}{3}x^2 + \tfrac{17}{81}x^4 + \tfrac{1}{5!}M_2^{(5)}(s_x)x^5,$$

for some s_x between 0 and x. Hence, for $x > 0$,

$$M_2(x) \geq \tfrac{4}{3} - \tfrac{2}{3}x^2 + \tfrac{17}{81}x^4 > 0,$$

as we wanted to prove. As a consequence of Theorem A, we conclude that the system (3–1) has at most one limit cycle. Moreover, when this limit cycle exists, it is hyperbolic and stable.

The van der Pol equation has a unique critical point located at the origin and it is repelling for $\varepsilon > 0$. The point at infinity is also a repeller for $\varepsilon > 0$. Therefore, the system has at least one limit cycle. Combining both statements we have proved the existence, uniqueness and hyperbolicity of the limit cycle.

We remark that in order to find an upper bound for the number of limit cycles by using our approach, it has not been necessary to explicitly solve the linear differential equation satisfied by the function $g_2(x)$. Only general properties of this function, which can be easily obtained from the linear equation, have been employed. In the next section we analyze a more difficult case.

4. A second example

We start this section with some motivation for the system of ordinary differential equations that we will study. In [Cima et al. 1997] it is proved that there are systems of the form

$$\dot{x} = P_{2n+1}(x, y), \quad \dot{y} = Q_{2m+1}(x, y),$$

with P_{2n+1} and Q_{2m+1} homogeneous polynomials of degrees $2n+1$ and $2m+1$ respectively ($n \neq m$), possessing at least $n+m+1$ limit cycles surrounding the origin. These examples are constructed by studying the perturbations of the Hamiltonian system $\dot{x} = y^{2n+1}$, $\dot{y} = -x^{2m+1}$. Inside this family, the simplest case, $n = 0$ and $m = 1$, gives a system of the form

$$\dot{x} = ax + by, \quad \dot{y} = cx^3 + dx^2y + exy^2 + fy^3$$

with at least two limit cycles. We would like to investigate whether there can be more than two. This seems to be a hard problem, and so we start by considering the simplest case:

$$\dot{x} = y, \quad \dot{y} = -x^3 + dx^2y + y^3, \tag{4–1}$$

for which it is not difficult to prove that there is at least one limit cycle [Cima et al. 1997]. This section is devoted to trying to prove that in fact one is the maximum number of limit cycles that the system can have. Before starting our study, we want to comment that we have not been able to prove the uniqueness of the limit cycles of system (4–1) by using standard results in the literature, such as those in [Ye et al. 1986; Zhang et al. 1992]. Our results are summarized in Section 5. As a starting point we prove a previous result that reduces the study to the case $d < 0$.

Lemma 4.1. (i) *The origin is the only critical point of the system* (4–1).

(ii) *If* $d \geq 0$, *the system has no limit cycles.*

(iii) *For $d < 0$ is close enough to zero, the system has at least one limit cycle.*

(iv) *If for some $\bar{d} < 0$ the system has no limit cycles, the same holds for any $d \le \bar{d}$.*

Proof. Part (i) is trivial.

(ii) The divergence of $X = (P, Q)$, the vector field associated with (4–1), equals dx^2+3y^2. For $d \ge 0$ this is always positive or zero, so using the divergence criterion we deduce that system (4–1) has no limit cycles.

(iii) Notice that (4–1) is a semicomplete family of rotated vector fields with respect to the parameter d, or SCFRVF for short (see [Duff 1953; Perko 1975]). This follows from the next computations, where we denote the vector field associated with (4–1) by $X_d(x, y) = (P_d(x, y), Q_d(x, y))$,

$$\frac{\partial}{\partial d} \arctan \frac{Q_d(x, y)}{P_d(x, y)} = \frac{P_d(x, y)\partial Q_d(x, y)/\partial d - Q_d(x, y)\partial P_d(x, y)/\partial d}{P_d^2(x, y) + Q_d^2(x, y)}$$
$$= \frac{x^2y^2}{P_d^2(x, y) + Q_d^2(x, y)} \ge 0.$$

In [Cima et al. 1997] it is proved that the origin is a repeller when $d \ge 0$, and an attractor when $d < 0$. Combining this with the fact that our system is an SCFRVF, we see that a repelling limit cycle bifurcates from the origin when d is negative and small. Hence, item (iii) follows.

To prove item (iv) we need to recall more properties of an SCFRVF. The first is the *nonintersection property*, which asserts that limit cycles corresponding to different values of d are disjoint.

The second is the *planar termination principle* [Perko 1990a; 1990b], which asserts the following for polynomial families of an SCFRVF: If d varies and we consider the continuous evolution of some limit cycle born at a critical point p (allowing for the possibility that the limit cycle goes to a multiple limit cycle, in which case we continue with the other limit cycle that has collided with it), then the union of this one-parameter family of limit cycles is a 1-connected open set K whose boundaries are p and a cycle of separatrices of X_d. The corners of this cycle of separatrices are finite or infinite critical points of X_d. In our case, because the only finite critical point of X_d is the origin, K is unbounded.

If for some value of $d = \bar{d} < 0$ the system has no limit cycles, this means that the limit cycles starting at the origin for $d = 0$ have disappeared for some d^*, where $\bar{d} < d^* < 0$, covering the set K. Since K covers from a neighborhood of the origin until infinity, then by the nonintersection property, the limit cycle cannot exist for $d \le \bar{d}$ either, as we wanted to prove. □

Nonexistence of limit cycles. We will now find a value $d = \bar{d}$, as sharp as possible, that determines parameter values for which there are no limit cycles. In a later section (page 289) we will study when there is a unique limit cycle.

From Lemma 4.1, to complete the study of the nonexistence of limit cycles for (4–1), the case $d < 0$ remains:

$$\dot{x} = y, \quad \dot{y} = -x^3 - a^2x^2y + y^3. \tag{4–2}$$

To prove the nonexistence of limit cycles, it suffices to apply Theorem A with $n = 1$, because for this value of n all of the connected components of $\mathbb{R}^2 \setminus \{f_1 = 0\}$ are simply connected. With the notation introduced in this theorem we have

$$M_1(x) = -x^2\big(xg_1(x) - \tfrac{1}{3}a^2g_1'(x)\big),$$

where $g_1(x)$ is any solution of the linear ordinary differential equation

$$y''(x) - \tfrac{2}{3}a^2x^2y(x) = 0. \tag{4–3}$$

To conclude the nonexistence of limit cycles, it suffices to give a concrete solution g_1 of (4–3) such that the corresponding M_1 does not change sign. In fact it suffices to ensure that

$$h_a(x) := xg_1(x) - \tfrac{1}{3}a^2g_1'(x) \tag{4–4}$$

does not change sign. Furthermore, from Theorem A we also know that M_1 satisfies a second order linear differential equation. Hence, the same happens with h_a. The expression $h_a(x)$ satisfies the equation

$$\begin{aligned}\big((6a^6 - 81)x^2 - 27a^2\big)\, y''(x) - 6(2a^6 - 27)xy'(x)&\\ - \big((4a^8 - 54a^2)x^4 - 18a^4x^2 + 162\big)\, y(x) &= 0.\end{aligned} \tag{4–5}$$

Fortunately, we can solve (4–3) in terms of the modified Bessel functions. More concretely, for all $x \in \mathbb{R}^+$, define

$$I_\nu(x) := \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{\Gamma(\nu+k+1)\Gamma(k+1)}. \tag{4–6}$$

This function is a solution of the Bessel equation

$$x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0.$$

Hence, it is easy to check that

$$g_1(x) = \sqrt{x}\, I_{1/4}\left(\frac{ax^2}{\sqrt{6}}\right) \tag{4–7}$$

is a solution of (4–3) for $x > 0$, and it can be extended to an odd solution of (4–3) for all $x \in \mathbb{R}$.

From now on we will fix this choice for g_1. In Lemma 4.2 we collect some known properties of the modified Bessel functions that are useful for our study of h_a. Before stating them we need to introduce some standard notations.

Following Poincaré's definition, given a function $f : \mathbb{R} \to \mathbb{R}$, its *asymptotic expansion at* ∞ is $\sum_{k=0}^{\infty} b_k x^{-k}$ if

$$\lim_{x\to\infty} x^n \left(f(x) - \sum_{k=0}^{n} \frac{b_k}{x^k} \right) = 0 \quad \text{for all } n \in \mathbb{N}.$$

The usual notation is $f(x) \sim \sum_{k=0}^{\infty} b_k x^{-k}$. Furthermore, the notation

$$f(x) \sim g(x) \sum_{k=0}^{\infty} b_k x^{-k}$$

means $f(x)/g(x) \sim \sum_{k=0}^{\infty} b_k x^{-k}$. In this case,

$$\lim_{x\to\infty} \left(\frac{f(x)}{g(x)} - b_0 \right) = 0.$$

This fact is also denoted as $f(x) \sim b_0 g(x)$ at ∞, and $b_0 g(x)$ is said to be the *dominant term* or to represent the leading behavior of $f(x)$ at infinity.

Lemma 4.2. *The modified Bessel function* $I_\nu(x)$ *of* (4–6) *satisfies*

$$I'_\nu(x) = I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x), \qquad I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{at } \infty.$$

Lemma 4.3. *Let* h_a *be given by expression* (4–4) *with* g_1 *given in* (4–7).

(i) *The function* h_a *is an even function.*

(ii) *In a neighborhood of zero,*

$$h_a(x) = -\frac{2^{13/8}}{3^{9/8}} \frac{a^{9/4}}{\Gamma\left(\frac{1}{4}\right)} + m(a)x^2 + O(x^4),$$

where $m(a)$ *is positive for* $a > 0$.

(iii) *For* $a > \sqrt[6]{27/2}$, *we have* $\lim_{x\to+\infty} h_a(x) = -\infty$.

(iv) *For* $a < \sqrt[6]{27/2}$, *we have* $\lim_{x\to+\infty} h_a(x) = +\infty$.

(v) *If the system of equations* $h_a(x) = h'_a(x) = 0$ *has some positive solution, it has to be at the point*

$$x = x^*(a) := \frac{3a}{\sqrt{2a^6 - 27}}.$$

(vi) *For* $x > 0$ *and sufficiently large values of* a, *the function* $h_a(x)$ *does not vanish.*

Proof. (i) From its definition, it is clear that h_a is even. Hence, from now on we will consider only positive values of x.

(ii) By Lemma 4.2, it is easy to verify that

$$h_a(x) = \frac{x^{3/2}}{9}\left(9I_{1/4}\left(\frac{ax^2}{\sqrt{6}}\right) - \sqrt{6}a^3 I_{-3/4}\left(\frac{ax^2}{\sqrt{6}}\right)\right). \tag{4–8}$$

By using this last expression and (4–6), the Taylor expansion of h_a at the origin follows.

(iii), (iv) By applying the second part of Lemma 4.2 to the last expression of h_a and by the asymptotic expansion properties, we obtain

$$h_a(x) \sim \frac{1}{\sqrt{2\pi}} \frac{(18 - 2\sqrt{6}a^3)x^2 e^{ax^2/\sqrt{6}}}{18x^{3/2}} \quad \text{at } \infty.$$

Hence the results about the behavior of h_a at infinity are proved.

(v) Let $\bar{x}(a)$ be a solution of the system of equations $h_a(x) = h_a'(x) = 0$. Since h_a satisfies the linear ordinary differential equation (4–5), we have when $x = \bar{x}(a)$, then $\big((6a^6 - 81)x^2 - 27a^2\big)\, h_a''(x)$ has to be zero. Assume that at this point h_a'' also vanishes, and that $\bar{x}(a) \neq x^*(a)$. Then, since $\bar{x}(a)$ is not a singular point for (4–5), this implies that $h_a(x) \equiv 0$, which gives a contradiction. Therefore $\bar{x}(a)$ has to be the positive root of $\big((6a^6 - 81)x^2 - 27a^2\big) = 0$, say $x^*(a)$, as we wanted to prove.

(vi) Let $\tilde{x}(a)$ be a positive solution of $h_a(x) = 0$. Then $x = \tilde{x}(a)$ is also a solution of the equation

$$\frac{I_{1/4}\left(\frac{ax^2}{\sqrt{6}}\right)}{I_{-3/4}\left(\frac{ax^2}{\sqrt{6}}\right)} = \frac{\sqrt{6}a^3}{9}. \tag{4–9}$$

If for sufficiently large a such a solution $\tilde{x}(a)$ exists, there are two possibilities:

(a) either $a\tilde{x}(a)^2$ is bounded above when a tends to infinity,

(b) or $a\tilde{x}(a)^2$ is unbounded.

In the first case we can construct a sequence $\{a_n\}_n$ tending to infinity and such that $\lim_{n\to\infty} a_n\tilde{x}(a_n)^2 = k \geq 0$. By replacing these values in (4–9) we arrive at a contradiction.

In the case (b), by Lemma 4.2, $I_{1/4}$ and $I_{-3/4}$ have the same behavior at infinity. Therefore, the left hand side of Equation (4–9) tends to 1 when $a_n\tilde{x}(a_n)^2$ tends to infinity. However, the right hand side tends to infinity, again reaching a contradiction. Hence, for sufficiently large a, the function h_a does not vanish, as we wanted to prove. □

The main result of this subsection is this:

Proposition 4.4. *Let φ be the function*

$$\varphi(a) := 3\sqrt{3}I_{1/4}\left(\frac{3\sqrt{3/2}a^3}{2a^6-27}\right) - \sqrt{2}a^3 I_{-3/4}\left(\frac{3\sqrt{3/2}a^3}{2a^6-27}\right). \tag{4–10}$$

Denote by $\bar{a}$ the largest positive solution of the equation $\varphi(a) = 0$. Then the differential equation (4–2) *has no limit cycles for $a \geq \bar{a}$.*

Proof. To prove that system (4–2) has no limit cycles for some value of a, it suffices to show that its corresponding h_a does not change sign. For sufficiently large a, by Lemma 4.3(vi), we already know that this is true. On the other hand, by studying the behavior of h_a near zero and infinity (see again Lemma 4.3), it is clear that h_a changes sign for $a < \sqrt[6]{27/2}$. Hence the case $a \geq \sqrt[6]{27/2}$ remains to be studied. Let $\tilde{a}$ be the biggest value of a for which the function h_a has some zero. Denote any of these zeros by $z(\tilde{a})$. From the behavior of h_a near zero and infinity, and from the regularity of h_a with respect to x and a, we have $h_{\tilde{a}}(z(\tilde{a})) = h'_{\tilde{a}}(z(\tilde{a})) = 0$. Hence, by Lemma 4.3(v),

$$z(\tilde{a}) = x^*(\tilde{a}) := \frac{3\tilde{a}}{\sqrt{2\tilde{a}^6 - 27}}.$$

By imposing that $h_{\tilde{a}}(x^*(\tilde{a})) = 0$ in the expression for h_a given in (4–8), we get the desired expression for φ. □

Remark 4.5. Although we have not been able to perform the analytic study of the zeros of φ, it is not difficult to make a numerical study. The equation $\varphi(a) = 0$ has a unique positive solution $\bar{a} \simeq 1.636$. For this value of a, the corresponding value of $d = -a^2$ in (4–1) is $\bar{d} \simeq -2.678$.

Uniqueness of limit cycles. From Lemma 4.1, we have to study (4–1) only in the case $d < 0$. As in the van der Pol equation, for obtaining the uniqueness of the limit cycle with our procedure, it suffices to apply Theorem A with $n = 2$. Following the notation introduced in Theorem A we get

$$M_2(x) = 3dg_2''(x) + 9xg_2'(x) + 4d^2x^2g_2(x), \tag{4–11}$$

where $g_2(x)$ is any solution of the linear ordinary differential equation

$$y'''(x) + 2dx^2y'(x) + \left(\tfrac{8}{3}d - 4x^2\right)xy(x) = 0. \tag{4–12}$$

Hence, to prove the existence of at most one limit cycle and its hyperbolicity, it suffices to choose a concrete g_2 as the solution of (4–12), such that its associated M_2, given in (4–11), does not change sign.

All the solutions of (4–12) are analytic for all $x \in \mathbb{R}$, with an infinite radius of convergence. We will choose the even solution of (4–12) defined by the initial conditions

$$y(0)=1, \quad y'(0)=y''(0)=0. \tag{4–13}$$

Notice that this function g_2 produces an even function M_2. Before studying the sign of M_2 we need to study the function g_2 near infinity in some detail. Recall that g_2 is a solution of Equation (4–12). It is easy to see that infinity is a singularity for Equation (4–12). Although unfortunately not regular,[1] the singularity turns out to be of *normal irregular* type. For this kind of singularity there are some powerful results, which in most cases give the asymptotic expansions of a fundamental set of solutions. The result we need here is a generalization of a theorem of Poincaré; a proof can be found in [Horn 1901]. See also [Wasow 1965, Theorem 12.3].

Theorem 4.6. *Consider the linear ordinary differential equation*

$$y^{(n)}(x)+b_1(x)y^{(n-1)}(x)+\cdots+b_n(x)y(x)=0, \tag{4–14}$$

where the functions $b_s(x)$ are either rational functions or admit asymptotic expansions at infinity of the form

$$b_s(x)\sim x^{sk}\sum_{i=0}^{\infty}\frac{b_{s,i}}{x^i}, \quad s=1,2,\dots,n, \tag{4–15}$$

where k is a positive integer or zero. Assume that the algebraic equation

$$m^n+b_{1,0}m^{n-1}+\cdots+b_{n-1,0}m+b_{n,0}=0$$

associated with (4–14) *has n different roots $m_1, m_2, \dots, m_n$. Then* (4–14) *has n linearly independent solutions $y_1, y_2, \dots, y_n$ whose asymptotic expansions at infinity are of the form*

$$y_s(x)\sim e^{f_s(x)}x^{\alpha_s}\sum_{i=0}^{\infty}\frac{B_{s,i}}{x^i}, \quad s=1,2,\dots,n,$$

where α_s and $B_{s,i}$ are constants with $B_{s,0}=1$, and the f_s are polynomials in x of degree $k+1$, vanishing at zero and having leading coefficient $m_s/(k+1)$. The asymptotic expansions of the functions $y_s(x)$ can be uniquely determined by formal substitutions in (4–14).

(A more general result about irregular singularities of linear equations, including also the case of multiple roots, can be found in [Wasow 1965, Theorem 19.1]. See [Bender and Orszag 1999, Chapter 3; Erdélyi 1956, Chapter III; Ince 1927,

[1] When infinity is a regular singularity for a linear equation, generically the solutions of the equation for sufficiently large x are $x^{\alpha}\sum_{k=0}^{\infty} b_k x^{-k}$, where α is not necessarily an integer, and the series has a positive radius of convergence; in nongeneric cases, some logarithms can appear in the expressions of the solutions.

Chapters VII and XVII] for more examples of irregular singularities, both normal and not; for a more recent point of view about linear differential equations, see [Varadarajan 1996] and the references therein.)

The next lemma gives the desired properties of g_2 at infinity.

Lemma 4.7. *Consider* $-\sqrt[3]{27/2} < d < 0$. *Let* g_2 *be the solution of* (4–12) *with initial conditions given in* (4–13). *Then the following conditions hold*:

(i) *The function* g_2 *is defined for all* $x \in \mathbb{R}$, *and it is an even positive function.*

(ii) *The functions* $g_2'(x)$ *and* $g_2''(x)$ *are positive for all* $x > 0$.

(iii) *The dominant term of the asymptotic behavior of* g_2 *at infinity is*

$$g_2(x) \sim c_1 e^{r_1 x^2} x^{\alpha_1}, \tag{4–16}$$

where r_1 *is the positive root of* $2r^3 + dr - 1 = 0$, $\alpha_1 = -\dfrac{2}{3}\dfrac{2d+9r_1^2}{d+6r_1^2}$, *and* c_1 *is a positive constant.*

Proof. (i) From the initial conditions that g_2 satisfies, we get

$$g_2(x) = \sum_{n=0}^{\infty} a_n (x^2)^n,$$

where $a_0 = 1$, $a_1 = 0$, $a_2 = -d/9$, and

$$a_n = \frac{1}{n(n-1)(2n-1)}\left(a_{n-3} - \tfrac{1}{3}d(3n-4)a_{n-2}\right), \quad \text{for } n \geq 3.$$

Furthermore, since $a_n > 0$ for $n \geq 2$, item (i) holds. Item (ii) follows by taking derivatives of the expression of g_2.

We prove item (iii) in two steps: first we find a basis of formal solutions of (4–12), then we use Theorem 4.6 and formal computations to get the leading term of the asymptotic behavior of g_2.

We start our first step by using a heuristic method, called *the method of dominant balance*, to get the leading terms of the basis of formal solutions; see for instance [Bender and Orszag 1999, p. 76]. Apply to (4–12) the change of dependent variable $y(x) = e^{S(x)}$, which yields

$$3S'''(x) + 9S'(x)S''(x) + 3S'(x)^3 + 6dx^2S'(x) - 12x^3 + 8dx = 0. \tag{4–17}$$

The leading behavior of $g_2(x)$ will be determined by those contributions to $S(x)$ that do not tend to zero when x approaches the irregular singularity. We suppose that the dominant terms in this equation when x is sufficiently large are $3S'(x)^3$, $6dx^2S'(x)$ and $-12x^3$. Then we obtain the simplified equation $3S'(x)^3 + 6dx^2S'(x) - 12x^3 = 0$, whose solutions are $S_s(x) = r_s x^2 + p_s$ $(s = 1, 2, 3)$, where the r_s are the three roots of the equation $2r^3 + dr - 1 = 0$ and the p_s

are arbitrary constants that we take as zero for simplicity. Afterwards, we can verify that when we replace these expressions of $S_s(x)$ in (4–17), the terms of the equation that we have neglected are, at infinity, of smaller order than those we have kept in the simplified equation. This fact validates the first step of the procedure. Once a value of $s = 1, 2, 3$ is fixed, to obtain the next contribution to the leading behavior we introduce the new change of variable $y(x) = e^{r_s x^2 + S(x)}$, with $\lim_{x\to\infty} S(x)/x^2 = 0$. We apply the method of dominant balance again (for brevity we omit the full differential equation satisfied for this new S). We propose the simplified equation $3(2d + 12r_s^2)x^2 S'(x) + (36r_s^2 + 8d)x = 0$, whose solution is $S(x) = \alpha_s \log x$, with $\alpha_s = -(4d + 18r_s^2)/(3d + 18r_s^2)$, where again the additive constant is not taken into account. As before, we can verify the self-consistency of the calculations by replacing this expression of $S(x)$ in the complete equation. A third term is obtained by using the change of variable $y(x) = e^{r_s x^2 + \alpha_s \log x + S(x)}$, with $\lim_{x\to\infty} S(x)/\log x = 0$. Repeating the same procedure, we obtain a solution that does not contribute to the leading behavior. Hence our candidates to be the leading behaviors at infinity of a basis of formal solutions of (4–12) are $e^{r_s x^2} x^{\alpha_s}$, where $s = 1, 2, 3$. To end this step we show that (4–12) admits three formal solutions of the form

$$\hat{y}_s(x) = e^{r_s x^2} x^{\alpha_s} \sum_{i=0}^{\infty} \frac{C_{s,i}}{x^{2i}}, \tag{4–18}$$

where the $C_{s,i}$, for $s = 1, 2, 3$ and $i = 1, 2, \dots$, are constant, with $C_{s,0} = 1$. To prove this last assertion, fix a value of s and introduce in (4–12) the change of variables $y(x) = x^{\alpha_s} e^{r_s x^2} h(u)$, with $u = 1/x$. Then the function $h(u)$ must satisfy an equation with the structure

$$b_1 u^9 h'''(u) + u^6(b_2 + b_3 u^2)h''(u) + u^3(b_4 + b_5 u^2 + b_6 u^4)h'(u) + u^4(b_7 + b_8 u^2)h(u) = 0,$$

where the coefficients b_i depend on the parameter d and on r_s. It is straightforward to verify that this equation admits a formal solution of the form $h(u) = \sum_{n=0}^{\infty} h_n (u^2)^n$ if and only if $b_4 = 54((30d^3 - 324)r_s^2 - 162d^2 r_s + 108d - d^4) \neq 0$. Since r_s satisfies the equation $2r^3 + dr - 1 = 0$ and $d \in (-\sqrt[3]{27/2}, 0)$, we have $b_4 \neq 0$. Hence, for each s, (4–12) admits the formal solution given in (4–18) as we wanted to prove. This is precisely the definition of a normal irregular singular point [Ince 1927, p. 168]. (The radius of convergence of these formal series is generally difficult to determine, and may be zero.)

For the second step in the proof of (iii), we check that (4–12) satisfies the assumptions of Theorem 4.6. Notice that

$$b_1(x) \equiv 0, \quad b_2(x) = 2dx^2, \quad b_3(x) = x^3\Big(-4 + \frac{8d}{3x^2}\Big).$$

Hence $k = 1$, and the equation associated with (4–12) is $m^3 + 2dm - 4 = 0$. Therefore, the polynomials given in (4–15) are of the form $f_s(x) = m_s x^2/2 + n_s x$, where m_s, for $s = 1, 2, 3$ are the three different roots of the previous cubic equation (notice that we need to use the interval of values where d varies, and that in the left boundary of this interval, Theorem 4.6 is no longer applicable, because the cubic polynomial has a double root). If we write $f_s(x) = r_s x^2 + n_s x$, then the values r_s are in fact the roots of the cubic equation $2r^3 + dr - 1 = 0$ obtained above by the dominant balance method. Notice also the approach above gives $n_s = 0$, for $s = 1, 2, 3$. Hence, by using Theorem 4.6 we can assure that Equation (4–12) has a basis of solutions y_s, for $s = 1, 2, 3$, such that

$$y_s(x) \sim \hat{y}_s(x) = e^{r_s x^2} x^{\alpha_s} \sum_{i=0}^{\infty} \frac{C_{s,i}}{x^{2i}}, \tag{4–19}$$

with $C_{s,0} = 1$, and some constants $C_{s,i}$, where $s = 1, 2, 3$ and $i = 1, 2, \ldots$.

Now the function g_2 can be written as $g_2(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x)$, for some constants c_s ($s = 1, 2, 3$). From (4–19), we have the leading behaviors at infinity of this basis of solutions. Since from items (i) and (ii) we know that the function $g_2(x)$ is a solution of (4–12) that tends to infinity when $x \to +\infty$, we conclude that the leading behavior of the asymptotic expansion of g_2 is $g_2(x) \sim c_1 e^{r_1 x^2} x^{\alpha_1}$ for the value $c_1 > 0$ given above, where r_1 is the positive root of $2r^3 + dr - 1 = 0$ and $\alpha_1 = -(4d + 18r_1^2)/(3d + 18r_1^2)$, as we wanted to prove. □

The next lemma gives some properties for the function M_2. To stress its dependence with respect to d, we will rename it $M_{2,d}$. Hence

$$M_{2,d}(x) := M_2(x) = 3dg_2''(x) + 9xg_2'(x) + 4d^2x^2g_2(x). \tag{4–20}$$

Lemma 4.8. *Consider* $-\sqrt[3]{27/2} < d < 0$. *Let* $M_{2,d}$ *be given in* (4–20), *where* g_2 *is the solution of* (4–12), *with initial conditions given in* (4–13). *Then*

(i) *The function* $M_{2,d}(x)$ *is positive for* $x \neq 0$ *near* 0, *and* $M_{2,d}(0) = 0$.

(ii) $\lim_{x\to+\infty} M_{2,d}(x) = +\infty$.

(iii) *If* $M'_{2,d}(x_1) = 0$ *for some* $x_1 > 0$, *then* $M_{2,d}(x_1) = ((\frac{2d^3+27}{3})x_1 - \frac{3d}{x_1})g_2'(x_1) > 0$.

Proof.

(i) Using the series expansion of $g_2(x)$ we obtain $M_{2,d}(x) = -dx^4 + (\frac{9}{5} + \frac{4}{27}d^3)x^6 + 0(x^8)$. Since d is negative, we have $M_{2,d}(x)$ is positive for $x \neq 0$ near 0, as we wanted to prove.

(ii) By using Lemma 4.7(iii), the leading term of the asymptotic expansion of $M_{2,d}$ at infinity is

$$M_{2,d}(x) \sim c_1 e^{r_1 x^2} x^{2+\alpha_1} \left(4d^2 + 18r_1 + 12dr_1^2\right),$$

where c_1, r_1 and α_1 are given also in that Lemma. Hence, because we have $4d^2 + 18r_1 + 12dr_1^2 > 0$, for the values of d considered here, the result follows.

(iii) By taking the derivative of (4–20) with respect to x, we get an expression that also involves $g_2'''(x)$. Because $g_2(x)$ satisfies (4–12), we can simplify this expression. Finally, by evaluating the resulting expression at x_1, item (iii) follows since x_1 satisfies $M'_{2,d}(x_1) = 0$. □

Using Lemmas 4.7 and 4.8, we get the main result of this section:

Proposition 4.9. *System* (4–1) *has at most one limit cycle when* $d \in (-\sqrt[3]{27/2}, 0)$. *When it exists*, *the limit cycle is hyperbolic and unstable.*

Proof. Recall that, for each value of d, by using Theorem A and the results of this subsection, we have reduced the problem to proving that the function $M_{2,d}(x)$ does not change sign. Recall also that $M_{2,d}$ is an even function vanishing at the origin, so it suffices to study it for positive values of x.

Consider, from now on, that d is a fixed value in $(-\sqrt[3]{27/2}, 0)$. As straightforward consequences of Lemma 4.8, we have:

(a) For x positive and small enough, $M_{2,d}$ is positive.

(b) For x positive and large enough, $M_{2,d}$ is positive.

(c) If $M_{2,d}$ has a local minimum at some value $\bar{x} > 0$, the function evaluated at this minimum $\bar{x}$ takes a positive value.

We claim that $M_{2,d}$ is positive for $x > 0$. Assume for a contradiction that it takes a negative value for some $x^* > 0$. By items (a) and (b), an absolute minimum $\bar{x} > 0$ exists, at which of course $M_{2,d}(\bar{x}) < 0$. But this inequality contradicts (c). Hence $M_{2,d}$ is always positive or zero, as we wanted to prove. □

5. Conclusions

Collecting the results obtained earlier, we conclude that system (4–1) has:

(i) No limit cycles when $d \geq 0$ or $d < -2.679$ (this value being obtained numerically by solving a nonlinear equation).

(ii) At most one limit cycle when $0 > d > -\sqrt[3]{27/2} \simeq -2.381$.

We have not been able to cover all the values of the parameter d. There is a small gap for which we do not know the maximum number of limit cycles of system (4–1). By a numerical study, we conclude that the limit cycle is unique and exists only when $-2.198 < d < 0$. Hence, although we have not completely solved the problem, it seems the method presented in this paper gives reasonably good estimates for the regions of nonexistence of limit cycles, and for the regions of uniqueness of limit cycles.

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Received November 20, 2004.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

ON THE CMC FOLIATION OF FUTURE ENDS OF A SPACETIME

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We consider spacetimes with compact Cauchy hypersurfaces and with Ricci tensor bounded from below on the set of timelike unit vectors, and prove that the results known for spacetimes satisfying the timelike convergence condition, namely, foliation by CMC hypersurfaces, are also valid in the present situation, if corresponding further assumptions are satisfied.

In addition we show that the volume of any sequence of spacelike hypersurfaces, which run into the future singularity, decays to zero provided there exists a time function covering a future end, such that the level hypersurfaces have nonnegative mean curvature and decaying volume.

1. Introduction

Let N be a $(n+1)$-dimensional spacetime with a compact Cauchy hypersurface, so that N is topologically a product, $N = I \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold and $I = (a, b)$ an interval. The metric in N can then be expressed in the form

$$d\bar{s}^2 = e^{2\psi}\big(-(dx^0)^2 + \sigma_{ij}(x^0, x)\,dx^i\,dx^j\big); \tag{1–1}$$

x^0 is the time function and (x^i) are local coordinates for $\mathcal{S}_0$.

If N satisfies a future mean curvature barrier condition and the timelike convergence condition, then a future end $N_+ = (x^0)^{-1}\big([a_0, b)\big)$ can be foliated by constant mean curvature (CMC) spacelike hypersurfaces and the mean curvature of the leaves can be used as a new time function [Gerhardt 1983; 2003]. Moreover, one of Hawking's singularity results implies that N is future timelike incomplete with finite Lorentzian diameter for the future end.

In this paper we want to extend these results to the case when the Ricci tensor is only bounded from below on the set of timelike unit vectors

$$\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \ge -\Lambda \qquad \text{for all } \langle\nu, \nu\rangle = -1 \tag{1–2}$$

MSC2000: 35J60, 53C21, 53C44, 53C50, 58J05.
Keywords: Lorentzian manifold, timelike incompleteness, CMC foliation, general relativity.

for some $\Lambda \ge 0$, and in addition, we want to show that the volume of the CMC leaves decays to zero, if the future singularity is approached.

We summarize our results:

Theorem 1.1. *Suppose that in a future end N_+ of N the Ricci tensor satisfies the estimate* (1–2) *of the preceding page*, *and suppose that a future mean curvature barrier exists* (*Definition 2.2*). *Then a slightly smaller future end $\tilde{N}_+$ can be foliated by CMC spacelike hypersurfaces*, *and there exists a smooth time function x^0 such that the slices*

$$M_\tau = \{x^0 = \tau\}, \qquad \tau_0 < \tau < \infty,$$

have mean curvature τ for some $\tau_0 > \sqrt{n\Lambda}$. The precise value of τ_0 depends on the mean curvature of a lower barrier.

Recall that a subset $M \subset N$ is said to be *achronal* if any timelike piecewise C^1-curve intersects M at most once.

Theorem 1.2. *Suppose that a future end $N_+ = (x^0)^{-1}\big([a_0, b)\big)$ of N can be covered by a time function x^0 such that the mean curvature of the slices $M_t = \{x^0 = t\}$ is nonnegative and the volume of M_t decays to zero*:

$$\lim_{t\to b} |M_t| = 0.$$

Then the volume $|M_k|$ of any sequence of spacelike achronal hypersurfaces M_k such that

$$\lim_k \inf_{M_k} x^0 = b$$

decays to zero. Thus, *if the additional conditions of Theorem 1.1 are also satisfied*, *the volume of the CMC hypersurfaces M_τ converges to zero*:

$$\lim_{\tau\to\infty} |M_\tau| = 0.$$

N is also future timelike incomplete if there is a compact spacelike hypersurface M with mean curvature H satisfying

$$H \ge H_0 > \sqrt{n\Lambda},$$

due to a result in [Andersson and Galloway 2002].

2. Notations and definitions

The main objective of this section is to state the equations of Gauss, Codazzi, and Weingarten for space-like hypersurfaces M in a (n+1)-dimensional Lorentzian manifold N. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), (R_{ijkl}), etc. Greek indices range from 0 to n and Latin from

1 to n; the summation convention is always used. Generic coordinate systems in N and M will be denoted by (x^α) and (ξ^i), respectively. Covariant differentiation will simply be indicated by indices; only in cases of possible ambiguity they will be preceded by a semicolon. For example, for a function u in N, the gradient will be (u_α) and the Hessian $(u_{\alpha\beta})$, but the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a *spacelike* hypersurface, i.e., the induced metric is Riemannian, with a differentiable normal ν which is timelike.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the spacelike hypersurface M are connected through the Gauss formula,

$$x_{ij}^\alpha = h_{ij} \nu^\alpha. \tag{2–1}$$

Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.,

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma .$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to ν.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha ,$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauss equation*

$$R_{ijkl} = -(h_{ik} h_{jl} - h_{il} h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta .$$

Now assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. N is then a topological product $\mathbb{R} \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that the metric in N has the form

$$d\bar{s}_N^2 = e^{2\psi}\big(-(dx^0)^2 + \sigma_{ij}(x^0, x)\, dx^i\, dx^j\big),$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the spacelike components (x^i); see [Geroch 1970; Hawking and Ellis 1973, p. 212; Geroch and Horowitz 1979, p. 252; Gerhardt 1983, Section 6]. We also assume that the coordinate system is *future-oriented*, that is, the time coordinate x^0 increases on future-directed curves. Hence, the *contravariant* timelike vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future-directed, and so is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \operatorname{graph} u|_{\mathcal{S}_0}$ be a spacelike hypersurface

$$M = \{ (x^0, x) : x^0 = u(x),\ x \in \mathcal{S}_0 \}.$$

Then the induced metric has the form

$$g_{ij} = e^{2\psi}(-u_i u_j + \sigma_{ij}),$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$g^{ij} = e^{-2\psi}\left(\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\right),$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$u^i = \sigma^{ij} u_j, \quad v^2 = 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2.$$

Hence, $\operatorname{graph} u$ is spacelike if and only if $|Du| < 1$.

The covariant and contravariant forms of a normal vector of a graph look like

$$(\nu_\alpha) = \pm v^{-1} e^{\psi}(1, -u_i), \quad (\nu^\alpha) = \mp v^{-1} e^{-\psi}(1, u^i),$$

respectively. Thus:

Remark 2.1. *Let M be spacelike graph in a future-oriented coordinate system. The contravariant future-directed and past-directed normal vectors have the respective forms*

$$(2\text{–}2) \qquad (\nu^\alpha) = v^{-1} e^{-\psi}(1, u^i), \qquad (\nu^\alpha) = -v^{-1} e^{-\psi}(1, u^i).$$

In the Gauss formula (2–1) of the preceding page, we are free to choose the future- or past-directed normal, but we stipulate that we always use the past-directed normal for reasons explained in [Gerhardt 2000a, Section 2].

Look at the component $\alpha = 0$ in (2–1) and obtain, in view of (2–2) above,

$$(2\text{–}3) \qquad e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{ij}.$$

Here, the covariant derivatives are taken relative to the induced metric of M and

$$-\bar{\Gamma}^0_{ij} = e^{-\psi} \bar{h}_{ij},$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$\bar{h}_{ij}e^{-\psi} = -\tfrac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij},$$

where the dot indicates differentiation with respect to x^0.

Finally, we define what we mean by a future mean curvature barrier.

Definition 2.2. Let N be a globally hyperbolic spacetime with compact Cauchy hypersurface $\mathcal{S}_0$ so that N can be written as a topological product $N = \mathbb{R} \times \mathcal{S}_0$ and its metric expressed as

$$d\bar{s}^2 = e^{2\psi}\big(-(dx^0)^2 + \sigma_{ij}(x^0, x)\,dx^i\,dx^j\big).$$

Here, x^0 is a globally defined future-directed time function and (x^i) are local coordinates for $\mathcal{S}_0$. N is said to have a *future mean curvature barrier* if there is a sequence M_k^+ of closed spacelike achronal hypersurfaces such that

$$\lim_{k\to\infty} H|_{M_k^+} = \infty \qquad \text{and} \qquad \limsup \inf_{M_k^+} x^0 > x^0(p) \quad \text{for all } p \in N$$

Likewise, N is said to have a *past mean curvature barrier* if there is a sequence M_k^- such that

$$\lim_{k\to\infty} H|_{M_k^-} = -\infty \qquad \text{and} \qquad \liminf \sup_{M_k^-} x^0 < x^0(p) \quad \text{for all } p \in N.$$

A future mean curvature barrier certainly represents a singularity, at least if N satisfies (1–2) on page 297, because of the future timelike incompleteness, but these singularities need not be crushing; see [Gerhardt 2004, Introduction].

3. Proof of Theorem 1.1

We start with some simple but very useful observations. If, for a given coordinate system (x^α), the metric has the form (1–1) of page 297, then the coordinate slices $M(t) = \{x^0 = t\}$ can be looked at as a solution of the evolution problem

$$\dot{x} = -e^{\psi}\nu, \tag{3–1}$$

where $\nu = (\nu^\alpha)$ is the past-directed normal vector. The embedding $x = x(t, \xi)$ is then given as $x = (t, x^i)$, where (x^i) are local coordinates for $\mathcal{S}_0$.

From (3–1) we can immediately derive evolution equations for the geometric quantities g_{ij}, h_{ij}, ν and $H = g^{ij}h_{ij}$ of $M(t)$; see [Gerhardt 2000a, Section 3].

To avoid confusion with notations for the geometric quantities of other hypersurfaces, we occasionally denote the induced metric and second fundamental of coordinate slices by $\bar{g}_{ij}$, $\bar{h}_{ij}$ and $\bar{H}$. Thus, the evolution equations

$$\dot{\bar{g}}_{ij} = -2e^{\psi}\bar{h}_{ij} \tag{3–2}$$

and

$$\dot{\bar{H}} = -\Delta e^{\psi} + (|\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)e^{\psi} \tag{3–3}$$

are valid.

The last equation is closely related to the derivative of the mean curvature operator: Let M_0 be a smooth spacelike hypersurface and in a tubular neighborhood $\mathcal{U}$ of M_0, consider hypersurfaces M that can be written as graph u over M_0 in the corresponding normal Gaussian coordinate system. Then the mean curvature of M can be expressed as

$$H = -\Delta u + \bar{H} + v^{-2}u^i u^j \bar{h}_{ij}, \tag{3–4}$$

(see (2–3) on page 300), and hence, choosing $u = \epsilon\varphi$, $\varphi \in C^2(M_0)$, we deduce

$$\frac{d}{d\epsilon}H|_{\epsilon=0} = -\Delta\varphi + \dot{\bar{H}}\varphi = -\Delta\varphi + (|\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\varphi. \tag{3–5}$$

Next we shall prove that CMC hypersurfaces are monotonically ordered, if the mean curvatures are sufficiently large.

Lemma 3.1. *Let* $M_1 = \operatorname{graph} u_1$ *and* $M_2 = \operatorname{graph} u_2$ *be spacelike hypersurfaces such that the mean curvatures* H_1 *and* H_2 *satisfy* $H_1 < H_2 = \tau_2$, *where* H_2 *is constant*, *and* $\sqrt{n\Lambda} < \tau_2$. *Then*

$$u_1 < u_2. \tag{3–6}$$

Proof. We first observe that the weaker conclusion $u_1 \le u_2$ is as good as the $u_1 < u_2$, in view of the maximum principle. Now suppose for a contradiction that $u_1 \le u_2$ is not valid, so that

$$E(u_1) = \{\, x \in \mathcal{S}_0 : u_2(x) < u_1(x) \,\} \ne \varnothing.$$

Then there exist points $p_i \in M_i$ such that

$$0 < d_0 = d(M_2, M_1) = d(p_2, p_1) = \sup\{\, d(p, q) : (p, q) \in M_2 \times M_1 \,\},$$

where d is the Lorentzian distance function. Let φ be a maximal geodesic from M_2 to M_1 realizing this distance with endpoints p_2 and p_1, and parametrized by arc length.

Denote by $\bar{d}$ the Lorentzian distance function to M_2, i.e., for $p \in I^+(M_2)$

$$\bar{d}(p) = \sup_{q \in M_2} d(q, p).$$

Since φ is maximal, $\Gamma = \{\, \varphi(t) : 0 \le t < d_0 \,\}$ contains no focal points of M_2 [O'Neill 1983, Theorem 34, p. 285]. Hence there exists an open neighborhood

$\mathcal{V} = \mathcal{V}(\Gamma)$ such that $\bar{d}$ is smooth in $\mathcal{V}$ [O'Neill 1983, Proposition 30], because $\bar{d}$ is a component of the inverse of the normal exponential map of M_2.

Now, M_2 is the level set $\{\bar{d} = 0\}$, and the level sets

$$M(t) = \{ p \in \mathcal{V} : \bar{d}(p) = t \}$$

are smooth hypersurfaces; $x^0 = \bar{d}$ is a time function in $\mathcal{V}$ and generates a normal Gaussian coordinate system, since $\langle D\bar{d}, D\bar{d} \rangle = -1$. Hence, by Equation (3–3) on page 302, the mean curvature $\bar{H}(t)$ of $M(t)$ satisfies

$$\dot{\bar{H}} = |\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta,$$

and therefore we have

$$\dot{\bar{H}} \geq \frac{1}{n}|\bar{H}|^2 - \Lambda > 0, \tag{3–7}$$

in view of the assumption $\sqrt{n\Lambda} < \tau_2$.

Next, consider a tubular neighborhood $\mathcal{U}$ of M_1 with corresponding normal Gaussian coordinates (x^α). The level sets

$$\tilde{M}(s) = \{x^0 = s\}, \qquad -\epsilon < s < 0,$$

lie in the past of $M_1 = \tilde{M}(0)$ and are smooth for small ϵ.

Since the geodesic φ is normal to M_1, it is also normal to $\tilde{M}(s)$ and the length of the geodesic segment of φ from $\tilde{M}(s)$ to M_1 is exactly $-s$, i.e., equal to the distance from $\tilde{M}(s)$ to M_1. Hence we deduce

$$d(M_2, \tilde{M}(s)) = d_0 + s;$$

that is, $\{ \varphi(t) : 0 \leq t \leq d_0 + s \}$ is also a maximal geodesic from M_2 to $\tilde{M}(s)$. We conclude further that, for fixed s, the hypersurface $\tilde{M}(s) \cap \mathcal{V}$ is contained in the past of $M(d_0 + s)$ and touches $M(d_0 + s)$ in $p_s = \varphi(d_0 + s)$. The maximum principle then implies

$$H|_{\tilde{M}(s)}(p_s) \geq H|_{M(d_0+s)}(p_s) > \tau_2,$$

in view of (3–7) above.

On the other hand, the mean curvature of $\tilde{M}(s)$ converges to the mean curvature of M_1 if s tends to zero; hence

$$H_1(\varphi(d_0)) \geq \tau_2,$$

contradicting the assumption that $H_1 < H_2$. □

Corollary 3.2. *The CMC hypersurfaces with mean curvature*

$$\tau > \sqrt{n\Lambda}$$

are uniquely determined.

Proof. Let $M_1 = \operatorname{graph} u_1$ and $M_2 = \operatorname{graph} u_2$ be hypersurfaces with mean curvature τ and suppose that, say,

$$\{\, x \in \mathcal{S}_0 : u_1(x) < u_2(x) \,\} \neq \emptyset.$$

Consider a tubular neighborhood of M_1 with a corresponding future-oriented normal Gaussian coordinate system (x^α). Then the evolution of the mean curvature of the coordinate slices satisfies

$$\dot{\bar{H}} = |\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \geq \frac{1}{n}|\bar{H}|^2 - \Lambda > 0$$

in a neighborhood of M_1; i.e., the coordinate slices $M(t) = \{x^0 = t\}$ with $t > 0$ all have mean curvature $\bar{H}(t) > \tau$. Using now M_1 and $M(t)$, $t > 0$, as barriers, we infer that for any $\tau' \in \mathbb{R}$, $\tau < \tau' < \bar{H}(t)$, there exists a spacelike hypersurface $M_{\tau'}$ with mean curvature τ' such that $M_{\tau'}$ can be expressed as $\operatorname{graph} u$ over M_1, where

$$0 < u < t.$$

For a proof see [Gerhardt 1983, Section 6]; a different more transparent proof of this result has been given in [Gerhardt 2000b].

Writing $M_{\tau'}$ as graph over $\mathcal{S}_0$ in the original coordinate system without changing the notation for u, we obtain

$$u_1 < u,$$

and by choosing t small enough, we may also conclude that

$$E(u) = \{\, x \in \mathcal{S}_0 : u(x) < u_2(x) \,\} \neq \emptyset,$$

which is impossible, in view of the preceding result. □

Lemma 3.3. *Under the assumptions of Theorem 1.1, if $M_{\tau_0} = \operatorname{graph} u_{\tau_0}$ is a CMC hypersurface with mean curvature $\tau_0 > \sqrt{n\Lambda}$, then the future of M_{τ_0} can be foliated by CMC hypersurfaces*

$$I^+(M_{\tau_0}) = \bigcup_{\tau_0 < \tau < \infty} M_\tau. \tag{3–8}$$

Each set M_τ can be written over $\mathcal{S}_0$ as

$$M_\tau = \operatorname{graph} u(\tau, \cdot),$$

such that u is strictly monotone increasing with respect to τ and continuous in $[\tau_0, \infty) \times \mathcal{S}_0$.

Proof. The monotonicity and continuity of u follow from Lemma 3.1 and Corollary 3.2, in view of the a priori estimates.

It remains to verify the relation (3–8). Letting $p = (t, y^i) \in I^+(M_{\tau_0})$, we have to show $p \in M_\tau$ for some $\tau > \tau_0$.

In [Gerhardt 1983, Theorem 6.3] it is proved that there exists a family

$$\{ M_\tau : \tau_0 \le \tau < \infty \},$$

of CMC hypersurfaces M_τ if there is a future mean curvature barrier.

Define $u(\tau, \cdot)$ by

$$M_\tau = \operatorname{graph} u(\tau, \cdot).$$

Then $u(\tau_0, y) < t < u(\tau^*, y)$ for some large τ^*, because the mean curvature barrier condition together with Lemma 3.1 implies that the CMC hypersurfaces run into the future singularity, if τ goes to infinity.

In view of the continuity of $u(\cdot, y)$ we conclude that there exists τ_1 such that $\tau_0 < \tau_1 < \tau^*$ and

$$u(\tau_1, y) = t.$$

Hence $p \in M_{\tau_1}$. □

Remark 3.4. *The continuity and monotonicity of u holds in any coordinate system* (x^α), *even in those that do not cover the future completely like the normal Gaussian coordinates associated with a spacelike hypersurface, which are defined in a tubular neighborhood.*

The proof of Theorem 1.1 is now almost finished. The remaining arguments are identical to those in [Gerhardt 2003, Section 2], but for the convenience of the reader, we shall briefly summarize the main steps.

We have to show that the mean curvature parameter τ can be used as a time function in $\{\tau_0 < \tau < \infty\}$, i.e., τ should be smooth with a nonvanishing gradient. Both properties are local.

First step: Fix an arbitrary $\tau' \in (\tau_0, \infty)$, and consider a tubular neighborhood $\mathcal{U}$ of $M' = M_{\tau'}$. Each set $M_\tau \subset \mathcal{U}$ can then be written as $\operatorname{graph} u(\tau, \cdot)$ over M'. For small $\epsilon > 0$ we have

$$M_\tau \subset \mathcal{U} \qquad \text{for all } \tau \in (\tau' - \epsilon, \tau' + \epsilon),$$

and with the help of the implicit function theorem we now show that u is smooth. Define the operator G by

$$G(\tau, \varphi) = H(\varphi) - \tau,$$

where $H(\varphi)$ is an abbreviation for the mean curvature of $\operatorname{graph} \varphi|_{M'}$. Then G is smooth and from (3–5) (page 302) we deduce that $D_2 G(\tau', 0)\varphi$ equals

$$-\Delta\varphi + (\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\varphi,$$

where the Laplacian, the second fundamental form and the normal correspond to M'. Hence $D_2 G(\tau', 0)$ is an isomorphism and the implicit function theorem implies that u is smooth.

Second step: Still in the tubular neighborhood of M', define the coordinate transformation

$$\Phi(\tau, x^i) = (u(\tau, x^i), x^i);$$

note that $x^0 = u(\tau, x^i)$. Then

$$\det D\Phi = \frac{\partial u}{\partial \tau} = \dot{u}.$$

We know that $\dot{u}$ is nonnegative. If it is strictly positive, Φ is a diffeomorphism, and hence τ is smooth with nonvanishing gradient. A proof that $\dot{u} > 0$ is given in [Gerhardt 2003, Lemma 2.2], but we give a simpler one: The CMC hypersurfaces in $\mathcal{U}$ satisfy an equation

$$H(u) = \tau,$$

where the left hand-side can be expressed as in Equation (3–4), page 302. Differentiating both sides with respect to τ and evaluating for $\tau = \tau'$, i.e., on M', where $u(\tau', \cdot\,) = 0$, we get

$$-\Delta\dot{u} + (|A|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\dot{u} = 1.$$

In a point where $\dot{u}$ attains its minimum, the maximum principle implies

$$(|A|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\dot{u} \geq 1.$$

Hence $\dot{u} \neq 0$ and $\dot{u}$ is therefore strictly positive.

4. Proof of Theorem 1.2

Let x^0 be a time function satisfying the assumptions of Theorem 1.2. In other words, $N_+ = \{a_0 < x^0 < b\}$, the mean curvature of the slices $M(t) = \{x^0 = t\}$ is nonnegative, and

$$\lim_{t\to b} |M(t)| = 0.$$

Also let M_k be a sequence of spacelike achronal hypersurfaces such that

$$\lim \inf_{M_k} x^0 = b.$$

Write M_k as graph u_k over $\mathcal{S}_0$. Then

$$g_{ij} = e^{2\psi}(u_iu_j + \sigma_{ij}(u, x))$$

is the induced metric, where we dropped the index k for better readability, and the volume element of M_k has the form

$$d\mu = v\sqrt{\det(\bar{g}_{ij}(u, x))}\,dx,$$

where

$$v^2 = 1 - \sigma^{ij} u_i u_j < 1, \tag{4–1}$$

and $(\bar{g}_{ij}(t,\cdot))$ is the metric of the slices $M(t)$.

From (3–2) we deduce

$$\frac{d}{dt}\sqrt{\det(\bar{g}_{ij}(t,\cdot))} = -e^{\psi}\bar{H}\sqrt{\det(\bar{g}_{ij})} \le 0. \tag{4–2}$$

Now, let $a_0 < t < b$ be fixed. Then for almost every k we have

$$t < u_k \tag{4–3}$$

and hence

$$\begin{aligned} |M_k| &= \int_{\mathcal{S}_0} v\sqrt{\det(\bar{g}_{ij}(u_k,x))}\,dx \\ &\le \int_{\mathcal{S}_0} \sqrt{\det(\bar{g}_{ij}(t,x))}\,dx = |M(t)|, \end{aligned}$$

in view of (4–1), (4–2) and (4–3). We conclude that $\limsup |M_k| \le |M(t)|$ for all $a_0 < t < b$, and thus $\lim |M_k| = 0$.

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Received October 31, 2004.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

THE AMBIENT OBSTRUCTION TENSOR AND THE CONFORMAL DEFORMATION COMPLEX

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We construct here a conformally invariant differential operator on algebraic Weyl tensors that gives special curved analogues of certain operators related to the deformation complex and that, upon application to the Weyl curvature, yields the (Fefferman–Graham) ambient obstruction tensor. This new definition of the obstruction tensor leads to simple direct proofs that the obstruction tensor is divergence-free and vanishes identically for conformally Einstein metrics. Our main constructions are based on the ambient metric of Fefferman–Graham and its relation to the conformal tractor connection. We prove that the obstruction tensor is an obstruction to finding an ambient metric with curvature harmonic for a certain (ambient) form Laplacian. This leads to a new ambient formula for the obstruction in terms of a power of this form Laplacian acting on the ambient curvature. This result leads us to construct Laplacian-type operators that generalise the conformal Laplacians of Graham–Jenne–Mason–Sparling. We give an algorithm for calculating explicit formulae for these operators, and this is applied to give formulae for the obstruction tensor in dimensions 6 and 8. As background to these issues, we give an explicit construction of the deformation complex in dimensions $n \geq 4$, construct two related (detour) complexes, and establish essential properties of the operators in these.

1. Introduction

The Bach tensor [1921] has long been considered an important natural invariant in 4-dimensional Riemannian and pseudo-Riemannian geometry and continues to play an interesting role. See [Anderson 2005; Tian and Viaclovsky 2005], for example. It is conformally invariant, vanishes for metrics that are conformal to Einstein metrics, and arises as the total metric variation of the action $\int |C|^2$, where

MSC2000: primary 53A55; secondary 22E70, 53A30, 58J10.
Keywords: ambient metric, conformal deformations, conformal geometry, detour complexes, differential complexes.
ARG gratefully acknowledges support from the Royal Society of New Zealand via Marsden Grant no. 02-UOA-108, and from the New Zealand Institute of Mathematics and its Applications for support via a Maclaurin Fellowship.

C denotes the Weyl curvature. From the latter and the conformal invariance of the Weyl curvature, it follows that it is a symmetric trace-free 2-tensor involving 4 derivatives of the metric. An explicit formula for the Bach tensor in terms of the Weyl curvature C, the Ricci tensor, and the Levi-Civita connection is very simple:

$$B_{ab} = \nabla^c \nabla^d C_{acbd} + \tfrac{1}{2} \operatorname{Ric}^{cd} C_{acbd}. \tag{1}$$

In higher even dimensions n, an analogue of the Bach tensor was discovered by Fefferman and Graham [1985]; it arose as an obstruction to their ambient metric construction. This *Fefferman–Graham obstruction tensor*, which we denote $\mathcal{O}_{ab}$ (or sometimes $\mathcal{O}^n_{ab}$), shares many of the properties of the Bach tensor. It is a trace-free symmetric 2-tensor that vanishes for conformally Einstein metrics. The obstruction tensor has the form $\Delta^{n/2-2}\nabla^c\nabla^d C_{acbd} + \text{lots}$. Here "lots" indicates lower order terms. There is strong evidence that the obstruction tensor will be as important in each even dimension as the Bach tensor is in dimension 4. Very recently Graham and Hirachi [2005] have shown that $\mathcal{O}_{ab}$ is the total metric variation of $\int Q$, where Q is Branson's Q-curvature [Branson 1995; Branson and Ørsted 1991]. This generalises the situation in dimension 4, since in that case $\int Q$ and $\int |C|^2$ agree up to a multiple. There is a direct link between the obstruction tensor and the nonexistence of certain operators on conformal manifolds which also generalises the 4-dimensional setting [Gover and Hirachi 2004] and further indicates the critical role of the obstruction tensor.

Despite this progress, the obstruction tensor has remained somewhat mysterious, partly due to the lack of a general formula. In the next section we explain that there is a fundamental difference between the Bach tensor in dimension 4 and the obstruction tensor in even dimensions 6 and greater. The idea is as follows. From the Bianchi identities, the expression (1) for the Bach tensor can be written as

$$\nabla^{(c}\nabla^{d)} C_{acbd} + \tfrac{1}{2} \operatorname{Ric}^{cd} C_{acbd},$$

where the parentheses indicate symmetrisation over the index pair cd. The differential operator $\nabla^{(c}\nabla^{d)} + \frac{1}{2}\operatorname{Ric}^{cd}$ is a *conformally invariant operator* which acts on the bundle of "algebraic Weyl tensors" (the bundle whose sections are 4-tensor fields with the same conformal weight and algebraic symmetries as the Weyl curvature) and takes values in a (density weighted) irreducible tensor bundle. One might hope that a similar result would hold in higher dimensions. This is not the case. In Proposition 2.1, we establish that in dimensions $n \geq 6$, the obstruction tensor cannot arise in this manner from a conformally invariant operator that acts between irreducible tensor bundles. This is an easy consequence of representation theory results of Boe and Collingwood [1985] that give a classification of conformally invariant operators on the sphere. (See [Eastwood and Slovák 1997] and references therein.) One focus of this article is to describe the correct generalisation of the described construction of the Bach tensor. This is Theorem 2.3, one of the main results.

In the conformally flat setting, the conformally invariant operator defined in the previous paragraph is the formal adjoint of an operator in the so-called (conformal) deformation complex. This is a complex of conformally invariant differential operators arising in connection with infinitesimal deformations of a conformal structure based at a conformally flat metric. The linearisation of the obstruction tensor, which we denote by B, is an operator in a class of conformally invariant operators acting between bundles in the complex. These "long operators" are predicted by the Boe–Collingwood classification. In Proposition 2.2, we show that the linearised obstruction operator and another long operator denoted by L factor through operators from the complex. For example, we obtain that $\mathsf{B} = \mathsf{G}\mathsf{C}$, where C is the linearised Weyl curvature operator and G is a gauge companion operator for L. That is, L and G have the same domain space (algebraic Weyl tensors), the system (L, G) gives a conformally invariant equation, and in Riemannian signature this system is elliptic. Theorem 2.3 gives a curved analogue of this picture. The theorem describes a conformally invariant differential operator $\mathbb{B}$ which, on general conformal manifolds of even dimension, acts on algebraic Weyl tensors and takes values in a reducible bundle. In dimensions $n \geq 6$, composing this with projection to a quotient gives a conformally invariant operator L which takes algebraic Weyl tensors to weighted algebraic Weyl tensors; L generalises L to conformally curved manifolds. This operator annihilates the Weyl curvature C, and $\mathbb{B}(C)$ is the obstruction tensor. An application of these results is given in Proposition 2.4, which in the conformally flat setting relates the conformally invariant null space of the system (L, G) to the cohomology of the deformation complex.

For a conformal structure of dimension n, the ambient metric is an associated, suitably homogeneous, and Ricci-flat metric on an $(n+2)$-manifold. In [Fefferman and Graham 1985], $\mathbb{O}_{ab}$ arose as an obstruction in even dimensions to the existence of a formal power series solution for this ambient metric. In Section 3B, we show that the obstruction tensor may equivalently be viewed as a formal obstruction to having the ambient curvature harmonic for a certain ambient form-Laplacian $\not{\boldsymbol{\Delta}}$. This leads to a new proof that the obstruction tensor is an obstruction to the ambient metric (see (v) of Theorem 4.4) and a very simple ambient formula for the obstruction. Let $\boldsymbol{R}$ denote the curvature of the ambient metric. Then $\not{\boldsymbol{\Delta}}^{n/2-2}\boldsymbol{R}$ is a disguised form of the obstruction. This is also established in Theorem 4.4 and in the same place used to give a new proof that the obstruction is divergence-free, i.e. that $\nabla^a \mathbb{O}_{ab} = 0$. (An alternative proof of this last result is given in [Graham and Hirachi 2005], and it also follows from the variational characterisation given in the same work. See [Branson 2005].)

An interpretation of these results on the underlying conformal manifold can be achieved via tractor bundles. The standard tractor bundle is a vector bundle with a conformally invariant connection that we may view as arising as an induced structure from the Cartan bundle with its normal conformal Cartan connection. On the other hand, this rank $n+2$ vector bundle also arises in a simple way from the tangent bundle of the ambient manifold. Using this observation, in Theorem 4.1 and Proposition 4.8 we construct families of conformally invariant operators with leading term a power of the Laplacian; these act between arbitrary tractor bundles of an appropriate density weight and generalise the GJMS operators of [Graham et al. 1992]. In Theorem 4.2, we show that the obstruction tensor is obtained by applying one of these operators, namely $\not\Box_{n/2-2}$, which has the form $\Delta^{n/2-2}+\text{lots}$, to the tractor field W that corresponds to $\boldsymbol{R}$. Thus the problem of finding formulae for the obstruction tensor is reduced to understanding the special case $\not\Box_{n/2-2}$ of the generalised GJMS-type operators $\not\Box_k$.

There is a one-to-one correspondence between Einstein metrics and a class of parallel standard tractors [Gauduchon 1990; Gover and Nurowski 2006]. With the tractor formula for the obstruction $\not\Box_{n/2-2}W$, this forms the basis of the proof of Theorem 4.3, which shows that the obstruction vanishes for conformally Einstein metrics.

Theorem 4.1 constructs a very general class of Laplace type conformal operators. The inductive steps leading to Theorem 4.1 yield a simple and effective algorithm for calculating explicit formulae for the conformal Laplacian operators of that theorem. Hence by Theorem 4.2, they give an algorithm for calculating explicit formulae for the obstruction. This algorithm is efficient in the sense that it does not entail constructing the ambient manifold but uses just its existence; the algorithm recovers only those invariants of the ambient metric that actually turn up in the final formula for the operator. In Section 4B, explicit tractor formulae for conformal Laplacian operators are given. See expressions (59) and (62). These are then applied to the W-tractor to give formulae for the obstruction in dimensions 6 and 8. Tractor formulae are given in (60) and (64), and formulae in terms of the Levi-Civita connection and its curvature are given in (61) and on page 348.

The next section establishes the basic background and notation before constructing the conformal deformation complex and introducing some related operators. It is a pleasure to thank Tom Branson and Robin Graham for helpful discussions.

2. Relationship to the conformal deformation complex

We first sketch here notation and background for conformal structures. Further details may be found in [Čap and Gover 2003], [Gover and Peterson 2003] or [Branson and Gover 2005]. We mainly follow the notational conventions of the last of these. Let M be a smooth manifold of dimension $n \geq 3$. To simplify our discussions we assume M is orientable. Recall that a *conformal structure* on M is a smooth ray subbundle $\mathcal{Q} \subset S^2T^*M$ whose fibre over x consists of conformally related metrics at the point x. The principal bundle $\pi : \mathcal{Q} \to M$ has structure group $\mathbb{R}_+$, and so each representation $\mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \mathrm{End}(\mathbb{R})$ induces a natural line bundle on $(M, [g])$ that we term the conformal density bundle $E[w]$. We shall write $\mathcal{E}[w]$ for the space of sections of this bundle. Here and throughout the article, sections, tensors, and functions are always smooth. When no confusion is likely to arise, we will use the same notation for a bundle and its section space.

We write $\boldsymbol{g}$ for the *conformal metric*, the tautological section of $S^2T^*M \otimes E[2]$ determined by the conformal structure. This will be used to identify TM with $T^*M[2]$. For many calculations we will use abstract indices in an obvious way. Given a choice of metric g from the conformal class, we write ∇ for the corresponding Levi-Civita connection. With these conventions the Laplacian Δ is given by $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b = \nabla^b\nabla_b$. Note that $E[w]$ is trivialised by a choice of metric g from the conformal class, and we write ∇ for the connection corresponding to this trivialisation. It follows immediately that (the coupled) ∇_a preserves the conformal metric.

The curvature $R_{ab}{}^c{}_d$ of the Levi-Civita connection is known as the Riemannian curvature and is defined by

$$[\nabla_a, \nabla_b]v^c = R_{ab}{}^c{}_d v^d,$$

where $[\,\cdot\,,\cdot\,]$ indicates the usual commutator bracket. The Riemannian curvature can be decomposed into the totally trace-free Weyl curvature C_{abcd} and a remaining part described by the symmetric *Schouten tensor* P_{ab} using $R_{abcd} = C_{abcd} + 2\boldsymbol{g}_{c[a}P_{b]d} + 2\boldsymbol{g}_{d[b}P_{a]c}$, where $[\cdots]$ indicates the antisymmetrisation over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor Ric_{ab} and vice versa: writing J for the trace $P_a{}^a$ of P, then $\mathrm{Ric}_{ab} = (n-2)P_{ab} + J\boldsymbol{g}_{ab}$. Under a *conformal transformation* we replace a choice of metric g by the metric $\hat{g} = e^{2\omega}g$, where ω is a smooth function. Explicit formulae for the corresponding transformation of the Levi-Civita connection and its curvatures are given, for example, in [Bailey et al. 1994a; Gover and Peterson 2003]. We recall that in particular the Weyl curvature is conformally invariant, that is, $\widehat{C}_{abcd} = C_{abcd}$.

A tensor $T^{a\cdots b}{}_{c\cdots d}$ of weight w and with k contravariant indices and ℓ covariant indices has *total order* $\ell - k - w$. For example, the Weyl curvature, the Schouten tensor, and the scalar curvature all have total order 2. The conformal metric $\boldsymbol{g}_{ab}$ has total order zero, and so the total order of any tensor is unchanged by the raising and lowering of indices using the conformal metric.

A differential operator P is a *natural differential operator* if it can be written as a universal polynomial in covariant derivatives with coefficients depending polynomially on the metric, its inverse, the curvature tensor, and its covariant derivatives. The coefficients of natural operators are called *natural tensors*. In the case that they are scalar they are often also called *Riemannian invariants*. Note that if T is a tensor with total order t then ∇T has total order $t+1$. It follows immediately that for any natural differential operator P that has T in its domain, the total order of PT is at least t. We say P is a *conformally invariant differential operator* if it is well-defined on conformal structures, i.e., is independent of a choice of conformal scale.

We will use E^k as a convenient alternative notation for $\wedge^k T^*M$. The tensor product of $E^k \otimes E^\ell$, $\ell \le n/2$, $k \le \lceil n/2 \rceil$, decomposes into irreducibles. We denote the highest weight component by $E^{k,\ell}$. (Here "weight" does not refer to conformal weight, but rather to the weight of the inducing $\mathrm{O}(n)$-representation.) We realise the tensors of $E^{k,\ell}$ as trace-free covariant $(k+\ell)$-tensors $T_{a_1\cdots a_k b_1 \cdots b_\ell}$ which are skew on the indices $a_1 \cdots a_k$ and also on the set $b_1 \cdots b_\ell$. Skewing over more than k indices annihilates T, as does symmetrising over any 3 indices. Then we write, for example, $E^{k,\ell}[w]$ as a shorthand for the tensor product $E^{k,\ell} \otimes E[w]$. The space of sections of each of these bundles is indicated by replacing E with $\mathcal{E}$. The sections of $\mathcal{E}^{2,2}[2]$ are the *algebraic Weyl tensors* as discussed in the introduction, that is, tensors u_{abcd} with the same symmetries and weight as the Weyl curvature. In particular, the Weyl curvature itself is a section in $\mathcal{E}^{2,2}[2]$. We will also often use the notation $E_{k,\ell}[w]$ as a shorthand for $E^{k,\ell}[w+2k+2\ell-n]$. This notation is suggested by the duality between $\mathcal{E}^{k,\ell}[w]$ and $\mathcal{E}_{k,\ell}[-w]$; for $\varphi \in \mathcal{E}^{k,\ell}[w]$ and $\psi \in \mathcal{E}_{k,\ell}[-w]$, with one of these compactly supported, there is the natural conformally invariant global pairing

$$\varphi, \psi \mapsto \langle \varphi, \psi \rangle := \int_M \varphi \cdot \psi \, d\mu_{\boldsymbol{g}},$$

where $\varphi \cdot \psi \in \mathcal{E}[-n]$ denotes a complete contraction between φ and ψ.

Since the Weyl curvature is conformally invariant, it follows easily that the linearisation (at a conformally flat metric) of the nonlinear operator $g \mapsto C^g \in \mathcal{E}^{2,2}[2]$, with C^g the Weyl curvature of the metric g, is a conformally invariant operator $\mathsf{C} : \mathcal{E}^{1,1}[2] \to \mathcal{E}^{2,2}[2]$. The formal adjoint of a conformally invariant operator is again conformally invariant. In particular, the formal adjoint of C is conformally invariant:

$$\mathsf{C}^* : \mathcal{E}_{2,2}[-2] \to \mathcal{E}_{1,1}[-2].$$

Now observe that in dimension 4 we have $\mathcal{E}^{2,2}[2] = \mathcal{E}_{2,2}[-2]$, and so C^* acts on the space of algebraic Weyl tensors $\mathcal{E}^{2,2}[2]$. It is given explicitly up to a multiple by $U_{abcd} \mapsto (\nabla^{(a}\nabla^{c)} + P^{ac})U_{abcd}$. It is straightforward to verify directly using (34) or the transformation formulae from [Gover and Peterson 2003] that this is also conformally invariant in the general curved case, and this operator applied to the Weyl curvature gives the Bach tensor.

On conformally flat structures of dimension at least 4, the null space of C locally agrees with the range of the conformal Killing operator $\mathsf{K} : \mathscr{E}^1[2] \to \mathscr{E}^{1,1}[2]$ given by $v_a \mapsto \nabla_{(a}v_{b)_0}$, where $(\cdots)_0$ indicates the symmetric trace-free part. These operators give the initial sequence of the *conformal deformation complex*. On oriented structures of dimension 4 this complex is simply

$$\mathscr{E}^1[2] \xrightarrow{\mathsf{K}} \mathscr{E}^{1,1}[2] \xrightarrow{\mathsf{C}} \mathscr{E}^{2,2}[2] \xrightarrow{\mathsf{C}^*\star} \mathscr{E}_{1,1}[-2] \xrightarrow{\mathsf{K}^*} \mathscr{E}_1[-2],$$

where $\star$ is the (conformal) Hodge star operator. Recall that in even dimensions this gives an isomorphism on the space of middle forms $\star : \mathscr{E}^{n/2} \to \mathscr{E}^{n/2}$, and so it also gives an isomorphism $\star : \mathscr{E}^{n/2,2}[2] \to \mathscr{E}^{n/2,2}[2]$.

The situation is more complicated in higher dimensions. In the deformation complex, the operator C is followed by the Weyl–Bianchi operator Bi from $\mathscr{E}^{2,2}[2]$ to $\mathscr{E}^{3,2}[2]$, given (in a conformal scale) by

$$U_{abcd} \mapsto (n-3)\nabla_{[a}U_{bc]de} - g_{d[a}\nabla_{|s|}U_{bc]}{}^{s}{}_{e} + g_{e[a}\nabla_{|s|}U_{bc]}{}^{s}{}_{d}. \tag{2}$$

Here $|\cdot|$ indicates that the enclosed indices are omitted from the skew-symmetrisation process. (Note that an easy consequence of its symmetries is that the operator (2) is trivial in dimension 4.) On oriented structures the formal adjoints of these operators conclude the complex, and so we have the picture

$$\cdot \xrightarrow{\mathsf{K}} \mathscr{E}^{1,1}[2] \xrightarrow{\mathsf{C}} \mathscr{E}^{2,2}[2] \xrightarrow{\mathsf{Bi}} \mathscr{E}^{3,2}[2] \to \cdots \to \mathscr{E}_{3,2}[-2] \xrightarrow{\overline{\mathsf{Bi}}} \mathscr{E}_{2,2}[-2] \xrightarrow{\mathsf{C}^*} \mathscr{E}_{1,1}[-2] \xrightarrow{\mathsf{K}^*} \cdot$$

Here we have omitted the initial section space $\mathscr{E}^1[2]$ and terminal section space $\mathscr{E}_1[-2]$, since they are outside the main focus of our discussions. In dimensions other than 6, $\overline{\mathsf{Bi}}$ is Bi^*. In dimension 6, $\overline{\mathsf{Bi}}$ means the composition $\mathsf{Bi}^*\star$. The Hodge star is also implicitly used in interpreting the diagram in dimension 5. In this case it gives isomorphisms $\star : E^{2,2}[2] \to E_{3,2}[-2]$ and $\star : E^{3,2}[2] \to E_{2,2}[-2]$, and under these Bi is identified modulo a sign with Bi^*. In dimensions at least 5, C^* is given by $U_{abcd} \mapsto (\nabla^{(a}\nabla^{c)} + P^{ac})U_{abcd}$, the same formula as in dimension 4. In even dimensions $n \geq 8$, the centre of the pattern consists in an obvious way of operators $\mathsf{Bi}_{(k)} : \mathscr{E}^{k,2}[2] \to \mathscr{E}^{k+1,2}[2]$ for $k = 3, \ldots, n/2-1$, their formal adjoints $\mathsf{Bi}^*_{(k)} : \mathscr{E}_{k+1,2}[-2] \to \mathscr{E}_{k,2}[-2]$ for $k = 3, \ldots, n/2-2$, and $\mathsf{Bi}^*_{(n/2-1)}\star : \mathscr{E}^{n/2,2}[2] \to \mathscr{E}_{n/2-1,2}[-2]$. The operators $\mathsf{Bi}_{(k)}$ generalise (2), which can be viewed up to a constant multiple as the "$k = 2$ case". For $U \in \mathscr{E}^{k,2}$, an explicit formula is $(\mathsf{Bi}_{(k)}U)_{a_0a_1\cdots a_kb_1b_2} = \mathrm{Proj}(\nabla_{a_0}U_{a_1\cdots a_kb_1b_2})$, where Proj is the bundle morphism which executes the projection into $\mathscr{E}^{k+1,2}[2]$. In odd dimensions at least 7, we have the operators $\mathsf{Bi}_{(k)}$ for $k = 3, \ldots, \lfloor n/2-1 \rfloor$ and their formal adjoints for $k = 3, \ldots, \lfloor n/2-2 \rfloor$. (The operator $\mathsf{Bi}_{(\lfloor n/2-1 \rfloor)}$ is formally self-adjoint).

In each dimension, the operators of the deformation complex are all conformally invariant, and the complex is locally exact and extends to give a resolution (on the sheaves of germs of smooth sections) of the sheaf of conformal Killing fields. This is a particular generalised Bernstein–Gelfand–Gelfand (gBGG) resolution. These resolutions are well understood and classified through the dual theory of generalised Verma modules, and the explicit construction of the complex above is an immediate consequence of the local uniqueness of the operators in the relevant gBGG resolution, along with explicit verification of the conformal invariance and nontriviality of the operators mentioned. See [Gasqui and Goldschmidt 1984] for an alternative construction of the complex via a theory of overdetermined systems of partial differential equations based around Spencer cohomology.

According to the results of [Boe and Collingwood 1985], in even dimensions the operators of the deformation complex are not the only conformally invariant operators between the bundles involved. There are also "long operators" from $\mathcal{E}^{k,\ell}[2]$ to $\mathcal{E}_{k,\ell}[-2]$, and an additional pair of operators about the centre of the pattern. We obtain the operator diagram

$$\cdot \xrightarrow{\mathsf{K}} \mathcal{E}^{1,1}[2] \xrightarrow{\mathsf{C}} \mathcal{E}^{2,2}[2] \xrightarrow{\mathsf{Bi}} \mathcal{E}^{3,2}[2] \to \cdots \to \mathcal{E}_{3,2}[-2] \xrightarrow{\overline{\mathsf{Bi}}} \mathcal{E}_{2,2}[-2] \xrightarrow{\mathsf{C}^*} \mathcal{E}_{1,1}[-2] \xrightarrow{\mathsf{K}^*} \cdot$$

$$\mathcal{E}^{3,2}[2] \longrightarrow \mathcal{E}_{3,2}[-2], \qquad \mathcal{E}^{2,2}[2] \xrightarrow{\mathsf{L}} \mathcal{E}_{2,2}[-2], \qquad \mathcal{E}^{1,1}[2] \xrightarrow{\mathsf{B}} \mathcal{E}_{1,1}[-2]$$

for dimensions 10 or greater. The operators in this diagram are unique up to multiplying by a constant, and the diagram indicates by arrows all the operators between the bundles explicitly presented. Thus, by implication, all compositions vanish. The same diagram applies in dimensions 8 and 6 with minor adjustments. In dimension 8 there are two "short" operators with domain $\mathcal{E}^{3,2}[2]$ and two with range $\mathcal{E}_{3,2}[-2]$. From these there is one nontrivial composition $\mathcal{E}^{3,2}[2] \to \mathcal{E}_{3,2}[-2]$. Similiarly in dimension 6 we have $\star\mathsf{Bi} : \mathcal{E}^{2,2}[2] \to \mathcal{E}^{3,2}[2]$ and $\mathsf{Bi}^* : \mathcal{E}^{3,2}[2] \to \mathcal{E}_{2,2}[-2]$, as well as the operators indicated, and $\mathsf{L} = \mathsf{Bi}^*\mathsf{Bi}$. In dimension 4 the corresponding diagram is

$$\cdot \xrightarrow{\mathsf{K}} \mathcal{E}^{1,1}[2] \underset{\star\mathsf{C}}{\overset{\mathsf{C}}{\rightrightarrows}} \mathcal{E}^{2,2}[2] \underset{\mathsf{C}^*}{\overset{\mathsf{C}^*\star}{\rightrightarrows}} \mathcal{E}_{1,1}[-2] \xrightarrow{\mathsf{K}^*} \cdot \qquad \mathcal{E}^{1,1}[2] \xrightarrow{\mathsf{B}} \mathcal{E}_{1,1}[-2]$$

and in this case $\mathsf{B} := \mathsf{C}^*\mathsf{C}$. Evidently on even-dimensional conformally flat structures there are *detour complexes* [Branson and Gover 2005], where one shortcuts the deformation complex via a long operator. The examples relevant here are

$$\mathcal{E}^1[2] \xrightarrow{\mathsf{K}} \mathcal{E}^{1,1}[2] \xrightarrow{\mathsf{B}} \mathcal{E}_{1,1}[-2] \xrightarrow{\mathsf{K}^*} \mathcal{E}_1[-2] \tag{3}$$

and in dimensions $n \geq 6$,

$$\mathscr{E}^1[2] \xrightarrow{\mathsf{K}} \mathscr{E}^{1,1}[2] \xrightarrow{\mathsf{C}} \mathscr{E}^{2,2}[2] \xrightarrow{\mathsf{L}} \mathscr{E}_{2,2}[-2] \xrightarrow{\mathsf{C}^*} \mathscr{E}_{1,1}[-2] \xrightarrow{\mathsf{K}^*} \mathscr{E}_1[-2].$$

These have applications in constructing torsion quantities which generalise Cheeger's de Rham half-torsion [Branson and Gover, in progress].

According to [Fefferman and Graham 1985], the obstruction tensor $\mathcal{O}_{ab}$ is a trace-free symmetric 2-tensor of weight $2-n$. In other words, it is a section of $\mathscr{E}_{1,1}[-2] = \mathscr{E}^{1,1}[2-n]$. From the general theory in [Eastwood and Slovák 1997], we know that all the operators indicated explicitly by arrows in the diagrams above admit curved analogues, that is, generalisations to general conformal structures. (In fact, the formulae given above for K, C^*, and Bi give conformally invariant operators on general structures. We will continue to use this notation for these operators even in the conformally curved setting.) From the diagrams, however, the difference between dimension 4 and higher even dimensions is clear. In dimension 4 there is a conformal operator $\mathscr{E}^{2,2}[2] \to \mathscr{E}_{1,1}[-2]$ that yields the Bach tensor, as described above. In higher dimensions the conformally invariant C^* does not have $\mathscr{E}^{2,2}[2]$ as its domain. These observations establish the following key point.

Proposition 2.1. *In even dimensions $n \geq 6$, there can be no conformally invariant differential operator $\mathscr{E}^{2,2}[2] \to \mathscr{E}_{1,1}[-2]$ that recovers the obstruction tensor upon application to the Weyl curvature C.*

If there were such an operator, then by Theorem 4.4 or by [Graham and Hirachi 2005], it would necessarily have as highest order term $\Delta^{n/2-2}\nabla^a\nabla^c U_{abcd}$. Its linearisation would therefore be an operator $\mathscr{E}^{2,2}[2] \to \mathscr{E}_{1,1}[-2]$. But there is no operator between these bundles in the diagram.

This brings us to the question of whether there can be any conformally invariant operator that yields the obstruction tensor in dimensions $n \geq 6$. We will see that there is, and we will construct the operator. To understand how this works, it is helpful to expose some properties of the operators B and L.

Proposition 2.2. *The operators $\mathsf{B} : \mathscr{E}^{1,1}[2] \to \mathscr{E}_{1,1}[-2]$ and $\mathsf{L} : \mathscr{E}^{2,2}[2] \to \mathscr{E}_{2,2}[-2]$ are formally self-adjoint. In each even dimension $n \geq 6$, there is a natural linear differential operator $\mathsf{H} : \mathscr{E}^{2,2}[2] \to \mathscr{E}_{2,2}[-2]$ such that B is given by the composition*

$$\mathsf{B} = \mathsf{C}^*\mathsf{H}\mathsf{C},$$

and there is a natural linear differential operator $\mathsf{N} : \mathscr{E}^{3,2}[2] \to \mathscr{E}_{3,2}[-2]$ such that L is given by the composition

$$\mathsf{L} = \mathsf{Bi}^*\mathsf{N}\mathsf{Bi}.$$

We prove this in Section 4 using the geometric tools developed below. The factorisations described in the proposition can also be established via central character arguments; see also [Branson and Gover 2005]. Note that L is only defined in even dimensions $n \geq 6$. In dimension 6, N is the identity. Otherwise, from the classification of conformally invariant operators on conformally flat manifolds, as discussed above, it follows that the operators H and N are not conformally invariant.

On conformally flat structures the operator $\mathsf{G} := \mathsf{C}^*\mathsf{H}$ is not conformally invariant ($n \neq 4$). It *is*, however, conformally invariant on the range of the linearised Weyl curvature, and we have $\mathsf{B} = \mathsf{GC}$. On the other hand, L annihilates the range of C. The next theorem gives special curved analogues of these operators.

We need some further notation. On conformal manifolds of dimension n there is a natural reducible, but indecomposable, bundle $W_{2,2}$ that has the composition series $E_{2,2}[-2] \oplus\!\!\!\!- E_{2,1}[-2] \oplus\!\!\!\!- E_{1,1}[-2]$. This means that $E_{1,1}[-2]$ is a (conformally invariant) subbundle and that $E_{2,1}[-2]$ is a subbundle of the quotient $W_{2,2}/E_{1,1}[-2]$. The bundle $W_{2,2}$, which is a subbundle of a certain tractor bundle, is constructed explicitly in proof of Theorem 2.3 in Section 4. It decomposes as $[W_{2,2}]_g = E_{2,2}[-2] \oplus E_{2,1}[-2] \oplus E_{1,1}[-2]$, given a choice of metric g from the conformal class. Let us write I^* and P for the respective canonical bundle maps $W_{2,2} \to E_{2,2}[-2]$ and $W_{2,2} \to E_{2,2}[-2] \oplus\!\!\!\!- E_{2,1}[-2]$ which are unique up to a constant multiple.

Theorem 2.3. *On conformal manifolds of even dimension $n \geq 6$ there is a natural nontrivial conformally invariant linear differential operator*

$$\mathbb{B} : \mathcal{E}^{2,2}[2] \to \mathcal{W}_{2,2} = \mathcal{E}_{2,2}[-2] \oplus\!\!\!\!- \mathcal{E}_{2,1}[-2] \oplus\!\!\!\!- \mathcal{E}_{1,1}[-2]$$

with the following properties:

(i) *The composition* $(\mathrm{I}^*\mathbb{B} =: L) : \mathcal{E}^{2,2}[2] \to \mathcal{E}_{2,2}[-2]$ *is a nontrivial conformally invariant differential operator of order $n-4$.*

(ii) *There is a linear differential operator $\bar{\mathbb{B}}$ such that* $\bar{\mathbb{B}}\mathbb{B} = \Delta^\ell + \mathrm{lots}$. *Thus on Riemannian signature conformal structures, $\mathbb{B}$ is graded injectively elliptic.*

(iii) *For the Weyl curvature $C \in \mathcal{E}^{2,2}[2]$ we have $\mathbb{B}(C) \in \mathcal{E}_{1,1}[-2]$. The natural conformal invariant $\mathcal{O}_{ab} \in \mathcal{E}_{1,1}[-2]$ given this way agrees with the obstruction tensor.*

We prove the theorem in Section 4. Note that there is a degenerate version of the theorem for dimension 4; see expression (28) and the comments that follow it.

From the uniqueness of L it is clear that on conformally flat manifolds L recovers L up to a constant multiple. However L is a special curved generalisation of L, since the property $L(C) = 0$ generalises to arbitrary conformal structures the vanishing of the composition LC. Since $L(C)$ vanishes, it follows from the composition series for $\mathcal{W}_{2,2}$ that the component of $\mathbb{B}(C)$ in $\mathcal{E}_{2,1}[-2]$ is a natural conformal invariant. That this also vanishes is a special property of $\mathbb{B}$ that, in a sense, carries to general structures the nonexistence of an operator $\mathcal{E}^{1,1}[2] \to \mathcal{E}_{2,1}[-2]$. It follows that on conformally flat structures the composition $\mathbb{B}\mathsf{C}$ determines a nontrivial operator $\mathcal{E}^{1,1}[2] \to \mathcal{E}_{1,1}[-2]$ which therefore agrees with B. If, for each metric g in the conformal class, we write G for the composition of $\mathbb{B}$ followed by projection to the component $\mathcal{E}_{1,1}[-2]$ (we have such a projection since, recall, $\mathcal{W}_{2,2}$ completely decomposes given a conformal scale), then by construction, G is a curved analogue of the operator G. That is, the restriction of G to conformally flat structures is G. Note that G has the special property that $G(C) = \mathbb{O}$, and as we will see from the construction of $\mathcal{W}_{2,2}$, although G is not conformally invariant, the conformal variation of G under $g \mapsto e^{2\omega} g$ is only quadratic in ω. Since G also has this sort of variation, this is optimal.

In the conformally flat case, it is easily shown that $\mathsf{P}\mathbb{B}$ can be reexpressed as a composition $\mathbb{U}\mathsf{L}$. Here $\mathbb{U}$ is the operator (35) below, except with w set to $6-n$. This result follows from the nonexistence of a nontrivial conformal operator $\mathcal{E}^{2,2}[2] \to \mathcal{E}_{2,1}[-2]$. It follows from this and (ii) that in even dimensions $n \geq 6$, (L, G) is a right factor of a Laplacian. That is, there are linear differential operators $\bar{\mathsf{L}}$ and $\bar{\mathsf{G}}$ such that

$$(\bar{\mathsf{L}}\ ,\ \bar{\mathsf{G}}) \begin{pmatrix} \mathsf{L} \\ \mathsf{G} \end{pmatrix} = \Delta^{\ell} + \text{lots}.$$

Since also G is conformally invariant on the null space of L, it follows that G is a conformal gauge companion operator in the sense of [Branson and Gover 2002]; see also [Branson and Gover 2005]. Thus in Riemannian signature, the operator pair (L, G) is an elliptic system. Since L has Bi as a right factor, the system $(\mathsf{Bi}, \mathsf{G})$ is also elliptic and has a conformally invariant null space. Let us denote this by $\mathcal{H}^2_G$, and note that on compact manifolds, $\mathcal{H}^2_G$ is finite-dimensional. This is closely related to the second cohomology of the deformation complex. For example, from Proposition 2.2 and an easy adaption of the proof of Proposition 2.5 in [Branson and Gover 2005], we obtain the following result, which suggests that $\mathcal{H}^2_G$ is a candidate for a space of conformal harmonics. Here we write H^i, $i = 1, 2$, for the first and second cohomology spaces in the deformation complex, and H^1_B for the first cohomology of the detour complex (3).

Proposition 2.4. *On even-dimensional conformally flat manifolds of dimension* $n \geq 6$, *there is an exact sequence*

$$0 \to H^1 \to H^1_B \to \mathcal{H}^2_G \to H^2,$$

where the last map $\mathcal{H}^2_G \to H^2$ *is simply* $\Phi \mapsto [\Phi]$, *the middle* $H^1_B \to \mathcal{H}^2_G$ *is the map on the quotient* $\mathcal{N}(\mathsf{B})/\mathcal{R}(\mathsf{K})$ *induced by the restriction of* C *to* $\mathcal{N}(\mathsf{B})$, *the null space of* B, *and the first map* $H^1 \to H^1_B$ *is inclusion.*

There are further results concerning the relationship of H^1_B to H^1 and $\mathcal{H}^2_G$ to H^2, but this will be taken up elsewhere; see also [Branson and Gover 2002].

3. The ambient construction and tractor calculus

In the subsequent sections we will explore the relationship between the Fefferman–Graham ambient metric construction [Fefferman and Graham 1985] and tractor calculus as derived in [Čap and Gover 2003; Gover and Peterson 2003]. The notation and conventions for the ambient metric closely follow [Branson and Gover 2005].

For $\pi : \mathcal{Q} \to M$ a conformal structure of signature (p, q), use ρ to denote the $\mathbb{R}_+$ action on $\mathcal{Q}$ given by $\rho(s)(x, g_x) = (x, s^2 g_x)$. An *ambient manifold* is a smooth $(n{+}2)$-manifold $\tilde{M}$ endowed with a free $\mathbb{R}_+$-action ρ and an $\mathbb{R}_+$-equivariant embedding $i : \mathcal{Q} \to \tilde{M}$. We write $\boldsymbol{X} \in \mathfrak{X}(\tilde{M})$ for the fundamental field generating the $\mathbb{R}_+$-action: for $f \in C^\infty(\tilde{M})$ and $u \in \tilde{M}$, we have $\boldsymbol{X} f(u) = (d/dt) f(\rho(e^t)u)|_{t=0}$. For an ambient manifold $\tilde{M}$, an *ambient metric* is a pseudo–Riemannian metric $\boldsymbol{h}$ of signature $(p+1, q+1)$ on $\tilde{M}$ satisfying the conditions: (i) $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$, where $\mathcal{L}_{\boldsymbol{X}}$ denotes the Lie derivative by $\boldsymbol{X}$; (ii) for $u = (x, g_x) \in \mathcal{Q}$ and $\xi, \eta \in T_u\mathcal{Q}$, we have $\boldsymbol{h}(i_*\xi, i_*\eta) = g_x(\pi_*\xi, \pi_*\eta)$; (iii) $\operatorname{Ric}(\boldsymbol{h}) = 0$ up to the addition of terms vanishing to order $n/2 - 1$ if n is even or $\operatorname{Ric}(\boldsymbol{h}) = 0$ to all orders if n is odd; and (iv) $\boldsymbol{h}(\boldsymbol{X}, \cdot) = \frac{1}{2} dQ$ to all orders.

If M is locally conformally flat, then there is a canonical solution to the ambient metric problem to all orders. This is simply a flat ambient metric. This is forced by (i)–(iii) in odd dimensions, but in even dimensions this extends the solution; see comments in [Branson and Gover 2005]. When discussing the conformally flat case, we assume this solution.

We write $\boldsymbol{\nabla}$ for the ambient Levi-Civita connection and use uppercase abstract indices A, B, etc., for tensors on $\tilde{M}$. The ambient Riemann tensor will be denoted $\boldsymbol{R}_{AB}{}^C{}_D$. Since $\mathcal{L}_{\boldsymbol{X}} \boldsymbol{h} = 2\boldsymbol{h}$, it follows that

$$\boldsymbol{\nabla} \boldsymbol{X} = \boldsymbol{h}, \tag{4}$$

$$\boldsymbol{X}^A \boldsymbol{R}_{ABCD} = 0. \tag{5}$$

Equalities without qualification, as here, indicate that the results hold to all orders or identically on the ambient manifold.

3A. ***Tractor bundles.*** Let $\tilde{\mathcal{E}}(w)$ denote the space of functions on $\tilde{M}$ which are homogeneous of degree $w \in \mathbb{R}$ with respect to the action ρ. More generally, a tensor field F on $\tilde{M}$ is said to be *homogeneous of degree w* if $\rho(s)^* F = s^w F$, that is, $\mathcal{L}_{\boldsymbol{X}} F = wF$. Just as sections of $\mathcal{E}[w]$ are equivalent to functions in $\tilde{\mathcal{E}}(w)|_{\mathcal{Q}}$, the restriction of a homogeneous tensor field to $\mathcal{Q}$ has an interpretation on M. Denote by $\boldsymbol{\mathcal{T}}$ the space of sections of $T\tilde{M}$ which are homogeneous of degree -1 and write $\boldsymbol{\mathcal{T}}(w)$ for sections in $\boldsymbol{\mathcal{T}} \otimes \tilde{\mathcal{E}}(w)$, where here $\otimes$ indicates a tensor product over $\tilde{\mathcal{E}}(0)$. From [Čap and Gover 2003] we have the following results: We may identify the standard tractor bundle $\mathbb{T}$ with $T\tilde{M}|_{\mathcal{Q}}$ modulo a suitable $\mathbb{R}_+$-action, so that sections of $\mathbb{T}$ are in one-one correspondence with sections in $\boldsymbol{\mathcal{T}}$. Thus we write $\mathcal{T}$ for the space of sections of the standard tractor bundle. The filtration of $\mathbb{T}$, which we traditionally indicate by a composition series,

$$\mathbb{T} = E[1] \oplus\!\!\!\!\!\!\supset E^1[1] \oplus\!\!\!\!\!\!\supset E[-1], \tag{6}$$

reflects the vertical subbundle of $T\mathcal{Q}$ and $T\mathcal{Q}$ as a subbundle of $T\tilde{M}|_{\mathcal{Q}}$. Then since the ambient metric $\boldsymbol{h}$ is homogeneous of degree 2, it descends to give a metric on $\mathbb{T}$. This is the usual tractor metric. Sections of $\boldsymbol{\mathcal{T}}$ may be characterised as those sections of $T\tilde{M}$ which are covariantly parallel along the integral curves of $\boldsymbol{X}$, which on $\mathcal{Q}$ are exactly the fibres of π. The normal tractor connection agrees with the ambient connection as follows. For $U \in \mathcal{T}$, let $\tilde{U}$ be the corresponding section of $\boldsymbol{\mathcal{T}}|_{\mathcal{Q}}$. A tangent vector field ξ on M has a lift to a field $\tilde{\xi} \in \boldsymbol{\mathcal{T}}(1)$ on $\mathcal{Q}$, which is everywhere tangent to $\mathcal{Q}$. This is unique up to adding $f\boldsymbol{X}$, where $f \in \tilde{\mathcal{E}}(0)$. We extend $\tilde{U}$ and $\tilde{\xi}$ smoothly and homogeneously to fields on $\tilde{M}$ and form $\boldsymbol{\nabla}_{\tilde{\xi}}\tilde{U}|_{\mathcal{Q}}$; this section is independent of the extensions and independent of the choice of $\tilde{\xi}$ as a lift of ξ and is exactly the section of $\boldsymbol{\mathcal{T}}(0)|_{\mathcal{Q}}$ corresponding to $\nabla_\xi U$ where ∇ here indicates the tractor connection.

When abstract indices are required, the section spaces of the tractor bundle and its dual can also be denoted $\mathcal{T}^A$ and $\mathcal{T}_A$. A choice of metric g from the conformal class determines a canonical splitting of the composition series (6), by [Bailey et al. 1994a; Čap and Gover 2000]. Via this splitting, direct sums $\oplus$ replace the semidirect sums $\oplus\!\!\!\!\!\!\supset$ in that series, and we introduce g-dependent sections Z^{Ab} in $\mathcal{T}^{Ab}[-1]$ and Y^A in $\mathcal{T}^A[-1]$ that describe the decomposition of $\mathbb{T}$ into the direct sum $E[1] \oplus E_a[1] \oplus E[-1]$. According to $V^A = Y^A\sigma + Z^{Ab}\mu_b + X^A\rho$, a section $V \in \mathcal{T}$ then corresponds to a triple $[V]_g = (\sigma, \mu, \rho)$ of sections, and in these terms the tractor metric is given by $h(V, V) = \boldsymbol{g}^{ab}\mu_a\mu_b + 2\sigma\rho$. Thus the tractor contractions of the projectors are

$$X^A Y_A = 1, \quad Z^{Ab} Z_{Aa} = \delta_a^b, \tag{7}$$

and 0 for the other pairings.

If $\hat{Y}^A$ and $\hat{Z}^A{}_b$ are the projectors for the metric $\hat{g} = e^{2\omega}g$, then we have

$$\hat{Z}^{Ab} = Z^{Ab} + \Upsilon^b X^A, \;\; \hat{Y}^A = Y^A - \Upsilon_b Z^{Ab} - \tfrac{1}{2}\Upsilon_b\Upsilon^b X^A. \tag{8}$$

Here $\Upsilon := d\omega$. In terms of this splitting, determined by g, the tractor connection is given by

$$\nabla_a X_A = Z_{Aa}\,, \;\; \nabla_a Z_{Ab} = -P_{ab}X_A - Y_A\boldsymbol{g}_{ab}\,, \;\text{ and }\; \nabla_a Y_A = P_{ab}Z_A{}^b. \tag{9}$$

We use the notation $\tilde{\mathbb{T}}^\Phi$ to denote an arbitrary ambient tensor bundle (with $\tilde{\mathbb{T}}^0$ meaning the trivial bundle) and write $\boldsymbol{\mathcal{T}}^\Phi(w)$, $w \in \mathbb{R}$, for the subspace of $\Gamma(\tilde{\mathbb{T}}^\Phi)$

consisting of sections S satisfying $\boldsymbol{\nabla}_{\boldsymbol{X}} S = wS$; we will say such sections are homogeneous of *weight* w. From the constructions above, it follows that the sections in $\boldsymbol{\mathcal{T}}^{\Phi}(w)|_{\mathcal{Q}}$ are equivalent to sections of a tractor bundle that we denote $\mathbb{T}^{\Phi}[w]$. We write $\mathcal{T}^{\Phi}[w]$ for the section space of the latter.

A basic example of interest is the bundle of k-form tractors $\mathbb{T}^k$, which is the k-th exterior power of the bundle of standard tractors. It is straightforward to verify that this has a composition series which, in terms of section spaces, is given by

$$\mathcal{T}^k = \Lambda^k \mathcal{T} \cong \mathcal{E}^{k-1}[k] \lpl \{\mathcal{E}^k[k] \oplus \mathcal{E}^{k-2}[k-2]\} \lpl \mathcal{E}^{k-1}[k-2]. \tag{10}$$

Also of direct relevance to our constructions below are the bundles denoted by $\mathbb{T}^{2,2}[w]$, which are the subbundles of $\mathbb{T}^2 \otimes \mathbb{T}^2 \otimes E[w]$ consisting of tractors of weight w and Weyl-tensor-type symmetries, that is, trace-free Riemann-tensor-type symmetries. We write $\mathcal{T}^{2,2}[w]$ for the section space of $\mathbb{T}^{2,2}[w]$ and note that (with notation as in Section 2) it has the composition series

$$\mathcal{E}^{1,1}[w+4] \lpl \begin{array}{c} \mathcal{E}^{2,1}[w+4] \\ \oplus \\ \mathcal{E}^{1}[w+2] \end{array} \lpl \begin{array}{c} \mathcal{E}^{2,2}[w+4] \\ \oplus \\ \mathcal{E}^{2}[w+2] \\ \oplus \\ \mathcal{E}^{1,1}[w+2] \\ \oplus \\ \mathcal{E}[w] \end{array} \lpl \begin{array}{c} \mathcal{E}^{2,1}[w+2] \\ \oplus \\ \mathcal{E}^{1}[w] \end{array} \lpl \mathcal{E}^{1,1}[w]. \tag{11}$$

A comment on punctuation is in order: here the columns represent composition factors, decomposed into $O(g)$-irreducibles, and these are separated by $\lpl$'s which indicate the composition structure. This series may be obtained by any $\mathfrak{so}(n+2)$ to $\mathfrak{so}(n)$ branching-rule algorithm or, alternatively, by simply considering the possible contractions of the projectors X, Y, and Z into a typical element of $\mathcal{T}^{2,2}[w]$.

3B. ***Operators and invariants via the ambient metric.*** An operator P acting between ambient tensor bundles is said to be *homogeneous* of weight $u \in \mathbb{R}$ if $[\boldsymbol{\nabla}_{\boldsymbol{X}}, P] = uP$. Operators homogeneous in this sense map homogeneous tensors of weight w to homogeneous tensors of weight $w+u$. On the other hand, a differential operator P is said to act *tangentially* along $\mathcal{Q}$, as an operator on some domain space, if we have $PQ = QP'$ for some operator P' (or equivalently $[P, Q] = QP''$ for some P''). Of particular interest are linear differential operators P which are both homogeneous and also act on some homogeneous tensor space $\boldsymbol{\mathcal{T}}^{\Phi}(w)$ as domain *tangentially* along $\mathcal{Q}$. Each such operator P clearly determines a well-defined operator on $\boldsymbol{\mathcal{T}}^{\Phi}(w)|_{\mathcal{Q}}$, and hence determines an operator on the equivalent weighted tractor bundle section space $\mathcal{T}^{\Phi}[w]$. If the operator P is natural as an operator on the ambient manifold, then since the ambient construction is not dependent on a choice of metric from the conformal class, it follows that the induced operator on weighted tractor fields is conformally invariant. The remaining issue is whether this induced operator is natural for the underlying conformal structure. For the operators we are interested in here, we solve this by giving an algorithm for expressing the induced operator as a formula in terms of known natural operators. This solves two problems, since one of our aims is to obtain explicit formulae for the operators concerned.

Before we construct examples of such operators, we require some further background. First note that from (4), we have

$$[\boldsymbol{\Delta}, \boldsymbol{X}] = 2\boldsymbol{\nabla}, \qquad \text{where} \quad \boldsymbol{\Delta} := \boldsymbol{\nabla}^A \boldsymbol{\nabla}_A, \tag{12}$$

and $\boldsymbol{\nabla}_A Q = 2\boldsymbol{X}_A$. Both identities hold to all orders. Thus $\boldsymbol{\nabla}_{\boldsymbol{X}} Q = 2Q$; Q is homogeneous of weight 2. A short computation shows that if U is an ambient tensor field, then

$$[\boldsymbol{\Delta}, Q]U = 2(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}} + 2)U. \tag{13}$$

It follows that for any positive integer ℓ, if an ambient tensor field U is $O(Q^\ell)$, then $\boldsymbol{\Delta}U$ and $\boldsymbol{\nabla}U$ are both $O(Q^{\ell-1})$.

Now we define an operator $\boldsymbol{D}$ (or $\boldsymbol{D}_A$ when indices are used). Let

$$\boldsymbol{D}V := \boldsymbol{\nabla}(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}} - 2)V - \boldsymbol{X}\boldsymbol{\Delta}V, \tag{14}$$

for any ambient tensor field V. It is readily verified that $\boldsymbol{D}$ is homogeneous of weight -1. By (12) we also have the equivalent formula

$$\boldsymbol{D}V = \boldsymbol{\nabla}(n + 2\boldsymbol{\nabla}_{\boldsymbol{X}})V - \boldsymbol{\Delta}\boldsymbol{X}V. \tag{15}$$

Using either of these with the computations above, we obtain $\boldsymbol{D}QV = Q\boldsymbol{D}V + 4Q\boldsymbol{\nabla}V$, and so $\boldsymbol{D}$ acts tangentially. For later use we note that for any integer $\ell \geq 2$, if V is $O(Q^\ell)$, then $\boldsymbol{D}_A V$ is $O(Q^{\ell-1})$.

Since $\boldsymbol{D}$ acts tangentially on *any* ambient tensor bundle, it follows that for every tractor bundle $\mathcal{T}^\Phi$ and $w \in \mathbb{R}$ we obtain an operator

$$D : \mathcal{T}^\Phi[w] \to \mathcal{T} \otimes \mathcal{T}^\Phi[w-1]$$

equivalent to $\boldsymbol{D}$ as an operator $\boldsymbol{\mathcal{T}}^\Phi(w)|_{\mathcal{Q}} \to \boldsymbol{\mathcal{T}} \otimes \boldsymbol{\mathcal{T}}^\Phi(w-1)|_{\mathcal{Q}}$. It is straightforward to prove (see [Čap and Gover 2003; Gover and Peterson 2003]) that D is the usual tractor-D operator of [Thomas 1926; Bailey et al. 1994a]. For a choice of metric g from the conformal class and for any $V \in \mathcal{T}^\Phi[w]$, D is given explicitly by

$$D^A V := (n + 2w - 2)wY^A V + (n + 2w - 2)Z^{Aa}\nabla_a V - X^A \Box V, \tag{16}$$

where $\Box V := \Delta V + wJV$. We note that D is a natural differential operator. A differential operator taking values in a tractor bundle (or acting between tractor bundles) is said to be natural if the $\mathfrak{so}(g)$-irreducible components of the operator are natural.

Acting on $\boldsymbol{\mathcal{T}}^{\Phi}(1-n/2)$, the operator $\boldsymbol{D}$ is simply $-\boldsymbol{X}\boldsymbol{\Delta}$, and correspondingly D simplifies to $-X\Box$ on $\mathcal{T}^{\Phi}[1-n/2]$. Thus $\boldsymbol{\Delta}$ acts tangentially on $\boldsymbol{\mathcal{T}}^{\Phi}(1-n/2)$ and as an operator on the restriction of this space to $\mathcal{Q}$, is equivalent to the tractor-coupled conformal Laplacian

$$\Box : \mathcal{T}^{\Phi}[1-n/2] \to \mathcal{T}^{\Phi}[-1-n/2]. \tag{17}$$

Many identities involving D are obtained most easily by calculating with $\boldsymbol{D}$ on $\tilde{M}$. For example, a short calculation using (4) and (12) shows that

$$\boldsymbol{D}_A \boldsymbol{X}^A V = (n+2w+2)(n+w)V - Q\boldsymbol{\Delta} V \tag{18}$$

for any $V \in \boldsymbol{\mathcal{T}}^{\Phi}(w)$. Hence for any $V \in \mathcal{T}^{\Phi}[w]$, we have

$$D_A X^A V = (n+2w+2)(n+w)V. \tag{19}$$

An observation key to the next section is that the ambient curvature $\boldsymbol{R}$ is "harmonic" for a certain Laplacian, at least at low orders. Before we construct this Laplacian we need some further notation. Write $\sharp$ (*hash*) for the natural tensorial action of sections A of $\mathrm{End}(T\tilde{M})$ on ambient tensors. For example, on an ambient covariant 2-tensor T_{AB}, we have $A\sharp T_{AB} = -A^C{}_A T_{CB} - A^C{}_B T_{AC}$. If A is skew for $\boldsymbol{h}$, then at each point, A is $\mathfrak{so}(\boldsymbol{h})$-valued. The hash action thus commutes with the raising and lowering of indices and preserves the $\mathrm{SO}(\boldsymbol{h})$-decomposition of tensors. For example, $A\sharp$ maps trace-free symmetric tensors to trace-free symmetric tensors. As a section of the tensor square of the $\boldsymbol{h}$-skew bundle endomorphisms of $T\tilde{M}$, the ambient curvature has a double hash action on ambient tensors; we write $\boldsymbol{R}\sharp\sharp T$. As a point on punctuation, it should be noted that we will treat tensors in composite expressions as multiplication operators. A composition of operators L, M, and N acting on S denoted $LMNS$ means $L(M(N(S)))$. For example, $\nabla\boldsymbol{R}\sharp\sharp T$ has the same interpretation as $\nabla(\boldsymbol{R}\sharp\sharp T)$.

From the Bianchi identities, we have that on any Riemannian or pseudo-Riemannian manifold,

$$\begin{aligned} &4\nabla_{A_1}\nabla_{B_1}\mathbf{Ric}_{A_2B_2} \\ &\quad = \boldsymbol{\Delta}\boldsymbol{R}_{A_1A_2B_1B_2} + \tfrac{1}{2}\boldsymbol{R}\sharp\sharp\boldsymbol{R}_{A_1A_2B_1B_2} - \mathbf{Ric}_{CA_1}\boldsymbol{R}^C{}_{A_2B_1B_2} + \mathbf{Ric}_{CB_1}\boldsymbol{R}^C{}_{B_2A_1A_2}. \end{aligned} \tag{20}$$

Remark. In (20) we adopt the convention that sequentially labeled indices in the subscript position (such as A_1 and A_2) are implicitly skew-symmetrised. This convention applies throughout this paper unless noted otherwise.

Let us define a Laplacian operator $\not{\boldsymbol{\Delta}}$ by the formula

$$\not{\boldsymbol{\Delta}} := \boldsymbol{\Delta} + \tfrac{1}{2}\boldsymbol{R}\sharp\sharp.$$

Then from (20) and the conditions on $\mathrm{Ric}(\boldsymbol{h})$ for the ambient metric, we have

$$4\boldsymbol{\nabla}_{A_1}\boldsymbol{\nabla}_{B_1}\mathbf{Ric}_{A_2B_2} = \not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2} + O(Q^{n/2-1}) \tag{21}$$

in even dimensions. Therefore

$$\not{\boldsymbol{\Delta}}\boldsymbol{R}_{BCDE} = 0 \tag{22}$$

modulo $O(Q^{n/2-3})$ in even dimensions and to infinite order in odd dimensions.

Remarks. 1. The operator $\not{\boldsymbol{\Delta}}$ is a type of form-Laplacian. On a Riemannian or pseudo-Riemannian manifold, suppose U is any tensor with Riemann tensor type symmetries. A short calculation shows that

$$\not{\boldsymbol{\Delta}} U = -\tfrac{1}{2}\left(\boldsymbol{\delta}_1^{\nabla}\boldsymbol{d}_1^{\nabla} + \boldsymbol{d}_1^{\nabla}\boldsymbol{\delta}_1^{\nabla} + \boldsymbol{\delta}_2^{\nabla}\boldsymbol{d}_2^{\nabla} + \boldsymbol{d}_2^{\nabla}\boldsymbol{\delta}_2^{\nabla}\right) U,$$

where $\boldsymbol{d}_i^{\nabla}$ is the Levi-Civita connection-coupled exterior derivative, $\boldsymbol{\delta}_i^{\nabla}$ is its formal adjoint, and the index i is 1 or 2 according to whether we regard U as a 2-form with values in a tensor bundle on the first pair of indices or the last pair. In terms of the Levi-Civita connection ∇, we have $(\boldsymbol{d}_1^{\nabla}U)_{A_0A_1A_2B_1B_2} = 3\boldsymbol{\nabla}_{A_0}U_{A_1A_2B_1B_2}$ and $(\boldsymbol{\delta}_2^{\nabla}U)_{A_1A_2B_2} = -\boldsymbol{\nabla}^{B_1}U_{A_1A_2B_1B_2}$, for example.

Returning to the ambient manifold, note that from these observations, the results concerning the degree to which the ambient curvature is $\not{\boldsymbol{\Delta}}$-harmonic are manifest, since on the one hand $\boldsymbol{d}_1^{\nabla}$ and $\boldsymbol{d}_2^{\nabla}$ annihilate $\boldsymbol{R}$ by the Bianchi identity and on the other hand $\boldsymbol{\delta}_1^{\nabla}\boldsymbol{R}$ and $\boldsymbol{\delta}_2^{\nabla}\boldsymbol{R}$ are $O(Q^{n/2-2})$ (or $O(Q^{\infty})$ in odd dimensions) by dint of the contracted Bianchi identity and the condition (iii) on the ambient Ricci curvature.

2. Note that from (20), if **Ric** vanishes to all orders on the ambient manifold, then it is immediate that $\not{\boldsymbol{\Delta}}\boldsymbol{R}$ vanishes to all orders. Conversely, if $\not{\boldsymbol{\Delta}}\boldsymbol{R}$ vanishes to all orders, then so does $4\boldsymbol{\nabla}_{A_1}\boldsymbol{\nabla}_{B_1}\mathbf{Ric}_{B_2A_2} + \mathbf{Ric}_{CA_1}\boldsymbol{R}^{C}{}_{A_2B_1B_2} - \mathbf{Ric}_{CB_1}\boldsymbol{R}^{C}{}_{B_2A_1A_2}$. On the other hand, contracting the latter with $\boldsymbol{X}^{A_1}\boldsymbol{X}^{B_1}$ and using (4) and (5) yields $2\mathbf{Ric}_{A_2B_2}$. Thus on the ambient manifold, the vanishing of **Ric** to all orders is equivalent to the vanishing of $\not{\boldsymbol{\Delta}}\boldsymbol{R}$ to all orders.

We may view the operator $\not{\boldsymbol{\Delta}}$ as the special case $\alpha = \frac{1}{2}$ of the family of ambient Laplacians

$$\boldsymbol{\Delta}_{\alpha} := \boldsymbol{\Delta} + \alpha\boldsymbol{R}\sharp\sharp, \quad \alpha \in \mathbb{R}, \tag{23}$$

which also includes the ambient form Laplacian at $\alpha = 1$ and the usual ambient Bochner Laplacian at $\alpha = 0$. While the latter was used in the constructions of

[Graham et al. 1992] giving conformal operators between densities, the generalisation to the ambient form Laplacian proved appropriate in [Branson and Gover 2005] for the study of conformal operators on (weighted) differential forms. It seems likely that others in the family will also have important roles, and so much of the discussion in the next section allows for the possibility of any $\alpha \in \mathbb{R}$. Certain key identities for $\boldsymbol{\Delta}$ are unaffected by the addition of the $\boldsymbol{R}\sharp\sharp$ term. In particular, since $\boldsymbol{X}^A \boldsymbol{R}_{ABCD} = 0$ it follows that

$$[\boldsymbol{\Delta}_\alpha, \boldsymbol{X}] = [\boldsymbol{\Delta}, \boldsymbol{X}] = 2\boldsymbol{\nabla}. \tag{24}$$

Using this, or even more simply by noting that $[\boldsymbol{R}\sharp\sharp, Q] = 0$, we obtain

$$[\boldsymbol{\Delta}_\alpha, Q] = [\boldsymbol{\Delta}, Q] = 2(n + 2\boldsymbol{\nabla}_X + 2). \tag{25}$$

A point of departure is $[\boldsymbol{\Delta}_\alpha, \boldsymbol{\nabla}]$. Observe that if $V_{BC\cdots E}$ is any ambient tensor, then by the Ricci-flatness of the ambient metric,

$$\begin{aligned}(26)\quad & [\boldsymbol{\Delta}, \boldsymbol{\nabla}_A] V_{BC\cdots E} \\ &= -2\boldsymbol{R}_A{}^P{}_B{}^Q \boldsymbol{\nabla}_P V_{QC\cdots E} - 2\boldsymbol{R}_A{}^P{}_C{}^Q \boldsymbol{\nabla}_P V_{BQ\cdots E} - \cdots - 2\boldsymbol{R}_A{}^P{}_E{}^Q \boldsymbol{\nabla}_P V_{BC\cdots Q}.\end{aligned}$$

This equality holds modulo $O(Q^{n/2-2})$ in even dimensions and to infinite order in odd dimensions.

Using the results above and the Bianchi identities, it is straightforward to verify that if we define the ambient homogeneous (of weight -2) tensor field

$$\boldsymbol{W}_{A_1 A_2 B_1 B_2} := \frac{3}{n-2} \boldsymbol{D}^{A_0} \boldsymbol{X}_{A_0} \boldsymbol{R}_{A_1 A_2 B_1 B_2}, \tag{27}$$

then in dimensions other than 4, we have $\boldsymbol{W}|_{\mathcal{Q}} = (n-4)\boldsymbol{R}|_{\mathcal{Q}}$. Note that $\boldsymbol{W}$ is well-defined in all dimensions and by construction is conformally invariant. Thus the equivalent tractor field W_{ABCE} is conformally invariant and of weight -2. In dimensions other than 4, it is immediate that this has Weyl tensor type symmetries. (Recall that $\boldsymbol{R}|_{\mathcal{Q}}$ is trace-free.) In fact, it has these symmetries in all dimensions and is a natural tractor field. In a choice of conformal scale, W_{ABCE} is given by

$$\begin{aligned}(28)\quad (n-4)\big(Z_A{}^a Z_B{}^b Z_C{}^c Z_E{}^e C_{abce} - 2Z_A{}^a Z_B{}^b X_{[C} Z_{E]}{}^e A_{eab} \\ - 2X_{[A} Z_{B]}{}^b Z_C{}^c Z_E{}^e A_{bce}\big) + 4X_{[A} Z_{B]}{}^b X_{[C} Z_{E]}{}^e B_{eb},\end{aligned}$$

where A_{abc} is the *Cotton tensor*,

$$A_{abc} := 2\nabla_{[b} P_{c]a}, \tag{29}$$

and

$$B_{ab} := \nabla^c A_{acb} + P^{dc} C_{dacb}. \tag{30}$$

Note that from (28) it follows that in dimension 4, B_{eb} is conformally invariant. This is the Bach tensor: from the contracted Bianchi identity, we have

$$(n-3)A_{abc} = \nabla^d C_{dabc}, \tag{31}$$

and so in dimension 4, (30) agrees with (1). In other dimensions $n \geq 3$ we also refer to B_{ab} as defined in (30) as the *Bach tensor*. The tractor field W first appeared in [Gover 1999, 2001]. The connection to the ambient curvature was derived in [Čap and Gover 2003], where the above results are treated in detail.

4. Conformal Laplacians and the ambient obstruction

In this section we show how one can obtain the ambient obstruction tensor by applying a conformally invariant operator $\not\Box_{n/2-2}$ of the form $\Delta^{n/2-2}$ + lots to the natural tractor field W defined above. For any integer $m \geq 1$, we let

$$\not\Box_m := \Box_m^{1/2},$$

where $\Box_m^{1/2}$ is the case $\alpha = 1/2$ of the operator $\Box_m^\alpha$ of Theorem 4.1. We prove Theorem 4.1 in Section 4A. The inductive nature of this proof will show that one can construct explicit tractor formulae for the operators $\Box_m^\alpha$ in terms of X, D, W, h, and h^{-1}. One may thus use Theorem 4.2 together with a choice of conformal scale and the formula for W given in (28) to construct a tractor formula for $\mathbb{O}_{ab}$. It is then easy to expand this tractor formula to a formula in terms of the Levi-Civita connection and its curvature.

In what follows, the phrase "generic n-even case" refers to the case in which n is even and M is conformally curved.

Theorem 4.1. *For every integer $m \geq 1$ and for every $\alpha \in \mathbb{R}$, there exists a conformally invariant operator $\Box_m^\alpha : \mathcal{T}^\Phi[m-n/2] \to \mathcal{T}^\Phi[-m-n/2]$ having leading term Δ^m. The operator $\Box_m^\alpha$ is natural in the following cases: in odd dimensions and for conformally flat M for all $m \geq 1$; in the generic n-even case for $1 \leq m \leq n/2-2$, if $\alpha = 0$ for $1 \leq m \leq n/2-1$, if $\mathcal{T}^\Phi[m-n/2] = \mathcal{T}[m-n/2]$ for $1 \leq m \leq n/2-1$, or if $\mathcal{T}^\Phi[m-n/2] = \mathcal{T}^0[m-n/2]$ for $1 \leq m \leq n/2$. In these cases there is a tractor formula for $\Box_m^\alpha$ given by a partial contraction polynomial in $\Box$, D, W, X, h, and h^{-1}, and this polynomial is linear in U. In the tractor formula for $\Box_m^\alpha U$, each free index appears either on U or on a W-tractor.*

We believe the operators $\not\Box_m$ will be important for many problems. For our current purposes, we are primarily interested in them when n is even, $m = n/2-2$, and the domain bundle is $\mathcal{T}^{2,2}[-2]$. In particular, we have the following result, which is an immediate consequence of Theorem 4.4.

Theorem 4.2. *Let M be a conformal manifold of dimension n even. Then*

$$\not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2} = K(n) X_{A_1} Z_{A_2}{}^{a} X_{B_1} Z_{B_2}{}^{b} \mathcal{O}_{ab}. \tag{32}$$

Here $K(n)$ is a known nonzero constant depending on n, and $\mathcal{O}_{ab} \in \mathcal{E}_{(ab)_0}[2-n]$ is the Fefferman–Graham obstruction tensor. It is conformally invariant and natural.

We have $K(4) = -8$. In dimensions at least 6, the constant $K(n)$ is given by $(n-4)k(n)$, where $k(n)$ is given in (38). Note that $\not{\Box}_{n/2-2} W \in \mathcal{T}^{2,2}[2-n]$. The theorem states that its components vanish in all factors of the composition series (11) for $\mathcal{T}^{2,2}[2-n]$, except for the (injecting) factor $\mathcal{E}_{1,1}[-2] = \mathcal{E}^{1,1}[2-n]$, and the term here is the obstruction, up to scale.

Theorem 4.3. *The obstruction tensor $\mathcal{O}_{ab}$ vanishes on conformally Einstein manifolds.*

Proof. A conformally Einstein manifold M admits a parallel standard tractor $\mathbb{I}$ (see [Gover and Nurowski 2006]) such that $\sigma := \mathbb{I}^A X_A \neq 0$ is an Einstein scale. It follows immediately that $\mathbb{I}$ annihilates the tractor curvature $\Omega_{bc}{}^{D}{}_{E}$:

$$\nabla_c \mathbb{I}^D = 0 \implies \Omega_{bc}{}^{D}{}_{E} \mathbb{I}^E = [\nabla_b, \nabla_c] \mathbb{I}^D = 0.$$

Also since $\mathbb{I}$ is parallel, viewing it as a multiplication operator, it is clear that $[D, \mathbb{I}] = 0$. From (27) (see also [Čap and Gover 2003]) we have $W_{A_1 A_2}{}^{D}{}_{E} = (3/(n-2)) D^{A_0} X_{A_0} Z_{A_1}{}^{b} Z_{A_2}{}^{c} \Omega_{bc}{}^{D}{}_{E}$. Thus $W_{BCDE} \mathbb{I}^E = 0$.

By Theorem 4.1 there is a formula for $\not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2}$ which is polynomial in $\Box$, D, W, X, h, and h^{-1}, and in this formula each of the indices A_1, A_2, B_1, and B_2 appears on a W tractor. On the other hand, since $\mathbb{I}$ is parallel and of weight 0, it commutes with the operators in this expression for $\not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2}$. Thus

$$\mathbb{I}^{B_1} \not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2} = 0, \tag{33}$$

since $\mathbb{I}^A W_{ABCD} = 0$.

From [Gover and Nurowski 2006] we have $\mathbb{I}^A = (1/n)\, D^A \sigma$. Thus from the expression (16) for the tractor-D operator, we have the expression

$$[\mathbb{I}^A]_g = \sigma Y^A - (1/n) J \sigma X^A$$

for $\mathbb{I}^A$ in terms of the (Einstein) metric $g := \sigma^{-2} \boldsymbol{g}$. (Recall that if ∇ is the Levi-Civita connection determined by $g = \sigma^{-2}\boldsymbol{g}$, then tautologically $\nabla \sigma = 0$.) In particular, in this scale, we have $\mathbb{I}^A Z_A{}^a = 0$. Thus from Theorem 4.2 above,

$$4(K(n))^{-1} Z^{A_2}{}_{a} Z^{B_2}{}_{b} \mathbb{I}^{A_1} \mathbb{I}^{B_1} \not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2} = \sigma^2 \mathcal{O}_{ab}.$$

But from (33), the left-hand side vanishes, and hence $\mathcal{O}_{ab} = 0$ on M. □

Obtaining the obstruction tensor via a conformally invariant operator on a tractor field as in Theorem 4.2 enables us to relate it to other conformally invariant operators associated with the deformation complex, by ideas along the lines of the *curved translation principle* of Eastwood [1996] and collaborators. This is the idea behind Theorem 2.3, which we are now ready to prove. Related generalisations of the curved translation principle have been explored in depth in the setting of operators on differential forms [Branson and Gover 2005].

Proof of Theorem 2.3. We first construct $\mathbb{B}$ and prove (iii). Let $W^{2,2}$ denote the quotient of $\mathbb{T}^{2,2}[-2]$ by the subbundle which is the kernel of the bundle map $\mathbb{T}^{2,2}[-2] \to \mathbb{T}^3 \otimes \mathbb{T}^3$ given by

$$U_{A_2A_3B_2B_3} \mapsto X_{A_1}X_{B_1}U_{A_2A_3B_2B_3}.$$

We write $W_{2,2}$ for the subbundle of $\mathbb{T}^{2,2}[2-n]$ consisting of tractors which are annihilated by any contraction with X, and write $\mathcal{W}^{2,2}$ and $\mathcal{W}_{2,2}$ for the section spaces of $W^{2,2}$ and $W_{2,2}$, respectively. Note that complete contractions between elements of $\mathbb{T}^{2,2}[-2]$ and sections of $\mathbb{T}^{2,2}[2-n]$ take values in $E[-n]$. Hence there is a conformally invariant pairing between $\mathcal{T}^{2,2}[-2]$ and $\mathcal{T}^{2,2}[2-n]$. It is clear that the contractions between elements of $\mathbb{T}^{2,2}[-2]$ and sections of $\mathbb{T}^{2,2}[2-n]$ induce a well-defined bundle map

$$\langle\,\cdot\,,\cdot\,\rangle : W^{2,2} \otimes W_{2,2} \to E[-n],$$

and so there is also a conformally invariant pairing between $\mathcal{W}^{2,2}$ and $\mathcal{W}_{2,2}$.

Given a section $U_{ABCD} \in \mathcal{T}^{2,2}[-2]$, let $[U_{ABCD}]$ be its image in the quotient space $\mathcal{W}^{2,2}$. From the tractor composition series (6) (see also (10) and the discussion there), it follows easily that the space $\mathcal{W}^{2,2}$ has a composition series

$$\mathcal{E}^{1,1}[2] \oplus\!\!\!\!+ \mathcal{E}^{2,1}[2] \oplus\!\!\!\!+ \mathcal{E}^{2,2}[2]$$

and that the injection $\mathrm{I} : \mathcal{E}^{2,2}[2] \to \mathcal{W}^{2,2}$ is given by

$$u_{abcd} \mapsto [Z_A{}^a Z_B{}^b Z_C{}^c Z_D{}^d u_{abcd}].$$

The differential operator $\mathbb{D} : \mathcal{W}^{2,2} \to \mathcal{T}^{2,2}[-2]$ given by

$$[U_{A_2A_3B_2B_3}] \mapsto \frac{9}{n(n-2)} \mathbb{Y}_{2,2} D^{A_1} D^{B_1} X_{A_1} X_{B_1} U_{A_2A_3B_2B_3}$$

is clearly well-defined and conformally invariant. Here $\mathbb{Y}_{2,2}$ is the bundle map executing the projection of $\mathbb{T}^2[-1] \otimes \mathbb{T}^2[-1]$ onto the direct summand $\mathbb{T}^{2,2}[-2]$. We write $\mathbb{D}^*$ for the formal adjoint of $\mathbb{D}$. This is a conformally invariant operator

$$\mathbb{D}^* : \mathcal{T}^{2,2}[2-n] \to \mathcal{W}_{2,2}.$$

On the other hand, from Theorem 4.1 there is a conformally invariant Laplacian-type operator $\not\Box_{n/2-2} : \mathcal{T}^{2,2}[-2] \to \mathcal{T}^{2,2}[2-n]$. Thus we have the composition

$$\mathbb{D}^* \not\Box_{n/2-2} \mathbb{D} : \mathcal{W}^{2,2} \to \mathcal{W}_{2,2}.$$

The operator $\mathbb{B}$ in the theorem is (up to a constant multiple) simply the composition

$$(\mathbb{D}^* \not\Box_{n/2-2} \mathbb{D}\mathrm{I} =: \mathbb{B}) : \mathcal{E}^{2,2}[2] \to \mathcal{W}_{2,2}.$$

By construction this is natural and conformally invariant.

Now in a conformal scale, $(\mathbb{D}\mathrm{I}(u))_{BCEF}$ is given explicitly by

$$\begin{aligned}(34)\quad (n-4)\big((n-3) Z_B{}^b Z_C{}^c Z_E{}^e Z_F{}^f u_{bcef} - 2 Z_B{}^b Z_C{}^c X_{[E} Z_{F]}{}^f \nabla^e u_{efbc} \\ - 2 X_{[B} Z_{C]}{}^c Z_E{}^e Z_F{}^f \nabla^b u_{bcef}\big) \\ + 4 X_{[B} Z_{C]}{}^c X_{[E} Z_{F]}{}^f (\nabla^{(b} \nabla^{e)} u_{bcef} + (n-3) P^{be} u_{bcef}).\end{aligned}$$

Thus from (28) and a minor calculation, $\mathbb{D}(\mathrm{I}(C))_{ABCD} = (n-3) W_{ABCD}$, where C is the Weyl curvature. So by Theorem 4.2, we have

$$(\not\Box_{n/2-2} \mathbb{D}\mathrm{I}C)_{A_2 A_3 B_2 B_3} = (n-3) K(n) X_{A_2} Z_{A_3}{}^a X_{B_2} Z_{B_3}{}^b \mathcal{O}_{ab}.$$

That is, $\not\Box_{n/2-2}\mathbb{D}\mathrm{I}C$ takes values in the factor $\mathcal{E}_{1,1}[-2]$ in the composition series for $\mathcal{T}^{2,2}[2-n]$. (Note that this factor is a conformally invariant subspace.) Now the formal adjoint of the tractor-D operator is again the tractor-D operator [Branson and Gover 2001]. So

$$\mathbb{D}^* X_{A_2} Z_{A_3}{}^a X_{B_2} Z_{B_3}{}^b \mathcal{O}_{ab} = \frac{9}{n(n-2)} X^{B_1} X^{A_1} D_{B_1} D_{A_1} X_{A_2} Z_{A_3}{}^a X_{B_2} Z_{B_3}{}^b \mathcal{O}_{ab}.$$

But a short calculation using (9) and (16) shows that this operation just returns $4(n-4)(n-3) X_{A_2} Z_{A_3}{}^a X_{B_2} Z_{B_3}{}^b \mathcal{O}_{ab}$, and this proves part (iii) of the theorem. All nonvanishing multiples can be absorbed into the definition of $\mathbb{B}$.

We treat now part (i). We need to show that $\mathrm{I}^*\mathbb{B}$ has order $n-4$ and is nontrivial. Since by construction there is a universal natural expression for the operator L, it is sufficient to establish this on the standard conformal sphere. Recall that $\not\Box_{n/2-2}$ has leading term $\Delta^{n/2-2}$. Thus $\not\Box_{n/2-2}$ is elliptic, since the sphere has Riemannian signature. From (34) it is clear that $\mathbb{D}\mathrm{I} : \mathcal{E}^{2,2}[2] \to \mathcal{T}^{2,2}[-2]$ is a differential splitting operator; there is a bundle homomorphism $\mathrm{J} : \mathbb{T}^{2,2}[-2] \to E^{2,2}[2]$ such that $\mathrm{J}\mathbb{D}\mathrm{I}$ is the identity on $\mathcal{E}^{2,2}[2]$. Thus on any manifold, $\mathcal{R}(\mathbb{D}\mathrm{I} : \mathcal{E}^{2,2}[2] \to \mathcal{T}^{2,2}[-2])$ is infinite-dimensional, and it follows immediately that $\not\Box_{n/2-2}\mathbb{D}\mathrm{I}$ is nontrivial on the standard conformal sphere. The action of $\not\Box_{n/2-2}\mathbb{D}\mathrm{I}$ on $\mathcal{E}^{2,2}[2]$ takes values in $\mathcal{T}^{2,2}[2-n]$. The composition series for $\mathcal{T}^{2,2}[2-n]$ is given by (11) with $w = 2-n$. From this we see, for example, that there is a canonical projection from $\mathcal{T}^{2,2}[2-n]$ to $\mathcal{E}^{1,1}[6-n] = \mathcal{E}_{1,1}[2]$ with which one can compose the operator

$\not\Box_{n/2-2}\mathbb{D}\mathrm{I} : \mathcal{E}^{2,2}[2] \to \mathcal{T}^{2,2}[2-n]$. By construction, this is a conformally invariant operator $\mathcal{E}^{2,2}[2] \to \mathcal{E}_{1,1}[2]$. On the other hand, from the classification of operators on conformally flat structures discussed in Section 2, the only conformally invariant operators on $\mathcal{E}^{2,2}[2]$ taking values in irreducible bundles are as follows: there is an operator $\mathcal{E}^{2,2}[2] \to \mathcal{E}^{3,2}[2]$ and an operator $\mathcal{E}^{2,2}[2] \to \mathcal{E}_{2,2}[-2]$. From elementary weight considerations, we know the latter has order $n-4$. Thus the composition described must be trivial. Continuing in this fashion and using (8), one concludes that $\not\Box_{n/2-2}\mathbb{D}\mathrm{I}$ takes values in the subspace $\mathcal{W}_{2,2} = \mathcal{E}_{2,2}[-2] \oplus \mathcal{E}_{2,1}[-2] \oplus \mathcal{E}_{1,1}[-2]$, and the composition of $\not\Box_{n/2-2}\mathbb{D}\mathrm{I}$ with projection to $\mathcal{E}_{2,2}[-2]$ is necessarily nontrivial. This composition is thus up to scale the unique conformally invariant operator between these bundles (on the conformal sphere). To finish, note that on the one hand, $\mathrm{I}^*\mathbb{D}^*$ is the formal adjoint of a splitting operator for $\mathcal{E}^{2,2}[2]$ and therefore acts as a multiple of the identity on the component $\mathcal{E}_{2,2}[-2]$. On the other hand, $\mathrm{I}^*\mathbb{D}^*$ must annihilate the components $\mathcal{E}_{2,1}[-2]$ and $\mathcal{E}_{1,1}[-2]$, since these have higher total order than the target bundle $\mathcal{E}_{2,2}[-2]$ for the composition and a natural differential operator cannot lower order.

Finally, we consider (ii). Let us first consider the case of a flat Riemannian or pseudo-Riemannian structure. Thus all curvature will vanish, until we note otherwise. If $F_{a_1a_2}$ is a 2-form, then $\Delta^{n/2-2}D^{A_0}X_{A_0}Z_{A_1}{}^{a_1}Z_{A_2}{}^{a_2}F_{a_1a_2}$ is well understood as a special case of [Branson and Gover 2005, Proposition 4.6]. The nonzero components of this have values in a subbundle of $\mathcal{T}^2[2-n]$ with composition series $\mathcal{E}_2 \oplus \mathcal{E}_1$. These components are (up to an overall nonzero constant multiple $((\delta d)^{n/2-2}F,\ a\delta(d\delta)^{n/2-2}F)$, where d is the exterior derivative, δ its formal adjoint, and a is a nonzero constant. Composing these components on the left with $(\delta d\ ,\ (1/a)\,d)$ yields $(\delta d + d\delta)^{n/2-1}F = (-1)^{n/2-1}\Delta^{n/2-1}F$. Now on flat structures we have the identity

$$(n-2)\big(\mathbb{D}\mathrm{I}(u)\big)_{A_1A_2B_1B_2} = 3D^{A_0}X_{A_0}Z_{A_1}{}^{a_1}Z_{A_2}{}^{a_2}U_{a_1a_2B_1B_2},$$

where $u \in \mathcal{E}^{2,2}[2]$ and $U_{a_1a_2B_1B_2}$ is the conformally invariant form-tractor given in scale by letting w equal 2 in the formula

$$U_{a_1a_2B_1B_2} = (n+w-5)Z_{B_1}{}^{b_1}Z_{B_2}{}^{b_2}u_{a_1a_2b_1b_2} + 2X_{B_1}Z_{B_2}{}^{b_2}\nabla^{b_1}u_{a_1a_2b_2b_1}. \tag{35}$$

Thus by viewing U as a 2-form with values in a tractor bundle and replacing ∇, d, and δ with their tractor connection coupled variants in the argument above, we conclude that there is an operator $\overline{\mathbb{A}}$ such that

$$\overline{\mathbb{A}}\Delta^{n/2-2}\mathbb{D}\mathrm{I}(u) = \frac{3}{n-2}\,\overline{\mathbb{A}}\Delta^{n/2-2}D^{A_0}X_{A_0}Z_{A_1}{}^{a_1}Z_{A_2}{}^{a_2}U_{a_1a_2B_1B_2} = \Delta^{n/2-1}U.$$

We continue with similar considerations, except that now we view $u^{a_1}{}_{a_2b_1b_2}$ as a 2-form on the b_1b_2 index pair that takes values in $\mathrm{End}(TM)$. If F now indicates a

2-form of weight w', we have

$$\begin{aligned}\mathbb{K}(F) &:= (3/(n+2w'-2))\, D^{A_0} X_{A_0} Z_{A_1}{}^{a_1} Z_{A_2}{}^{a_2} F_{a_1a_2} \\ &= (n+w'-4) Z_{A_1}{}^{a_1} Z_{A_2}{}^{a_2} F_{a_1a_2} + 2X_{A_1} Z_{A_2}{}^{a_2} \nabla^{a_1} F_{a_2a_1}.\end{aligned}$$

So if $w' = 1$ in particular, then the formula on the right-hand side agrees with (35) with $w = 2$. In formally calculating $\Delta^{n/2-1} U^{a_1}{}_{a_2B_1B_2}$ using the identities (9) and the Leibniz rule to obtain a formula polynomial in u, ∇, the metric g, its inverse, and the projectors X, Y, and Z, we may ignore the a_1 and a_2. Their contribution is buried in the meaning of the Levi-Civita connection ∇. Now for a 2-form F of weight 1, we have that on flat structures, $\Delta^{n/2-1}\mathbb{K}(F)$ takes values in $\mathcal{E}_2[-1] \oplus \mathcal{E}_1[-1]$ and has the form $\big(((3-n)(\delta d)^{n/2-1} + (d\delta)^{n/2-1})F,\, *\big)$ up to an overall nonzero multiple [Branson and Gover 2005]. Here $*$ indicates some term, the details of which will not concern us. We note that the first entry gives an elliptic operator on F; we may act on this with the operator $\delta d + (3-n)d\delta$ to yield $(3-n)(-1)^{n/2}\Delta^{n/2}F$. Thus there is a linear differential operator $\overline{\mathbb{A}}_2$ such that $\overline{\mathbb{A}}_2\Delta^{n/2-1}U = \Delta^{n/2}u$.

Combining these observations, we see that there is a linear differential operator $\overline{\mathbb{A}}_3$ such that $\overline{\mathbb{A}}_3\Delta^{n/2-2}\mathbb{D}\mathrm{I}(u) = \Delta^{n/2}u$. Finally, one can easily verify directly that $\mathbb{D}^*$ is differentially invertible as a graded differential operator on the subspace $\mathcal{W}_{2,2}$. (That is, its inverse is also a graded differential operator. The point is that in terms of a splitting of $\mathcal{W}_{2,2}$ determined by a choice of conformal scale, a straightforward calculation shows that $\mathbb{D}^*$ takes (u, v, w) to $(ku, \ell v + \delta\cdot u, mw + \delta\cdot v + \delta\cdot\delta\cdot u)$, where k, ℓ, and m are nonzero integers, $\delta\cdot$ indicates a divergence operator, and $\delta\cdot\delta\cdot$ a double divergence operator.) Thus with $\bar{\mathbb{B}}$ defined to be the necessary multiple of $\overline{\mathbb{A}}_3(\mathbb{D}^*)^{-1}$, we have (ii) for flat structures. But now the result follows in general, since moving to curved structures yields the same formal calculation, except that at each stage the differential operators concerned may have additional lower order terms involving curvature. It is easily checked that these terms can only yield terms of order lower than n in the final calculation of $\bar{\mathbb{B}}\mathbb{B}$. □

Proof of Proposition 2.2. We treat L first. We already have $\mathsf{L} = \mathsf{Bi}^*\mathsf{Bi}$ in dimension 6, and so we shall assume that $n \geq 8$. Let us denote by

$$\mathbb{U} : \mathcal{E}^{2,2}[2] \to \mathcal{E}^2 \otimes \mathcal{T}^2$$

the conformally invariant operator given by (35). We write d^∇ for the tractor connection coupled exterior derivative and δ^∇ for its formal adjoint. Thus for example for $U \in \mathcal{E}^2 \otimes \mathcal{T}^2$ we have $(d^\nabla U)_{a_0a_1a_2B_1B_2} = 3\nabla_{a_0} U_{a_1a_2B_1B_2}$. It is straightforward using (9) to verify that on conformally flat structures, the composition $d^\nabla\mathbb{U}$ can be reexpressed in the form $\mathbb{M}\mathsf{Bi}$, where $\mathbb{M} : \mathcal{E}^{3,2}[2] \to \mathcal{E}^3 \otimes \mathcal{T}$ is a conformally invariant first-order differential splitting operator.

There are conformally invariant formally self-adjoint operators $L_k : \mathcal{E}^k \to \mathcal{E}_k$, $0 \le k \le n$, with leading term $(\delta d)^k$. These are the "long operators" for the de Rham complex given in [Branson and Gover 2005]. It is shown there that there are natural linear differential operators Q_{k+1} such that $L_k = \delta Q_{k+1} d$.

Now suppose we are on a contractible (conformally flat) manifold. This suffices for our present purposes. Then the tractor bundle is flat and trivial. It follows that there are conformally invariant and formally self-adjoint tractor-coupled variants of the L_k,

$$L_k^{\nabla} : \mathcal{E}^k \otimes \mathcal{T}^2 \to \mathcal{E}_k \otimes \mathcal{T}^2.$$

These are obtained by starting with the natural formulae for $L_k = \delta Q_{k+1} d$ and formally replacing each instance of d, δ, and the Levi-Civita connection with, respectively, d^{∇}, δ^{∇}, and the Levi-Civita tractor-coupled connection. By construction the result has a factorisation $L_k^{\nabla} = \delta^{\nabla} Q_{k+1}^{\nabla} d^{\nabla}$ for some differential operator Q_{k+1}^{∇}.

Observe that by composition, we have a formally self-adjoint conformally invariant operator $\mathbb{U}^* L_2^{\nabla} \mathbb{U} : \mathcal{E}^{2,2}[2] \to \mathcal{E}_{2,2}[-2]$, where $\mathbb{U}^*$ is the formal adjoint of $\mathbb{U}$. We will reexpress this. By taking formal adjoints, we have $\mathbb{U}^* \delta^{\nabla} = \mathsf{Bi}^* \mathbb{M}^*$ from $d^{\nabla} \mathbb{U} = \mathbb{M} \mathsf{Bi}$. Thus we obtain an operator

$$\mathsf{Bi}^* \mathbb{M}^* Q_3^{\nabla} \mathbb{M} \mathsf{Bi} : \mathcal{E}^{2,2}[2] \to \mathcal{E}_{2,2}[-2].$$

The result follows from the uniqueness of L, provided the displayed operator is nontrivial. It is clearly sufficient to establish this for Riemannian signature structures and at a flat metric within the conformal class. We use the alternative expression $\mathbb{U}^* \delta^{\nabla} Q_3^{\nabla} d^{\nabla} \mathbb{U} = \mathsf{Bi}^* \mathbb{M}^* Q_3^{\nabla} \mathbb{M} \mathsf{Bi}$. On flat structures, $Q_3 = (d\delta)^{n/2-3}$, and so $Q_3^{\nabla} = (d^{\nabla} \delta^{\nabla})^{n/2-3}$. It follows that for $u \in \mathcal{E}^{2,2}[2]$ of compact support, $\delta^{\nabla} (d^{\nabla} \delta^{\nabla})^{n/2-3} d^{\nabla} \mathbb{U}(u)$ vanishes if and only if $d^{\nabla} \mathbb{U}(u)$ vanishes, since the tractor connection is flat. (Suppose $d^{\nabla} \mathbb{U}(u) \neq 0$. Then there exists a parallel $T \in \mathcal{T}^2$ such that $T^{B_1 B_2} (d^{\nabla} \mathbb{U}(u))_{a_0 a_1 a_2 B_1 B_2} \neq 0$. In other words, if $f_{a_1 a_2} := T^{B_1 B_2} \mathbb{U}(u)_{a_1 a_2 B_1 B_2}$, then $df \neq 0$. But on the other hand, if $0 = T^{B_1 B_2} (\delta^{\nabla} (d^{\nabla} \delta^{\nabla})^{n/2-3} d^{\nabla} \mathbb{U}(u))_{a_0 a_1 a_2 B_1 B_2}$, then $(\delta d)^{n/2-2} f = 0$ which implies $df = 0$.) This is equivalent to $\mathbb{M} \mathsf{Bi}(u)$ vanishing. Since $\mathbb{M}$ is a differential splitting operator, this in turn is equivalent to $\mathsf{Bi}(u) = 0$. Thus the composition $\delta^{\nabla} Q_3^{\nabla} d^{\nabla} \mathbb{U} : \mathcal{E}^{2,2}[2] \to \mathcal{E}_2 \otimes \mathcal{T}^2$ is nontrivial. Now it is easily verified that $\mathcal{E}_{2,2}[-2]$ turns up with multiplicity 1 in the composition series for $\mathcal{E}_2 \otimes \mathcal{T}^2$. It follows by an exact analogue of the argument used on page 331 that $\delta^{\nabla} Q_3^{\nabla} d^{\nabla} \mathbb{U}$ only takes values in $\mathcal{E}_{2,2}[-2]$ and composition factors of higher total order. Thus on the range of this operator, $\mathbb{U}^*$ acts as a nonzero multiple of the projection to the component $\mathcal{E}_{2,2}[-2]$. (Recall that $\mathbb{U}^*$ is the formal adjoint of a differential splitting operator $\mathbb{U} : \mathcal{E}^{2,2}[2] \to \mathcal{E}^2 \otimes \mathcal{T}^2$, and so it must act as a nonzero multiple of the identity on the component $\mathcal{E}_{2,2}[-2]$. On the other hand, it is differential, so it cannot lower total order.)

Now we consider the situation for B. We require a conformally invariant differential splitting operator $\Gamma : \mathcal{E}^{1,1}[2] \to \mathcal{E}^1 \otimes \mathcal{T}^2$ that will in this case play a role analogous to $\mathbb{U}$ above. This is easily constructed explicitly and directly, and can be obtained from a composition of the related operators in Section 5.1 of [Branson and Gover 2002], so we omit the details. Since Γ has values in a weight zero adjoint tractor-valued bundle of 1-forms it is clear that the composition $d^\nabla\Gamma$ is conformally invariant. This is easily verified nontrivial. On the other hand, in terms of a metric g, the tractor curvature is given by

$$Z_{B_1}{}^{b_1} Z_{B_2}{}^{b_2} C_{a_1a_2b_1b_2} + \frac{2}{n-3} X_{B_1} Z_{B_2}{}^{b_2} \nabla^{b_1} C_{a_1a_2b_2b_1}.$$

Thus the linearisation, at a conformally flat metric g_0, of the tractor curvature is $(1/(n-3))\mathbb{U}\mathsf{C}$. This is manifestly nontrivial, and so via arguments used several times already concerning the uniqueness of irreducible conformally invariant operators, it is straightforward to verify that this operator must agree with $d^\nabla\Gamma$ (on conformally flat structures), at least up to scale. We set the scale of Γ so that $d^\nabla\Gamma = (1/(n-3))\mathbb{U}\mathsf{C}$. On flat manifolds, $Q_2 = (d\delta)^{n/2-2}$, and so by almost the same argument as for L, we conclude that on conformally flat manifolds, the formally self-adjoint conformally invariant operator $\mathsf{C}^*\mathbb{U}^* Q_2^\nabla \mathbb{U}\mathsf{C}$ is nontrivial. $\square$

The next theorem shows that for n even, if the ambient curvature is formally Ricci-flat to $O(Q^{n/2-1})$, then a tensor part of the coefficient of $Q^{n/2-1}$ is a natural conformal invariant of the underlying manifold and so is an obstruction to finding an ambient metric which is Ricci-flat to higher order. For our purposes, the main point is that this is achieved by Theorem 4.4(iii), which recovers this obstruction via a tangential operator acting on the ambient curvature.

Theorem 4.4. *For a conformal manifold M of even dimension n, let $\boldsymbol{h}$ be an associated ambient metric satisfying* $\mathrm{Ric}(\boldsymbol{h}) = Q^{n/2-1}\boldsymbol{B}$. *Then we have*

(i) $\boldsymbol{B}|_{\mathcal{Q}}$ *is equivalent to a tractor* $B_{AB} \in \mathcal{E}_{(AB)_0}[-n]$ *such that* $X^A B_{AB} = 0$.

(ii) *The weighted tensor* $Z^A{}_a Z^B{}_b B_{AB} =: \mathcal{O}_{ab}$ *is a section of* $\mathcal{E}_{(ab)_0}[2-n]$.

(iii) *For* $n \geq 6$, *we have*

$$\not{\Delta}^{n/2-2} \boldsymbol{R}_{A_1A_2B_1B_2} = k(n) \boldsymbol{X}_{A_1} \boldsymbol{X}_{B_1} \boldsymbol{B}_{A_2B_2} + O(Q),$$

where $k(n)$ is the dimension dependent nonzero constant given above. In dimension 4,

$$3\boldsymbol{D}^{A_0} \boldsymbol{X}_{A_0} \boldsymbol{R}_{A_1A_2B_1B_2} = 16 \boldsymbol{X}_{A_2} \boldsymbol{X}_{B_1} \boldsymbol{B}_{A_1B_2} + O(Q).$$

(iv) *The tensor* $\mathcal{O}_{ab}$ *is divergence-free.*

(v) *The weighted tensor* $\mathcal{O}_{ab}$ *is a nontrivial natural conformal invariant of the form* $\Delta^{n/2-2}\nabla^c\nabla^d C_{cadb} + \text{lots} = (n-3)\Delta^{n/2-2}(\Delta P_{ab} - \nabla_a\nabla_b J) + \text{lots}$ *(up* *to a constant multiple*), *and so is an obstruction to finding an ambient metric which is Ricci-flat modulo* $O(Q^{n/2})$.

Remarks. 1. The statement of the theorem up to the definition of $\mathcal{O}_{ab}$ in (ii) is a characterisation of the Fefferman–Graham obstruction tensor (Graham, private communication; see also [Fefferman and Graham, in progress]). This gives a complete obstruction to the ambient metric in the sense that if this vanishes, then the ambient construction may be continued to all orders [Fefferman and Graham 1985]. Hence $\mathcal{O}_{ab}$ is the usual obstruction tensor, as claimed in Theorem 4.2. Thus part (iii), above, gives a new ambient formula for the Fefferman–Graham obstruction tensor.

2. From (25) it follows easily that $\mathcal{O}_{ab}$ may be equally viewed as an obstruction to obtaining an ambient metric which is harmonic for $\not{\boldsymbol{\Delta}}$ in the sense that $\not{\boldsymbol{\Delta}}\boldsymbol{R}$ vanishes to all orders. See also the remark on page 325.

3. It should be pointed out that

$$\boldsymbol{\Delta}^{n/2-3}\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2} = k(n)\boldsymbol{X}_{A_1}\boldsymbol{X}_{B_1}\boldsymbol{B}_{A_2B_2} + O(Q) \tag{36}$$

is an alternative ambient formula for the obstruction, and we could replace the $\boldsymbol{\Delta}^{n/2-3}$ by $\boldsymbol{\Delta}_{\alpha}^{n/2-3}$ in this formula.

Proof of Theorems 4.2 and 4.4. As above, we write **Ric** for Ric($\boldsymbol{h}$). It is immediate that $\boldsymbol{B}$ is symmetric and homogeneous of weight $-n$. Also from (5) it follows that $\boldsymbol{X}^A\boldsymbol{B}_{AB} = 0$. So $\boldsymbol{B}|_{\mathcal{Q}}$ is equivalent to a tractor field $B_{AB} \in \mathcal{T}_{(AB)}[-n]$ satisfying $X^A B_{AB} = 0$. From this last equality and (8), it is clear that $\mathcal{O}_{ab}$ is conformally invariant, while from the weight and symmetry of B_{AB}, it follows that $\mathcal{O}_{ab} \in \mathcal{E}_{(ab)}[2-n]$. For parts (i) and (ii), it remains to show that both B_{AB} and $\mathcal{O}_{ab}$ are trace-free.

First we consider the case $n \neq 4$. Note that since $\boldsymbol{\nabla}_A Q = 2\boldsymbol{X}_A$, we have

$$\boldsymbol{\nabla}_{A_1}\boldsymbol{\nabla}_{B_1}\mathbf{Ric}_{A_2B_2} = (n-2)(n-4)Q^{n/2-3}\boldsymbol{X}_{A_1}\boldsymbol{X}_{B_1}\boldsymbol{B}_{A_2B_2} + O(Q^{n/2-2}). \tag{37}$$

From (21) and (25) together with a short computation, it follows that

$$\not{\boldsymbol{\Delta}}^{n/2-2}\boldsymbol{R}_{A_1A_2B_1B_2} = k(n)\boldsymbol{X}_{A_1}\boldsymbol{X}_{B_1}\boldsymbol{B}_{A_2B_2} + O(Q),$$

as claimed in (iii), where

$$k(n) = (n-2)(n-4)(-1)^{n/2-3}2^{n-4}\big((n/2-3)!\big)^2. \tag{38}$$

(Note that (21) and (25) also give the alternative formula in Remark 3, above.)

Since $(n-4)\boldsymbol{R}|_{\mathcal{Q}}$ is equivalent to the tractor field W, it follows from Proposition 4.8 that $(n-4)\not{\boldsymbol{\Delta}}^{n/2-2}\boldsymbol{R}|_{\mathcal{Q}}$ descends to the natural tractor field $\not\Box_{n/2-2}W$. On the other hand, using $\delta_B{}^A = X^A Y_B + Y^A X_B + Z^A{}_a Z_B{}^b \delta_b{}^a$ and the fact that $X^A B_{AB} = 0$, we see that

$$X_{A_1} X_{B_1} B_{A_2 B_2} = X_{A_1} Z_{A_2}{}^a X_{B_1} Z_{B_2}{}^b \mathbb{O}_{ab}. \tag{39}$$

Therefore, $\boldsymbol{X}_{A_1} \boldsymbol{X}_{B_1} \boldsymbol{B}_{A_2 B_2}|_{\mathcal{Q}}$ is equivalent to the tractor field $X_{A_1} Z_{A_2}{}^a X_{B_1} Z_{B_2}{}^b \mathbb{O}_{ab}$. This establishes (32) of Theorem 4.2.

Since the left-hand side of (32) is natural, it follows that $X_{A_1} X_{B_1} B_{A_2 B_2}$ is natural. Hence $\mathbb{O}_{ab} = Z^{A_2}{}_a Z^{B_2}{}_b B_{A_2 B_2} = 4 Y^{A_1} Y^{B_1} Z^{A_2}{}_a Z^{B_2}{}_b X_{A_1} X_{B_1} B_{A_2 B_2}$ is likewise natural, as claimed in (v) and Theorem 4.2.

Next we show that B_{AB} and $\mathbb{O}_{ab}$ are trace-free. According to Theorem 4.1, the operators $\not{\Box}_m$ preserve tensor type. Since $W_{A_1 A_2 B_1 B_2}$ is trace-free, it follows that $\not{\Box}_{n/2-2} W_{A_1 A_2 B_1 B_2}$ is completely trace-free. Thus $h^{A_1 B_2} X_{A_1} X_{B_1} B_{B_2 A_2} = 0$, by (32) and (39). Since B_{CD} is symmetric and $X^A B_{AB} = 0$, it follows that $h^{AB} B_{AB} = 0$ as claimed. Now using (7) and again that $X^A B_{AB} = 0$, we see that $\boldsymbol{g}^{ab} \mathbb{O}_{ab} = 0$.

We must obtain the corresponding results in dimension 4. First observe that in any dimension,

$$3 \boldsymbol{D}^{A_0} \boldsymbol{X}_{A_0} \boldsymbol{R}_{A_1 A_2 B_1 B_2} = (n-2)\big((n-4) \boldsymbol{R}_{A_1 A_2 B_1 B_2} + 2 \boldsymbol{X}_{A_1} \nabla^C \boldsymbol{R}_{A_2 C B_1 B_2}\big) + O(Q),$$

by (15) and (18). From the contracted Bianchi identity, we have for $n = 4$,

$$3 \boldsymbol{D}^{A_0} \boldsymbol{X}_{A_0} \boldsymbol{R}_{A_1 A_2 B_1 B_2} = 8 \boldsymbol{X}_{A_2} \nabla_{B_1} \mathbf{Ric}_{A_1 B_2} + O(Q) = 16 \boldsymbol{X}_{A_2} \boldsymbol{X}_{B_1} \boldsymbol{B}_{A_1 B_2} + O(Q).$$

Relating W to the left-hand side via (27), we conclude that in dimension 4,

$$W_{A_1 A_2 B_1 B_2} = -8 X_{A_1} X_{B_1} B_{A_2 B_2}.$$

Comparing this with (28), we have $-2 X_{A_1} X_{B_1} B_{A_2 B_2} = X_{A_1} Z_{A_2}{}^a X_{B_1} Z_{B_2}{}^b B_{ab}$. Thus $\mathbb{O}_{ab}$ is a scalar multiple of the Bach tensor, $\mathbb{O}_{ab} = -\frac{1}{2} B_{ab}$, which is natural and trace-free, by (1). Also, since W is trace-free and $X^A B_{AB} = 0$, it follows that B_{AB} is trace-free.

It is well known (and easily verified) that the Bach tensor in dimension 4 is divergence-free. For (iv) we need the analogous result in other dimensions. First note that a short calculation using the formula (16) for the tractor-D operator and the identities (9) for the connection shows that

$$2 D^{A_1} X_{A_1} Z_{A_2}{}^a X_{B_1} Z_{B_2}{}^b \mathbb{O}_{ab} = (n-4) X_{A_2} X_{B_1} Z_{B_2}{}^b \nabla^a \mathbb{O}_{ab}.$$

So in dimensions other than 4, it follows that $D^{A_1} X_{A_1} X_{B_1} B_{A_2 B_2}$ and equivalently $(\boldsymbol{D}^{A_1} \boldsymbol{X}_{A_1} \boldsymbol{X}_{B_1} \boldsymbol{B}_{A_2 B_2})|_{\mathcal{Q}}$, vanish if and only if $\nabla^a \mathbb{O}_{ab} = 0$. On the ambient manifold, by (36), $\boldsymbol{D}^{A_1} \boldsymbol{X}_{A_1} \boldsymbol{X}_{B_1} \boldsymbol{B}_{A_2 B_2}$ is

$$\boldsymbol{D}^{A_1} \boldsymbol{\Delta}^{n/2-3} \not{\boldsymbol{\Delta}} \boldsymbol{R}_{A_1 A_2 B_1 B_2} + O(Q),$$

up to a nonzero multiple, since $\boldsymbol{D}$ acts tangentially. We ignore terms $O(Q)$ for much of the remainder of this calculation. The preceding display expands to

$$(4-n)\nabla^{A_1}\boldsymbol{\Delta}^{n/2-3}\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2} - \boldsymbol{\Delta}\boldsymbol{X}^{A_1}\boldsymbol{\Delta}^{n/2-3}\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2}.$$

From (5) and (24) we obtain

$$\begin{aligned}(40)\quad (4-n)[\nabla^{A_1},\boldsymbol{\Delta}]\boldsymbol{\Delta}^{n/2-4}\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2}& \\ +(6-n)\boldsymbol{\Delta}[\nabla^{A_1},\boldsymbol{\Delta}]\boldsymbol{\Delta}^{n/2-5}\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2}& \\ +\cdots-4\boldsymbol{\Delta}^{n/2-4}[\nabla^{A_1},\boldsymbol{\Delta}]\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2}& \\ -2\boldsymbol{\Delta}^{n/2-3}\big([\nabla^{A_1},\boldsymbol{\Delta}]\boldsymbol{R}_{A_1A_2B_1B_2}+\tfrac{1}{2}\nabla^{A_1}(\boldsymbol{R}\sharp\sharp\boldsymbol{R}_{A_1A_2B_1B_2})\big)&,\end{aligned}$$

after some reorganisation. It remains only to observe that all the terms in this sum are $O(Q)$. First we note that from (21) and (37), it is clear that

$$\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2} = KQ^{n/2-3}\boldsymbol{X}_{A_1}\boldsymbol{X}_{B_1}\boldsymbol{B}_{A_2B_2} + O(Q^{n/2-2}),$$

for some constant K. Thus by (25), each term $\boldsymbol{\Delta}^k[\nabla^{A_1},\boldsymbol{\Delta}]\boldsymbol{\Delta}^\ell\not{\boldsymbol{\Delta}}\boldsymbol{R}_{A_1A_2B_1B_2}$, for $k+\ell=n/2-4$, is some number times

$$(41)\qquad \boldsymbol{\Delta}^k[\nabla^{A_1},\boldsymbol{\Delta}]Q^{n/2-3-\ell}\boldsymbol{X}_{A_1}\boldsymbol{X}_{B_1}\boldsymbol{B}_{A_2B_2}+O(Q),$$

since $[\nabla^{A_1},\boldsymbol{\Delta}]$ is a first-order operator. Now consider the identity obtained from (26) by including the $O(Q^{n/2-2})$ terms which were omitted. From this identity, (5), and the fact that $\nabla Q = 2\boldsymbol{X}$, it follows that $[[\nabla^{A_1},\boldsymbol{\Delta}],Q]=0$ identically on the ambient manifold. Thus (41) is $O(Q)$.

Now consider the last term in (40). By direct calculation, we have

$$[\nabla^{A_1},\boldsymbol{\Delta}]\boldsymbol{R}_{A_1A_2B_1B_2} = -\tfrac{1}{2}\nabla^{A_1}(\boldsymbol{R}\sharp\sharp\boldsymbol{R}_{A_1A_2B_1B_2})+O(Q^{n/2-2}),$$

and so as required,

$$\boldsymbol{\Delta}^{n/2-3}\big([\nabla^{A_1},\boldsymbol{\Delta}]\boldsymbol{R}_{A_1A_2B_1B_2}+\tfrac{1}{2}\nabla^{A_1}(\boldsymbol{R}\sharp\sharp\boldsymbol{R}_{A_1A_2B_1B_2})\big)=O(Q).$$

Finally, we must show that in general $\mathbb{O}_{ab}$ is nontrivial. Up to scale, $\mathbb{O}_{ab}$ is given by $4Y^{A_1}Y^{B_1}Z^{A_2}{}_aZ^{B_2}{}_b\not{\Box}_{n/2-2}W_{A_1A_2B_1B_2}$. From (28) and (31), it is clear that $4Y^{A_1}Y^{B_1}Z^{A_2}{}_aZ^{B_2}{}_bW_{A_1A_2B_1B_2}$ is at leading order a nonzero multiple of $\nabla^d\nabla^cC_{cadb}$. Using that $\not{\Box}_{n/2-2}$ has leading term $\Delta^{n/2-2}$ and (9) to verify that the commutator of $\Delta^{n/2-2}$ with $4Y^{A_1}Y^{B_1}Z^{A_2}{}_aZ^{B_2}{}_b$ generates only lower order terms, we conclude that $\mathbb{O}_{ab}=\ell(n)\Delta^{n/2-2}\nabla^d\nabla^cC_{cadb}+\text{lots}$, where $\ell(n)$ is a nonzero constant. Given the form of the leading term, an elementary exercise shows that this natural tensor cannot vanish in general. $\square$

4A. ***Conformal Laplacian operators on tractor fields.*** It remains to prove Theorem 4.1. Our strategy is to first define the operators $\Box_m^\alpha$, which we do via powers of the ambient Laplacian $\boldsymbol{\Delta}_\alpha$ in Proposition 4.8, and then rewrite each such power as a combination of compositions of low order tangential operators, each of which has an immediate interpretation as an operator on a tractor bundle. This leads to a simple algorithm for rewriting any operator of this form in terms of basic tractor operators using only the existence of an ambient metric. Two of the key tools are Theorem 4.7, which explains how ambient derivatives of the ambient curvature can be reexpressed in terms of low order tangential operators, and Proposition 4.10, which describes harmonic extensions of tensor fields along $\mathcal{Q}$.

Almost all of the subsequent discussion concerns the ambient manifold $\tilde{M}$ with metric as discussed in Section 3. Occasionally we pause to interpret results on the underlying conformal manifold M.

In the generic n-even case, some identities, such as (22) and (26), hold to only finite order in Q. In many proofs, we will apply the operators $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}$ to both sides of an identity, and this will reduce the order to which the identity holds. Thus we must keep track of the number of times that we apply $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}$. In odd dimensions and in the conformally flat case, this is unnecessary since the identities hold to all orders. For simplicity, many of the proofs that follow explicitly treat only the generic n-even case. The proofs in the other cases are essentially the same, except that they do not require the operator counts. In addition, we have stated some of the results themselves in the generic n-even case only. All results hold as stated. Propositions 4.5 and 4.6, Theorem 4.7, and Lemma 4.11 also hold in general; they hold to all orders in both the odd-dimensional case and the conformally flat case, where the upper bounds stated in the *hypotheses* of the results no longer apply.

We will often use abbreviated notations. We may abbreviate (26) by writing $[\boldsymbol{\Delta}, \boldsymbol{\nabla}]V = \sum \boldsymbol{R}\boldsymbol{\nabla} V$. It is easily verified that (26) generalises to

$$[\boldsymbol{\Delta}_\alpha, \boldsymbol{\nabla}]V = \sum \boldsymbol{R}\boldsymbol{\nabla} V + \alpha \sum (\boldsymbol{\nabla}\boldsymbol{R})V, \tag{42}$$

which also holds modulo $O(Q^{n/2-2})$ in even dimensions and to infinite order in odd dimensions. For example, let V be any symmetric ambient 2-tensor. In this case (42) stands for

$$[\boldsymbol{\Delta}_\alpha, \boldsymbol{\nabla}_A]V_{BC} = 2(\alpha-1)\boldsymbol{R}_A{}^P{}_B{}^Q\boldsymbol{\nabla}_P V_{QC} \\ + 2(\alpha-1)\boldsymbol{R}_A{}^P{}_C{}^Q\boldsymbol{\nabla}_P V_{BQ} - 2\alpha(\boldsymbol{\nabla}_A\boldsymbol{R}_B{}^P{}_C{}^Q)V_{PQ},$$

which holds to the appropriate order. If the V on the left-hand side of (42) has any free indices, then in every term of the right-hand side of (42), each such index either remains attached to V in its original position or moves onto an $\boldsymbol{R}$. Some of the proofs in Section 4 will use this fact, which follows immediately from (26) and the definition of $\boldsymbol{R}\sharp\sharp$. The expressions we treat will often involve iterations of operators. To indicate how many operators we are composing in such an iteration, we will use exponents. For example, we might indicate $\nabla_A\nabla_B\boldsymbol{R}_{CDEF}$ by writing $\nabla^2\boldsymbol{R}$. We will often use the symbol $\mathcal{P}$ to denote a partial contraction polynomial. The same $\mathcal{P}$ may denote different polynomials in different parts of a discussion.

We often use the identities (13) and $\nabla Q = 2\boldsymbol{X}$ without explicit mention.

The proof of Theorem 4.1 begins with the development of a useful ambient calculus. This involves a sequence of results.

Proposition 4.5. *Suppose that n is even and M is generic. Let an integer ℓ be given, and suppose that $0 \le \ell \le n/2-4$. Then on the ambient manifold,*

$$\boldsymbol{\Delta}\nabla^{\ell}\boldsymbol{R} = \sum(\nabla^p\boldsymbol{R})(\nabla^q\boldsymbol{R}) + O(Q^{n/2-3-\ell}), \tag{43}$$

where $p+q=\ell$. If the $\boldsymbol{R}$ on the left-hand side of (43) *has any free indices, then for every term in the summation, these indices appear on an $\boldsymbol{R}$* (*as opposed to a* ∇).

Proof. We use induction. The case $\ell = 0$ follows from (22). Suppose next that $0 \le m \le n/2-5$ and that the result holds for $\ell = m$. From this assumption and (26), we have

$$\begin{aligned}\boldsymbol{\Delta}\nabla^{m+1}\boldsymbol{R} &= \nabla\boldsymbol{\Delta}\nabla^m\boldsymbol{R} + \sum\boldsymbol{R}(\nabla^{m+1}\boldsymbol{R}) + O(Q^{n/2-2})\\ &= \nabla\Big(\sum(\nabla^p\boldsymbol{R})(\nabla^q\boldsymbol{R}) + O(Q^{n/2-3-m})\Big) + \sum\boldsymbol{R}(\nabla^{m+1}\boldsymbol{R}) + O(Q^{n/2-2})\\ &= \sum(\nabla^s\boldsymbol{R})(\nabla^t\boldsymbol{R}) + O(Q^{n/2-3-(m+1)}).\end{aligned}$$

Here $p+q=m$ and $s+t=m+1$. The use of the inductive assumption and (26) never moves a free index from an $\boldsymbol{R}$ onto a ∇. □

Proposition 4.6. *Suppose that n is even and M is generic. Let an integer ℓ be given, and suppose that $0 \le \ell \le n/2-3$. Then*

$$\boldsymbol{\Delta}^{\ell}\boldsymbol{R} = \sum(\nabla^{v_1}\boldsymbol{R})\cdots(\nabla^{v_j}\boldsymbol{R}) + O(Q^{n/2-2-\ell}).$$

The number of factors in a term may vary from term to term, but in any case, $v_i \le \ell$ for $1 \le i \le j$. If A, B, C, and D denote the indices of the $\boldsymbol{R}$ on the left-hand side, then for each term in the sum, these indices are on an $\boldsymbol{R}$.

Proof. We again use induction. Suppose that $0 \le m \le n/2-4$ and that the result holds for $\ell = m$. Then

$$\boldsymbol{\Delta}^{m+1}\boldsymbol{R} = \boldsymbol{\Delta}\left(\sum(\nabla^{v_1}\boldsymbol{R})\cdots(\nabla^{v_j}\boldsymbol{R}) + O(Q^{n/2-2-m})\right). \tag{44}$$

By expanding the right-hand side above using the Leibniz rule and the formula $\boldsymbol{\Delta} = \nabla^A\nabla_A$, we obtain an expression $\sum(\nabla^{u_1}\boldsymbol{R})\cdots(\nabla^{u_k}\boldsymbol{R}) + O(Q^{n/2-2-(m+1)})$ plus a sum of the form $\sum(\boldsymbol{\Delta}\nabla^{t_0}\boldsymbol{R})(\nabla^{t_1}\boldsymbol{R})\cdots(\nabla^{t_s}\boldsymbol{R})$. In each case, $u_i \le m+1$ and $t_i \le m$. But by Proposition 4.5, $\boldsymbol{\Delta}\nabla^{t_0}\boldsymbol{R} = \sum(\nabla^p\boldsymbol{R})(\nabla^q\boldsymbol{R}) + O(Q^{n/2-3-t_0})$, where $p+q=t_0 \le m$. Thus $\boldsymbol{\Delta}\boldsymbol{\nabla}^{t_0}\boldsymbol{R} = \sum(\boldsymbol{\nabla}^p\boldsymbol{R})(\boldsymbol{\nabla}^q\boldsymbol{R}) + O(Q^{n/2-2-(m+1)})$. The use of the inductive assumption and Proposition 4.5 never moves an index from an $\boldsymbol{R}$ onto a $\boldsymbol{\nabla}$. □

Theorem 4.7. *Suppose that n is even and M is generic. Let $\boldsymbol{h}$ be an ambient metric for a conformal manifold of dimension n. Given $t \ge 0$ and $u \ge 0$, suppose that $t+u \le n/2-3$. Then there is a partial contraction $\mathcal{P}$, polynomial in $\boldsymbol{D}_A$, $\boldsymbol{R}_{ABCD}$, $\boldsymbol{X}_A$, $\boldsymbol{h}_{AB}$, and its inverse $\boldsymbol{h}^{AB}$, such that*

$$\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R} = \mathcal{P} + O(Q^{n/2-2-t-u}). \tag{45}$$

Each term of $\mathcal{P}$ is of degree at least 1 *in $\boldsymbol{R}_{ABCD}$. If, in* (45), *$\boldsymbol{R}$ has any free indices, then in $\mathcal{P}$ these indices always appear on an $\boldsymbol{R}$.*

Proof. By Proposition 4.6, we may write

$$\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R} = \sum(\boldsymbol{\nabla}^{v_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_j}\boldsymbol{R}) + O(Q^{n/2-2-u-t}),$$

where $v_i \le t+u$ for each i. If the $\boldsymbol{R}$ on the left-hand side has any free indices, then for each term in the sum, these indices always appear on an $\boldsymbol{R}$; this follows from Proposition 4.6. To complete the proof, we show that if $0 \le \ell \le n/2-3$, then $\boldsymbol{\nabla}^\ell\boldsymbol{R} = \mathcal{P} + O(Q^{n/2-2-\ell})$. We use induction. Suppose that $1 \le m \le n/2-3$, and suppose that $\boldsymbol{\nabla}^\ell\boldsymbol{R} = \mathcal{P} + O(Q^{n/2-2-\ell})$ whenever $0 \le \ell \le m-1$. By (14) we have

$$\boldsymbol{D}_A\boldsymbol{\nabla}^{m-1}\boldsymbol{R} = (n-2m-4)\boldsymbol{\nabla}_A\boldsymbol{\nabla}^{m-1}\boldsymbol{R} - \boldsymbol{X}_A\boldsymbol{\Delta}\boldsymbol{\nabla}^{m-1}\boldsymbol{R}.$$

Note that $n-2m-4>0$. Also observe that each $\boldsymbol{R}$ has the same indices. From this equation and Proposition 4.5, we conclude that

$$\boldsymbol{\nabla}^m\boldsymbol{R} = \boldsymbol{D}_A\boldsymbol{\nabla}^{m-1}\boldsymbol{R} + \boldsymbol{X}_A\big(\sum(\boldsymbol{\nabla}^p\boldsymbol{R})(\boldsymbol{\nabla}^q\boldsymbol{R}) + O(Q^{n/2-3-(m-1)})\big),$$

where $p+q \le m-1$. Also note that if $\boldsymbol{R}$ on the left-hand side of this equation has any free indices, then in each term of the right-hand side, these indices always appear on an $\boldsymbol{R}$. We now see that $\boldsymbol{\nabla}^m\boldsymbol{R} = \mathcal{P} + O(Q^{n/2-2-m})$, from our inductive assumption. □

Remark. Theorem 4.7 shows that when $n \ne 4$, an ambient partial contraction $\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R}|_{\mathcal{Q}}$ is equivalent to a conformal invariant which is obtained by taking a partial contraction polynomial in D, W, X, h, and its inverse h^{-1}. Moreover in each case, via the inductive steps of the proof, one obtains the explicit formula for the invariant as a partial contraction of these quantities. More generally, this shows that any "Weyl invariant" [Bailey et al. 1994b; Fefferman 1979] arising from a complete (partial) contraction of ambient tensors of the form (45) is contained in the space of invariants generated by complete (partial) contractions of the expressions polynomial in the tractor operators and fields D, W, X, h, and h^{-1}. Furthermore, there is an explicit algorithm for finding the tractor formula, given the formula for the ambient invariant. This is a slight generalisation of a result along these lines obtained in [Čap and Gover 2003].

The next proposition is a simple generalisation of results in [Branson and Gover 2005; Graham et al. 1992].

Proposition 4.8. *For every integer $m \geq 1$ and every ambient homogeneous tensor space $\boldsymbol{\mathcal{T}}^{\Phi}(m-n/2)$,*

$$\boldsymbol{\Delta}_{\alpha}^{m} : \boldsymbol{\mathcal{T}}^{\Phi}(m-n/2) \to \boldsymbol{\mathcal{T}}^{\Phi}(-m-n/2)$$

is tangential and so determines a conformally invariant operator

$$\Box_{m}^{\alpha} : \mathcal{T}^{\Phi}[m-n/2] \to \mathcal{T}^{\Phi}[-m-n/2].$$

Proof. By construction, the operators $\boldsymbol{\Delta}_{\alpha}$ preserve tensor type (with respect to pointwise $\mathrm{SO}(\boldsymbol{h})$ tensor decompositions) and lower homogeneity weight by 2. Hence $\boldsymbol{\Delta}_{\alpha}^{m}$ maps $\boldsymbol{\mathcal{T}}^{\Phi}(m-n/2)$ to $\boldsymbol{\mathcal{T}}^{\Phi}(-m-n/2)$.

To show that $\boldsymbol{\Delta}_{\alpha}^{m}$ acts tangentially, we calculate $\boldsymbol{\Delta}_{\alpha}^{m} QA$ for A of homogeneity $m-2-n/2$. Without any homogeneity assumption, we have

$$[\boldsymbol{\Delta}_{\alpha}^{m}, Q] = \textstyle\sum_{p=0}^{m-1} \boldsymbol{\Delta}_{\alpha}^{m-1-p}[\boldsymbol{\Delta}_{\alpha}, Q]\boldsymbol{\Delta}_{\alpha}^{p}.$$

Acting on $\boldsymbol{\mathcal{T}}^{\Phi}(w)$, the p-th term on the right acts as $2(2(w-2p)+n+2)\boldsymbol{\Delta}_{\alpha}^{m-1}$ by (25). Hence $[\boldsymbol{\Delta}_{\alpha}^{m}, Q]$ acts as $2m(2w-2m+n+4)\boldsymbol{\Delta}_{\alpha}^{m-1}$. This vanishes identically if $w = m-2-n/2$. Thus $\boldsymbol{\Delta}_{\alpha}^{m}$ is tangential on $\boldsymbol{\mathcal{T}}^{\Phi}(m-n/2)$ as desired. $\square$

The remainder of this section is concerned with obtaining tractor formulae for the operators in the previous theorem. A key idea is to assume that the ambient tensor field being acted on is suitably "harmonic" as in the following lemma. Since tangential operators do not depend on how the field is extended off $\mathcal{Q}$, this involves no loss of generality.

Lemma 4.9. *Suppose $k \geq 2$ is an integer. In the generic n-even case, suppose $k \leq n/2-1$ or that $\alpha = 0$ and $k \leq n/2$. Given $S \in \boldsymbol{\mathcal{T}}^{\Phi}(k-n/2)$, suppose $\boldsymbol{\Delta}_{\alpha} S$ is $O(Q^{k-1})$. Finally, let v, $0 \leq v \leq k-1$, be given. Then there is a linear differential operator $\mathcal{P}$ of order at most $2v$ given by a partial contraction formula polynomial in $\boldsymbol{X}_{A}$, $\boldsymbol{D}_{A}$, $\boldsymbol{R}_{ABCD}$, $\boldsymbol{h}_{AB}$, and $\boldsymbol{h}^{AB}$, such that*

$$\boldsymbol{\nabla}^{v} S = \mathcal{P} S + O(Q^{k-v}). \tag{46}$$

If, on the left-hand side of (46), *S has any free indices, then in every term of $\mathcal{P}S$, each of them appears either on S in its natural position or on $\boldsymbol{R}$.*

Proof. We will assume that n is even and M is generic. For $v = 1$, observe that by (14) and (23) we have

$$2(k-1)\boldsymbol{\nabla} S = \boldsymbol{D} S - \alpha \boldsymbol{X} \boldsymbol{R} \sharp\sharp S + \boldsymbol{X} \boldsymbol{\Delta}_{\alpha} S.$$

This is in the required form, since $\boldsymbol{\Delta}_\alpha S = O(Q^{k-1})$.

We now proceed by induction on v. Suppose that $1 \leq m < k-1$ and that (46) holds for $1 \leq v \leq m$. By (14) it follows that

$$2(k-m-1)\nabla^{m+1} S = \boldsymbol{D}\nabla^m S - \alpha \boldsymbol{X}\boldsymbol{R}\sharp\sharp\nabla^m S + \boldsymbol{X}\boldsymbol{\Delta}_\alpha \nabla^m S. \tag{47}$$

If S on the left-hand side has any free indices, then in every term of the right-hand side, each of these indices appears on an S in its natural position or on an $\boldsymbol{R}$. From the inductive assumption and the properties of $\boldsymbol{D}$, it then follows that $\boldsymbol{D}\nabla^m S - \alpha \boldsymbol{X}\boldsymbol{R}\sharp\sharp\nabla^m S$ is of the form $\mathcal{P}S + O(Q^{k-(m+1)})$, where $\mathcal{P}$ is as described in the statement of the lemma. On the other hand, by (42),

$$\begin{aligned}\boldsymbol{\Delta}_\alpha \nabla^m S = \nabla^m \boldsymbol{\Delta}_\alpha S + \textstyle\sum(\nabla^p \boldsymbol{R})(\nabla^q S)& \\ + \alpha \textstyle\sum(\nabla^{p+1}\boldsymbol{R})(\nabla^{q-1} S) + O(Q^{n/2-2-(m-1)}),&\end{aligned} \tag{48}$$

where $p+q=m$, $p \geq 0$, and $q \geq 1$. When we use (42) to construct (48), each index attached to S on the left-hand side of (48) either remains fixed or moves onto an $\boldsymbol{R}$. Note that $\nabla^m \boldsymbol{\Delta}_\alpha S$ is $O(Q^{k-(m+1)})$ and that $n/2-2-(m-1) \geq k-(m+1)$. Thus $\boldsymbol{\Delta}_\alpha \nabla^m S = \sum(\nabla^x \boldsymbol{R})(\nabla^y S) + O(Q^{k-(m+1)})$. Here $x+y=m$, $x \leq m$, and $y \leq m$. If $\alpha = 0$, then we have $1 \leq y$ and $x \leq m-1$. By Theorem 4.7 and by our inductive assumption, it follows that $\boldsymbol{\Delta}_\alpha \nabla^m S = \mathcal{P}S + O(Q^{k-(m+1)})$, where $\mathcal{P}$ is as in the statement of the lemma. □

The usefulness of Lemma 4.9 results from the next proposition, which generalises to ambient tensors and $\boldsymbol{\Delta}_\alpha$-Laplacians a result of [Graham et al. 1992].

Proposition 4.10. *Let $k \geq 1$ be an integer. Then for any $T \in \boldsymbol{\mathcal{T}}^\Phi(k-n/2)$, there is an $S \in \boldsymbol{\mathcal{T}}^\Phi(k-n/2)$ such that $T-S$ is $O(Q)$ and $\boldsymbol{\Delta}_\alpha S$ is $O(Q^{k-1})$.*

Proof. Let $w := k-n/2$. Suppose that $S_{m-1} \in \boldsymbol{\mathcal{T}}^\Phi(w)$ is such that $T - S_{m-1}$ is $O(Q)$ and $\boldsymbol{\Delta}_\alpha S_{m-1} = Q^{m-1}E$. (Then $E \in \boldsymbol{\mathcal{T}}^\Phi(w-2m)$.) If $A \in \boldsymbol{\mathcal{T}}^\Phi(w-2m)$, then $S_m := S_{m-1} + Q^m A \in \boldsymbol{\mathcal{T}}^\Phi(w)$ and $T - S_m$ is $O(Q)$. We have $\boldsymbol{\Delta}_\alpha S_m = Q^{m-1}E + \boldsymbol{\Delta}_\alpha Q^m A$. Now $\boldsymbol{\Delta}_\alpha Q^m A = \sum_{i=0}^{m-1} Q^i[\boldsymbol{\Delta}_\alpha, Q]Q^{m-i-1}A + O(Q^m)$, and from (25) and the homogeneity of A and Q this becomes

$$\begin{aligned}\boldsymbol{\Delta}_\alpha Q^m A &= \textstyle\sum_{i=0}^{m-1} 2(n+2w-4i-2)Q^{m-1}A + O(Q^m)\\ &= 4m(w+n/2-m)Q^{m-1}A + O(Q^m).\end{aligned}$$

Thus if $m \neq w+n/2$ (i.e. $m \neq k$), then setting $A = -[4m(w+n/2-m)]^{-1}E$ gives $\boldsymbol{\Delta}_\alpha S_m = O(Q^m)$. □

Note that the proof establishes much more than we require in the proposition. It shows that the $\boldsymbol{\Delta}_\alpha$-harmonic extension of $T|_{\mathcal{Q}}$ only fails at $O(Q^k)$, and that past this the extension continues. Also, if we allow w such that $w+n/2 \notin \{1, 2, \dots\}$, then for any $T \in \boldsymbol{\mathcal{T}}^{\Phi}(w)$ and any integer $\ell \geq 0$, there is $S \in \boldsymbol{\mathcal{T}}^{\Phi}(w)$ such that $T - S$ is $O(Q)$ and $\boldsymbol{\Delta}_\alpha S$ is $O(Q^\ell)$.

Remark. Recall that one of our central aims (at least for $n \geq 6$) is to understand the result of applying $\not{\boldsymbol{\Delta}}^{n/2-2}$ to the ambient curvature $\boldsymbol{R}$. For this it would appear that we do not need Proposition 4.10, since by (22), the ambient curvature already has the property we require of S, namely, that $\boldsymbol{\Delta}_{1/2}\boldsymbol{R} = \not{\boldsymbol{\Delta}}\boldsymbol{R} = O(Q^{n/2-3})$. On the other hand, we prefer here to treat $\not{\boldsymbol{\Delta}}^{n/2-2}\boldsymbol{R}$ in two steps. First, we derive a tractor formula for the conformally invariant operator $\not{\Box}_{n/2-2}$ on $\mathcal{T}^{2,2}[-2]$. For this we will use Proposition 4.10. This operator arises from $\not{\boldsymbol{\Delta}}^{n/2-2}$ on $\boldsymbol{\mathcal{T}}^{2,2}(-2)$. Then finally we may apply the operator $\not{\Box}_{n/2-2}$ to the tractor field W; see (28). Proceeding in this way, we can be sure that the tractor formula that we obtain for the ambient quantity $\not{\boldsymbol{\Delta}}^{n/2-2}\boldsymbol{R}|_{\mathcal{Q}}$ is precisely the tractor formula for $\not{\Box}_{n/2-2}$ on $\mathcal{T}^{2,2}[-2]$ applied to W.

Next, we need to understand how powers of the $\boldsymbol{\Delta}_\alpha$-Laplacian are related to iterations of $\boldsymbol{D}$. We begin with a lemma which indicates the impact of moving Laplacians to the right of $\boldsymbol{\nabla}$'s.

Lemma 4.11. *Suppose that n is even and M is generic. Let $\alpha \in \mathbb{R}$, $w \in \mathbb{R}$, and $T \in \boldsymbol{\mathcal{T}}^{\Phi}(w)$ be given. Let*

$$S = \boldsymbol{\Delta}_\alpha^{t_1}\boldsymbol{\nabla}^{u_1}\cdots\boldsymbol{\Delta}_\alpha^{t_p}\boldsymbol{\nabla}^{u_p}T, \tag{49}$$

where $t_i + u_i \geq 1$ for each i. Suppose that $k := \sum_{i=1}^{p}(t_i + u_i) \leq n/2 - 1$. Then

$$S = \sum(\boldsymbol{\nabla}^{v_1}\boldsymbol{\Delta}_\alpha^{w_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_q}\boldsymbol{\Delta}_\alpha^{w_q}\boldsymbol{R})(\boldsymbol{\nabla}^{v_{q+1}}\boldsymbol{\Delta}_\alpha^{w_{q+1}}T) + O(Q^{n/2-k}), \tag{50}$$

where $v_j + w_j \leq k$ for each j. If T has any free indices in (49), *then in* (50) *these indices appear either on T in their original position or on an $\boldsymbol{R}$.*

Proof. We proceed by induction on k. Suppose that $1 \leq m \leq n/2 - 2$. Suppose the result holds whenever $1 \leq k \leq m$, and let S be as in (49) with $k = m + 1$. If $t_1 = 0$, then by our inductive assumption we see immediately that (50) holds modulo $O(Q^{n/2-(m+1)})$. On the other hand, suppose $t_1 > 0$. Then by our inductive assumption, $S = \boldsymbol{\Delta}_\alpha\big(\sum(\boldsymbol{\nabla}^{v_1}\boldsymbol{\Delta}_\alpha^{w_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_q}\boldsymbol{\Delta}_\alpha^{w_q}\boldsymbol{R})(\boldsymbol{\nabla}^{v_{q+1}}\boldsymbol{\Delta}_\alpha^{w_{q+1}}T) + O(Q^{n/2-m})\big)$, where $v_j + w_j \leq m$ for each j. Suppose we use the Leibniz rule to expand $\boldsymbol{\Delta}(\boldsymbol{\nabla}^{v_1}\boldsymbol{\Delta}_\alpha^{w_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_q}\boldsymbol{\Delta}_\alpha^{w_q}\boldsymbol{R})(\boldsymbol{\nabla}^{v_{q+1}}\boldsymbol{\Delta}_\alpha^{w_{q+1}}T)$. Then each term in the resulting sum will either contain two factors of the form $\boldsymbol{\nabla}^{v_j+1}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P}$ or one factor of the form $\boldsymbol{\Delta}\boldsymbol{\nabla}^{v_j}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P}$, where $\boldsymbol{P}$ denotes $\boldsymbol{R}$ or T in each case. But $\boldsymbol{\Delta}\boldsymbol{\nabla}^{v_j}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P}$ equals $\boldsymbol{\Delta}_\alpha\boldsymbol{\nabla}^{v_j}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P} - \alpha\boldsymbol{R}\sharp\sharp\boldsymbol{\nabla}^{v_j}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P}$, and by (42) we may write $\boldsymbol{\Delta}_\alpha\boldsymbol{\nabla}^{v_j}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P}$ in the form $\boldsymbol{\nabla}^{v_j}\boldsymbol{\Delta}_\alpha^{w_j+1}\boldsymbol{P} + \sum(\boldsymbol{\nabla}^{v'_\ell}\boldsymbol{R})\boldsymbol{\nabla}^{v''_\ell}\boldsymbol{\Delta}_\alpha^{w_j}\boldsymbol{P} + O(Q^{n/2-(m+1)})$. Here $v'_\ell + v''_\ell = v_j$. When we use (42), any given index attached to $\boldsymbol{P}$ either remains fixed or moves onto an $\boldsymbol{R}$. This completes the induction. $\square$

Lemma 4.12. *Suppose ℓ is an integer and $\ell \geq 1$. In the generic n-even case, suppose also that $\ell \leq n/2-1$. Let $T \in \boldsymbol{\mathcal{T}}^{\Phi}(\ell - n/2)$ be given. Then*

$$\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{D}T = -\boldsymbol{X}\boldsymbol{\Delta}_\alpha^{\ell}T + \sum(\nabla^{v_1}\boldsymbol{\Delta}^{w_1}\boldsymbol{R})\cdots(\nabla^{v_p}\boldsymbol{\Delta}^{w_p}\boldsymbol{R})(\nabla^{v_{p+1}}\boldsymbol{\Delta}_\alpha^{w_{p+1}}T) + \alpha\boldsymbol{X}\sum(\nabla^{r_1}\boldsymbol{\Delta}^{s_1}\boldsymbol{R})\cdots(\nabla^{r_q}\boldsymbol{\Delta}^{s_q}\boldsymbol{R})(\nabla^{r_{q+1}}\boldsymbol{\Delta}_\alpha^{s_{q+1}}T) + O(Q). \tag{51}$$

Here $v_i + w_i \leq \ell-1$ for $1 \leq i \leq p+1$, and $r_i + s_i \leq \ell-1$ for $1 \leq i \leq q+1$. If $\alpha = 0$, then $v_i + w_i \leq \ell - 2$ for $1 \leq i \leq p$, and $v_{p+1} + w_{p+1} \leq \ell - 1$. If T on the left-hand side has any free indices, then on the right-hand side these indices always appear on $\boldsymbol{R}$ or in their natural positions on T.

Proof. Suppose that n is even and M is generic. If $\ell = 1$, the result follows from (14). Now suppose that $\ell \geq 2$. From (14) and (24) we have

$$\begin{aligned}
\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{D}_A T &= 2(\ell-1)\boldsymbol{\Delta}_\alpha^{\ell-1}\nabla_A T - \boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{X}_A\boldsymbol{\Delta}_\alpha T + \alpha\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{X}_A\boldsymbol{R}\sharp\sharp T\\
&= 2(\ell-1)\boldsymbol{\Delta}_\alpha^{\ell-1}\nabla_A T - [\boldsymbol{\Delta}_\alpha^{\ell-1}, \boldsymbol{X}_A]\boldsymbol{\Delta}_\alpha T - \boldsymbol{X}_A\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{\Delta}_\alpha T\\
&\qquad + \alpha[\boldsymbol{\Delta}_\alpha^{\ell-1}, \boldsymbol{X}_A]\boldsymbol{R}\sharp\sharp T + \alpha\boldsymbol{X}_A\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{R}\sharp\sharp T\\
&= -\boldsymbol{X}_A\boldsymbol{\Delta}_\alpha^{\ell}T + 2(\ell-1)\boldsymbol{\Delta}_\alpha^{\ell-1}\nabla_A T - \textstyle\sum_{i=0}^{\ell-2}\boldsymbol{\Delta}_\alpha^{\ell-2-i}[\boldsymbol{\Delta}_\alpha, \boldsymbol{X}_A]\boldsymbol{\Delta}_\alpha^{i}\boldsymbol{\Delta}_\alpha T\\
&\qquad + \alpha\Big(\textstyle\sum_{i=0}^{\ell-2}\boldsymbol{\Delta}_\alpha^{\ell-2-i}[\boldsymbol{\Delta}_\alpha, \boldsymbol{X}_A]\boldsymbol{\Delta}_\alpha^{i}\Big)\boldsymbol{R}\sharp\sharp T + \alpha\boldsymbol{X}_A\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{R}\sharp\sharp T\\
&= -\boldsymbol{X}_A\boldsymbol{\Delta}_\alpha^{\ell}T + 2(\ell-1)\boldsymbol{\Delta}_\alpha^{\ell-1}\nabla_A T - 2\textstyle\sum_{i=0}^{\ell-2}\boldsymbol{\Delta}_\alpha^{\ell-2-i}\nabla_A\boldsymbol{\Delta}_\alpha^{i}\boldsymbol{\Delta}_\alpha T\\
&\qquad + 2\alpha\Big(\textstyle\sum_{i=0}^{\ell-2}\boldsymbol{\Delta}_\alpha^{\ell-2-i}\nabla_A\boldsymbol{\Delta}_\alpha^{i}\Big)\boldsymbol{R}\sharp\sharp T + \alpha\boldsymbol{X}_A\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{R}\sharp\sharp T.
\end{aligned}$$

Each of the original indices on T remains fixed in this calculation except in the terms of $\boldsymbol{R}\sharp\sharp T$, where it may either remain in its original position on T or move onto an $\boldsymbol{R}$. By (42), we may reexpress this in the form

$$\begin{aligned}
\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{D}T = -\boldsymbol{X}\boldsymbol{\Delta}_\alpha^{\ell}T &+ \textstyle\sum\boldsymbol{\Delta}_\alpha^{s_j}\boldsymbol{R}\nabla\boldsymbol{\Delta}_\alpha^{t_j}T + \alpha\textstyle\sum\boldsymbol{\Delta}_\alpha^{s_j}(\nabla\boldsymbol{R})\boldsymbol{\Delta}_\alpha^{t_j}T\\
&+ \alpha\boldsymbol{\Delta}_\alpha^{\ell-2}\nabla\boldsymbol{R}\sharp\sharp T + \alpha\textstyle\sum\boldsymbol{\Delta}_\alpha^{p_i}\boldsymbol{R}\nabla\boldsymbol{\Delta}_\alpha^{q_i}\boldsymbol{R}\sharp\sharp T\\
&+ \alpha\textstyle\sum\boldsymbol{\Delta}_\alpha^{p_i}(\nabla\boldsymbol{R})\boldsymbol{\Delta}_\alpha^{q_i}\boldsymbol{R}\sharp\sharp T + \alpha\boldsymbol{X}\boldsymbol{\Delta}_\alpha^{\ell-1}\boldsymbol{R}\sharp\sharp T + O(Q),
\end{aligned} \tag{52}$$

where $s_j + t_j = \ell - 2$ for each j and $p_i + q_i = \ell - 3$ for each i. When we use (42) to construct (52), each index on T or $\boldsymbol{R}$ either remains fixed or moves onto an $\boldsymbol{R}$. In the right-hand side of (52) the coefficient of $\boldsymbol{X}\boldsymbol{\Delta}_\alpha^{\ell}T$ is exact. Otherwise, no attempt has been made to present the coefficients precisely. At this point we need only the general form of the expression. Where α appears as a coefficient, this means as usual that all terms of this form have coefficient a multiple of α.

For ambient tensors U and V,

$$\boldsymbol{\Delta}_\alpha UV = (\boldsymbol{\Delta}U)V + (\nabla U)\nabla V + U\boldsymbol{\Delta}_\alpha V + \boldsymbol{R}UV.$$

Thus by using the definition of $\boldsymbol{\Delta}_\alpha$ together with the Leibniz rule, we may reexpress the right-hand side of (52) in the form given on the right-hand side of (51), except that on each $\boldsymbol{R}$ or T, the operators $\boldsymbol{\nabla}$, $\boldsymbol{\Delta}$, and $\boldsymbol{\Delta}_\alpha$ may not be in the order given in (51). But by Lemma 4.11, we may indeed reexpress the right-hand side of (52) in the form given on the right-hand side of (51). In doing this, we may move an index that was originally attached to an $\boldsymbol{R}$ or a T, but we always move the index onto an $\boldsymbol{R}$. In the new expression, we have $v_i + w_i \leq \ell - 1$ for $1 \leq i \leq p+1$ and $r_i + s_i \leq \ell - 1$ for $1 \leq i \leq q+1$; this follows from Lemma 4.11. In the $\alpha = 0$ case, the fact that $v_i + w_i \leq \ell - 2$ for $1 \leq i \leq p$ follows from the fact that (52) simplifies to $\boldsymbol{\Delta}^{\ell-1}\boldsymbol{D}T = -\boldsymbol{X}\boldsymbol{\Delta}^{\ell}T + \sum \boldsymbol{\Delta}^{s_j}\boldsymbol{R}\boldsymbol{\nabla}\boldsymbol{\Delta}^{t_j}T + O(Q)$ when $\alpha = 0$. □

We are now ready to show that the powers of the $\boldsymbol{\Delta}_\alpha$-Laplacian can be reexpressed as a sum of compositions of tangential operators.

Proposition 4.13. *Suppose $k \geq 1$ is an integer. Let $w = k - n/2$, and let $V \in \mathcal{T}^{\Phi}(w)$ be given. In the generic n-even case, suppose that $k \leq n/2 - 2$, or $\alpha = 0$ and $k \leq n/2 - 1$, or $\mathcal{T}^{\Phi}(w) = \mathcal{T}(w)$ and $k \leq n/2 - 1$, or $\mathcal{T}^{\Phi}(w) = \mathcal{T}^{0}(w)$ and $k \leq n/2$. Then*

$$(-1)^{k-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta}_\alpha^k V = \boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{k-1}}V + \mathcal{P}V + O(Q),$$

where $\mathcal{P}$ is a linear differential operator of order less than $2k$ given as a partial contraction polynomial in $\boldsymbol{X}_A$, $\boldsymbol{D}_A$, $\boldsymbol{R}_{ABCD}$, $\boldsymbol{h}_{AB}$, and $\boldsymbol{h}^{AB}$. If V has any free indices, then for every term of $\mathcal{P}V$, these indices appear either on $\boldsymbol{R}$ or in their natural position on V. The indices A_i are not skew-symmetrised.

Proof. The case of $V \in \mathcal{T}^{0}(w)$ is treated in [Gover and Peterson 2003]. For the remaining cases, we assume, as usual, that we are in the generic n-even setting.

We begin with the case $k \leq n/2 - 2$ and the case $\alpha = 0$ and $k \leq n/2 - 1$; we use induction on k. Suppose that $1 \leq m \leq n/2 - 3$ or that $\alpha = 0$ and $1 \leq m \leq n/2 - 2$, and suppose the result holds whenever $k = m$. Let $V \in \mathcal{T}^{\Phi}(m + 1 - n/2)$. By Proposition 4.10, there exists an $S \in \mathcal{T}^{\Phi}(m + 1 - n/2)$ such that $V - S$ is $O(Q)$ and $\boldsymbol{\Delta}_\alpha S$ is $O(Q^m)$. Then by our inductive assumption,

$$(-1)^{m-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{m-1}}\boldsymbol{\Delta}_\alpha^m(\boldsymbol{D}_{A_m}S) = \boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{m-1}}(\boldsymbol{D}_{A_m}S) + \mathcal{P}S + O(Q), \tag{53}$$

where $\mathcal{P}$ is of order less than $2m$. If S on the left-hand side of (53) has any free indices, then in each term of $\mathcal{P}S$, these indices appear either on $\boldsymbol{R}$ or in their natural position on S. Now apply Lemma 4.12 with $\ell = m + 1$ and $T = S$. We find that

$$\begin{aligned}\boldsymbol{\Delta}_\alpha^m\boldsymbol{D}_{A_m}S = -\boldsymbol{X}_{A_m}\boldsymbol{\Delta}_\alpha^{m+1}S + O(Q)& \\ + \textstyle\sum(\boldsymbol{\nabla}^{v_1}\boldsymbol{\Delta}^{w_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{v_p}\boldsymbol{\Delta}^{w_p}\boldsymbol{R})(\boldsymbol{\nabla}^{v_{p+1}}\boldsymbol{\Delta}_\alpha^{w_{p+1}}S)& \\ + \alpha\boldsymbol{X}\textstyle\sum(\boldsymbol{\nabla}^{r_1}\boldsymbol{\Delta}^{t_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{r_q}\boldsymbol{\Delta}^{t_q}\boldsymbol{R})(\boldsymbol{\nabla}^{r_{q+1}}\boldsymbol{\Delta}_\alpha^{t_{q+1}}S).&\end{aligned} \tag{54}$$

Here $v_i + w_i \leq m$ for $1 \leq i \leq p+1$, and $r_i + t_i \leq m$ for $1 \leq i \leq q+1$. If $\alpha = 0$, then $v_i + w_i \leq m - 1$ for $1 \leq i \leq p$ and $v_{p+1} + w_{p+1} \leq m$. If, on the left-hand side of (54), S has any free indices, then on the right-hand side of this equation these indices appear on $\boldsymbol{R}$ or in their natural positions on S. Since $\boldsymbol{\Delta}_\alpha S$ is $O(Q^m)$, we may assume that $w_{p+1} = t_{q+1} = 0$ in (54). Thus by Theorem 4.7 and Lemma 4.9, we have

$$\boldsymbol{\Delta}_\alpha^m \boldsymbol{D}_{A_m} S = -\boldsymbol{X}_{A_m} \boldsymbol{\Delta}_\alpha^{m+1} S + \mathcal{P} S + O(Q). \tag{55}$$

Since $v_{p+1} \leq m$ and $r_{q+1} \leq m$ in (54), it follows that the order of $\mathcal{P}$ is at most $2m$ in (55). If S in (55) has free indices, then in $\mathcal{P}S$ these appear either on $\boldsymbol{R}$ or in their natural positions on S. From (53) and (55) it now follows that

$$(-1)^m \boldsymbol{X}_{A_1} \cdots \boldsymbol{X}_{A_m} \boldsymbol{\Delta}_\alpha^{m+1} S = \boldsymbol{\Delta} \boldsymbol{D}_{A_1} \cdots \boldsymbol{D}_{A_m} S + \mathcal{P} S + O(Q). \tag{56}$$

But $\boldsymbol{D}_A$ acts tangentially along $\mathcal{Q}$, and $\boldsymbol{\Delta}$ acts tangentially on fields homogeneous of degree $1-n/2$. Thus $\boldsymbol{\Delta} \boldsymbol{D}_{A_1} \cdots \boldsymbol{D}_{A_m} + \mathcal{P}$ acts tangentially on S. By Proposition 4.8, $\boldsymbol{\Delta}_\alpha^{m+1}$ also acts tangentially on S, and so we may replace S with V on both sides of (56). This completes the induction.

Finally, suppose that $\boldsymbol{\mathcal{T}}^\Phi(w) = \boldsymbol{\mathcal{T}}(w)$. By the Ricci-flatness of the ambient metric, it follows that $\boldsymbol{R}\sharp\sharp V$ is $O(Q^{n/2-1})$. Thus for $1 \leq k \leq n/2 - 1$ we see that $\boldsymbol{\Delta}_\alpha^k V = \boldsymbol{\Delta}^k V + O(Q)$, and the result follows from the case $\alpha = 0$. $\square$

We are now ready to prove Theorem 4.1 and at the same time describe tractor formulae for the operators $\Box_m^\alpha$. We begin with the tractor formulae.

Theorem 4.14. *Via the algorithm implicit in the inductive steps above, the operators $\Box_m^\alpha$ have tractor formulae (for m in the ranges given in Theorem 4.1) as follows:*

$$(-1)^{m-1} X_{A_1} \cdots X_{A_{m-1}} \Box_m^\alpha U = \Box D_{A_1} \cdots D_{A_{m-1}} U + \mathcal{P}^{\Phi,m}_{A_1 \cdots A_{m-1}} U, \tag{57}$$

where the differential operator $\mathcal{P}^{\Phi,m}$ is a partial contraction polynomial in X, D, W, h, and h^{-1}. Thus for $m \neq n/2$,

$$\begin{aligned}(m-1)!\Big(\textstyle\prod_{i=2}^m (n-2i)\Big)\Box_m^\alpha U \\ = D^{A_{m-1}} \cdots D^{A_1} \Box D_{A_1} \cdots D_{A_{m-1}} U + D^{A_{m-1}} \cdots D^{A_1} \mathcal{P}^{\Phi,m}_{A_1 \cdots A_{m-1}} U.\end{aligned} \tag{58}$$

The indices attached to U on the left-hand side appear, in each term of $\mathcal{P}^{\Phi,m} U$, on U in their original position or on W. The indices A_i in (57) and (58) are not skew-symmetrised.

Proof of theorems 4.1 and 4.14. Recall that $\boldsymbol{\Delta} : \boldsymbol{\mathcal{T}}^\Phi(1-n/2) \to \boldsymbol{\mathcal{T}}^\Phi(-1-n/2)$ descends to the generalised conformal Laplacian operator $\Box$ and $\boldsymbol{D}$ descends to D; see (17). Thus (57) is an immediate consequence of Proposition 4.13. From this the claims of naturality are immediate from the naturality of X, $\Box$, D, W, h, and h^{-1}. That the $\Box_m^\alpha$ have leading term Δ^m follows easily from the expression (16) for D and the identities (9) for the tractor connection. Then note that (58) follows from (57) and (19). $\square$

4B. ***Calculating explicit formulae; examples.*** One can easily compute explicit formulae for the obstruction tensors in low dimensions. From the proof of theorems 4.2 and 4.4, we know that in dimension 4, $\mathcal{O}_{ab}$ is simply $-(1/2)\, B_{ab}$, where B_{ab} is the Bach tensor as given in (1).

In dimension 6, we have $m = 1$, and the relevant ambient operator from Proposition 4.8 is $\not{\Delta} : \boldsymbol{\mathcal{T}}^{2,2}(-2) \to \boldsymbol{\mathcal{T}}^{2,2}(-4)$, which descends to

$$\Box + \tfrac{1}{4} W \sharp\sharp =: \not{\Box}_1 : \mathcal{T}^{2,2}[-2] \to \mathcal{T}^{2,2}[-4]. \tag{59}$$

The left-hand side of (59) is the tractor formula for $\not{\Box}_1$. By Theorem 4.2, applying this to W yields the obstruction tensor via the identity (32); see (28). That is, $2^6 X_{A_1} Z_{A_2}{}^{a} X_{B_1} Z_{B_2}{}^{b} \mathcal{O}^6_{ab} = \Box W + \frac{1}{4} W \sharp\sharp W$, where we have used the fact that $k(6) = 2^6$. Thus

$$\begin{aligned} &64\,\mathcal{O}_{ce} X_{[B} Z_{C]}{}^{c} Z_{[E}{}^{e} X_{D]} \\ &\qquad = \Box W_{BCDE} - W^{A}{}_{CB}{}^{F} W_{FADE} - W^{A}{}_{CD}{}^{F} W_{BAFE} - W^{A}{}_{CE}{}^{F} W_{BADF}. \end{aligned}$$

But $4 Y^B Y^D Z^C{}_a Z^E{}_b X_{[B} B_{C][E} X_{D]} = \mathcal{O}_{ab}$. Thus, in any conformal scale, $\mathcal{O}_{ab}$ is given by

$$\begin{aligned} &\tfrac{1}{16} Y^B Y^D Z^C{}_a Z^E{}_b \\ &\qquad \times (\Box W_{BCDE} - W^{A}{}_{CB}{}^{F} W_{FADE} - W^{A}{}_{CD}{}^{F} W_{BAFE} - W^{A}{}_{CE}{}^{F} W_{BADF}). \end{aligned} \tag{60}$$

If one expands using (9), (28) and the definitions of $\Box$ and the tractor metric, it is an entirely mechanical process to rewrite (60) in terms of the Levi-Civita connection and its curvature (with metric contractions). A computation using this process together with *Mathematica* and J. Lee's *Ricci* software package [Lee 1998] shows that

$$\begin{aligned} \mathcal{O}^6_{ab} = \tfrac{1}{16}\Delta B_{ab} - \tfrac{1}{4} J B_{ab} + \tfrac{1}{8} B_{cd} C_a{}^c{}_b{}^d - \tfrac{1}{2} P_{cd} \nabla^c A_{(ab)}{}^d \\ + \tfrac{1}{4} A_{cad} A^c{}_b{}^d - \tfrac{1}{2} A_{cad} A^d{}_b{}^c - \tfrac{1}{4} A_{(ab)c} \nabla^c J + \tfrac{1}{4} P_{cd} P^d{}_e C_a{}^c{}_b{}^e, \end{aligned} \tag{61}$$

where A and B are respectively the Cotton and Bach tensors as given in (29) and (30). This formula for $\mathcal{O}^6_{ab}$ agrees up to a constant factor with the formula given in [Graham and Hirachi 2005].

In dimension 8, we find that $\mathcal{O}_{ab} = (1/384)\, \mathsf{T}_{(ab)}$, where T_{ab} is as given on the next page. To see that $\mathcal{O}^8_{ab} = \mathsf{T}_{(ab)}$, up to a constant factor, we begin by constructing a tractor formula for $\not{\Box}_2$ on $\mathcal{T}^{2,2}[-2]$. Let $T \in \boldsymbol{\mathcal{T}}^{\Phi}(-2)$ be an extension of any element of $\mathcal{T}^{2,2}[-2]$. By Proposition 4.10 we may assume that $\not{\Delta} T = O(Q)$. Thus

$$
\begin{aligned}
&-B_{ab|c}{}^{c}{}_{d}{}^{d} + 10B_{ab|c}{}^{c}J - 28B_{ab|cd}P^{cd} + 24B_{ac|bd}P^{cd}\\
&\quad -4B_{cd|e}{}^{e}C_{a}{}^{c}{}_{b}{}^{d} - 24B_{ac|d}P_{b}{}^{c}{}_{|}{}^{d} - 24B_{cd|a}P_{b}{}^{c}{}_{|}{}^{d} + 56B_{ac|d}P_{b}{}^{d}{}_{|}{}^{c}\\
&\quad -6B_{ab|c}J_{|}{}^{c} + 12B_{ac|b}J_{|}{}^{c} + 24B_{cd|a}P^{cd}{}_{|b} - 32B_{ac|d}P^{cd}{}_{|b}\\
&\quad -4B_{cd|e}C_{a}{}^{c}{}_{b}{}^{d}{}_{|}{}^{e} + 4B_{ab}J_{|c}{}^{c} - 16B_{cd}P^{cd}{}_{|ab} - 40B_{cd}P_{ab|}{}^{cd}\\
&\quad +56B_{cd}P_{a}{}^{c}{}_{|b}{}^{d} - 8B_{ac}B_{b}{}^{c} + 3B_{cd}B^{cd}\boldsymbol{g}_{ab} - 24B_{ab}J^{2} - 64B_{ac}P_{bd}P^{cd}\\
&\quad +76B_{ab}P_{cd}P^{cd} + 28B_{cd}\boldsymbol{g}_{ab}P^{c}{}_{e}P^{de} + 16B_{cd}JC_{a}{}^{c}{}_{b}{}^{d} + 32B_{cd}P_{ae}C_{b}{}^{cde}\\
&\quad -24B_{cd}P^{c}{}_{e}C_{a}{}^{d}{}_{b}{}^{e} + 4B_{cd}C_{ae}{}^{c}{}_{i}C_{b}{}^{edi} - 8B_{cd}C_{ae}{}^{c}{}_{i}C_{b}{}^{ide} - 8A_{acb}J_{|d}{}^{dc}\\
&\quad -32A_{acb|de}P^{cd}{}_{|}{}^{e} - 16A_{acd|e}A_{b}{}^{cd}{}_{|}{}^{e} + 16A_{cda|e}A^{cd}{}_{b|}{}^{e} - 32A_{acb|d}J_{|}{}^{cd}\\
&\quad +32A_{cad|e}P_{b}{}^{e}P^{cd} - 64A_{abc|d}P^{cd}J - 128A_{acd|e}P_{b}{}^{d}P^{ce} - 128A_{cad|e}P_{b}{}^{d}P^{ce}\\
&\quad -608A_{acb|d}P^{c}{}_{e}P^{de} - 32A_{cad|b}P^{c}{}_{e}P^{de} + 32A_{acd|e}P^{e}{}_{i}C_{b}{}^{cdi}\\
&\quad +32A_{cad|e}P^{e}{}_{i}C_{b}{}^{cdi} + 32A_{acd|e}P^{d}{}_{i}C_{b}{}^{cei} + 32A_{cad|e}P^{d}{}_{i}C_{b}{}^{cei}\\
&\quad -64A_{abc|d}P_{ei}C^{cedi} + 32P_{cd}P_{ei}C_{a}{}^{c}{}_{b}{}^{e}{}_{|}{}^{di} + 32P_{cd}J_{|e}C_{a}{}^{c}{}_{b}{}^{e}{}_{|}{}^{d}\\
&\quad +32A_{cde}P^{d}{}_{i}C_{a}{}^{c}{}_{b}{}^{e}{}_{|}{}^{i} + 32A_{cde}P^{d}{}_{i}C_{a}{}^{c}{}_{b}{}^{i}{}_{|}{}^{e} + 64A_{acd}P_{ei}C_{b}{}^{cde}{}_{|}{}^{i}\\
&\quad +64A_{cad}P_{ei}C_{b}{}^{cde}{}_{|}{}^{i} + 8J_{|c}J_{|d}C_{a}{}^{c}{}_{b}{}^{d} - 16P_{cd|e}P^{c}{}_{i|}{}^{e}C_{a}{}^{d}{}_{b}{}^{i}\\
&\quad +32A_{cad}J_{|e}C_{b}{}^{cde} - 32A_{cad}J_{|e}C_{b}{}^{ecd} - 16A_{cde}A^{cd}{}_{i}C_{a}{}^{i}{}_{b}{}^{e}\\
&\quad +32A_{cde}A^{dc}{}_{i}C_{a}{}^{i}{}_{b}{}^{e} - 32A_{acd}A_{e}{}^{d}{}_{i}C_{b}{}^{eci} - 32A_{cad}A^{d}{}_{ei}C_{b}{}^{eci}\\
&\quad -32A_{cad}A_{ei}{}^{c}C_{b}{}^{ide} + 64A_{cad}A_{ebi}C^{cdei} - 32A_{cad}A_{ebi}C^{cedi} - 64A_{acd}P_{b}{}^{d}J_{|}{}^{c}\\
&\quad -64A_{cad}P_{b}{}^{d}J_{|}{}^{c} - 32A_{abc}JJ_{|}{}^{c} - 16A_{cda}P^{cd}J_{|b} - 224A_{acb}P^{c}{}_{d}J_{|}{}^{d}\\
&\quad -96A_{cad}A_{e}{}^{cd}P_{b}{}^{e} - 192A_{cad}A^{cd}{}_{e}P_{b}{}^{e} - 224A_{acb}P_{de}P^{cd}{}_{|}{}^{e}\\
&\quad -96A_{abc}A_{d}{}^{c}{}_{e}P^{de} - 320A_{cad}A_{eb}{}^{d}P^{ce} + 736A_{cad}A^{d}{}_{be}P^{ce}\\
&\quad -96A_{acd}P_{b}{}^{d}{}_{|e}P^{ce} - 96A_{cad}P_{b}{}^{d}{}_{|e}P^{ce} - 192A_{cad}A^{c}{}_{be}P^{de}\\
&\quad +16P_{cd}P^{c}{}_{e}C_{ai}{}^{d}{}_{j}C_{b}{}^{iej} - 32P_{cd}P^{c}{}_{e}C_{ai}{}^{d}{}_{j}C_{b}{}^{jei} - 32P_{cd}P_{ei}C_{ajb}{}^{c}C^{deij}\\
&\quad +4\boldsymbol{g}_{ab}P_{cd}P_{ei}C^{c}{}_{j}{}^{e}{}_{k}C^{dijk} - 4\boldsymbol{g}_{ab}P_{cd}P_{ei}C^{ce}{}_{jk}C^{djik} - 32P_{cd}P_{ei}P^{ei}C_{a}{}^{c}{}_{b}{}^{d}\\
&\quad +32P_{cd}P^{c}{}_{e}JC_{a}{}^{d}{}_{b}{}^{e} - 224P_{cd}P_{ei}P^{ce}C_{a}{}^{d}{}_{b}{}^{i} + 150\boldsymbol{g}_{ab}P_{cd}P_{ei}P^{e}{}_{j}C^{cidj}\\
&\quad +150\boldsymbol{g}_{ab}P_{cd}P_{ei}P^{c}{}_{j}C^{deij} - 32P_{ac}P_{de}P^{c}{}_{i}C_{b}{}^{dei} - 64P_{ac}P_{de}P^{d}{}_{i}C_{b}{}^{eci}
\end{aligned}
$$

A tensor T_{ab} such that $\mathbb{O}^{8}_{ab} = \frac{1}{384}\mathsf{T}_{(ab)}$. Here $B_{ab|cd} := \nabla_d \nabla_c B_{ab}$, etc.

---

by (14), (22), and (24), we first obtain

$$
\begin{aligned}
\not{\boldsymbol{\Delta}}\boldsymbol{D}_A T_{BCDE} = -\boldsymbol{X}_A \not{\boldsymbol{\Delta}}^2 T_{BCDE} + 2[\boldsymbol{\Delta}, \nabla_A] T_{BCDE} + \boldsymbol{R}\sharp\sharp\nabla_A T_{BCDE}\\
-\tfrac{1}{4}\boldsymbol{X}_A(\boldsymbol{R}\sharp\sharp\boldsymbol{R})\sharp\sharp T_{BCDE} + \boldsymbol{X}_A(\nabla_{|I|}\boldsymbol{R})\sharp\sharp\nabla^{|I|} T_{BCDE} + O(Q).
\end{aligned}
$$

Here the $|\cdot|$ indicates that the enclosed index is not involved in the hash action, and $(\boldsymbol{R}\sharp\sharp\boldsymbol{R})\sharp\sharp T_{BCDE}$ denotes the double hash of $\boldsymbol{R}\sharp\sharp\boldsymbol{R}$ with T_{BCDE}. From this equation together with (5), (14), (22), and (26), it follows that

$$\begin{aligned}\not{\boldsymbol{\Delta}}\boldsymbol{D}_A T_{BCDE} = -\boldsymbol{X}_A \not{\boldsymbol{\Delta}}^2 T_{BCDE} &- 2\boldsymbol{R}_A{}^P{}_B{}^Q \boldsymbol{D}_P T_{QCDE} - 2\boldsymbol{R}_A{}^P{}_C{}^Q\boldsymbol{D}_P T_{BQDE}\\ &- 2\boldsymbol{R}_A{}^P{}_D{}^Q \boldsymbol{D}_P T_{BCQE} - 2\boldsymbol{R}_A{}^P{}_E{}^Q\boldsymbol{D}_P T_{BCDQ}\\ &+ \tfrac12 \boldsymbol{R}\sharp\sharp\boldsymbol{D}_A T_{BCDE} - \tfrac14 \boldsymbol{R}\sharp\sharp\boldsymbol{X}_A\boldsymbol{R}\sharp\sharp T_{BCDE}\\ &- \tfrac14 \boldsymbol{X}_A(\boldsymbol{R}\sharp\sharp\boldsymbol{R})\sharp\sharp T_{BCDE} + \tfrac14\boldsymbol{X}_A(\boldsymbol{D}_{|I|}\boldsymbol{R})\sharp\sharp\boldsymbol{D}^{|I|}T_{BCDE}\\ &- \tfrac18 \boldsymbol{X}_A\boldsymbol{X}_I(\boldsymbol{R}\sharp\sharp\boldsymbol{R})\sharp\sharp\boldsymbol{D}^{|I|}T_{BCDE}\\ &- \tfrac18\boldsymbol{X}_A\boldsymbol{X}^I(\boldsymbol{D}_{|I|}\boldsymbol{R})\sharp\sharp\boldsymbol{R}\sharp\sharp T_{BCDE} + O(Q).\end{aligned}$$

Since the dimension is 8, it follows from (14) that $\boldsymbol{X}^A\boldsymbol{D}_A V = -4V + O(Q)$ for all $V \in \boldsymbol{\mathcal{T}}^{\Phi}(-2)$. Thus from the definition of $\not{\boldsymbol{\Delta}}$ we see that

$$\begin{aligned}\boldsymbol{X}_A \not{\boldsymbol{\Delta}}^2 T_{BCDE} = -\boldsymbol{\Delta}\boldsymbol{D}_A T_{BCDE} &- 2\boldsymbol{R}_A{}^P{}_B{}^Q \boldsymbol{D}_P T_{QCDE} - 2\boldsymbol{R}_A{}^P{}_C{}^Q\boldsymbol{D}_P T_{BQDE}\\ &- 2\boldsymbol{R}_A{}^P{}_D{}^Q \boldsymbol{D}_P T_{BCQE} - 2\boldsymbol{R}_A{}^P{}_E{}^Q\boldsymbol{D}_P T_{BCDQ}\\ &- \tfrac14 \boldsymbol{R}\sharp\sharp\boldsymbol{X}_A\boldsymbol{R}\sharp\sharp T_{BCDE} + \tfrac14 \boldsymbol{X}_A(\boldsymbol{R}\sharp\sharp\boldsymbol{R})\sharp\sharp T_{BCDE}\\ &+ \tfrac14\boldsymbol{X}_A(\boldsymbol{D}_{|I|}\boldsymbol{R})\sharp\sharp\boldsymbol{D}^{|I|}T_{BCDE} + \tfrac12\boldsymbol{X}_A\boldsymbol{R}\sharp\sharp\boldsymbol{R}\sharp\sharp T_{BCDE}\\ &+ O(Q).\end{aligned}$$

We restrict this to $\mathcal{Q}$ and then attach Y^A. The result is that for any $T \in \mathcal{T}^{2,2}[-2]$,

$$\begin{aligned}(62)\quad \not\Box_2 T_{BCDE} = -Y^A\Box D_A T_{BCDE} &- \tfrac12 Y^A W_A{}^P{}_B{}^Q D_P T_{QCDE}\\ &- \tfrac12 Y^A W_A{}^P{}_C{}^Q D_P T_{BQDE} - \tfrac12 Y^A W_A{}^P{}_D{}^Q D_P T_{BCQE}\\ &- \tfrac12 Y^A W_A{}^P{}_E{}^Q D_P T_{BCDQ} - \tfrac1{64} Y^A W\sharp\sharp X_A W\sharp\sharp T_{BCDE}\\ &+ \tfrac1{64}(W\sharp\sharp W)\sharp\sharp T_{BCDE} + \tfrac1{16}(D_{|I|}W)\sharp\sharp D^{|I|}T_{BCDE}\\ &+ \tfrac1{32} W\sharp\sharp W\sharp\sharp T_{BCDE}.\end{aligned}$$

We use this to construct a tractor formula for $\mathcal{O}^8_{ab}$. From Theorem 4.2 we have

$$(63)\qquad \mathcal{O}^8_{ce} = -\tfrac{1}{384} Y^B Z^C{}_c Y^D Z^E{}_e \not\Box_2 W_{BCDE}.$$

A short computation shows that $W\sharp\sharp W\sharp\sharp W_{BCDE} = (W\sharp\sharp W)\sharp\sharp W_{BCDE}$. Thus from (62) and (63) we have

$$\begin{aligned}(64)\quad \mathcal{O}^8_{ab} = \tfrac{1}{24576} Y^B Z^C{}_a Y^D Z^E{}_b \Big(&64 Y^A \Box D_A W_{BCDE}\\ &+ 32 Y^A W_A{}^P{}_B{}^Q D_P W_{QCDE} + 32 Y^A W_A{}^P{}_C{}^Q D_P W_{BQDE}\\ &+ 32 Y^A W_A{}^P{}_D{}^Q D_P W_{BCQE} + 32 Y^A W_A{}^P{}_E{}^Q D_P W_{BCDQ}\\ &+ Y^A W\sharp\sharp X_A W\sharp\sharp W_{BCDE} - 3 W\sharp\sharp W\sharp\sharp W_{BCDE}\\ &- 4(D_{|I|}W)\sharp\sharp D^{|I|} W_{BCDE}\Big).\end{aligned}$$

By using the same techniques as in our derivation of (61), we see that $\mathcal{O}^8_{ab} = \mathsf{T}_{(ab)}$.

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Received November 4, 2004. Revised March 21, 2005.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

THE UNITARY DUAL OF THE HERMITIAN QUATERNIONIC GROUP OF SPLIT RANK 2

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We classify the irreducible noncuspidal representations of the hermitian quaternionic group of split rank 2. We also find the complete noncuspidal unitary dual of this nonquasisplit group.

1. Introduction and preliminaries

We are interested in studying the composition series and unitary dual of the p-adic hermitian quaternionic group of semisimple rank 2, denoted by $G_2(D)$. This group is not quasisplit, so the calculations of the unitary dual cannot be obtained by the direct application of Shahidi's methods. The group has an interesting feature: it has an isolated unitary representation, a phenomenon that occurs, for example, in the case of the exceptional group G_2.

We also calculate the unitary dual supported on the non-Siegel maximal parabolic subgroup using global methods similar to those used in [Muić and Savin 2000] for the Siegel case, but resolving some obstacles related to the Langlands correspondence between the hermitian quaternionic group of semisimple rank 1 and its split form. Similar classifications were obtained for classical split groups by Sally and Tadić [1993] for p-adic $\mathrm{GSp}(2, F)$ and $\mathrm{Sp}(2, F)$, and by Konno [2001] for the quasisplit unitary group. Regarding the exceptional groups, the classification for the group G_2 was done by Muić [1997]. In the classification of the subquotients of the principal series of the hermitian quaternionic group we use the structure of the Ψ-Hopf module on the Grothendieck group of the representations of the finite length. This structure in the case of the split connected groups with the root system of types C_n and B_n was observed by Tadić [1995] and then, in the case of $O(2n, F)$, the similar result was obtained by Ban [1999].

In this section, we recall the structure of the hermitian quaternionic groups, state a result about the aforementioned structure of the Ψ-Hopf module on the Grothendieck group, and state the Langlands' classification and the criterion for square integrability. We resolve the questions of the reducibility of the induced

MSC2000: 22E35.
Keywords: unitary dual, p-adic groups, hermitian quaternionic groups.

representations for the hermitian quaternionic group of the semisimple rank 1 and make some observations about its structure.

In Section 2 we deal with the reducibility and composition series for the principal series for the group $G_2(D)$. Sections 3 and 4 are devoted to the determination of reducibility in the case of the induction from the maximal parabolic subgroups. Section 5 is devoted to the classification of the noncuspidal part of the unitary dual of the group $G_2(D)$.

For the admissible representation σ of any group we consider, we denote by ω_σ its central character (if it exists). We will denote the Steinberg representation of the group G by St_G. If H is a subgroup of the group G and $g \in G$ normalizes H, for the representation σ of the group H, we denote by ${}^g\sigma$ the representation of the group H defined by ${}^g\sigma(h) = \sigma(g^{-1}hg)$. We denote by $\{\alpha, \beta\}$ the basis of the root system corresponding to the maximal F-split torus in $G_2(D)$. The choice of the maximal F-split torus will be given in the next subsection. Also $\{\alpha, \beta\}$ will denote the basis of the root system with respect to the diagonal subgroup in SO(4, F).

Hermitian quaternionic groups. Let F be a nonarchimedean local field of characteristic zero, having residual field with q elements. We choose a uniformizer of the field and denote it by ϖ. Let D be a quaternionic algebra central over F and let τ be an involution (of the first kind) fixing the center of D. By [Mœglin et al. 1987], the division algebra D defines a reductive group G over F as follows. Let

$$V_n = e_1 D \oplus \cdots \oplus e_n D \oplus e_{n+1} D \oplus \cdots \oplus e_{2n} D$$

be a right vector space over D. The relations $(e_i, e_{2n-j+1}) = \delta_{ij}$ for $i = 1, 2, \ldots, n$ define a hermitian form on V_n:

$$\begin{aligned} (v, v') &= \varepsilon\tau((v', v)) && \text{for } v, v' \in V_n,\ \varepsilon \in \{-1, 1\}, \\ (vx, v'x') &= \tau(x)(v, v')x' && \text{for } x, x' \in D. \end{aligned}$$

We extend the involution τ on $M(k, D)$, denoting it by * :

$$g^* = (g_{ij})^* = \tau(g_{ij})^t.$$

For a smooth representation τ of the group GL(n, D), we define the representation

$$\tau^*(g) = \tau(g^{-*}).$$

By the observation in [Muić and Savin 2000], for the irreducible smooth representation τ of the group GL($\cdot$, D), the relation $\tau^* \cong \tilde{\tau}$ holds. Let $G_n(D, \varepsilon)$ be the group of the isometries of the form $(\cdot\,, \cdot)$. We can also describe $G_n(D)$ as

$$G_n(D) = \left\{ g \in \mathrm{GL}(2n, D) : g^* \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\},$$

where $J_n = \begin{pmatrix} & & 1 \\ & 1 & \\ \iddots & & \\ 1 & & \end{pmatrix}$. Using this characterization, the isomorphism

$$D \otimes_F \bar{F} \cong M(2, \bar{F}),$$

and the well-known explicit matrix realization of the algebra D, it is easy to see that $G_n(D, \varepsilon)$ is the group of F-rational points of a reductive algebraic group, an inner form of the split group $\mathrm{SO}(4n)$ or $\mathrm{Sp}(4n)$, if $\varepsilon = -1$ or $\varepsilon = 1$, respectively. In this paper, we will do explicit calculations for the case $\varepsilon = -1$, so we drop ε from the notation and, unless otherwise specified, assume $\varepsilon = -1$. Having in mind a matrix representation of the group of $\bar{F}$-rational points, there is a maximal (diagonal) split torus A_0 which has the following set of F-rational points:

$$A_0(F) = \left\{ \begin{pmatrix} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ & & & & \lambda_n^{-1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_2^{-1} & \\ & & & & & & & \lambda_1^{-1} \end{pmatrix} : \lambda_i \in F^* \right\}.$$

The element of $A_0(F)$ shown is denoted by $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$. $X(A_0)$ denotes the group of F-rational characters on A_0, which can be identified with characters on $A_0(F)$. The root system corresponding to the maximal F-split torus is of type C_n, with the set of simple roots $\{\alpha_i,\ i = 1, \dots, n\}$, where

$$\alpha_i(\mathrm{diag}(\lambda_1, \dots, \lambda_n)) = \lambda_i \lambda_{i+1}^{-1}, \quad \text{for } i = 1, \dots, n-1,$$
$$\alpha_n(\mathrm{diag}(\lambda_1, \dots, \lambda_n)) = \lambda_n^2.$$

The standard Levi F-subgroups correspond to the subsets θ of the set of simple roots $\Delta = \Delta(G_n(D), A_0)$ in the root system and we denote them by M_θ. To describe the Levi subgroups $M_\theta(F)$, we set $B_l = J_l(A_l^{-1})^* J_l$ if A_l is a quadratic matrix of order l.

(i) If $\alpha_n \notin \theta$ there are positive integers $n_1, n_2, \dots, n_k$ such that $\sum n_i = n$ and

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & & \\ & \ddots & & & & \\ & & A_{n_k} & & & \\ & & & B_{n_k} & & \\ & & & & \ddots & \\ & & & & & B_{n_1} \end{pmatrix} : A_{n_i} \in \mathrm{GL}(n_i, D) \right\}.$$

(ii) If $\alpha_n \in \theta$ there are positive integers $n_1, n_2, \ldots, n_k, r$ such that $\sum n_i + r = n$ and

$$M_\theta(F) = \left\{ \begin{pmatrix} A_{n_1} & & & & & & \\ & \ddots & & & & & \\ & & A_{n_k} & & & & \\ & & & G_r(D) & & & \\ & & & & B_{n_k} & & \\ & & & & & \ddots & \\ & & & & & & B_{n_1} \end{pmatrix} : A_{n_i} \in \mathrm{GL}(n_i, D) \right\}.$$

We describe the Langlands classification, following [Borel and Wallach 2000]. Let $\nu(x) = |x|_F$ if $x \in F$ and $\nu(x) = |\det x|_F$ if $x \in D$; here det is the norm homomorphism. For any essentially square integrable representation (mod center) δ of the group $\mathrm{GL}(m, D)$, there exists a unique real number $e(\delta)$ and a unique square-integrable representation δ^u such that $\delta = \nu^{e(\delta)}\delta^u$. We say that an (ordered) multiset $(\delta_1, \delta_2, \ldots, \delta_k)$ of irreducible essentially square-integrable representations of $\mathrm{GL}(\,\cdot\,, D)$-groups is in *standard order* if $e(\delta_1) \geq e(\delta_2) \geq \cdots \geq e(\delta_k)$. For the representations δ_i of $\mathrm{GL}(\,\cdot\,, D)$ groups and representation τ of the group $G_r(D)$, we write

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau = \mathrm{Ind}_P^{G_n(D)}(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau),$$

where P is a corresponding standard parabolic subgroup of $G_n(D)$. Suppose $(\delta_1, \delta_2, \ldots, \delta_k)$ is a multiset of irreducible essentially square-integrable representations of $\mathrm{GL}(\,\cdot\,, D)$-groups which is in the standard order, and assume that $e(\delta_k) > 0$. If τ is an irreducible tempered representation of $G_r(D)$, we consider the representation $\delta = \delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau$ of the corresponding standard Levi subgroup and let

$$e(\delta) = \big(e(\delta_1), e(\delta_1), \ldots, e(\delta_k), e(\delta_k), 0, \ldots, 0\big) \in X(A_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n.$$

Here the number $e(\delta_i)$ appears in $e(\delta)$ exactly n_i times if δ_i is a representation of the group $\mathrm{GL}(n_i, D)$, and 0 appears r times. We introduce a partial order on $X(A_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ related to the root system of type $\mathrm{C_n}$. We say that $(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n)$ if and only if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for $k = 1, \ldots, n$. This order is the one obtained in general as follows. Let $(\,\cdot\,,\cdot\,)$ be the Weyl group-invariant scalar product on $X(A_0) \otimes \mathbb{R}$ and let $(\beta_1, \beta_2, \ldots, \beta_n)$ be the basis bidual to $(\alpha_1, \ldots, \alpha_n)$. Then for $\nu_1, \nu_2 \in X(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$ we say $\nu_1 \leq \nu_2$ if and only if $(\nu_1, \beta_i) \leq (\nu_2, \beta_i)$ for all $i = 1, \ldots, n$.

Lemma 1.1 (Langlands' classification [Borel and Wallach 2000]). *The induced* (*standard*) *representation* $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, *where the irreducible essentially square-integrable representations* $(\delta_1, \delta_2, \ldots, \delta_k)$ *are in the standard order with* $e(\delta_k) > 0$ *and where* τ *is an irreducible tempered representation of the group*

$G_r(D)$, has a unique irreducible quotient, denoted $L(\delta_1, \delta_2, \dots, \delta_k; \tau)$, which is of the multiplicity one in the induced representation. In this way, we obtain every irreducible representation of the group $G_n(D)$. If the standard representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ has an irreducible subquotient $\sigma = L(\delta'_1, \dots, \delta_{k'}; \tau')$ other than its Langlands quotient, then $e(\delta'_1 \otimes \delta'_2 \otimes \cdots \otimes \delta'_{k'} \otimes \tau') < e(\delta_1 \otimes \delta_2 \otimes \cdots \delta_k \otimes \tau)$.

Given an admissible irreducible representation π of $G_n(D)$ and an ordered partition $\alpha = (n_1, n_2, \dots, n_k)$ of $n - r$, let $s_{(\alpha)}(\pi)$ denote the normalized Jacquet module of π with respect to the standard parabolic subgroup P_α with Levi subgroup isomorphic to $\mathrm{GL}(n_1, D) \times \mathrm{GL}(n_2, D) \times \cdots \times \mathrm{GL}(n_k, D) \times G_r(D)$. Let P_α denote a standard parabolic subgroup minimal with the property that $s_{(\alpha)}(\pi) \neq 0$. Each irreducible subquotient of $s_{(\alpha)}(\pi)$ is necessarily cuspidal. The square integrability criterion from [Casselman 1995] for general p-adic reductive groups readily applies, and we obtain:

Lemma 1.2 (Square integrability criterion). *A necessary and sufficient condition for an irreducible admissible representation π to be square-integrable is that, for every ordered partition $\alpha = (n_1, n_2, \dots, n_k)$ of $n - r$ minimal with the property $s_\alpha(\pi) \neq 0$ and every irreducible subquotient subquotient σ of $s_{(\alpha)}(\pi)$, we have*

$$(e(\sigma), \beta_{n_1+\cdots+n_i}) > 0 \quad \textit{for all } i = 1, \dots, k.$$

Given an admissible representation σ of the standard Levi subgroup M_θ and an element w of the Weyl group such that $w(\theta) = \theta'$ is subset of the set of the simple roots, we set $N_w = N_0 \cap w\overline{N}_\theta w^{-1}$, where $\overline{N}_\theta$ is the unipotent radical of the parabolic subgroup opposite to P_θ. For $m \in M_{\theta'}$ we define the representation of $M_{\theta'}$ by ${}^w\sigma(m) = \sigma(w^{-1}mw)$. We define (formally), for $f \in \mathrm{Ind}_{M_\theta}^{G_n(D)}(\sigma)$,

$$A_w(\sigma)f(g) = \int_{N_w} f(w^{-1}ng)\,dn.$$

If this integral converges for every f, it defines the intertwining operator

$$A_w(\sigma) : \mathrm{Ind}_{M_\theta}^{G_n(D)}(\sigma) \to \mathrm{Ind}_{M_{\theta'}}^{G_n(D)}({}^w\sigma).$$

Often, the operator A_w will have some additional (complex) arguments, usually denoting the action of the family of intertwining operators on the family of the representations, which depends on these complex numbers in an obvious way. If w is the longest element in the relative Weyl group, we call the operator A_w the long intertwining operator. Sometimes we use a different definition for the long intertwining operator: we denote by $\delta_1 \underline{\times} \delta_2 \underline{\times} \cdots \underline{\times} \delta_k \underline{\rtimes} \tau$ the representation of M_θ induced from the opposite (lower-triangular) parabolic subgroup. The long-intertwining operator from the representation space of the standard representation

$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ to the representation space of the representation $\delta_1 \underline{\times} \delta_2 \underline{\times} \cdots \underline{\times} \delta_k \underline{\rtimes} \tau$ is denoted by R and defined (formally) by

$$R(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau, N_\theta, \overline{N}_\theta) f(g) = \int_{\overline{N}_\theta} f(\bar{n}g)\, d\bar{n}.$$

If this operator is injective or surjective (for the standard representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$), the representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$ is irreducible.

Reducibility of the induced representations. We recall briefly some results from [Bernstein et al. 1984; Tadić 1990] about the reducibility of the induced representations of $\mathrm{GL}(n, D)$. To the irreducible cuspidal representation σ of the group $\mathrm{GL}(n, D)$, Jacquet–Langlands correspondence attaches an irreducible essentially square-integrable representation σ' of the group $\mathrm{GL}(2n, F)$. If σ' is a cuspidal representation, we set $s(\sigma) = 1$, and if σ' is a subquotient of the induced representation $\tau \times \tau\nu$ for some irreducible cuspidal representation τ of the group $\mathrm{GL}(n, F)$, we set $s(\sigma) = 2$. These are the only possibilities. We then set $\nu_\sigma = \nu^{s(\sigma)}$. Then, for the irreducible cuspidal representations σ_i of $\mathrm{GL}(n_1, D)$ and $\mathrm{GL}(n_2, D)$, the representation $\sigma_1 \times \sigma_2$ is reducible if and only if $n_1 = n_2$, $s(\sigma_1) = s(\sigma_2)$ and $\sigma_1 = \nu_{\sigma_2}^{\pm 1} \sigma_2$.

Using the factorization of the long intertwining operator [Speh and Vogan 1980] we obtain the following lemma, for which see also [Tadić 1994].

Lemma 1.3 (Reducibility of the principal series). *For the irreducible admissible representations τ_i of the D^* the principal series representation $\tau_1 \times \cdots \times \tau_n \rtimes 1$ of the group $G_n(D)$ reduces if and only if*

(i) *there exists i such that $\tau_i \rtimes 1$ or $\tilde{\tau}_i \rtimes 1$ reduces in $G_1(D)$, or*

(ii) *there exist distinct i and j such that $\tau_i \times \tau_j$ or $\tilde{\tau}_i \times \tau_j$ or $\tau_i \times \tilde{\tau}_j$ or $\tilde{\tau}_i \times \tilde{\tau}_j$ reduce in* $\mathrm{GL}(2, D)$.

We will describe reducibility in $G_1(D)$ shortly.

We recall from [Zelevinsky 1981; Tadić 1990] the Hopf algebra structure on the Grothendieck group R_n of smooth representations of finite length of the group $\mathrm{GL}(n, D)$. Let $R(*)$ be the Grothendieck group related to the corresponding reductive group, and $R = \bigoplus_{n \geq 0} R_n$. The multiplication $m : R \otimes R \to R$ is defined by induction, and comultiplication $m^* : R \to R \otimes R$ by Jacquet modules:

$$m^*(\pi) = \sum_{k=0}^{n} s.s(r_{(k,n-k),(n)}(\pi)) \in R \otimes R.$$

Here π is a smooth representation of finite length of $\mathrm{GL}(n, D)$, and $r_{(k,n-k),(n)}(\pi) \in R_k \otimes R_{n-k}$ is the normalized Jacquet module with respect to the maximal standard parabolic subgroup with Levi subgroup $\mathrm{GL}(k, D) \times \mathrm{GL}(n-k, D)$. By linearity, we extend the definition of m and m^* to R. The tensor product $R \otimes R$ has an algebra structure in the usual way. The comultiplication m^* is a ring homomorphism; the proof can be found in [Zelevinsky 1981] for the case of general linear groups over the field F.

Set $R(G) = \bigoplus_{n \geq 0} R(G_n(D))$. This is obviously and R-module, and a comodule structure is defined like the one in the GL-case: for a smooth, finite length representation π of the group $G_n(D)$ we put

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s(s_{(k)}(\sigma)).$$

Denote by $s : R \otimes R \to R \otimes R$ the linear map such that $s(\pi_1 \otimes \pi_2) = \pi_2 \otimes \pi_1$ for representations π_1 and π_2. Define the ring homomorphism $\Psi : R \to R \otimes R$ by

$$\Psi = (m \otimes 1) \circ (\sim \otimes m^*) \circ s \circ m^*.$$

Proposition 1.4 (The Ψ-Hopf module structure on $R(G)$). *For the smooth, finite length representation π of the group* $\mathrm{GL}(m, D)$, *and smooth, finite length representation σ of the group $G_k(D)$ we have*

$$\mu^*(\pi \rtimes \sigma) = \Psi^*(\pi) \rtimes \mu^*(\sigma).$$

Proof. As in the split case in [Tadić 1995]. □

Proposition 1.5 (R-groups). *Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ denote discrete series representations of general linear groups over the division algebra D, and τ a discrete series representation of $G_r(D)$. The representation*

$$\sigma_1 \times \sigma_2 \times \cdots \times \sigma_k \rtimes \tau$$

is multiplicity-free and has length 2^d, where d is the number of mutually nonequivalent σ_i such that $\sigma_i \rtimes \tau$ reduces.

Proof. This is proved in [Hanzer 2004]. □

Structure and reducibility results for $G_1(D)$**.** It is easy to see that

$$(1) \quad G_1(D) = \left\{ \begin{bmatrix} ad_1 & bd_1 \\ cd_1 & dd_1 \end{bmatrix} ; a, b, c, d \in F,\ d_1 \in D,\ (ad - bc)d_1\tau(d_1) = 1 \right\}.$$

So there is an epimorphism

$$\phi : G_1(D) \to F^*/(F^*)^2,\ \phi(g) = (ad - bc)(F^*)^2$$

whose kernel is isomorphic to $\mathrm{SL}(2, F)D_1$, where D_1 denotes the subgroup of elements of norm 1 in D^*, and is realized as a subgroup of diagonal matrices in $G_1(D)$. Also $\mathrm{SL}(2, F) \cap D_1 = \{\pm I\}$.

From now on, for irreducible smooth representations of D^* (which are always finite-dimensional) we will use χ to denote (unitary) characters and τ to denote (unitary) higher-dimensional representations. The distinction is important, because, by the Jacquet–Langlands correspondence, characters correspond to twists of the Steinberg representation of $\mathrm{GL}(2, F)$, and higher-dimensional representations of D^* correspond to the cuspidal representations of $\mathrm{GL}(2, F)$ [Bernstein et al. 1984]. Each (continuous) character χ of D^* is of the form $\chi = \chi' \circ \det$, for some character χ' of the field F.

Proposition 1.6. *Assume τ and χ are irreducible admissible representations of D^*.*

(i) *If $\tau \not\cong \tilde{\tau}$, then $\tau\nu^s \rtimes 1$ is irreducible for all $s \in \mathbb{R}$.*

(ii) *If $\chi^2 \neq 1$, then $\chi\nu^s \rtimes 1$ is irreducible for all $s \in \mathbb{R}$*

(iii) *Assume $\tau \cong \tilde{\tau}$. Then $\tau\nu^s \rtimes 1$ reduces if and only if $s = \pm\frac{1}{2}$ and $\omega_\tau = 1$ or $s = 0$ and $\omega_\tau \neq 1$.*

(iv) *If $\chi^2 = 1$, $\chi\nu^s \rtimes 1$ reduces if and only if $s = \pm\frac{1}{2}$.*

In both cases, when we have reducibility at $s = \pm\frac{1}{2}$, the induced representation has length 2 and one of the subquotients is a square-integrable representation, denoted by $\delta[\chi\nu^{1/2}; 1]$ (or $\delta[\tau\nu^{1/2}; 1]$). When the representation $\tau \rtimes 1$ reduces, it is a direct sum of the two nonequivalent tempered representations. The square-integrable representations obtained this way are mutually inequivalent.

Proof. Let w_0 be the unique nontrivial element of the Weyl group of $G_1(D)$. Applying the standard result of Harish-Chandra [Ban 1999], and taking into account the action of the Weyl group, the representation $\tau\nu^s \rtimes 1$ of $G_1(D)$ reduces for some $s \in \mathbb{R}$ only if $\tau^* \cong \tilde{\tau} \cong \tau$. So we assume that $\tau \cong \tilde{\tau}$. Let

$$A_{w_0}(\tau, s) f_s(e) = \int_U f_s(w_0^{-1} u)\, du$$

be the action of the standard intertwining operator, where f denotes a function in the "compact" picture of the representation $\tau \rtimes 1$ and f_s is its analytic section. Make the identification $U \cong F$. We have the explicit calculation:

$$A_{w_0}(\tau, s) f_s(e) = \int_{|n| \leq 1} f_s(w_0^{-1} n)\, dn + \sum_{k=1}^{\infty} q^{-2ks} \omega_\tau(\varpi^k) \int_{O^*} f\left(\begin{bmatrix} u^{-1} & 0 \\ \varpi^k & u \end{bmatrix}\right) du.$$

We denote by K_m the m-th congruence subgroup in $\mathrm{GL}(2, D)$. If we denote the first integral above, which always converges, by I_1, and if f is $(K_m \cap G_1(D))$-right-invariant, we get

$$A_{w_0}(\tau, s) f_s(e) = I_1 + \sum_{k=1}^{m-1} q^{-2ks} \omega_\tau(\overline{\omega}^k) \int_{O^*} \omega_\tau(u) f\left(\begin{bmatrix} 1 & 0 \\ u^{-1}\overline{\omega}^k & 1 \end{bmatrix}\right) du + \sum_{k=m}^{\infty} q^{-2ks} \omega_\tau(\overline{\omega}^k) f(e) \int_{O^*} \omega_\tau(u)\, du.$$

We conclude that the operator $A_{w_0}(\tau, s)$ is holomorphic at $s = 0$ if and only if the central character ω_τ of the representation τ is nontrivial. By the results of Harish-Chandra, in this situation the induced representation is reducible at $s = 0$, and this is the only point of the reducibility [Silberger 1980].

Now consider the self-contragredient representations τ (or χ) with trivial central character so the induced representation is irreducible at $s = 0$.

We determine the poles of the Plancherel measure by computing the composition of the intertwining operators $A_{w_0}(\tau, s) A_{w_0}(\tau, -s)$. Set $\tilde{f}_s = A_{w_0}(\tau, -s) f_{-s}$. Then

$$\begin{aligned} A_{w_0}(\tau, s) \tilde{f}_s(e) &= \int_F |\xi_0|^{-2s-1} \tilde{f}_s\left(\begin{bmatrix} 1 & 0 \\ \xi_0^{-1} & 1 \end{bmatrix}\right) d\xi_0 \\ &= \int_F |\xi_0|^{-2s-1} \int_F |\xi|^{2s-1} f_{-s}\left(\begin{bmatrix} 1 & 0 \\ \xi^{-1} + \xi_0^{-1} & 1 \end{bmatrix}\right) d\xi\, d\xi_0. \end{aligned}$$

To detect the poles of the Plancherel measure, it is enough to consider an f such that $\operatorname{supp} f_{-s} \subset P_0 \overline{U}_0$ and such that

$$f_{-s}|_{\overline{U}_0}\left(\begin{bmatrix} 1 & 0 \\ \xi & 1 \end{bmatrix}\right) = \begin{cases} 0 & \text{if } |\xi| > 1, \\ v_0 & \text{if } |\xi| \le 1. \end{cases}$$

The vector v_0 belongs to the representation space of τ. After some simple calculations, we conclude that the composition of those intertwining operators is trivial only for $s = \pm\frac{1}{2}$. (Note that the characters of D^* having order at most 2 are necessarily trivial on F^*.) □

2. The principal series representations

In this section we write down all the composition factors for the principal series representations, identifying the occurrence of square-integrable and tempered irreducible subquotients.

Recall that to each square-integrable representation of $\mathrm{GL}(n, D)$ is attached a segment of cuspidal representations [Tadić 1990]. So, the (essentially) unique square-integrable subquotient of the representation $\rho\nu_\rho^k \times \rho\nu_\rho^{k-1} \times \cdots \times \rho$ is denoted by $\delta(\rho\nu_\rho^k, \rho)$. Here ρ denotes the cuspidal representation of some $\mathrm{GL}(m, D)$. In our case, $\nu_\chi = \nu^2$ for segments of characters of D^* and $\nu_\tau = \nu$ for segments of higher dimensional irreducible cuspidal representations of D^*.

The next two propositions describe the composition series of all principal series induced from the characters. Lemma 1.3 and Proposition 1.6 give the reducibility points.

Proposition 2.1. *Let* χ_1 *be a unitary character of* D^* *and take* $\alpha \in \mathbb{R}$. *In the Grothendieck group* $R(G_2(D))$, *we have*

$$\chi_1\nu^{\alpha} \times \chi_1\nu^{\alpha+2} \rtimes 1 =$$
$$\begin{cases} L(\chi_1\nu^{3/2}\delta(\nu,\nu^{-1});1) + \pi_1 + L(\chi_1\nu^{5/2};\delta[\chi_1\nu^{1/2};1]) + L(\chi_1\nu^{5/2},\chi_1\nu^{1/2};1) \\ \hfill \text{if } \chi_1^2=1,\ \alpha=\frac{1}{2}, \\ L(\chi_1\nu^{1/2}\delta(\nu,\nu^{-1});1) + \pi_2 + L(\chi_1\nu^{3/2};\delta[\chi_1\nu^{1/2};1]) + L(\chi_1\nu^{3/2},\chi_1\nu^{1/2};1) \\ \hfill \text{if } \chi_1^2=1,\ \alpha=-\frac{1}{2}, \\ L(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1});1) + L(\chi_1\nu^{\alpha+2},\chi_1\nu^{\alpha};1) & \text{if } \alpha \in \mathbb{R}^+ \setminus \{\frac{1}{2}\}, \\ L(\chi_1\nu^{1}\delta(\nu,\nu^{-1});1) + L(\chi_1\nu^{2};\chi_1 \rtimes 1) & \text{if } \alpha = 0, \\ L(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1});1) + L(\chi_1\nu^{\alpha+2},\chi_1^{-1}\nu^{-\alpha};1) & \text{if } \alpha \in (-1,0) \setminus \{-\frac{1}{2}\}, \\ L(\chi_1\delta(\nu,\nu^{-1}) \rtimes 1) + L(\chi_1\nu,\chi_1^{-1}\nu;1) & \text{if } \alpha = -1, \\ L(\chi_1^{-1}\nu^{-1-\alpha}\delta(\nu,\nu^{-1});1) + L(\chi_1^{-1}\nu^{-\alpha},\chi_1\nu^{\alpha+2};1) & \text{if } \alpha \in (-2,-1), \\ L(\chi_1^{-1}\nu\delta(\nu,\nu^{-1});1) + L(\chi_1^{-1}\nu^{2};\chi_1 \rtimes 1) & \text{if } \alpha = -2, \\ L(\chi_1^{-1}\nu^{-1-\alpha}\delta(\nu,\nu^{-1});1) + L(\chi_1^{-1}\nu^{-\alpha},\chi_1^{-1}\nu^{-\alpha-2};1) & \text{if } \alpha < -2. \end{cases}$$

The representations π_1 *and* π_2 *are square-integrable and mutually inequivalent.*

Proof. In the course of the proof we will make extensive use of [Tadić 1998, Remark 3.2 and Lemma 3.7]. We have

$$\chi_1\nu^{\alpha} \times \chi_1\nu^{\alpha+2} \rtimes 1 = \chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1 + L(\chi_1\nu^{\alpha+2},\chi_1\nu^{\alpha}) \rtimes 1.$$

Using the Ψ-Hopf module structure (Proposition 1.4) we obtain

$$s_{(2)}(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1)$$
$$\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \otimes 1 + \chi_1^{-1}\nu^{-(\alpha+1)}\big(\delta(\nu,\nu^{-1})\big)^{\sim} \otimes 1 + \chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2} \otimes 1$$

and

$$s_{(1)}(\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1) = \chi_1\nu^{\alpha+2} \otimes \chi\nu^{\alpha} \rtimes 1 + \chi_1^{-1}\nu^{-\alpha} \otimes \chi\nu^{\alpha+2} \rtimes 1.$$

First, assume that all three expressions $\chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2} \otimes 1$, $\chi\nu^{\alpha} \rtimes 1$ and $\chi\nu^{\alpha+2} \rtimes 1$ are irreducible. Then, applying [Tadić 1998, Lemma 3.7] on Jacquet subquotients, we see that, in that case, the representation $\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1$ is irreducible. In general (without assumptions on the reducibility of those three expressions), with the aid of the Aubert involution [1995], we conclude that $\chi_1\nu^{\alpha+1}\delta(\nu,\nu^{-1}) \rtimes 1$ and

$L(\chi_1\nu^{\alpha+2}, \chi_1\nu^{\alpha}) \rtimes 1$ have the same length. So *with* the previous assumption, the representation $\chi_1\nu^{\alpha} \times \chi_1\nu^{\alpha+2} \rtimes 1$ has length 2.

Second, assume $\chi_1^{-1}\nu^{-\alpha} \times \chi_1\nu^{\alpha+2}$ is reducible. This implies $\chi_1^2 = 1$ and $\alpha \in \{0, -2\}$. For each choice of α from this set, we get representations which are the same in the Grothendieck group. We have

$$\chi_1\nu^2 \times \chi_1 \rtimes 1 = \chi_1\nu\delta(\nu, \nu^{-1}) \rtimes 1 + \chi_1\nu 1_{\mathrm{GL}(2,D)} \rtimes 1.$$

Lemma 2.2. *The representation* $\chi_1\nu\delta(\nu, \nu^{-1}) \rtimes 1$ *is irreducible.*

Proof. We can apply ideas from [Tadić 1998, Section 6]. □

Third, assume that $\chi\nu^{\alpha} \rtimes 1$ is reducible. It follows that $\chi_1^2 = 1$ and $\alpha \in \{\pm\frac{1}{2}\}$. The case $\alpha = \frac{1}{2}$ will be addressed first. We have (in the Grothendieck group)

$$(2) \quad \begin{aligned} \chi_1\nu^{5/2} \times \chi_1\nu^{1/2} \rtimes 1 &= \chi_1\nu^{3/2}\delta(\nu, \nu^{-1}) \rtimes 1 + \chi_1\nu^{3/2}1_{\mathrm{GL}(2,D)} \rtimes 1 \\ &= \chi_1\nu^{5/2} \rtimes \delta[\chi_1\nu^{1/2}; 1] + \chi_1\nu^{5/2} \rtimes L(\chi_1\nu^{1/2}; 1). \end{aligned}$$

We have

$$(3) \quad s_{(2)}(\chi_1\nu^{3/2}\delta(\nu, \nu^{-1}) \rtimes 1) = \chi_1\nu^{3/2}\delta(\nu, \nu^{-1}) \otimes 1 + \chi_1\nu^{-3/2}\delta(\nu, \nu^{-1}) \otimes 1 + \chi_1\nu^{5/2} \times \chi_1\nu^{-1/2} \otimes 1,$$

$$(4) \quad s_{(2)}(\chi_1\nu^{5/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]) = \chi_1\nu^{1/2} \times \chi_1\nu^{5/2} \otimes 1 + \chi_1\nu^{-5/2} \times \chi_1\nu^{1/2} \otimes 1.$$

From this, applying [Tadić 1998, Remark 3.2], it follows that both

$$\chi_1\nu^{3/2}\delta(\nu, \nu^{-1}) \rtimes 1 \quad \text{and} \quad \chi_1\nu^{5/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]$$

are reducible representations and that they have an irreducible subquotient in common. Examining Jacquet modules in (3) and (4), we conclude that there is only one such subquotient, denoted π_1, and it is a square-integrable representation. Analogously we conclude that $\chi_1\nu^{3/2}1_{\mathrm{GL}(2,D)} \rtimes 1$ and $\chi_1\nu^{5/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]$ have a common irreducible subquotient. We also conclude that each of the representations which appear on the right-hand side of (2) has length at most 3. If we explore Jacquet modules of the representation $\chi_1\nu^{5/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]$ with respect to the minimal parabolic subgroup, we see that this is impossible. So we obtain the decomposition of the principal series into 4 irreducible subquotients.

In the case $\alpha = -\frac{1}{2}$, the discussion is similar, but here we find a common square-integrable subquotient π_2 in $\chi_1\nu^{1/2}\delta(\nu, \nu^{-1}) \rtimes 1$ and $\chi_1\nu^{3/2} \rtimes L(\chi_1\nu^{1/2}; 1)$. Examining the Jacquet modules with respect to the minimal parabolic subgroup, we find that the principal series has length 4.

Finally, the reducibility of $\chi_1\nu^{\alpha+2} \rtimes 1$ leads to the representations already seen above. □

Proposition 2.3. *We assume* $\chi_1^2 = 1$. *Then*

$$\chi_2\nu^\alpha \times \chi_1\nu^{1/2} \rtimes 1 = \begin{cases} L(\chi_1\nu^{3/2}\delta(\nu,\nu^{-1}); 1) + \pi_1 + L(\chi_1\nu^{5/2}; \delta[\chi_1\nu^{1/2}; 1]) + L(\chi_1\nu^{5/2}, \chi_1\nu^{1/2}; 1) \\ \qquad\qquad \text{if } \chi_2 = \chi_1,\ \alpha = \frac{5}{2}, \\ L(\chi_1\nu^{1/2}\delta(\nu,\nu^{-1}); 1) + \pi_2 + L(\chi_1\nu^{3/2}; \delta[\chi_1\nu^{1/2}; 1]) + L(\chi_1\nu^{3/2}, \chi_1\nu^{1/2}; 1) \\ \qquad\qquad \text{if } \chi_2 = \chi_1,\ \alpha = \frac{3}{2}, \\ L(\chi_2\nu^{1/2}; \delta[\chi_1\nu^{1/2}; 1]) + \pi_3 + L(\chi_1\nu^{1/2}, \chi_2\nu^{1/2}; 1) + L(\chi_1\nu^{1/2}; \delta[\chi_2\nu^{1/2}; 1]) \\ \qquad\qquad \text{if } \chi_2^2 = 1,\ \chi_2 \neq \chi_1,\ \alpha = \frac{1}{2}, \\ \chi_2 \rtimes \delta[\chi_1\nu^{1/2}; 1] + L(\chi_1\nu^{1/2}; \chi_2 \rtimes 1) \qquad \text{if } \alpha = 0, \\ \qquad\qquad \text{and in other cases:} \\ L(\chi_2\nu^\alpha; \delta[\chi_1\nu^{1/2}; 1]) + L(\chi_2\nu^\alpha, \chi_1\nu^{1/2}; 1) \qquad \text{if } \alpha > 0, \\ L(\chi_2^{-1}\nu^{-\alpha}; \delta[\chi_1\nu^{1/2}; 1]) + L(\chi_2^{-1}\nu^{-\alpha}, \chi_1\nu^{1/2}; 1) \qquad \text{if } \alpha < 0. \end{cases}$$

Moreover, π_1, π_2, π_3 *are mutually inequivalent*, *square-integrable representations.*

Proof. We have $\chi_2\nu^\alpha \times \chi_1\nu^{1/2} \rtimes 1 = \chi_2\nu^\alpha \rtimes \delta[\chi_1\nu^{1/2}; 1] + \chi_2\nu^\alpha \rtimes L(\chi_1\nu^{1/2}; 1)$. Also $s_{(2)}(\chi_2\nu^\alpha \rtimes \delta[\chi_1\nu^{1/2}; 1]) = \chi_2\nu^\alpha \times \chi_1\nu^{1/2} + \chi_2^{-1}\nu^{-\alpha} \times \chi_1\nu^{1/2}$ and

$$s_{(1)}(\chi_2\nu^\alpha \rtimes \delta[\chi_1\nu^{1/2}; 1]) = \chi_1\nu^{1/2} \otimes \chi_2\nu^\alpha \rtimes 1 + \chi_2\nu^\alpha \otimes \delta[\chi_1\nu^{1/2}; 1] + \chi_2^{-1}\nu^{-\alpha} \otimes \delta[\chi_1\nu^{1/2}; 1].$$

The assumption that $\chi_2\nu^\alpha \times \chi_1\nu^{1/2}$, $\chi_2^{-1}\nu^{-\alpha} \times \chi_1\nu^{1/2}$ and $\chi_2\nu^\alpha \rtimes 1$ are irreducible, together with [Tadić 1998, Lemma 3.7], lead to the conclusion that $\chi_2\nu^\alpha \rtimes \delta[\chi_1\nu^{1/2}; 1]$ and $\chi_2\nu^\alpha \rtimes L(\chi_1\nu^{1/2}; 1)$ are irreducible. If we drop these assumptions, the only new case to consider is $\chi_2\nu^{1/2} \times \chi_1\nu^{1/2} \rtimes 1$, with $\chi_2^2 = 1$.

First, suppose that $\chi_2 = \chi_1$. The representations $\chi_1\nu^{1/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]$ and $\chi_1\nu^{1/2} \rtimes L(\chi_1\nu^{1/2}; 1)$ are irreducible. Namely, the representation $\chi_1\nu^{1/2} \times \chi_1\nu^{-1/2}$ is an irreducible unitarizable representation of $\mathrm{GL}(2, D)$, so the representation $\chi_1\nu^{1/2} \times \chi_1\nu^{-1/2} \rtimes 1$ is also unitarizable and $\chi_1\nu^{1/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]$ is its quotient. But the latter is also a standard representation, so it is irreducible.

Next suppose that $\chi_2 \neq \chi_1$. By examining $s_{(2)}(\chi_2\nu^{1/2} \rtimes \delta[\chi_1\nu^{1/2}; 1])$ we see that $\chi_2\nu^{1/2} \rtimes \delta[\chi_1\nu^{1/2}; 1]$ has length at most 2. Also we see from [Tadić 1998, Remark 3.2] that this representation and $\chi_1\nu^{1/2} \rtimes L(\chi_2\nu^{1/2}; 1)$ have one common subquotient, a square-integrable representation denoted π_3. □

Now, we describe the composition factors of all principal series induced from higher-dimensional representations. The principal series representation of the form $\tau_1\nu^\alpha \times \tau_2\nu^\beta \rtimes 1$ where τ_1, τ_2 have dimension greater than 1, are reducible only in the situations covered by the next four propositions.

Proposition 2.4. *Let τ_1 denote an irreducible, admissible, unitary representation of D^* of dimension greater than* 1. *If τ_1 is not a selfdual representation, we have*

$$\tau_1\nu^{\alpha+1}\times\tau_1\nu^{\alpha}\rtimes 1=\begin{cases} L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu^{\alpha+1},\tau_1\nu^{\alpha};1) & \text{if } \alpha>0,\\ L(\nu^{1/2}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu;\tau_1\rtimes 1) & \text{if } \alpha=0,\\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu^{\alpha+1},\tilde\tau_1\nu^{-\alpha};1) & \text{if } \alpha\in(-\frac{1}{2},0),\\ \delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2})\rtimes 1+L(\tau_1\nu^{1/2},\tilde\tau_1\nu^{-1/2};1) & \text{if } \alpha=-\frac{1}{2},\\ L(\nu^{-\alpha-\frac{1}{2}}\delta(\tilde\tau_1\nu^{1/2},\tilde\tau_1\nu^{-1/2});1)+L(\tilde\tau_1\nu^{-\alpha},\tau_1\nu^{\alpha+1};1) & \text{if } \alpha\in(-1,-\frac{1}{2}),\\ L(\nu^{-\alpha-\frac{1}{2}}\delta(\tilde\tau_1\nu^{1/2},\tilde\tau_1\nu^{-1/2});1)+L(\tilde\tau_1\nu^{-\alpha},\tilde\tau_1^{-\alpha-1};1) & \text{if } \alpha<-1.\end{cases}$$

Proof. We have

$$\tau_1\nu^{\alpha+1}\times\tau_1\nu\rtimes 1=\nu^{1/2}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2}))\rtimes 1+\nu^{1/2}L(\tau_1\nu^{1/2},\tau_1\nu^{-1/2})\rtimes 1.$$

Analogously to the proof of the previous proposition, we examine

$$\begin{aligned} s_{(2)}(\nu^{1/2+\alpha}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2}))\rtimes 1)&=\nu^{1/2+\alpha}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2})\otimes 1\\ &\quad+\nu^{-1/2-\alpha}\big(\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2})\big)^{\sim}\otimes 1+\tilde\tau_1\nu^{-\alpha}\times\tau_1\nu^{\alpha+1}\otimes 1.\end{aligned}$$

Also $s_{(1)}(\nu^{1/2+\alpha}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2}))=\tau_1\nu^{\alpha+1}\otimes\tau_1\nu^{\alpha}\rtimes 1+\tilde\tau_1\nu^{-\alpha}\otimes\tau_1\nu^{\alpha+1}\rtimes 1$. With the assumptions that $\tilde\tau_1\nu^{-\alpha}\times\tau_1\nu^{\alpha+1}$, $\tau_1\nu^{\alpha}\rtimes 1$, and $\tau_1\nu^{\alpha+1}\rtimes 1$ are irreducible, and applying [Tadić 1998, Lemma 3.7], we obtain that the representation $(\nu^{1/2+\alpha}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2})\rtimes 1$ is irreducible. By the properties of the Aubert involution, also that the representation $\nu^{1/2}L(\tau_1\nu^{1/2},\tau_1\nu^{-1/2})\rtimes 1$ is irreducible. These assumptions are met when $\tau_1\not\cong\tilde\tau_1$. □

Proposition 2.5. *Let τ_1 denote an irreducible, unitary, selfdual representation of D^* of dimension greater than* 1. *Without loss of generality we can assume $\alpha\geq-\frac{1}{2}$.*

(i) *If $\chi_{\tau_1}=1$ we have*

$$\tau_1\nu^{\alpha+1}\times\tau_1\nu^{\alpha}\rtimes 1=\begin{cases} L(\nu^{1/2}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu;\tau_1\rtimes 1) & \text{if } \alpha=0,\\ L(\nu\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu^{3/2};\delta[\tau\nu^{1/2};1])+\pi_4+L(\tau_1\nu^{3/2},\tau_1\nu^{1/2};1) & \\ & \text{if } \alpha=\frac{1}{2},\\ L(\tau_1\nu^{1/2};\delta[\tau_1\nu^{1/2};1])+L(\tau\nu_1^{1/2},\tau\nu_1^{1/2};1)+T_1+T_2 & \text{if } \alpha=-\frac{1}{2},\\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu^{\alpha+1},\tau_1\nu^{-\alpha};1) & \text{if } \alpha\in(-\frac{1}{2},0),\\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{1/2},\tau_1\nu^{-1/2});1)+L(\tau_1\nu^{\alpha+1},\tau_1\nu^{\alpha};1) & \text{if } \alpha\in\mathbb{R}^+\setminus\{\frac{1}{2}\}.\end{cases}$$

(ii) *If* $\chi_{\tau_1} \neq 1$ *and* $\tau_1 \rtimes 1 = T_3' + T_4'$ *we have*

$$\tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1 =$$
$$\begin{cases} L(\tau_1\nu; T_3') + L(\tau_1\nu; T_4') + 2L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + \pi_5 + \pi_6 & \textit{if } \alpha = 0, \\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{\alpha}; 1) & \textit{if } \alpha > 0, \\ \delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1 + L(\tau_1\nu^{1/2}, \tau_1\nu^{1/2}; 1) & \textit{if } \alpha = -\frac{1}{2}, \\ L(\nu^{\alpha+\frac{1}{2}}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + L(\tau_1\nu^{\alpha+1}, \tau_1\nu^{-\alpha}; 1) & \textit{if } \alpha \in (-\frac{1}{2}, 0). \end{cases}$$

Moreover, π_4, π_5 *and* π_6 *are mutually inequivalent square-integrable representations*, *and* T_1, T_2 *and* $\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ *in the second case are mutually inequivalent tempered* (*not square-integrable*) *representations.*

Proof. Dropping the assumptions that $\tilde{\tau}_1\nu^{-\alpha} \times \tau_1\nu^{\alpha+1}$, $\tau_1\nu^{\alpha} \rtimes 1$, and $\tau_1\nu^{\alpha+1} \rtimes 1$ are irreducible (see proof of the previous proposition), we are left to deal with the following families of representations in (5)–(8) below:

$$\tau_1\nu \times \tau_1 \rtimes 1, \text{ when } \chi_{\tau_1} = 1. \tag{5}$$

Analogously to Lemma 2.2 we conclude that $\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ is irreducible. Another representation to consider is

$$\tau_1\nu \times \tau_1 \rtimes 1, \text{ when } \chi_{\tau_1} \neq 1. \tag{6}$$

Here we obtain a single case where multiplicity one fails; this is also the only induced representation of length 6. Examining the Jacquet modules we learn that the representation $\tau_1 \rtimes T_3'$ has length at most 3 and that it is reducible (because it has the same length as $\tau_1 \rtimes T_4'$). If we assume that it has length 2, then also

$$\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1 = L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + \pi, \text{ (in } R(G_2(D))),$$

where π is some subrepresentation. We see that then $L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1)$ has to be a subrepresentation of $\tau_1\nu \times \tau_1 \rtimes 1$, but $\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ is also a subrepresentation of $\tau_1\nu \times \tau_1 \rtimes 1$. This leads to conclusion that either $L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1)$ is a subrepresentation of $\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$, or the multiplicity of $L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1)$ in $\tau_1\nu \times \tau_1 \rtimes 1$ is greater then 1; both of them false (in this situation). So we conclude that $\tau_1 \rtimes T_3'$ and $\tau_1 \rtimes T_4'$ both have length 3, and both have unique subrepresentations which are square-integrable (denoted π_5 and π_6). By careful examination of the composition sequences of the Jacquet modules, we conclude that the representations $\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ and $\nu^{1/2}L(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ have one irreducible quotient in common. In the

Grothendieck group we have

$$\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1 = L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + \pi_5 + \pi_6,$$
$$\tau_1\nu \rtimes T_3' = L(\tau_1\nu; T_3') + \pi_5 + L(\nu^{1/2}\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1).$$

The next case is

$$\tau_1\nu^{3/2} \times \tau_1\nu^{1/2} \rtimes 1 \quad \text{if } \chi_{\tau_1} = 1. \tag{7}$$

Examining the length of Jacquet modules, similarly to the case of inducing from the characters, we see that the length of $\nu\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ can't be 3, because that's inconsistent with the associativity of Jacquet modules. The rest is straightforward. We use a similar analysis to deal with

$$\tau_1\nu^{1/2} \times \tau_1\nu^{1/2} \rtimes 1 \quad \text{if } \chi_{\tau_1} = 1. \tag{8}$$ □

Proposition 2.6. *Let τ_2 be a unitary, irreducible selfdual representation of D^* of dimension greater than* 1, *with trivial central character, and let τ_1 denote a unitary irreducible representation of D^* of dimension greater than* 1.

(a) *If $\tau_1 \not\cong \tilde{\tau}_1$, we have*

$$\tau_1\nu^{\alpha} \times \tau_2\nu^{1/2} \rtimes 1 = \begin{cases} L(\tau_1\nu^{\alpha}; \delta[\tau_2\nu^{1/2}; 1]) + L(\tau_1\nu^{\alpha}, \tau_2\nu^{1/2}; 1) & \text{if } \alpha > 0, \\ T_3 + L(\tau_2\nu^{1/2}; \tau_1 \rtimes 1) & \text{if } \alpha = 0, \end{cases}$$

where $T_3 = \tau_1 \rtimes \delta[\tau_2\nu^{1/2}; 1]$ is an irreducible tempered representation.

(b) *If $\tau_1 \cong \tilde{\tau}_1$, we consider two cases*:

(i) *If $\chi_{\tau_1} = 1$, then* $\tau_1\nu^{\alpha+1} \times \tau_1\nu^{\alpha} \rtimes 1 =$

$$\begin{cases} L(\tau_1\nu^{\alpha}; \delta[\tau_2\nu^{1/2}; 1]) + L(\tau_1\nu^{\alpha}, \tau_2\nu^{1/2}; 1) & \text{if } |\alpha| \notin \mathbb{R}_0^+ \setminus \{0, \frac{1}{2}, \frac{3}{2}\}, \\ L(\tau_1\nu^{1/2}; \delta[\tau_2\nu^{1/2}; 1]) + L(\tau_2\nu^{1/2}; \delta[\tau_1\nu^{1/2}; 1]) + L(\tau_1\nu^{1/2}, \tau_2\nu^{1/2}; 1) + \pi_7 & \\ & \text{if } |\alpha| = \frac{1}{2} \text{ and } \tau_1 \not\cong \tau_2, \\ L(\tau_1\nu^{1/2}; \delta[\tau_1\nu^{1/2}; 1]) + L(\tau\nu_1^{1/2}, \tau\nu_1^{1/2}; 1) + T_1 + T_2 & \text{if } |\alpha| = \frac{1}{2} \text{ and } \tau_1 \cong \tau_2, \\ \tau_1 \times \delta[\tau_2\nu^{1/2}; 1] + L(\tau_2\nu^{1/2}; \tau_1 \rtimes 1) & \text{if } \alpha = 0, \\ L(\nu\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + L(\tau_1\nu^{3/2}; \delta[\tau\nu^{1/2}; 1]) + \pi_4 + L(\tau_1\nu^{3/2}, \tau_1\nu^{1/2}; 1) & \\ & \text{if } |\alpha| = \frac{3}{2} \text{ and } \tau_1 \cong \tau_2, \\ L(\tau_1\nu^{3/2}; \delta[\tau_2\nu^{1/2}; 1]) + L(\tau_1\nu^{3/2}, \tau_2\nu^{1/2}; 1) & \text{if } |\alpha| = \frac{3}{2} \text{ and } \tau_1 \not\cong \tau_2. \end{cases}$$

The representation π_7 is square-integrable, and $\tau_1 \times \delta[\tau_2\nu^{1/2}; 1]$ is an irreducible tempered representation.

(ii) *If* $\chi_{\tau_1} \neq 1$ *and* $\tau_1 \rtimes 1 = T_3' + T_4'$, *then*

$$\tau_1\nu^\alpha \times \tau_2\nu^{1/2} \rtimes 1 = \begin{cases} L(\tau_1\nu^\alpha; \delta[\tau_2\nu^{1/2}; 1]) + L(\tau_1\nu^\alpha, \tau_2\nu^{1/2}; 1) & \text{if } \alpha > 0, \\ L(\tau_2\nu^{1/2}; T_3') + L(\tau_2\nu^{1/2}; T_4') + T_5 + T_4 & \text{if } \alpha = 0. \end{cases}$$

T_4 *and* T_5 *are irreducible tempered representations.*

Proof. The only new case left to check, after dealing with ones which are covered by [Tadić 1998, Lemma 3.7], is

$$\tau_1\nu^{1/2} \times \tau_2\nu^{1/2} \rtimes 1 \quad \text{if } \tau_1 \not\cong \tau_2,\ \chi_{\tau_1} = \chi_{\tau_2} = 1. \tag{9}$$

This case is resolved in the same way as for the characters. □

Proposition 2.7. *Let* τ_2 *be a unitary, irreducible, self-dual representation of* D^* *such that* $\chi_{\tau_2} \neq 1$, *so that* $\tau_2 \rtimes 1 = T_3' \oplus T_4'$, *and let* τ_1 *be an irreducible unitary representation of* D^*.

(a) *If* $\alpha > 0$, *then* $\tau_1\nu^\alpha \times \tau_2 \rtimes 1 =$

$$\begin{cases} L(\tau_1\nu^{1/2}; T_3') + L(\tau_1\nu^{1/2}; T_4') + T_4 + T_5 & \text{if } \alpha = \frac{1}{2},\ \tau_1 \cong \tilde{\tau}_1,\ \omega_{\tau_1} = 1, \\ L(\tau_2\nu; T_3') + L(\tau_2\nu; T_4') + 2L(\nu^{1/2}\delta(\tau_2\nu^{1/2}, \tau_2\nu^{-1/2}); 1) + \pi_5 + \pi_6 & \\ & \text{if } \tau_1 \cong \tau_2 \text{ and } \alpha = 1, \\ L(\tau_1\nu^\alpha; T_3') + L(\tau_1\nu^\alpha; T_4') & \text{in other cases.} \end{cases}$$

(b) *If* $\alpha = 0$, *then*

$$\tau_1 \times \tau_2 \rtimes 1 = \begin{cases} T_6 + T_7 + T_8 + T_9 & \text{if } \tau_1 \cong \tilde{\tau}_1,\ \omega_{\tau_1} \neq 1,\ \tau_1 \not\cong \tau_2, \\ T_{10} + T_{11} & \text{in other cases.} \end{cases}$$

The representations T_i, $i = 6, \ldots, 11$, *are mutually inequivalent, tempered (not square-integrable) representations.*

Proof. The cases in the first part of this Proposition were already covered, and the statements of the second part follow from [Hanzer 2004]. □

We now settle the mixed case of the principal series representations.

Proposition 2.8. *Let* χ *be a unitary character of* D^*, *and let* τ *be irreducible, admissible unitary representation of* D^* *of dimension greater than* 1. *Then the principal series representation* $\chi\nu^\alpha \times \tau\nu^\beta \rtimes 1$ (*for* $\alpha, \beta \in \mathbb{R}$) *reduces only in the following cases*:

(i) *If* $\chi^2 = 1$, *then* $\tau\nu^{\beta} \times \chi^{1/2} \rtimes 1 =$

$$\begin{cases} L(\tau\nu^{\beta}; \delta[\chi\nu^{1/2}; 1]) + L(\tau\nu^{\beta}, \chi\nu^{1/2}; 1) & \text{if } \beta > 0 \text{ and } \tau\nu^{\beta} \rtimes 1 \text{ is irreducible},\\ \tau \rtimes \delta[\chi\nu^{1/2}; 1] + L(\chi\nu^{1/2}; \tau \rtimes 1) & \text{if } \beta = 0 \text{ and } \tau \rtimes 1 \text{ is irreducible},\\ T_7 + T_8 + L(\chi\nu^{1/2}, T_3') + L(\chi\nu^{1/2}, T_4') & \text{if } \tau \cong \tilde{\tau},\ \omega_\tau \neq 1,\ \beta = 0,\\ L(\tau\nu^{1/2}; \delta[\chi\nu^{1/2}; 1]) + \pi_8 + L(\chi\nu^{1/2}; \delta[\tau\nu^{1/2}; 1]) + L(\tau\nu^{1/2}, \chi\nu^{1/2}; 1) & \\ & \text{if } \tau \cong \tilde{\tau},\ \omega_\tau = 1,\ |\beta| = \tfrac{1}{2}, \end{cases}$$

where, in the third case, $\tau \rtimes 1 = T_3' + T_4'$. *The tempered representation* $\tau \rtimes \delta[\chi\nu^{1/2}; 1]$ *from the second case is irreducible, and the representation* π_8 *is square-integrable.*

(ii) *If* $\tau\nu^{1/2} \rtimes 1$ *reduces, then* $\chi\nu^{\alpha} \times \tau\nu^{1/2} \rtimes 1 =$

$$\begin{cases} L(\chi\nu^{\alpha}; \delta[\tau\nu^{1/2}; 1]) + L(\chi\nu^{\alpha}, \tau\nu^{1/2}; 1) & \text{if } \alpha > 0 \text{ and } \chi\nu^{\alpha} \rtimes 1 \text{ is irreducible},\\ L(\tau\nu^{1/2}; \delta[\chi\nu^{1/2}; 1]) + L(\chi\nu^{1/2}; \delta[\tau\nu^{1/2}; 1]) + L(\tau\nu^{1/2}, \chi\nu^{1/2}; 1) + \pi_8 & \\ & \text{if } \chi^2 = 1 \text{ and } \alpha = \tfrac{1}{2},\\ \chi \rtimes \delta[\tau\nu^{1/2}; 1] + L(\tau\nu^{1/2}; \chi \rtimes 1) & \text{if } \alpha = 0. \end{cases}$$

The tempered representation $\chi \rtimes \delta[\tau\nu^{1/2}; 1]$ *in the third case is irreducible.*

(iii) *If* $\tau \rtimes 1 = T_3' \oplus T_4'$, *then* $\chi\nu^{\alpha} \times \tau \rtimes 1 =$

$$\begin{cases} L(\chi\nu^{\alpha}; T_3') + L(\chi\nu^{\alpha}; T_4') & \text{if } \alpha > 0 \text{ and } \chi\nu^{\alpha} \rtimes 1 \text{ is irreducible},\\ \chi \rtimes T_3' + \chi \rtimes T_4' & \text{if } \alpha = 0,\\ T_7 + T_8 + L(\chi\nu^{1/2}, T_3') + L(\chi\nu^{1/2}, T_4') & \text{if } \chi^2 = 1,\ |\alpha| = \tfrac{1}{2}. \end{cases}$$

Proof. Use [Tadić 1998, Lemma 3.7]. □

3. Induced representations of the group $G_2(D)$; the Siegel case

We now consider the reducibility of the representations of the form

$$\sigma\nu^{s} \rtimes 1,$$

where σ is an irreducible admissible cuspidal representation of GL(2, D) and s is a real number. By a result of Harish-Chandra [Ban 1999], if this induced representation is reducible for some s, σ must be self-contragredient. So, from now on, we assume that $\sigma \cong \tilde{\sigma}$. Let σ' be the square-integrable representation of GL(4, F) corresponding to σ by the Jacquet–Langlands correspondence. It is actually a cuspidal representation as well [Bernstein et al. 1984].

Proposition 3.1. *Let* σ *be an irreducible, admissible, selfdual cuspidal representation of the group* GL(2, D). *The representation* $\sigma \rtimes 1$ *is irreducible if and only if*

$L(s,\sigma',\Lambda^2\rho_4)$ has a pole at $s=0$. If this is so, the representation $\sigma\nu^s \rtimes 1$, where $s\in\mathbb{R}$, reduces only for $s=\pm\frac{1}{2}$.

Proof. Recall that the Plancherel measure is defined as

$$R(s,\sigma,\bar{N}(F),N(F))R(s,\sigma,N(F),\bar{N}(F))=\mu^{-1}(s,\sigma).$$

Our notation is as in [Muić and Savin 2000]. From that paper we know that

$$\mu(s,\sigma)=\mu(s,\sigma'),$$

where on the left-hand side we have the Plancherel measure in the group $G_2(D)$, and on the right-hand side the Plancherel measure corresponding to the representation induced from σ' in $\mathrm{SO}(8,F)$. Because σ' is cuspidal, the reducibility of $\sigma'\nu^s \rtimes 1$ can be obtained directly from the Plancherel measure: there exists a unique $s_0\ge 0$ such that $\sigma'\nu^{s_0}\rtimes 1$ reduces [Silberger 1980] and

$$\begin{aligned} s_0=0 &\quad \text{if and only if } \mu(0,\sigma')\neq 0,\\ s_0>0 &\quad \text{if and only if } \mu(s,\sigma') \text{ has a pole at } s=s_0. \end{aligned}$$

So, $\sigma\nu^s\rtimes 1$ is reducible if and only if $\sigma'\nu^s\rtimes 1$ is reducible, and $s_0\in\{\frac{1}{2},0\}$, by the results in [Shahidi 1990b]. Because σ' is generic, the Plancherel measure is expressible in terms of L-functions. To conclude, $\sigma'\rtimes 1$ is irreducible if and only if $L(s,\sigma',\Lambda^2\rho_4)$ has a pole at $s=0$; see [Shahidi 1992]. □

4. Induced representations of the group $G_2(D)$; the non-Siegel case

We now consider the representations of the form

$$\tau\nu^s\rtimes\delta,$$

where τ is an irreducible admissible unitary representation of D^*, δ is an irreducible cuspidal representation of $G_1(D)$, and $s\in\mathbb{R}$. As in the previous section, to examine the reducibility, it is enough to assume that $\tau\cong\tilde{\tau}$, and $s\ge 0$. Throughout this section we keep this assumption. For an algebraic number field k, we denote its ring of adeles by A_k. We consider the restriction

$$\delta|_{\mathrm{SL}(2,F)D_1}=\sum_{i=1}^{k}\tau_i\otimes\delta_i,$$

according to the observation about the structure of $G_1(D)$ — see Equation (1). The procedure we use is this: we choose a summand in the restriction above, such as $\tau_1\otimes\delta_1$, and lift it to the discrete series representation $\tau_1\otimes\delta_1'$ of the group $\mathrm{SL}(2,F)\times\mathrm{SL}(2,F)$. Then we find representations δ' and δ'' of $\mathrm{SO}(4,F)$ such that the representation $\tau_1\otimes\delta_1'$ is a component in the restrictions of the representations

δ' and δ'' to $\mathrm{SL}(2, F)\cdot\mathrm{SL}(2, F)$. Then, using global methods, we will prove that $\mu(s, \tau\otimes\delta)^2 = \mu(s, \tau'\otimes\delta')\mu(s, \tau'\otimes\delta'')$. The difficulty in applying the global methods lies in that there are global L-packets for the group $\mathrm{SL}(2, A_k)$ including both automorphic and nonautomorphic global representations [Labesse and Langlands 1979], so we have to make some adjustments. Also, in order to ensure that the representations δ' and δ'' differ only in quadratic character, i.e., they have the same restriction to $\mathrm{SL}(2, F)\cdot\mathrm{SL}(2, F)$, we have to be careful when varying representations in the local L-packets of τ_1 and δ_1. Before we proceed with the detailed exposition, briefly remind the reader how the group $\mathrm{SL}(2, F)\cdot\mathrm{SL}(2, F)$ sits in $\mathrm{SO}(4, F)$. Let $\{\alpha, \beta\}$ denote the basis of the root system $\Phi(\mathrm{SO}(4, F), T)$, where T is a diagonal subgroup in the standard matrix realization of $\mathrm{SO}(4, F)$. So, with the obvious meaning, we choose $\alpha = e_1 - e_2$ and $\beta = e_1 + e_2$. The Levi subgroup M_α corresponding to the root α is isomorphic to $\mathrm{GL}(2, F)$; the same is true for M_β. One copy of $\mathrm{SL}(2, F)$ is standardly embedded in M_α and the other in M_β; one is block-diagonal, and the other is not.

We can choose a number field k having two places v_1 and v_2 such that $k_{v_1}\cong k_{v_2}\cong F$, and a division algebra $\mathbf{D}$ of rank 4 over k that it splits only at v_1 and v_2, with $\mathbf{D}_{v_1}\cong D\cong\mathbf{D}_{v_1}$. Then we can define the reductive group $\mathbf{G}_1$ over k such that $\mathbf{G}_1(k_v)\cong\mathrm{SO}(4, k_v)$ for all $v\notin\{v_1, v_2\}$ and $\mathbf{G}_1(k_{v_1})\cong G_1(D)\cong\mathbf{G}_1(k_{v_2})$. Analogously, we define $\mathbf{G}_2$ over k. Also, we can define $\mathbf{D}_1$, the subgroup of elements of norm 1 in $\mathbf{D}^*$ such that $\mathbf{D}_1(k_v)\cong\mathrm{SL}(2, k_v)$ for $v\notin\{v_1, v_2\}$ and $\mathbf{D}_1(k_{v_1})\cong D_1\cong\mathbf{D}_1(k_{v_2})$. We choose any of the summands from the restriction of δ, e.g., $\tau_1\otimes\delta_1$.

First assume that $\dim\delta_1 > 1$. Consider the set of ideles $((\pm I)_v)$ that can be observed as a subgroup of $\mathbf{D}_1$. We can form the character $\omega = \prod\omega_v$ on that set such that $\omega_{v_1} = \omega_{\delta_1}$, $\omega_{v_2} = \omega_{\delta_1}$, and ω_v are almost everywhere trivial. Then we can introduce the space $L(\mathbf{D}_1(A_k))$ (and other notation) as in [Flicker 1987], and study the representations of the functions from $C(\mathbf{D}_1(A_k))$ on the space $L(\mathbf{D}_1(A_k))$. We choose a full tensor $f = \otimes f_v$ from the space $C(\mathbf{D}_1(A_k))$. We can choose f in such a way that f_{v_1} and f_{v_2} are the coefficients of the representation δ_1, and at all other nonarchimedean places f_v are spherical. Then we can adjust the support of f_v at the archimedean places in such way that we can reason analogously to [Flicker 1987, Proposition §3.3 and Theorem §4.3]. We obtain the existence of an automorphic cuspidal representation $\pi'_1 = \bigotimes_v\pi'_{1,v}$ of the group $\mathbf{D}_1(A_k)$ with central character ω such that $\pi'_{1,v_1}\cong\delta_1\cong\pi'_{1,v_2}$ Then there exists a grossencharacter $\omega' = \bigotimes_v\omega'_v$ such that $\omega'|_{((\pm I_v))} = \omega$. Also, we can find an automorphic cuspidal representation $\pi' = \bigotimes_v\pi'_v$ of the group $\mathbf{D}^*(A_k)$ with central character ω' such that π'_1 embeds in π'; the proof is analogous to one in [Flicker 1992]. Note that π'_{v_1} and π_{v_2} are cuspidal representations of D^* of dimension greater then 1, so by the Jacquet–Langlands correspondence, they correspond to cuspidal representations of

$\mathrm{GL}(2, F)$. This enables us to use results of [Flicker and Kazhdan 1988] about lifts of representations of $\mathbf{D}^*(A_k)$ to the representations of $\mathrm{GL}(2, A_k)$ with one fixed cuspidal place. So there exists an automorphic cuspidal representation $\pi = \bigotimes_v \pi_v$ of $\mathrm{GL}(2, A_k)$ such that $\pi_v \cong \pi'_v$ for all $v \notin \{v_1, v_2\}$, and π_{v_i} and π'_{v_i} correspond. Let $\pi_1 = \bigotimes_v \pi_{1,v}$ denote some automorphic cuspidal representation of $\mathrm{SL}(2, A_k)$ embedded in the representation $\pi|\mathrm{SL}(2, A_k)$. We can arrange that $\pi_{1,v} \cong \pi'_{1,v}$ for every place v different from v_1, v_2. Indeed, let $\{\phi\}$ be an admissible homomorphism $\{\phi\}: W_{K/k} \to PGL(2) \times W_{K/k}$ defined by the representation $\mathrm{Ind}(W_{K/k}, W_{K/E}, \theta)$, where K is some large, but finite Galois extension of k, E a quadratic extension of k contained in K, and θ a Grossencharacter of E that doesn't factor through $Nm_{E/k}$. Let π_2 be some automorphic cuspidal representation of $\mathrm{SL}(2, A_k)$ embedded in π. If π_2 does not belong to the L-packet parameterized by $\{\phi\}$, we define a representation π_1 of $\mathrm{SL}(2, A_k)$ in the following way:

$$\pi_{1,v} = \pi'_{1,v} \quad \text{for all } v \notin \{v_1, v_2\},$$
$$\pi_{1,v_i} = \pi_{2,v_i} \quad \text{for } i = 1, 2.$$

The representation π_1 is in the same L-packet as π_2 and it is automorphic; see [Labesse and Langlands 1979]. If π_2 corresponds to $\{\phi\}$ as above, we can form π_1 as above at split places, but at v_1 and v_2 we must adjust representations to obtain a representation which is in the same L-packet as π_2 but is also automorphic. We can do so because the multiplicity with which π_1 occurs in the space of cusp forms is

$$\frac{1}{[S^\circ_\phi \setminus S_\phi]} \sum_{s \in S^\circ_\phi \setminus S_\phi} \langle s, \pi_1 \rangle,$$

with notation as in [Labesse and Langlands 1979]. So we want to make

$$\langle s, \pi_1 \rangle = \prod \langle s, \pi_{1,v} \rangle$$

a trivial character. But we can easily do that fixing at the place v_1 the representation which defines the trivial character on the local group $S^\circ_{\phi_{v_1}} \setminus S_{\phi_{v_1}}$, and adjusting accordingly at the place v_2.

Second, if $\dim \delta_1 = 1$, i.e., $\delta_1 = 1$, we fix a nonarchimedean place u outside $\{v_1, v_2\}$ and fix some cuspidal representation π_u of $\mathrm{SL}(2, k_u)$ at that place. As before, we can choose an automorphic cuspidal representation π'_1 of $\mathbf{D}_1$ which has that component on the place u, and which is unramified at the places v_1 and v_2, i.e., equal to $\delta_1 = 1$. Now there exists a lift from the automorphic cuspidal representations of $\mathbf{D}^*$ to such representations of $\mathrm{GL}(2, A_k)$, with fixed place u with cuspidal component, and, as before, we obtain the representation of $\mathrm{SL}(2, A_k)$ having properties as in the previous case.

If the finite set $\left\{g_i = \left[\begin{smallmatrix} 1 & 0 \\ 0 & x_i \end{smallmatrix}\right]\right\}$ is a set of representatives of $\mathrm{GL}(2, F)/\mathrm{SL}(2, F)F^*$, then the set

$$g_i' = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x_i & 0 & 0 \\ 0 & 0 & x_i^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

is a set of representatives (in our realization) of $\mathrm{SO}(4, F)/\big(\mathrm{SL}(2, F) \cdot \mathrm{SL}(2, F)\big)$. Now, with the same character ω as before, we can find an automorphic cuspidal representation $\varepsilon = \bigotimes_v \varepsilon_v$ of $\mathrm{SL}(2, A_k)$ with the central character ω such that

$$\begin{aligned} &\varepsilon_{v_1} \cong \varepsilon_{v_2} \cong \tau_1 && \text{if } \pi_{1,v_1} \cong \pi_{1,v_2}, \\ &\varepsilon_{v_1} \cong \tau_1,\ \varepsilon_{v_2} \cong {}^{g_i}\tau_1 && \text{if } \pi_{1,v_2} \cong {}^{g_i}\pi_{1,v_1}. \end{aligned}$$

There exists an automorphic cuspidal representation σ' of the group $\mathbf{G}_1(A_k)$ in which $\varepsilon \otimes \pi_1'$ embeds as a representation of the group $\mathrm{SL}(2, A_k)\mathbf{D}_1(A_k)$. Analogously, there exists an automorphic cuspidal representation σ of $\mathrm{SO}(4, A_k)$ in which $\varepsilon \otimes \pi_1$ embeds as a representation of $\mathrm{SL}(2, A_k) \cdot \mathrm{SL}(2, A_k)$. We can arrange that

$$\sigma'_{v_1} \cong \sigma'_{v_2} \cong \delta \qquad \text{and} \qquad \sigma'_v \cong \sigma_v \quad \text{for all } v \notin \{v_1, v_2\}.$$

Let $\gamma' = \bigotimes_v \gamma_v'$ be an automorphic cuspidal representation of $\mathbf{D}^*(A_k)$ such that $\gamma'_{v_1} \cong \gamma'_{v_2} \cong \tau$ and let γ be its lift to $\mathrm{GL}(2; A_k)$ such that $\gamma_v' \cong \gamma_v$ for all $v \notin \{v_1, v_2\}$ and $\gamma_{v_1} \cong \gamma_{v_2} \cong \tau'$, where τ' corresponds to τ by Jacquet–Langlands correspondence. This can be arranged; see [Flicker and Kazhdan 1988].

We have to normalize measures on the unipotent radicals of the groups considered in order to get the global functional equation right. We can decompose D as $F \oplus D^-$, looking at the center F of the algebra D as the τ-hermitian part of D and D^- as the τ-antihermitian part. Now, the unipotent radical of the non-Siegel parabolic subgroup in the group $G_2(D)$ in the case $\varepsilon = -1$ (the case we are now considering) is $N(F) \cong D \oplus D \oplus F$, and in the case $\varepsilon = 1$ it is $N'(F) \cong D \oplus D \oplus D^-$. Let ψ_F denote a nontrivial additive character of F. Introduce self-dual measures on $N(F)$ and $N'(F)$ by the use of the F-form $\langle x, y\rangle = \sum_{i=1}^5 x_i y_i + \tau(x_i y_i)$ on D^5, and a character ψ_F so that the self-dual measure on D^5 is the product of a self-dual measure α_F on $N(F)$ and α'_F on $N'(F)$. Fix a nontrivial character $\psi = \bigotimes_v \psi_v$ of A_k trivial on k and such that $\psi_{v_1} = \psi_{v_2} = \psi_F$. As above, at each split place we can get a self-dual measure α_v on $N(k_v)$ with respect to ψ_v, and a self-dual measure α on $N(A_k)$. In this way, we get a coherent family of measures $\{\alpha_v\}$ such that $\alpha = \prod \alpha_v$ and α is actually the Tamagawa measure [Weil 1973, §VII.2, Corollary 1], meaning that $\alpha(N(A_k)/N(k)) = 1$. The Plancherel measure is defined analogously to the Siegel case.

Proposition 4.1. $\mu(s, \tau \otimes \delta)^2 = \mu(s, \tau' \otimes \sigma_{v_1})\mu(s, \tau' \otimes \sigma_{v_2})$.

Proof. Denote by S a finite set of places containing v_1 and v_2, all the places of residual characteristic 2, and all the places where γ_v, σ_v, and ψ_v are ramified. For every $v \notin S$, let $f_{v,s}$ denote the unique unramified vector in $\gamma_v' \nu^s \rtimes \sigma_v'$, normalized to be equal to 1 on the maximal compact subgroup K_v. Analogously, define $\bar{f}_{v,s}$ in $\gamma_v' \nu^s \underline{\rtimes} \sigma_v'$. In the L-group $SO(8, \mathbb{C})$ of $SO(8, k_v)$, the action (representation) of the $GL(2, \mathbb{C}) \times SO(4, \mathbb{C})$ on the unipotent radical is equal to $\Lambda^2(\mathbb{C}^2) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^4)$. So, for $v \notin S$ we can explicitly calculate the constants $c(v, s, \gamma_v' \otimes \sigma_v')$ in terms of L-functions, where

$$R(s, \gamma_v' \otimes \sigma_v', N(k_v), \bar{N}(k_v)) f_{v,s} = c(v, s, \gamma_v' \otimes \sigma_v') \bar{f}_{v,s}.$$

We then have

$$c(v, s, \gamma_v' \otimes \sigma_v') = \frac{L(s, \gamma_v' \otimes \sigma_v', \rho_2 \otimes \rho_4)(1 - \chi_{\gamma_v'}(\varpi) q_v^{-s})^{-1}}{L(1+s, \gamma_v' \otimes \sigma_v', \rho_2 \otimes \rho_4))(1 - \chi_{\gamma_v'}(\varpi) q_v^{-s-1})^{-1}}.$$

It is easily seen that the product

$$c_S(s, \gamma' \otimes \sigma') = \prod_{v \notin S} c(v, s, \gamma_v' \otimes \sigma_v')$$

converges in some right half-plane, and it continues to a meromorphic function on $\mathbb{C}$. Analogously we have

$$R(s, \gamma_v' \otimes \sigma_v', \bar{N}(k_v), N(k_v)) \bar{f}_{v,s} = c(v, -s, \tilde{\gamma}_v' \otimes \sigma_v') f_{v,s}.$$

We now take $f_s = \bigotimes_v f_{v,s} \in \gamma' \nu^s \rtimes \sigma'$, where for each $v \notin S$ we have chosen spherical $f_{v,s}$ as above. Because we have chosen the Tamagawa measure on the (global) unipotent radical we have the global functional equation

$$R(s, \gamma' \otimes \sigma', \bar{N}(A_k), N(A_k)) R(s, \gamma' \otimes \sigma', N(A_k), \bar{N}(A_k)) f_s = f_s;$$

see [Mœglin and Waldspurger 1995, Theorem IV.1.10]. When the right-hand side of this equation is written as a product of local intertwining operators, the local Plancherel measures appear. So we have

$$\prod_{v \in S} \mu(s, \gamma_v' \otimes \sigma_v') c_S(s, \gamma' \otimes \sigma') c_S(-s, \tilde{\gamma}' \otimes \sigma') = 1.$$

By analogy with the previous equation for $\mathbf{G}_2(A_k)$, we have the equation

$$\prod_{v \in S} \mu(s, \gamma_v \otimes \sigma_v) c_S(s, \gamma \otimes \sigma) c_S(-s, \tilde{\gamma} \otimes \sigma) = 1$$

in $SO(8, A_k)$. But at each split place we have an isomorphism $\mathbf{G}_2(k_v) \cong SO(8, k_v)$ that preserves unipotent radicals, and we have isomorphic representations, so we have equality of the Plancherel measures. From this, it follows that

$$\mu(s, \tau \otimes \delta)^2 = \mu(s, \tau' \otimes \sigma_{v_1})\mu(s, \tau' \otimes \sigma_{v_2}). \qquad \square$$

Remark. Because of our adjustment of the representation ε, we conclude that the representations σ_{v_1} and σ_{v_2} differ in the quadratic character that is trivial on $\mathrm{SL}(2, F) \cdot \mathrm{SL}(2, F)$.

We now compute the Plancherel measure above. Because τ_1 is a generic representation for some nontrivial character of F and π_{v_1} is a generic representation for some nontrivial character, we conclude that σ_{v_1} is a generic representation of the group $\mathrm{SO}(4, F)$. We can now use [Shahidi 1990b] and express the Plancherel measure in terms of γ-factors. We fix a nontrivial additive character ψ of F and obtain, up to the exponential factor,

$$\mu(s, \tau' \otimes \sigma_{v_1}) \approx \frac{\gamma(2s, \tau', \Lambda^2\rho_2, \psi)\gamma(s, \tau' \times \sigma_{v_1})}{\gamma(1+2s, \tau', \Lambda^2\rho_2, \psi)\gamma(1+s, \tau' \times \sigma_{v_1})}.$$

The only difficulty appears in the calculation of the Rankin–Selberg γ-factor of the groups $\mathrm{GL}(2, F) \times \mathrm{SO}(4, F)$. If σ_{v_1} is noncuspidal, the computation is straightforward, using the multiplicativity of γ-factors [Shahidi 1990a]. So, assume that the representations σ_{v_i} appearing in the previous proposition are square-integrable, noncuspidal representations. This is the case when, with the previous notations, $\delta_1 = 1$. Let π denote the cuspidal unitary representation of $\mathrm{GL}(2, F)$ with trivial central character such that $\tau_1 \hookrightarrow \pi|_{\mathrm{SL}(2,F)}$. Then π is a self-contragredient representation. Such π's differ mutually by a quadratic character. We denote the basis of the root system for $\mathrm{SO}(4, F)$ by $\{\alpha, \beta\}$. Now, the standard Levi subgroup M_α is diagonally embedded in $\mathrm{SO}(4, F)$ and contains the diagonal version of $\mathrm{SL}(2, F)$, and M_β isn't diagonal and also contains the other copy of $\mathrm{SL}(2, F)$. Consider the representation $\mathrm{Ind}_{M_\beta}^{\mathrm{SO}(4,F)} \pi\nu^{1/2}$. It is easy to see that this representation restricted to $\mathrm{SL}(2, F) \cdot \mathrm{SL}(2, F)$ decomposes as

$$\mathrm{Ind}_{M_\beta}^{\mathrm{SO}(4,F)} \pi\nu^{1/2}|_{\mathrm{SL}(2,F)\,\mathrm{SL}(2,F)} = \sum \nu \rtimes 1 \otimes \tau_i,$$

where the τ_i are components of the restriction of π to $\mathrm{SL}(2, F)$. Let δ' denote the unique square-integrable subquotient of $\mathrm{Ind}_{M_\beta}^{\mathrm{SO}(4,F)} \pi\nu^{1/2}$. Then $\mathrm{St}_{\mathrm{SL}(2,F)} \otimes \tau_1$ injects in δ', so σ_{v_1} and σ_{v_2} differ from δ' by a quadratic character. We can conclude:

Corollary 4.2. *Assume that the representations σ_{v_i} are not cuspidal and that σ_{v_1} injects in $\mathrm{Ind}_{P_\beta}^{\mathrm{SO}(4,F)} \pi\nu^{1/2}$.*

(i) *If* $\dim \tau > 1$ *then*

(a) *if $\chi_{\tau'}(\varpi) \neq 1$ the representation $\tau\nu^s \rtimes \delta$ reduces only for $s = 0$, and*

(b) *if $\chi_{\tau'}(\varpi) = 1$ then $\tau \rtimes \delta$ is irreducible, and $\tau\nu^s \rtimes \delta$ reduces at $s = \frac{1}{2}$ or at $s = \frac{3}{2}$, depending on whether $\tau' \ncong \pi$ or $\tau' \cong \pi$, respectively.*

(ii) *If* $\tau = \chi$ *is a quadratic character, the representation* $\chi\nu^s \rtimes \delta$ *reduces only at* $s = \frac{1}{2}$.

Proof. In the formula for Plancherel measure we include the expression for the Rankin–Selberg factor

$$\gamma(s, \tau' \times \sigma_{v_1}, \psi) = \gamma(s, \tau' \times \pi\nu^{1/2}, \psi)\gamma(s, \tau' \times \pi\nu^{-1/2}, \psi).$$

If τ' is cuspidal, we obtain the claim, and if $\tau' = \chi\mathrm{St}_{\mathrm{GL}(2,F)}$ we have

$$\gamma(s, \chi\mathrm{St}_{\mathrm{GL}(2,F)} \times \pi\nu^{\frac{1}{2}}, \psi) = \gamma(s, \chi\nu^{1/2} \times \pi\nu^{1/2}, \psi)\gamma(s, \chi\nu^{-1/2} \times \pi\nu^{1/2}, \psi) = 1.$$

□

We denote by $^\varepsilon$ the conjugation in SO(4, F) by the element

$$\varepsilon = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

of $O(4, F) \setminus \mathrm{SO}(4, F)$, and, accordingly, if π is the representation of SO(4, F) we denote the representation of SO(4, F) obtained using the involution $^\varepsilon$ by $^\varepsilon\pi$. Keeping the assumptions from the proposition, in the case of cuspidal σ_{v_i}'s we have

Corollary 4.3. (i) *Suppose* $\tau = \chi$ *is a quadratic character of* D^*.

(a) *If* $^\varepsilon\sigma_{v_1} \not\cong \sigma_{v_1}$ *the representation* $\chi\nu^s \rtimes \delta$ *reduces only at* $s = \frac{1}{2}$.

(b) *If* $^\varepsilon\sigma_{v_1} \cong \sigma_{v_1}$ *(so that also* $^\varepsilon\sigma_{v_2} \cong \sigma_{v_2}$*) then if at least one of the representations* $\chi \rtimes \tilde{\sigma}_{v_i}$, $i = 1, 2$, *of the group* SO(6, F) *is irreducible,* $\chi\nu^s \rtimes \delta$ *reduces (only) for* $s = \frac{3}{2}$. *On the other hand, if both of the representations* $\chi \rtimes \tilde{\sigma}_{v_i}$, $i = 1, 2$, *reduce,* $\chi\nu^s \rtimes \delta$ *reduces for* $s = \frac{1}{2}$.

(ii) *Suppose* $\dim \tau > 1$.

(a) *If* $\chi_\tau(\varpi) = 1$ *the representation* $\tau\nu^{1/2} \rtimes \delta$ *reduces.*

(b) *If* $\chi_\tau(\varpi) = -1$ *then the points of reducibility are at* $s = 0$ *or* $s = 1$. *We have reducibility at* $s = 1$ *if at least one of the representations* $\tau' \rtimes \sigma_{v_i}$ *of the group* SO(8, F) *is irreducible.*

Proof. We prove (i). Using the multiplicativity of γ-factors, we come to γ-factors $\gamma(s, \chi\nu^{1/2} \times \sigma_{v_i})$, which leads us to consider the reducibility of the representations $\chi\nu^s \rtimes \sigma_{v_i}$ of the group SO(6, F). It is well known that in order to have reducibility for some real number s, the nontrivial element of the Weyl group has to fix the representation $\chi \otimes \sigma_{v_i}$, which amounts to the statement that $^\varepsilon\sigma_{v_1} \cong \sigma_{v_1}$. If it isn't so, the aforementioned L-functions and γ-factors appearing in the case of SO(6, F)

are trivial, so the only pole of the Plancherel measure comes from the Hecke L-function for τ'. If ${}^{\varepsilon}\sigma_{v_1} \cong \sigma_{v_1}$, the irreducibility of $\chi \rtimes \tilde{\sigma}_{v_1}$ implies that $L(s, \chi \times \sigma_{v_1})$ has a pole at $s = 0$. This implies that $\mu(s, \tau \otimes \delta)$ has a pole at $s = \frac{3}{2}$, no matter the situation with $\chi \rtimes \tilde{\sigma}_{v_2}$. On the other hand, if both of the representations $\chi \rtimes \tilde{\sigma}_{v_i}$ reduce, the Rankin–Selberg L-functions are holomorphic for real s and the only poles of $\mu(s, \tau \otimes \delta)$ come from the Hecke L-function for $\chi \mathrm{St}_{\mathrm{GL}(2,F)}$.

The proof of (ii) follows from [Goldberg and Shahidi 2001, Theorem 4.8]. □

5. Unitary dual of the group $G_2(D)$

We are interested in finding the hermitian, and especially irreducible unitarizable representations of $G_2(D)$. We will list them by grouping together the ones with the same cuspidal support.

5.1. ***Unitary subquotients of the principal series.*** Let χ_1 and χ_2 denote unitary characters of D^*. Let $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. Without loss of generality we can assume that $s_1 \geq s_2 \geq 0$.

Proposition 5.1. *Assume that we have unitary characters χ_1 and χ_2 such that $\chi_1^2 \neq 1$ and $\chi_2^2 \neq 1$.*

(i) *If $\chi_1 \neq \chi_2^{\pm 1}$ then the representation π has a hermitian subquotient if and only if $s_1 = s_2 = 0$ and then it is an irreducible tempered representation.*

(ii) *If $\chi_1 = \chi_2$ then the representation π has a hermitian subquotient if and only if $s_1 = s_2 = 0$ and then it is an irreducible tempered representation.*

(iii) *Suppose if $\chi_1 = \chi_2^{-1}$. If $s_2 = 0$ the representation π has a hermitian subquotient only if $s_1 = 0$ and we obtain an irreducible tempered representation (isomorphic to one obtained in the previous case for the same χ_1).*

If $s_2 > 0$ the representation π has a hermitian subquotient only if $s_1 = s_2$; then for all $s_1 > 0$ all the subquotients of the representation π are hermitian. For $s_1 \in (0, 1)$ we have $\pi = \chi_1\nu^{s_1} \times \chi_1^{-1}\nu^{s_1} \rtimes 1 = L(\chi_1\nu^{s_1}, \chi_1^{-1}\nu^{s_1}; 1)$ and π is a unitary representation. For $s_1 > 1$, π is not a unitary representation. For $s_1 = 1$ we have

$$\pi = \chi_1\delta(\nu, \nu^{-1}) \rtimes 1 + L(\chi_1\nu, \chi_1^{-1}\nu; 1),$$

where the first subquotient is a tempered representation, and the other is unitary (nontempered).

Proof. The first two cases follow from the criterion for the hermiticity of the Langlands quotient. For the third case we observe that $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1}$ is the complementary series of the group $\mathrm{GL}(2, D)$ for $\alpha \in (0, 1)$. From this, it follows that $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1} \rtimes 1$ has exclusively unitarizable subquotients for $s_1 \in (0, 1]$. □

Proposition 5.2. *Let χ_1 and χ_2 be unitary characters such that $\chi_1^2 = 1$ and $\chi_2^2 \neq 1$. Again, let $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. The representation π has a hermitian subquotient only if $s_2 = 0$; then all of its subquotients are hermitian representations. In this case, for $s_1 = 0$, π is an irreducible tempered representation; for $s_1 = \frac{1}{2}$, the representation π has an irreducible tempered subquotient; for $s_1 \in (0, \frac{1}{2}]$, the representation $L(\chi_1\nu^{s_1}; \chi_2 \rtimes 1)$ is unitarizable; and for $s_1 > \frac{1}{2}$, the representation π is irreducible and nonunitarizable.*

Proof. We just comment on the case $s_2 = 0$. We have the standard intertwining operators $A_{w_{2\alpha+\beta}}(s_1) : \chi_1\nu^{s_1} \times \chi_2 \rtimes 1 \to \chi_1\nu^{-s_1} \times \chi_2 \rtimes 1$, which converge for $s_1 > 0$. These operators define, for $s_1 > \frac{1}{2}$, a continuous family, indexed by s_1, of nondegenerate hermitian forms on the compact picture X of the representation $\chi_1 \times \chi_2 \rtimes 1$ by means of

$$(f_1, f_2)_{s_1} = \int_K \langle f_{1,s_1}(k), A_{w_{2\alpha+\beta}}(s_1) f_{2,s_1}(k)\rangle \, dk.$$

Here f_1 and f_2 belong to X and f_{1,s_1}, f_{2,s_1} denote their holomorphic sections, which are identified with elements from $\chi_1\nu^{s_1} \times \chi_2 \rtimes 1$. If one of these forms were unitarizable, meaning that the irreducible representation $\chi_1\nu^{s_1} \times \chi_2 \rtimes 1$ is unitarizable, all the other forms would have to be unitarizable, too, because of the connectedness of the indexing set. But for $s_1 > \frac{5}{2}$ the representation $\chi_1\nu^{s_1} \times \chi_2 \rtimes 1$ has unbounded matrix coefficients, which implies nonunitarizability. The operator $A_{w_{2\alpha+\beta}}(s_1)$ has a pole at $s_1 = 0$, but we can normalize it by multiplying it by an appropriate real polynomial; we obtain, for $s_1 \in [0, \frac{1}{2})$, a family of (normalized) intertwining operators $A'_{w_{2\alpha+\beta}}(s_1)$ which also define a continuous family of nondegenerate hermitian forms on X. By the same argument, we obtain the unitarizability of the representations considered for $s_1 \in [0, \frac{1}{2})$. In this way, we obtain the complementary series representations, and, by the results in [Miličić 1973], the subquotients of the representation on the edge of the complementary series ($s_1 = \frac{1}{2}$) are unitarizable. □

Proposition 5.3. *With notation as before, assume that $\chi_1^2 \neq 1$ and $\chi_2^2 = 1$. Then the representation π has a hermitian subquotient only if $s_1 = s_2 = 0$, and π is then irreducible and tempered. This representation is already described in the previous proposition.*

The proof is left to the reader.

Proposition 5.4. *Assume that we have unitary characters χ_1 and χ_2 such that $\chi_1^2 = \chi_2^2 = 1$ and $\chi_1 \neq \chi_2$. Let $\pi = \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1$. Consider the regions defined on the s_1s_2 plane in Figure 1 (which also includes points that do not have $s_1 \geq s_2$).*

(i) *The representation π for (s_1, s_2) from the closed region I has all its subquotients unitarizable. The composition factors are given in Proposition 2.3.*

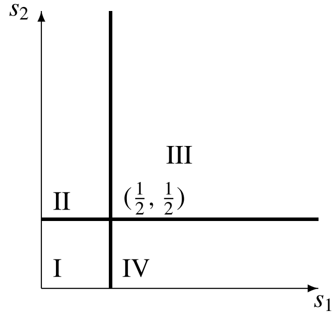

Figure 1

(ii) *For (s_1, s_2) in each of the open regions II, III, IV we obtain representations all of whose subquotients are hermitian, but none unitarizable.*

Proof. For (s_1, s_2) with $s_1 \geq s_2 > 0$ from open region I we have the family of standard long intertwining operators $A(s_1, s_2) : \chi_1\nu^{s_1} \times \chi_2\nu^{s_2} \rtimes 1 \to \chi_1\nu^{-s_1} \times \chi_2\nu^{-s_2} \rtimes 1$ which defines a continuous family of nondegenerate hermitian forms on the compact picture. So, we can fix such a pair (s_1, s_2) and form the one-parameter family $t \to (ts_1, ts_2)$ for $t \geq 0$. Because $\chi_1 \times \chi_2 \rtimes 1$ is irreducible, we can assume that we have unitarizable representations for $t \geq 0$, until $\chi_1\nu^{ts_1} \times \chi_2\nu^{ts_2} \rtimes 1$ starts being reducible. An analogous reasoning and the unboundedness of the matrix coefficients ensures the nonunitarizability of the representations on the unbounded regions. We, of course, could apply immediately [Tadić 1983] to conclude that. □

Proposition 5.5. *Given a quadratic character χ_1 of D^*, set $\pi = \chi_1\nu^{s_1} \times \chi_1\nu^{s_2} \rtimes 1$. All subquotients of π are hermitian representations. Let the notation be as in Figure 2.*

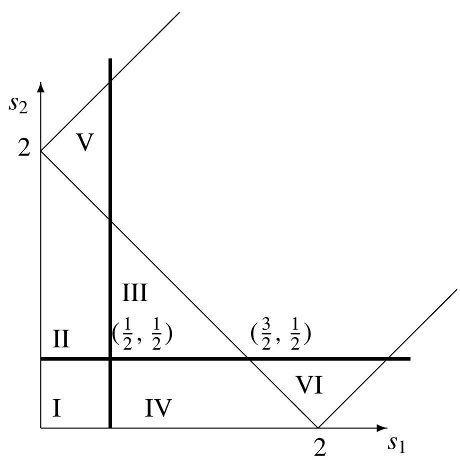

Figure 2

(i) *First we consider the open regions in the figure. Only for* (s_1, s_2) *lying in the* (*open*) *regions I or III is it the case that* π *is unitarizable* (*irreducible*) *nontempered representation.*

(ii) *Now we consider the boundaries. Besides the square-integrable and tempered subquotients, the other unitarizable subquotients that appear for* (s_1, s_2) *lying on the boundaries are the subquotients of* π *for* (s_1, s_2) *on the boundary of regions I and III and* $L(\chi_1\nu^{5/2}, \chi_1\nu^{1/2}; 1)$.

Proof. As in the previous proposition, we obtain the unitarizability of region I. Consider $\chi_1\nu^{s_1} \times \chi_1\nu^{-s_1}$ in $\mathrm{GL}(2, D)$ for $s_1 \in (0, 1)$. These (irreducible) representations belong to the complementary series of $\mathrm{GL}(2, D)$, so all the subquotients of the representation $\chi_1\nu^{s_1} \times \chi_1\nu^{s_1} \rtimes 1$ are unitarizable. In the open region III this gives unitarizability of some representations and the unitarizability on the entire region then follows. This gives, by [Miličić 1973], the unitarizability of all subquotients for π from the boundaries of I and III. Now, consider the representation $\chi_1\nu^{5/2} \times \chi_1\nu^{1/2} \rtimes 1$. If we prove that it has some nonunitarizable subquotients, this would imply the nonunitarizability in the open region VI (and, symmetrically, in V). We have

$$\chi_1\nu^{5/2} \times \chi_1\nu^{1/2} \rtimes 1 = L(\chi_1\nu^{3/2}\delta(\nu, \nu^{-1}); 1) + \pi_1 + L(\chi_1\nu^{5/2}; \delta[\chi_1\nu^{1/2}; 1]) + L(\chi_1\nu^{5/2}, \chi_1\nu^{1/2}; 1).$$

The unitarizability of $L(\chi_1\nu^{5/2}, \chi_1\nu^{1/2}; 1)$ is proved using global methods [Grbac 2004]. In the case $\chi_1 = 1$, $L(\chi_1\nu^{5/2}, \chi_1\nu^{1/2}; 1)$ is a trivial character, so we know from [Casselman 1981] that the only unitarizable subquotients of $\nu^{5/2} \times \nu^{1/2} \rtimes 1$ are the Steinberg representation and the trivial one. In general, the nonunitarizability of $L(\chi_1\nu^{3/2}\delta(\nu, \nu^{-1}); 1)$ and $L(\chi_1\nu^{5/2}; \delta[\chi_1\nu^{1/2}; 1])$ can be proved using the Howe–Moore theorem [Borel and Wallach 2000]. Denote $L(\chi_1\nu^{3/2}\delta(\nu, \nu^{-1}); 1)$ by π. Consider the unbounded set $S = \{a_0 \in A_0 : |\alpha_1(a_0)|_F \leq 1,\ |\alpha_2(a_0)| = 1\}$. We have

$$s_{(1,1)}(\pi) = \chi_1\nu^{1/2}\otimes\chi_1\nu^{5/2} + \chi_1\nu^{1/2}\otimes\chi_1\nu^{-5/2} + \chi_1\nu^{-5/2}\otimes\chi_1\nu^{1/2}.$$

Let v and $\tilde{v}$ be the canonical lifts of the vectors in Jacquet modules corresponding to the last summand in this sum. There exists $\varepsilon > 0$ such that for every a_0 from $A_0(\varepsilon)$ we have

$$\langle \pi(a_0)v, \tilde{v}\rangle = \delta_{P_0}^{1/2}(a_0)(\chi_1\nu^{-5/2} \otimes \chi_1\nu^{1/2})(a_0)\langle j_{P_0}(v), \tilde{j}_{P_0}(\tilde{v})\rangle.$$

Here $A_0(\varepsilon) = \{a_0 \in A_0 : |\alpha_1(a_0)|_F \leq \varepsilon,\ |\alpha_2(a_0)|_F \leq \varepsilon\}$. The vectors $j_{P_0}(v)$ and $\tilde{j}_{P_0}(\tilde{v})$ denote the projection on the corresponding Jacquet module. We fix an element a_0 from $A_0(\varepsilon)$. Then a_0S is a subset of $A_0(\varepsilon)$. So, we can find an unbounded sequence of elements in a_0S that defines a sequence of matrix coefficients of π not vanishing at infinity. By the Howe–Moore theorem π is not a uniformly bounded representation, hence is not unitarizable. Quite analogously we prove the nonunitarizability of $L(\chi_1\nu^{5/2}; \delta[\chi_1\nu^{1/2}; 1])$.

Now, consider the representation

$$\chi_1\nu^2 \times \chi_1 \rtimes 1 = L(\chi_1\nu\delta(\nu, \nu^{-1}); 1) + L(\chi_1\nu^2, \chi_1 \rtimes 1).$$

We prove that both subquotients on the right are nonunitarizable. We have a holomorphic family of the irreducible hermitian representations $\chi_1\nu^s\delta(\nu, \nu^{-1}) \rtimes 1$ for $s \in [1, \frac{3}{2})$, because we have a holomorphic family of nondegenerate hermitian forms obtained by standard intertwining operators on the compact picture of the representation $\chi_1\delta(\nu, \nu^{-1}) \rtimes 1$. If we assume unitarizability at $s = 1$, this would imply unitarizability on the whole interval, and the unitarizability of all the subquotients on the edge of the interval, at $s = \frac{3}{2}$, which is false by the preceding reasoning. Analogously, the unitarizability of $L(\chi_1\nu^2; \chi_1 \rtimes 1) = \chi_1\nu L(\nu, \nu^{-1}) \rtimes 1$ would imply the unitarizability of all the subquotients of $\chi_1\nu^{3/2}L(\nu, \nu^{-1}) \rtimes 1$; but we have shown that this is not the case. This proves nonunitarizability on the region IV and on the remaining boundaries. □

We continue with the examination of the principal series representations induced by the higher-dimensional representations of D^*. Let τ_1 and τ_2 be unitarizable representations of D^* of dimension greater then 1. Let $\pi = \tau_1\nu^{s_1} \times \tau_2\nu^{s_2} \rtimes 1$. We can assume that $s_1 \geq s_2 \geq 0$. The next four propositions are completely analogous to the first four propositions in the previous subsection, so we just note them.

Proposition 5.6. *With the notation as above, assume that* $\tau_1 \ncong \tilde{\tau}_1$ *and* $\tau_2 \ncong \tilde{\tau}_2$.

(i) *If* $\tau_1 \ncong \tau_2$ *and* $\tau_1 \ncong \tilde{\tau}_2$, *then the representation* π *has a hermitian subquotient if and only if* $s_1 = s_2 = 0$, *and then* π *is an irreducible tempered representation.*

(ii) *If* $\tau_1 \cong \tau_2$ *then* π *has a hermitian subquotient if and only if* $s_1 = s_2 = 0$, *and then* π *is an irreducible tempered representation.*

(iii) *If* $\tau_1 \cong \tilde{\tau}_2$ *the representation* π *has a hermitian subquotient if and only if* $s_1 = s_2$. *In that case all of its subquotients are hermitian. For* $s_1 \in (0, \frac{1}{2})$ *the representation* $L(\tau_1\nu^{s_1}, \tilde{\tau}_1\nu^{s_1}; 1)$ *is unitarizable, for* $s_1 > \frac{1}{2}$ *nonunitarizable. We also get tempered subquotients for* $s_1 \in \{0, \frac{1}{2}\}$.

Proposition 5.7. *Assume that* $\tau_1 \cong \tilde{\tau}_1$ *and* $\tau_2 \ncong \tilde{\tau}_2$. *Then the representation* π *has a hermitian subquotient only if* $s_2 = 0$. *In that case, if* $\omega_{\tau_1} = 1$, π *has a tempered subquotient only if* $s_1 = \frac{1}{2}$ *or* $s_1 = 0$. *On the other hand,* $L(\tau_1\nu^{s_1}, \tau_2 \rtimes 1)$ *is unitarizable for* $s_1 \in (0, \frac{1}{2}]$, *and for* $s_1 > \frac{1}{2}$ *it is a hermitian nonunitarizable representation. If* $\omega_{\tau_1} \neq 1$, π *has a tempered subquotient only if* $s_1 = 0$; *in that case* π *is a sum of two nonequivalent tempered representations and* $L(\tau_1\nu^{s_1}, \tau_2 \rtimes 1)$ *is hermitian nonunitarizable representation for every positive* s_1.

Proposition 5.8. *Assume that $\tau_1 \ncong \tilde{\tau}_1$ and $\tau_2 \cong \tilde{\tau}_2$. Then π has a hermitian subquotient only if $s_1 = s_2 = 0$. In this case π is an irreducible tempered representation or a sum of two nonequivalent tempered representations, depending on the central character of τ_1. These tempered representations are those described in the previous proposition.*

Proposition 5.9. *Assume that τ_1 and τ_2 are nonisomorphic, unitary, self-contragredient representations. We keep our assumption $s_1 \geq s_2 \geq 0$. Then all the subquotients of the representation π are hermitian and*

(i) *If $\omega_{\tau_1} = \omega_{\tau_2} = 1$, we have the same situation as in Figure 1, π has unitarizable subquotients only inside closed region I, the tempered (not square-integrable) subquotients appear for $(s_1, s_2) \in \{(\frac{1}{2}, 0), (0, 0)\}$, and square-integrable representation appears for $(s_1, s_2) = (\frac{1}{2}, \frac{1}{2})$.*

(ii) *If $\omega_{\tau_1} = 1$ and $\omega_{\tau_2} \neq 1$ π has unitarizable subquotients only for $s_2 = 0$ and $s_1 \in [0, \frac{1}{2}]$. Tempered (not square-integrable) representations appear for $(s_1, s_2) \in \{(\frac{1}{2}, 0), (0, 0)\}$.*

(iii) *If $\chi_{\tau_1} \neq 1$ and $\chi_{\tau_2} \neq 1$ π is unitarizable only for $s_1 = s_2 = 0$ and π (as we have already seen) is a sum of 4 nonequivalent tempered representations.*

Proposition 5.10. *Assume that $\tau_1 \cong \tau_2$ and τ_1 is self-contragredient, such that $\omega_{\tau_1} \neq 1$. Then all the irreducible subquotients of the representation π are hermitian, and the unitarizable subquotients appear only for (s_1, s_2) from the closed region I on Figure 3 (when all of them are unitarizable). The square-integrable subquotients appear for $(s_1, s_2) = (1, 0)$ and the tempered (not square-integrable) subquotients appear for $(s_1, s_2) = (0, 0)$.*

In the next proposition, we note an occurrence of the isolated unitary representation in the unitary dual, namely, the representation $L(\tau_1 \nu^{3/2}, \tau_1 \nu^{1/2}; 1)$.

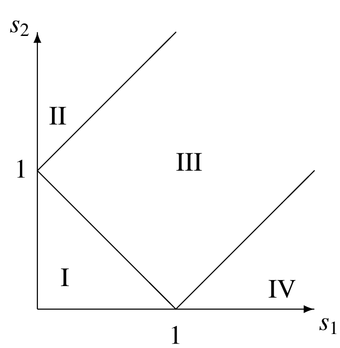

Figure 3

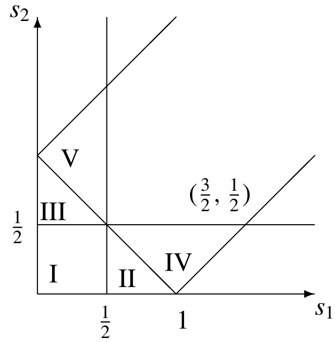

Figure 4

Proposition 5.11. *We keep all the assumptions of the previous proposition, except now* $\omega_{\tau_1} = 1$. *In Figure 4, considering the open regions, we have unitarizability only on region I, where we have nontempered representations. On the boundaries, we have a square-integrable subquotient for* $(s_1, s_2) = \left(\frac{3}{2}, \frac{1}{2}\right)$, *and tempered subquotients for* $(s_1, s_2) \in \left\{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0, 0)\right\}$. *We have other unitary subquotients on the boundary of region I, and* $L(\tau_1\nu^{3/2}, \tau_1\nu^{1/2}; 1)$ *is a unitarizable subquotient.*

Proof. We discuss only the more difficult cases. Consider the representation

$$\begin{aligned}&\tau_1\nu^{3/2} \times \tau_1\nu^{1/2} \rtimes 1\\ &\quad = L(\nu\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1) + \pi_4 + L(\tau\nu^{3/2}; \delta[\tau_1\nu^{1/2}; 1]) + L(\tau_1\nu^{3/2}, \tau_1\nu^{1/2}; 1).\end{aligned}$$

The unitarizability of the representation $L(\tau_1\nu^{3/2}, \tau_1\nu^{1/2}; 1)$ is proved by global methods [Grbac 2004]. This is an isolated unitary representation in the unitary dual. We will prove the nonunitarizability of $L(\nu\delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}); 1)$ and of $L(\tau\nu^{3/2}; \delta[\tau_1\nu^{1/2}; 1])$. We will do that in the following way: we will calculate the Plancherel measure $\mu(s, \delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}))$. Let us again denote by $A(s)$ the standard intertwining operator such that

$$A(s) : \delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2})\nu^s \rtimes 1 \to \delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2})\nu^{-s} \rtimes 1.$$

We will prove that the Plancherel measure has a simple pole at $s = 1$ and that $A(s)$ has no pole at $s = -1$. We will apply these observations to the calculation of Jantzen filtrations near $s = 1$. This will give us nonunitarizability. Let $\delta = \delta(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2})$. First, we use the previously mentioned result stating that $\mu(s, \delta) = \mu(s, \delta')$. The representation δ' is generic, so we can apply the results from [Shahidi 1990b] to compute the Plancherel measure in terms of the γ-factors. Up to an exponential factor, we have

$$\mu\left(\frac{s}{2}, \delta'\right) \approx \frac{\gamma(s, \delta', \Lambda^2\rho_4, \psi)}{\gamma(1+s, \delta', \Lambda^2\rho_4, \psi)},$$

where ψ is a nontrivial additive character of F, given in advance. By multiplicativity of γ-factors [Shahidi 1990a] we have

$$\gamma(s, \delta', \Lambda^2\rho_4, \psi) = \gamma(s, \tau_1\nu^{1/2}, \det, \psi)\gamma(s, \tau_1\nu^{-1/2}, \det, \psi)\gamma(s, \tau_1\nu^{1/2} \times \tau_1\nu^{1/2}, \psi).$$

The last factor is a Rankin–Selberg γ-factor. When a γ-factor is expressed in terms of L-functions, we obtain, up to an exponential factor,

$$\mu(\frac{s}{2}, \delta') \approx \frac{(1-q^{-1-s})(1-q^{1-s})(1-q^{1+s})(1-q^{-1+s})(1-q^{-rs})(1-q^{rs})}{(1-q^{s})(1-q^{-s})(1-q^{-2+s})(1-q^{-2-s})(1-q^{-r+rs})(1-q^{-r-rs})},$$

where r is a natural number satisfying $L(s, \tau_1 \times \tau_1) = (1-q^{rs})^{-1}$. Indeed $\mu(s, \delta')$ has a simple pole at $s = 1$. Denote by w_ε the reflection in the Weyl group with respect to the root ε. Then consider the intertwining operator

$$A_{w_{\alpha+\beta}}(s) : \tau_1\nu^{s+\frac{1}{2}} \times \tau_1\nu^{s-\frac{1}{2}} \rtimes 1 \to \tau_1\nu^{-s+\frac{1}{2}} \times \tau_1\nu^{-s-\frac{1}{2}} \rtimes 1.$$

The poles of the operator $A(s)$ are among the poles of the operator $A_{w_{\alpha+\beta}}(s)$ and $A_{w_{\alpha+\beta}}(s)|_{\delta\nu^s\rtimes 1} = A(s)$. But using the factorization of the operator $A_{w_{\alpha+\beta}}(s)$ [Shahidi 1981], we see that it has no poles at $s = -1$. Let X denotes the compact picture of the representation $\delta\nu^s \rtimes 1$. We will consider the Jantzen filtrations of the space X, for $s \in [0, 1]$. For $s \in (0, 1)$ the representations $\delta\nu^s \rtimes 1$ are irreducible, and the mentioned interval parameterizes a nondegenerate family of hermitian forms in the compact picture X. For s=0, $A(s)$ is holomorphic, and, normalized, generates the intertwining algebra of the representation $\delta \rtimes 1 = T_1 + T_2$ (follows from the proof of the Proposition 2.5). The operator $A(0)$ endows the space of this representation with the hermitian form which is of a different sign on each of the T_i's. This gives us the nonunitarizability of $\delta\nu^s \rtimes 1$ for $s \in (0, 1)$. By the theory of Jantzen filtrations [Vogan 1984], at $s = 1$ we consider filtrations $X = X_1^0 \supset X_1^1 \supset \cdots \supset 0$. Because we have a standard representation, we have $X_1^1 = \pi_4$, a square-integrable representation. We will prove that $X_1^2 = \{0\}$, i.e., that a hermitian form defined on X_1^1 by

$$< v, v' >_1 = \lim_{s\to 1} \int_K < v(k), \frac{1}{s-1} A(s)v'_s(k) > dk$$

is nondegenerate, so its radical, namely X_1^2, is trivial. Because of the simplicity of the pole of the Plancherel measure at $s = 1$, we have

$$A(-s)\frac{1}{s-1}A(s) = h(s),$$

where h is holomorphic function in the neighborhood of $s = 1$, and $h(1) \neq 0$. Hence, for nonzero $v' \in X$ such that $v'_1 \in \pi_4$, we have $\lim_{s\to 1} A(s)v'_s/(s-1) \notin L(\delta\nu, 1)$.

Now, we can choose $v \in X_1^1$ such that

$$\langle v, v' \rangle_1 = \lim_{s\to 1} \int_K \Big\langle v(k), \frac{A(s)v'_s(k)}{s-1} \Big\rangle dk \neq 0.$$

We can obtain the signature of $\delta\nu^s \rtimes 1$ for $s > 1$ and for $s < 1$ in terms of signatures (p_0, q_0) and (p_1, q_1). But we know that on these segments we have nonunitarizable representations. We conclude that $p_0 \neq 0$ and $q_0 \neq 0$, which is equivalent to nonunitarizability.

The proof of the nonunitarizability of $L(\tau_1\nu^{3/2}; \delta[\tau_1\nu^{1/2}; 1])$ follows the same pattern: We will compute the Plancherel measure of $\mu(s, \tau_1 \otimes \delta[\tau_1\nu^s; 1])$. We can easily extend the results from the fourth section to the case when we consider square-integrable representations instead of cuspidal ones. So we have

$$\mu(s, \tau_1 \otimes \delta[\tau_1\nu^s; 1])^2 = \mu(s, \tau_1' \otimes \sigma_1)\mu(s, \tau_1' \otimes \sigma_2).$$

The representations σ_i, $i = 1, 2$, from the above equation are obtained originally by considering the restrictions of the representations to the groups $\mathrm{SL}(2, F)D_1$ or $\mathrm{SL}(2, F) \cdot \mathrm{SL}(2, F)$. It is not hard to see that, in this case, $\sigma_i \hookrightarrow \mathrm{Ind}_{M_\alpha}^{\mathrm{SO}(4,F)}(\tau_1'\chi_i)$, where χ_i is a quadratic character on F^*. Here M_α is the standard Levi subgroup, which is diagonally positioned in SO(4, F), and τ_1' is a Langlands' lift of the representation τ_1. Because of the genericity of the representations σ_i, we can apply the results of Shahidi about multiplicativity of γ-factors. We obtain that the Plancherel measure $\mu(s, \tau_1' \otimes \sigma_i)$ can have a pole of order one at $s = \frac{3}{2}$. We obtain a pole there if and only if, $\tau_1' \cong \tau_1'\chi_i$. But $\mu(s, \tau_1 \otimes \delta[\tau_1\nu^{1/2}; 1])$ must have a pole there, so it is a pole of order one. As in the previous case, we conclude that the intertwining operators appearing in the definition of the Plancherel measure $\mu(s, \tau_1 \otimes \delta[\tau_1\nu^s; 1])$ are holomorphic near $s = \pm\frac{3}{2}$. Now we can conclude, as in the previous discussion, that $L(\tau_1\nu^{3/2}; \delta[\tau_1\nu^{1/2}; 1])$ is a nonunitarizable representation. The only Langlands quotient left to settle is $L(\tau_1\nu; \tau_1 \rtimes 1) = \nu^{1/2}L(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$. We obtain the hermiticity of the representations $\pi_s = \nu^s L(\tau_1\nu^{1/2}, \tau_1\nu^{-1/2}) \rtimes 1$ for $s \in (0, 1)$ using the action of the long intertwining operator acting on the space $\tau_1\nu^{s+\frac{1}{2}} \times \tau_1\nu^{s-\frac{1}{2}} \rtimes 1$. But unitarity of the representation π_s at $s = \frac{1}{2}$ would imply unitarizability of all the subquotients at $s = 1$, which contradicts what we have just proved. □

Again, let $\pi = \tau_1\nu^{s_1} \times \chi_1\nu^{s_2} \rtimes 1$. We can assume $s_1, s_2 \geq 0$. The proof of the next result is straightforward.

Proposition 5.12. (i) *If τ_1 is not a self-contragredient representation and $\chi_1^2 \neq 1$, π has a hermitian quotient only if $(s_1, s_2) = (0, 0)$, and then π is an irreducible, tempered representation.*

(ii) *If* τ_1 *is selfdual, but* $\chi_1^2 \neq 1$ π *has a hermitian quotient only if* $s_2 = 0$ (*then all the subquotients are hermitian*), *and has unitarizable subquotients for* s_1 *from the segment* $[0, \frac{1}{2}]$ *if* $\omega_{\tau_1} = 1$ *and only in the origin if* $\omega_{\tau_1} \neq 1$. *If* $\omega_{\tau_1} = 1$, π *is irreducible tempered for* $s_1 = 0$, *and has a tempered subquotient for* $s_1 = \frac{1}{2}$. *If* $\omega_{\tau_1} \neq 1$, *we obtain a tempered representation for* $s_1 = s_2 = 0$.

(iii) *If* $\chi_1^2 = 1$ *and* τ_1 *is not a selfdual representation*, π *has a hermitian subquotient only for* $s_1 = 0$ (*then all quotients are hermitian*). *For* $s_2 = 0$ *representation* π *is irreducible tempered*, *for* $s_2 = \frac{1}{2}$ *it has a tempered subquotient*, *and unitarizable subquotients appear for* $\{(0, s_2) : s_2 \in [0, \frac{1}{2}]\}$.

(iv) *If* $\chi_1^2 = 1$ *and* $\tau_1 \rtimes 1$ *reduces*, *every subquotient of* π *is hermitian. For* $s_1 = s_2 = 0$ *representation* π *is a sum of two-nonequivalent tempered representations*, *and for* $(s_1, s_2) = (0, \frac{1}{2})$ *has a tempered subquotient. Other unitarizable subquotients appear for* $s_1 = 0$ *and* $s_2 \in \left[0, \frac{1}{2}\right]$.

(iv) *If* $\chi_1^2 = 1$ *and* $\tau_1 \nu^{1/2} \rtimes 1$ *reduces*, *we have the analogous situation as for the characters*; *unitarizability of all subquotients of* π *on the closed region I in Figure 1.*

5.2. ***The unitary dual supported on the nonminimal parabolic subgroups.*** Once we have handled the reducibility questions in this case, the rest is straightforward. Assume that $s \geq 0$.

Proposition 5.13. (i) *We consider induction from the Siegel parabolic subgroup*: *If* τ *is an irreducible cuspidal representation of* $\mathrm{GL}(2, D)$, *let* $\pi_s = \tau\nu^s \rtimes 1$. *If* τ *is not self-dual*, π_s *is hermitian only when* $s = 0$; *then it is a tempered representation. If* τ *is self-dual*, π_s *reduces for some* $s_0 \in \{0, \frac{1}{2}\}$, *and all of it subquotients are always hermitian. In this case*, *if* $s_0 = 0$, π_0 *is the sum of two nonequivalent tempered representations*, *and otherwise*, π_s *are nonunitarizable. If* $s_0 = \frac{1}{2}$, π_s *is a nontempered irreducible unitary representation for* $s \in (0, \frac{1}{2})$, *tempered for* $s = 0$, *and for* $s = \frac{1}{2}$ *the representation* $\pi_{\frac{1}{2}}$ *has two irreducible unitary subquotients*; *one of them is nontempered and the other is a square-integrable representation.*

(ii) *We consider induction from the non-Siegel parabolic subgroup*: *the representation* $\tau\nu^s \rtimes \delta$ *for irreducible representation* τ *of the group* D^* *and irreducible cuspidal representation* δ *of the group* $G_1(D)$ *is unitarizable for nonselfdual* τ *only for* $s = 0$, *and then it is irreducible tempered representation. Otherwise*, *it reduces for some* $s_0 \in \{0, \frac{1}{2}, 1, \frac{3}{2}\}$ (*Corollaries 4.2 and 4.3*). *If* $s_0 = 0$, π_0 *is a sum of two nonequivalent tempered representations*, *otherwise* π_s *is nonunitarizable. If* $s_0 \in \left\{\frac{1}{2}, 1, \frac{3}{2}\right\}$, *the representation* π_0 *is tempered*, *nontempered unitarizable for* $s \in (0, s_0)$ *and for* $s = s_0$ *it has a nontempered unitarizable Langlands quotient*, *and a square-integrable subrepresentation.*

Acknowledgments. I express my gratitude to Profs. M. Tadić and G. Muić for their constant help, and to my colleague N. Grbac for helpful conversations.

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Received October 5, 2004. Revised May 17, 2005.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 226, No. 2, 2006

LOWER BOUNDARY HYPERPLANES OF THE CANONICAL LEFT CELLS IN THE AFFINE WEYL GROUP $W_a(\tilde{A}_{n-1})$

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Dedicated to Professor George Lusztig on his sixtieth birthday.

Let Γ be any canonical left cell of the affine Weyl group W_a of type $\tilde{A}_{n-1}$ for $n > 1$. We describe the lower boundary hyperplanes for Γ, answering two questions of Humphreys.

Let W_a be an affine Weyl group and let Φ be the root system of the corresponding Weyl group. Fix a positive root system Φ^+ of Φ. There is a bijection from W_a to the set of alcoves in the euclidean space E spanned by Φ. We identify the elements of W_a with the alcoves (also with the topological closure of the alcoves) of E. According to a result of Lusztig and Xi [1988], we know that the intersection of any two-sided cell of W_a with the dominant chamber of E is exactly a single left cell of W_a, called a canonical left cell. When W_a is of type $\tilde{A}_{n-1}$, with $n > 1$, there is a bijection ϕ from the set of two-sided cells of W_a to the set of partitions of n; see Remark 2.1 and subsequent paragraphs, as well as [Shi 1986].

From now on, unless otherwise specified, we always assume that W_a is an affine Weyl group of type $\tilde{A}_{n-1}$, where $n > 1$. This article answers two questions posed recently by J. E. Humphreys (private communication):

(1) Can one find the set $B(L)$ of all the lower boundary hyperplanes for any canonical left cell L of W_a?

(2) How does the partition $\phi(L)$ determine the set $B(L)$, and in which case does the set $B(L)$ determine the partition $\phi(L)$ also?

In the first two sections, we collect some concepts and known results for later use. In Section 3, we give criteria for a hyperplane to be the lower boundary of a canonical left cell of W_a. Then we prove our main results in Section 4.

MSC2000: 20F55.
Keywords: affine Weyl groups, canonical left cells, hyperplanes, alcoves, sign types.
Supported by the NSF of China, the SF of the University Doctoral Program of ME of China, the Shanghai Priority Academic Discipline, and the CST of Shanghai (No. 03JC14027).

1. Sign types

Let $\boldsymbol{n} = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. An $\boldsymbol{n}$*-sign type* (or just a sign type) is by definition a matrix $X = (X_{ij})_{i,j\in\boldsymbol{n}}$ over the symbol set $\{+, \bigcirc, -\}$, with

$$\{X_{ij}, X_{ji}\} \in \{\{+, -\}, \{\bigcirc, \bigcirc\}\} \qquad \text{for} \quad i, j \in \boldsymbol{n}.$$

X is determined entirely by its "upper unitriangular" part $X^{\Delta} = (X_{ij})_{i<j}$. We identify X with X^{Δ}. X is *dominant*, if $X_{ij} \in \{+, \bigcirc\}$ for any $i < j$ in $\boldsymbol{n}$, and is *admissible*, if

$$\begin{aligned} - \in \{X_{ij}, X_{jk}\} &\Longrightarrow X_{ik} \leqslant \max\{X_{ij}, X_{jk}\}, \\ - \notin \{X_{ij}, X_{jk}\} &\Longrightarrow X_{ik} \geqslant \max\{X_{ij}, X_{jk}\} \end{aligned} \tag{1–1}$$

for any $i < j < k$ in $\boldsymbol{n}$, where we set a total ordering: $- < \bigcirc < +$.

Lemma 1.1 ([Shi 1987b, Lemma 3.1; Shi 1999, Corollary 2.8]). (1) *A dominant sign type $X = (X_{ij})$ is admissible if and only if for any $i \leqslant h < k \leqslant j$, condition $X_{ij} = \bigcirc$ implies $X_{hk} = \bigcirc$.*

(2) *If an admissible sign type $X = (X_{ij})$ is not dominant, then there exists at least one k with $1 \leqslant k < n$ and $X_{k,k+1} = -$.*

Proof. This is an easy consequence of conditions (1–1). □

Let $E = \{(a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i = 0\}$. This is a euclidean space of dimension $n-1$ with inner product $\langle (a_1, \dots, a_n), (b_1, \dots, b_n)\rangle = \sum_{i=1}^{n} a_i b_i$. For $i \neq j$ in $\boldsymbol{n}$, let $\alpha_{ij} = (0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)$, with 1 and -1 at the i-th and j-th positions, respectively. Then $\Phi = \{\alpha_{ij} \mid 1 \leqslant i \neq j \leqslant n\}$ is the root system of type A_{n-1}, which spans E. $\Phi^+ = \{\alpha_{ij} \in \Phi \mid i < j\}$ is a positive root system of Φ with corresponding simple root system $\Pi = \{\alpha_{i,i+1} \mid 1 \leqslant i < n\}$. For any $\epsilon \in \mathbb{Z}$ and $i < j$ in $\boldsymbol{n}$, define a hyperplane

$$H_{ij;\epsilon} = \{(a_1, \dots, a_n) \in E \mid a_i - a_j = \epsilon\}. \tag{1–2}$$

Encode a connected component C of $E \setminus \bigcup_{1\leqslant i<j\leqslant n,\, \epsilon\in\{0,1\}} H_{ij;\epsilon}$ by a sign type $X = (X_{ij})_{i<j}$ as follows. Take any $v = (a_1, \dots, a_n) \in C$ and, for $i < j$ in $\boldsymbol{n}$, set

$$X_{ij} = \begin{cases} + & \text{if } a_i - a_j > 1, \\ - & \text{if } a_i - a_j < 0, \\ \bigcirc & \text{if } 0 < a_i - a_j < 1. \end{cases}$$

X only depends on C, but not on the choice of v; see [Shi 1986, Chapter 5]. Note that not all sign types can be obtained in this way.

Proposition 1.2 ([Shi 1986, Proposition 7.1.1 and §2]). *A sign type $X = (X_{ij})$ can be obtained in the above way if and only if it is admissible.*

Lemma 1.3. *Let $X = (X_{ij})$ be a dominant admissible sign type with $X_{p,p+1} = \bigcirc$ for some p with $1 \leqslant p < n$. Let $X' = (X'_{ij})$ be the sign type given by*

$$X'_{ij} = \begin{cases} X_{ij} & \text{if } (i, j) \neq (p, p+1), \\ - & \text{if } (i, j) = (p, p+1) \end{cases}$$

for $i < j$ in $\mathbf{n}$. Then X' is admissible if and only if $X_{ph} = X_{p+1,h}$ for all $h \in \mathbf{n}$.

Proof. This is an easy consequence of (1–1). □

For $\alpha \in \Phi$, let s_α be the reflection in α:

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Let T_α be the translation by α: $T(v) = v + \alpha$. Define $s_i = s_{\alpha_{i,i+1}}$ for $1 \leqslant i < n$, and $s_0 = T_{\alpha_{1n}} s_{\alpha_{1n}}$. Then $S = \{s_i \mid 0 \leqslant i < n\}$ forms a distinguished generator set of the affine Weyl group W_a of type $\tilde{A}_{n-1}$.

A connected component in

$$E \setminus \bigcup_{\substack{1 \leqslant i < j \leqslant n \\ k \in \mathbb{Z}}} H_{ij;k}$$

is called an *alcove*. The (right) action of W_a on E induces a simply transitive permutation on the set $\mathfrak{A}$ of alcoves in E. There exists a bijection $w \mapsto A_w$ from W_a to $\mathfrak{A}$ such that A_1 (where 1 is the identity element of W_a) is the unique alcove in the dominant chamber of E whose closure contains the origin and such that $(A_y)w = A_{yw}$ for $y, w \in W_a$; see [Shi 1987a, Proposition 4.2]. To each $w \in W_a$ we associate an admissible sign type $X(w)$ that contains the alcove A_w. An admissible sign type X can be identified with the set $\{w \in W_a \mid X(w) = X\}$.

2. Partitions and Kazhdan–Luzstig cells

Let $(P, \preceq)$ be a finite poset. By a *chain* of P, we mean a totally ordered subset of P (allow to be an empty set). Also, a *cochain* of P is a subset K of P whose elements are pairwise incomparable. A *k-(co)chain family* in P ($k \geqslant 1$) is a subset J of P which is a disjoint union of k (co)chains J_i ($1 \leqslant i \leqslant k$). We usually write $J = J_1 \cup \cdots \cup J_k$.

A *partition* of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ and $\sum_{i=1}^{r} \lambda_i = n$. In particular, when $\lambda_1 = \cdots = \lambda_r = a$, we also write $\lambda = (a^r)$, and call it a *rectangular* partition. Let Λ_n be the set of all partitions of n.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ be in Λ_n. Write $\lambda \leqslant \mu$ if the inequalities $\sum_{j=1}^{i} \lambda_j \leqslant \sum_{j=1}^{i} \mu_j$ hold for $i \geqslant 1$. We say that μ is *conjugate* to λ

if $\mu_i = |\{j \mid \lambda_j \geqslant i, 1 \leqslant j \leqslant r\}|$ for $1 \leqslant i \leqslant t$, where $|X|$ stands for the cardinality of the set X.

Let d_k be the maximal cardinality of a k-chain family in P for $k \geqslant 1$. Then $d_1 < d_2 < \cdots < d_r = n = |P|$ for some $r \geqslant 1$. Let $\lambda_1 = d_1$ and $\lambda_i = d_i - d_{i-1}$ for $1 < i \leqslant r$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ by [Greene 1976, Theorem 1.6]. We get $\phi(P) = (\lambda_1, \ldots, \lambda_r) \in \Lambda_n$, called the *partition associated to chains* in P. Replacing the word "k-chain family" by "k-cochain family", we can also define $\psi(P) = (\mu_1, \ldots, \mu_t) \in \Lambda_n$, again by [Greene 1976, Theorem 1.6], called the *partition associated to cochains* in P. Moreover, $\psi(P)$ is conjugate to $\phi(P)$.

Remark 2.1. Let $(P, \preceq)$ be a finite poset with $\psi(P) = (\mu_1, \ldots, \mu_t)$. For $1 \leqslant k \leqslant t$, let $P^{(k)} = P_1 \cup \cdots \cup P_k$ be a k-cochain family of P with $|P^{(k)}| = \sum_{h=1}^{k} \mu_h$. Then $\mu_1 \geqslant |P_i| \geqslant \mu_k$ for $1 \leqslant i \leqslant k$. In particular, when $\psi(P) = (a^t)$ is rectangular, we have $|P_1| = \cdots = |P_k| = a$. This fact will be used in the proof of Lemma 4.2.

For each admissible sign type $X = (X_{ij})$, we write $i \leqslant_X j$ in $\boldsymbol{n}$ if either $i = j$ or $X_{ij} = +$. By [Shi 1999, Lemma 2.2], the order $\leqslant_X$ is a partial order on $\boldsymbol{n}$. We associate to X two partitions $\phi(X)$ and $\psi(X)$ of n defined above.

Kazhdan and Lusztig [1979] defined certain equivalence classes in a Coxeter system (W, S), called a *left cell*, a *right cell* and a *two-sided cell*.

Let W_a be the affine Weyl group of type $\tilde{A}_{n-1}$ for $n > 1$. Each element w of W_a determines a sign type $X(w)$, and hence it in turn determines two partitions $\phi(w) := \phi(X(w))$ and $\psi(w) := \psi(X(w))$. This defines two maps $\phi, \psi : W_a \longrightarrow \Lambda_n$, each of which induces, by [Shi 1986, Theorem 17.4], a bijection from the set of two-sided cells of W_a to the set Λ_n.

To each $w \in W_a$, we associate a set $\mathcal{R}(w) = \{s \in S \mid ws < w\}$, where $\leqslant$ is the Bruhat order in the Coxeter system (W_a, S). Define

$$Y_0 = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s_0\}\}.$$

By [Lusztig and Xi 1988, Theorem 1.2], the intersection of Y_0 with any two-sided cell $\phi^{-1}(\lambda)$ ($\lambda \in \Lambda_n$) is a single left cell of W_a, written Γ_λ and called a *canonical left cell.*

3. Lower boundary of a canonical left cell

We now define a lower boundary hyperplane for any $F \subset W_a$, and give criteria for a hyperplane of E to be lower boundary for a canonical left cell of W_a.

For $i < j$ in $\boldsymbol{n}$ and $k \in \mathbb{Z}$, the hyperplane $H_{ij;k}$ divides the space E into three parts: $H_{ij;k}^{+} = \{v \in E \mid \langle v, \alpha_{ij}\rangle > k\}$, $H_{ij;k}^{-} = \{v \in E \mid \langle v, \alpha_{ij}\rangle < k\}$, and $H_{ij;k}$. For any set F of alcoves in E, call $H_{ij;k}$ a *lower boundary hyperplane* of F if $\bigcup_{A \in F} A \subset H_{ij;k}^{+}$ and if there exists some alcove C in F such that $\overline{C} \cap H_{ij;k}$ is a facet of C of dimension $n-2$, where $\bar{C}$ stands for the closure of C in E under the usual topology.

Let Γ be a canonical left cell of W_a. As a subset in W_a, Γ is a union of some dominant sign types, by [Shi 1986, Proposition 18.2.2]; denote by $S(\Gamma)$ the set of these sign types. Regarded as a union of alcoves, the topological closure of Γ in E is connected [Shi 1986, Theorem18.2.1] and is bounded by a certain set of hyperplanes in E of the form $H_{ij;\epsilon}$, for $1 \leqslant i < j \leqslant n$ and $\epsilon = 0, 1$, defined in (1–2). Then a lower boundary hyperplane of Γ must be one of such hyperplanes. Let $B(\Gamma)$ be the set of all the lower boundary hyperplanes of Γ. Given a hyperplane $H_{ij;\epsilon}$ with $1 \leqslant i < j \leqslant n$ and $\epsilon = 0, 1$, we see that $H_{ij;\epsilon} \in B(\Gamma)$ if and only if one of the following conditions holds.

Condition 3.1. $\epsilon = 1$, $X_{ij} = +$ for all $X = (X_{ab}) \in S(\Gamma)$, and there exists some $Y = (Y_{ab}) \in S(\Gamma)$ such that the sign type $Y' = (Y'_{ab})$ defined below is admissible:

$$Y'_{ab} = \begin{cases} Y_{ab} & \text{if } (a, b) \neq (i, j), \\ \bigcirc & \text{if } (a, b) = (i, j). \end{cases}$$

Condition 3.2. $\epsilon = 0$, and there exists some $X = (X_{ab}) \in S(\Gamma)$ with $X_{ij} = \bigcirc$ such that the sign type $X' = (X'_{ab})$ defined by

$$X'_{ab} = \begin{cases} X_{ab} & \text{if } (a, b) \neq (i, j), \\ - & \text{if } (a, b) = (i, j) \end{cases}$$

is admissible.

Remark 3.3. By Lemma 1.1(2), Condition 3.2 happens only if $j = i + 1$.

Proposition 3.4. (1) *$H_{i,i+1;0} \in B(\Gamma)$ if and only if there exists some $X = (X_{ab}) \in S(\Gamma)$ such that $X_{i,h} = X_{i+1,h}$ for all $h \in \boldsymbol{n}$. In particular, when these conditions hold, we have $X_{i,i+1} = \bigcirc$.*

(2) *If $H_{ij;1} \in B(\Gamma)$ and if either $i \leqslant k < l \leqslant j$ or $k \leqslant i < j \leqslant l$, then $H_{kl;1} \in B(\Gamma)$ if and only if $(i, j) = (k, l)$.*

Proof. Part (1) follows from Condition 3.2 and Lemma 1.3. Then part (2) is a direct consequence of Condition 3.1 and Lemma 1.1(1). □

4. Description of the sets $B_0(\Gamma_\lambda)$ and $B_1(\Gamma_\lambda)$

We now answer the two questions of Humphreys.

Let Γ_λ be the canonical left cell of W_a corresponding to $\lambda \in \Lambda_n$. Let $B_\epsilon(\Gamma_\lambda) = \{H_{ij;\epsilon} \mid H_{ij;\epsilon} \in B(\Gamma_\lambda)\}$ for $\epsilon = 0, 1$.

Lemma 4.1. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda_n$ contains at least two different parts. Then $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$.*

Proof. Let $\mu = (\mu_1, \ldots, \mu_t)$ be the conjugate partition of λ. Then μ also contains at least two different parts. Given any p with $1 \leqslant p < n$, there exists a permutation $a_1, a_2, \ldots, a_t$ of $1, 2, \ldots, t$ such that $m_s < p$ and $m_{s+1} > p$ for some s with $0 \leqslant s < t$, where $m_u := \sum_{k=1}^{u} \mu_{a_k}$ for $0 \leqslant u \leqslant t$ with the convention that $m_0 = 0$. Define a dominant sign type $X = (X_{ij})$ such that for any i, j with $1 \leqslant i < j \leqslant n$, $X_{ij} = \bigcirc$ if and only if $m_h < i < j \leqslant m_{h+1}$ for some h with $0 \leqslant h < t$. Clearly, X is admissible with $\psi(X) = \mu$. Hence $X \in S(\Gamma_\lambda)$. We see also that $X_{ph} = X_{p+1,h}$ for all h such that $1 \leqslant h \leqslant n$. So we conclude that $H_{p,p+1;0} \in B_0(\Gamma_\lambda)$ by Proposition 3.4(1). Our result follows by Remark 3.3. □

Lemma 4.2. *For a rectangular partition $(k^a) \in \Lambda$ with $a, k \in \mathbb{N}$, we have*

$$B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leqslant p < n, a \nmid p\}.$$

Proof. Let $X = (X_{ij})$ be a dominant admissible sign type. Then a maximal cochain in $\boldsymbol{n}$ with respect to $\leqslant_X$ must consist of consecutive numbers. Now suppose $\psi(X) = (a^k)$. Then by Remark 2.1, we can take a maximal k-cochain family $\boldsymbol{n} = P_1 \cup \cdots \cup P_k$ such that $P_h = \{a(h-1)+1, a(h-1)+2, \ldots, ah\}$ with $1 \leqslant h \leqslant k$ are the maximal cochains in $\boldsymbol{n}$ with respect to $\leqslant_X$. We have $X_{a(h-1)+1,ah} = \bigcirc$ and $X_{a(h-1)+1,ah+1} = +$, which are different. So by the arbitrariness of X and by Proposition 3.4(1), we see that

$$H_{ah,ah+1;0} \notin B_0(\Gamma_{(k^a)}) \quad \text{for } 1 \leqslant h < k. \tag{4–1}$$

On the other hand, let $Y = (Y_{ij})$ be a sign type defined by

$$Y_{ij} = \begin{cases} \bigcirc & \text{if } a(h-1) < i < j \leqslant ah \text{ for some } 1 \leqslant h \leqslant k, \\ + & \text{otherwise} \end{cases}$$

for $1 \leqslant i < j \leqslant n$. Then it is clear that Y is dominant admissible with $\psi(Y) = (a^k)$. Suppose $a(h-1) < p < ah$ for some $h \in [1, k]$. Then $Y_{p,p+1} = \bigcirc$. We see also that $Y_{ph} = Y_{p+1,h}$ for all $h \in [1, n]$. By Proposition 3.4(1), we have

$$H_{p,p+1;0} \in B_0(\Gamma_{(k^a)}) \quad \text{for all } p \text{ with } 1 \leqslant p < n \text{ and } a \nmid p.$$

The result follows from this, (4–1), and Remark 3.3. □

Theorem 4.3. $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$ *for all $\lambda \in \Lambda_n$ unless λ is a rectangular partition. In the latter case, say $\lambda = (k^a)$ for $k, a \in \mathbb{N}$, we have $B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leqslant p < n, a \nmid p\}$.*

Proof. We see that a partition is nonrectangular if and only if it contains at least two different parts. So our result follows immediately from Lemmas 4.1 and 4.2. □

Theorem 4.4. $B_1(\Gamma_\lambda) = \{H_{i,i+r;1} \mid 1 \leqslant i \leqslant n - r\}$ *for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda_n$.*

Proof. Let $\mu = (\mu_1, \dots, \mu_t) \in \Lambda_n$ be conjugate to λ. First we claim that, for any $X = (X_{ij}) \in S(\Gamma_\lambda)$,

$$X_{i,i+r} = + \quad \text{for } i = 1, \dots, n-r. \tag{4–2}$$

Otherwise, there would exist some $X = (X_{ij}) \in S(\Gamma_\lambda)$ with $X_{i,i+r} = \bigcirc$ for some i, $1 \leqslant i \leqslant n-r$. By Lemma 1.1(1), we would have $X_{hk} = \bigcirc$ for all h, k such that $i \leqslant h < k \leqslant i+r$. Then $\{i, i+1, \dots, i+r\}$ would be a cochain in $\boldsymbol{n}$ with respect to the partial order $\leqslant_X$, whose cardinality is $r+1 > \mu_1 = r$, contradicting the assumption $\psi(\Gamma_\lambda) = (\mu_1, \mu_2, \dots, \mu_t)$.

Next we want to find, for any p with $1 \leqslant p \leqslant n-r$, some $Y = (Y_{ij}) \in S(\Gamma_\lambda)$ such that the sign type $Y' = (Y'_{ij})$ defined by

$$Y'_{ij} = \begin{cases} Y_{ij} & \text{if } (i,j) \neq (p, p+r), \\ \bigcirc & \text{if } (i,j) = (p, p+r) \end{cases} \tag{4–3}$$

for $1 \leqslant i < j \leqslant n$, is admissible. If this happens, we automatically have $\psi(Y') \gneqq \mu$ by the proof of (4–2).

Take a permutation $a_1, a_2, \dots, a_t$ of $1, 2, \dots, t$ satisfying two conditions:

(1) Let $m_u = \sum_{k=1}^{u} \mu_{a_k}$ for $0 \leqslant u \leqslant t$ with the convention that $m_0 = 0$. Then there exists some $s \in [0, t)$ such that $a_{s+1} = 1$, $m_s < p$ and $m_{s+1} \geqslant p$.

(2) s is the largest possible number with the property (1) when $a_1, a_2, \dots, a_t$ ranges over all the permutations of $1, 2, \dots, t$.

Then we have $t - s \geqslant 2$, $p \leqslant m_{s+1} < p+r$ and $m_{s+2} \geqslant p+r$. Define a dominant sign type $Y = (Y_{ij})$ such that $Y_{ij} = \bigcirc$ if and only if either

$$m_u < i < j \leqslant m_{u+1} \text{ for } 0 \leqslant u < t, \quad \text{or} \quad p \leqslant i < j \leqslant p+r \text{ with } (i,j) \neq (p, p+r).$$

By Lemma 1.1(1), Y is admissible with $\psi(Y) = \mu$, i.e., $Y \in S(\Gamma_\lambda)$. Clearly, the sign type Y' obtained from Y as in (4–3) is also dominant admissible by Lemma 1.1(1). This implies by Condition 3.1 that $H_{p,p+r;1}$ belongs to $B_1(\Gamma_\lambda)$ for any $p = 1, \dots, n-r$. The result follows by Proposition 3.4(2). □

Remark 4.5. Theorems 4.3 and 4.4 answer the two questions of Humphreys. In particular, the canonical left cells of W_a associated to the rectangular partitions are determined entirely by the corresponding B_1-set of hyperplanes. From the above description of B_0-sets of hyperplanes, we see that compared with the other canonical left cells of W_a, the positions of the canonical left cells associated to rectangular partitions are farther from the walls of the dominant chamber.

Remark 4.6. When $\lambda = (n)$, we have $B_0(\Gamma_\lambda) = \varnothing$ and

$$B_1(\Gamma_\lambda) = \{H_{i,i+1;1} \mid 1 \leqslant i < n\}.$$

Actually, this is the unique canonical left cell whose B_1-set contains a hyperplane of the form $H_{i,i+1;1}$. Also, this is the unique canonical left cell whose B_0-set is empty. On the other hand, $B_0(\Gamma_{(1^n)}) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$ and $B_1(\Gamma_{(1^n)}) = \varnothing$. $\Gamma_{(1^n)}$ is the unique canonical left cell whose B_1-set is empty.

Remark 4.7. When $n \in \mathbb{N}$ is a prime number, the B_0-sets of all the canonical left cells Γ_λ of W_a are $\{H_{i,i+1;0} \mid 1 \leqslant i < n\}$, except for the case where $\lambda = (n)$.

Remark 4.8. Now assume that (W_a, S) is an irreducible affine Weyl group of arbitrary type with Δ a choice of simple roots system of Φ. We are unable to describe the lower boundary hyperplanes for a canonical left cell L of W_a in general. This is because L is not always a union of some sign types (as in the case of type $\tilde{B}_2$). But we know that L is a single sign type when L is in either the lowest or the highest two-sided cell of W_a (see [Shi 1987c; Shi 1988]) for which we can describe its lower boundary hyperplanes: if L is in the lowest two-sided cell of W_a, then $B_1(L) = \{H_{\alpha;1} \mid \alpha \in \Delta\}$ and $B_0(L) = \varnothing$, where $H_{\alpha;1} := \{v \in E \mid \langle v, \alpha^\vee\rangle = 1\}$ and $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$; if L is in the highest two-sided cell of W_a, then $B_1(L) = \varnothing$ and $B_0(L) = \{H_{\alpha;0} \mid \alpha \in \Delta\}$. This extends the result in Remark 4.6. We conjecture that any canonical left cell of W_a is a union of some sign types whenever W_a has a simply-laced type, namely $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$. If this is true, one would be able to describe the lower boundary hyperplanes for the canonical left cells of these groups.

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Received November 21, 2004.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 226 No. 2 August 2006

0030-8730(200608)226:2;1-C