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We consider spacetimes with compact Cauchy hypersurfaces and with Ricci tensor bounded from below on the set of timelike unit vectors, and prove that the results known for spacetimes satisfying the timelike convergence condition, namely, foliation by CMC hypersurfaces, are also valid in the present situation, if corresponding further assumptions are satisfied.

In addition we show that the volume of any sequence of spacelike hypersurfaces, which run into the future singularity, decays to zero provided there exists a time function covering a future end, such that the level hypersurfaces have nonnegative mean curvature and decaying volume.

1. Introduction

Let N be a $(n+1)$-dimensional spacetime with a compact Cauchy hypersurface, so that N is topologically a product, $N = I \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold and $I = (a, b)$ an interval. The metric in N can then be expressed in the form

$$d\bar{s}^2 = e^{2\psi}\big(-(dx^0)^2 + \sigma_{ij}(x^0, x)\, dx^i\, dx^j\big); \tag{1–1}$$

x^0 is the time function and (x^i) are local coordinates for $\mathcal{S}_0$.

If N satisfies a future mean curvature barrier condition and the timelike convergence condition, then a future end $N_+ = (x^0)^{-1}\big([a_0, b)\big)$ can be foliated by constant mean curvature (CMC) spacelike hypersurfaces and the mean curvature of the leaves can be used as a new time function [Gerhardt 1983; 2003]. Moreover, one of Hawking's singularity results implies that N is future timelike incomplete with finite Lorentzian diameter for the future end.

In this paper we want to extend these results to the case when the Ricci tensor is only bounded from below on the set of timelike unit vectors

$$\bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \ge -\Lambda \qquad \text{for all } \langle \nu, \nu\rangle = -1 \tag{1–2}$$

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for some $\Lambda \geq 0$, and in addition, we want to show that the volume of the CMC leaves decays to zero, if the future singularity is approached.

We summarize our results:

Theorem 1.1. *Suppose that in a future end N_+ of N the Ricci tensor satisfies the estimate* (1–2) *of the preceding page, and suppose that a future mean curvature barrier exists* (*Definition 2.2*). *Then a slightly smaller future end $\tilde{N}_+$ can be foliated by CMC spacelike hypersurfaces, and there exists a smooth time function x^0 such that the slices*

$$M_\tau = \{x^0 = \tau\}, \qquad \tau_0 < \tau < \infty,$$

have mean curvature τ for some $\tau_0 > \sqrt{n\Lambda}$. The precise value of τ_0 depends on the mean curvature of a lower barrier.

Recall that a subset $M \subset N$ is said to be *achronal* if any timelike piecewise C^1-curve intersects M at most once.

Theorem 1.2. *Suppose that a future end $N_+ = (x^0)^{-1}\big([a_0, b)\big)$ of N can be covered by a time function x^0 such that the mean curvature of the slices $M_t = \{x^0 = t\}$ is nonnegative and the volume of M_t decays to zero*:

$$\lim_{t\to b} |M_t| = 0.$$

Then the volume $|M_k|$ of any sequence of spacelike achronal hypersurfaces M_k such that

$$\lim_k \inf_{M_k} x^0 = b$$

decays to zero. Thus, if the additional conditions of Theorem 1.1 are also satisfied, the volume of the CMC hypersurfaces M_τ converges to zero:

$$\lim_{\tau\to\infty} |M_\tau| = 0.$$

N is also future timelike incomplete if there is a compact spacelike hypersurface M with mean curvature H satisfying

$$H \geq H_0 > \sqrt{n\Lambda},$$

due to a result in [Andersson and Galloway 2002].

2. Notations and definitions

The main objective of this section is to state the equations of Gauss, Codazzi, and Weingarten for space-like hypersurfaces M in a (n+1)-dimensional Lorentzian manifold N. Geometric quantities in N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$, etc., and those in M by (g_{ij}), (R_{ijkl}), etc. Greek indices range from 0 to n and Latin from

1 to n; the summation convention is always used. Generic coordinate systems in N and M will be denoted by (x^α) and (ξ^i), respectively. Covariant differentiation will simply be indicated by indices; only in cases of possible ambiguity they will be preceded by a semicolon. For example, for a function u in N, the gradient will be (u_α) and the Hessian $(u_{\alpha\beta})$, but the covariant derivative of the curvature tensor will be abbreviated by $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$. We also point out that

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon$$

with obvious generalizations to other quantities.

Let M be a *spacelike* hypersurface, i.e., the induced metric is Riemannian, with a differentiable normal ν which is timelike.

In local coordinates, (x^α) and (ξ^i), the geometric quantities of the spacelike hypersurface M are connected through the Gauss formula,

$$x_{ij}^\alpha = h_{ij}\nu^\alpha. \tag{2–1}$$

Here, and also in the sequel, a covariant derivative is always a *full* tensor, i.e.,

$$x_{ij}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma.$$

The comma indicates ordinary partial derivatives.

In this implicit definition the *second fundamental form* (h_{ij}) is taken with respect to ν.

The second equation is the *Weingarten equation*

$$\nu_i^\alpha = h_i^k x_k^\alpha,$$

where we remember that ν_i^α is a full tensor.

Finally, we have the *Codazzi equation*

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauss equation*

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Now assume that N is a globally hyperbolic Lorentzian manifold with a *compact* Cauchy surface. N is then a topological product $\mathbb{R} \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian manifold, and there exists a Gaussian coordinate system (x^α), such that the metric in N has the form

$$d\bar{s}_N^2 = e^{2\psi}\big(-(dx^0)^2 + \sigma_{ij}(x^0, x)\, dx^i\, dx^j\big),$$

where σ_{ij} is a Riemannian metric, ψ a function on N, and x an abbreviation for the spacelike components (x^i); see [Geroch 1970; Hawking and Ellis 1973, p. 212; Geroch and Horowitz 1979, p. 252; Gerhardt 1983, Section 6]. We also assume that the coordinate system is *future-oriented*, that is, the time coordinate x^0 increases on future-directed curves. Hence, the *contravariant* timelike vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future-directed, and so is its *covariant* version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Let $M = \operatorname{graph} u|_{\mathcal{S}_0}$ be a spacelike hypersurface

$$M = \{ (x^0, x) : x^0 = u(x),\ x \in \mathcal{S}_0 \}.$$

Then the induced metric has the form

$$g_{ij} = e^{2\psi}(-u_i u_j + \sigma_{ij}),$$

where σ_{ij} is evaluated at (u, x), and its inverse $(g^{ij}) = (g_{ij})^{-1}$ can be expressed as

$$g^{ij} = e^{-2\psi}\left(\sigma^{ij} + \frac{u^i}{v}\frac{u^j}{v}\right),$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$u^i = \sigma^{ij} u_j, \quad v^2 = 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2.$$

Hence, $\operatorname{graph} u$ is spacelike if and only if $|Du| < 1$.

The covariant and contravariant forms of a normal vector of a graph look like

$$(\nu_\alpha) = \pm v^{-1} e^{\psi}(1, -u_i), \quad (\nu^\alpha) = \mp v^{-1} e^{-\psi}(1, u^i),$$

respectively. Thus:

Remark 2.1. *Let M be spacelike graph in a future-oriented coordinate system. The contravariant future-directed and past-directed normal vectors have the respective forms*

$$(\nu^\alpha) = v^{-1} e^{-\psi}(1, u^i), \qquad (\nu^\alpha) = -v^{-1} e^{-\psi}(1, u^i). \tag{2–2}$$

In the Gauss formula (2–1) of the preceding page, we are free to choose the future- or past-directed normal, but we stipulate that we always use the past-directed normal for reasons explained in [Gerhardt 2000a, Section 2].

Look at the component $\alpha = 0$ in (2–1) and obtain, in view of (2–2) above,

$$e^{-\psi} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}^0_{00} u_i u_j - \bar{\Gamma}^0_{0j} u_i - \bar{\Gamma}^0_{0i} u_j - \bar{\Gamma}^0_{ij}. \tag{2–3}$$

Here, the covariant derivatives are taken relative to the induced metric of M and

$$-\bar{\Gamma}^0_{ij} = e^{-\psi} \bar{h}_{ij},$$

where $(\bar{h}_{ij})$ is the second fundamental form of the hypersurfaces $\{x^0 = \text{const}\}$.

An easy calculation shows

$$\bar{h}_{ij} e^{-\psi} = -\tfrac{1}{2}\dot{\sigma}_{ij} - \dot{\psi}\sigma_{ij},$$

where the dot indicates differentiation with respect to x^0.

Finally, we define what we mean by a future mean curvature barrier.

Definition 2.2. Let N be a globally hyperbolic spacetime with compact Cauchy hypersurface $\mathcal{S}_0$ so that N can be written as a topological product $N = \mathbb{R} \times \mathcal{S}_0$ and its metric expressed as

$$d\bar{s}^2 = e^{2\psi}\big(-(dx^0)^2 + \sigma_{ij}(x^0, x)\, dx^i\, dx^j\big).$$

Here, x^0 is a globally defined future-directed time function and (x^i) are local coordinates for $\mathcal{S}_0$. N is said to have a *future mean curvature barrier* if there is a sequence M_k^+ of closed spacelike achronal hypersurfaces such that

$$\lim_{k\to\infty} H|_{M_k^+} = \infty \qquad \text{and} \qquad \limsup \inf_{M_k^+} x^0 > x^0(p) \quad \text{for all } \ p \in N$$

Likewise, N is said to have a *past mean curvature barrier* if there is a sequence M_k^- such that

$$\lim_{k\to\infty} H|_{M_k^-} = -\infty \qquad \text{and} \qquad \liminf \sup_{M_k^-} x^0 < x^0(p) \quad \text{for all } \ p \in N.$$

A future mean curvature barrier certainly represents a singularity, at least if N satisfies (1–2) on page 297, because of the future timelike incompleteness, but these singularities need not be crushing; see [Gerhardt 2004, Introduction].

3. Proof of Theorem 1.1

We start with some simple but very useful observations. If, for a given coordinate system (x^α), the metric has the form (1–1) of page 297, then the coordinate slices $M(t) = \{x^0 = t\}$ can be looked at as a solution of the evolution problem

$$\dot{x} = -e^{\psi}\nu, \tag{3–1}$$

where $\nu = (\nu^\alpha)$ is the past-directed normal vector. The embedding $x = x(t, \xi)$ is then given as $x = (t, x^i)$, where (x^i) are local coordinates for $\mathcal{S}_0$.

From (3–1) we can immediately derive evolution equations for the geometric quantities g_{ij}, h_{ij}, ν and $H = g^{ij}h_{ij}$ of $M(t)$; see [Gerhardt 2000a, Section 3].

To avoid confusion with notations for the geometric quantities of other hypersurfaces, we occasionally denote the induced metric and second fundamental of coordinate slices by $\bar{g}_{ij}$, $\bar{h}_{ij}$ and $\bar{H}$. Thus, the evolution equations

$$\dot{\bar{g}}_{ij} = -2e^{\psi}\bar{h}_{ij} \tag{3–2}$$

and

$$\dot{\bar{H}} = -\Delta e^{\psi} + (|\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)e^{\psi} \tag{3–3}$$

are valid.

The last equation is closely related to the derivative of the mean curvature operator: Let M_0 be a smooth spacelike hypersurface and in a tubular neighborhood $\mathcal{U}$ of M_0, consider hypersurfaces M that can be written as graph u over M_0 in the corresponding normal Gaussian coordinate system. Then the mean curvature of M can be expressed as

$$H = -\Delta u + \bar{H} + v^{-2}u^i u^j \bar{h}_{ij}, \tag{3–4}$$

(see (2–3) on page 300), and hence, choosing $u = \epsilon\varphi$, $\varphi \in C^2(M_0)$, we deduce

$$\frac{d}{d\epsilon}H|_{\epsilon=0} = -\Delta\varphi + \dot{\bar{H}}\varphi = -\Delta\varphi + (|\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\varphi. \tag{3–5}$$

Next we shall prove that CMC hypersurfaces are monotonically ordered, if the mean curvatures are sufficiently large.

Lemma 3.1. *Let $M_1 = \operatorname{graph} u_1$ and $M_2 = \operatorname{graph} u_2$ be spacelike hypersurfaces such that the mean curvatures H_1 and H_2 satisfy $H_1 < H_2 = \tau_2$, where H_2 is constant, and $\sqrt{n\Lambda} < \tau_2$. Then*

$$u_1 < u_2. \tag{3–6}$$

Proof. We first observe that the weaker conclusion $u_1 \le u_2$ is as good as the $u_1 < u_2$, in view of the maximum principle. Now suppose for a contradiction that $u_1 \le u_2$ is not valid, so that

$$E(u_1) = \{\, x \in \mathcal{S}_0 : u_2(x) < u_1(x) \,\} \ne \varnothing.$$

Then there exist points $p_i \in M_i$ such that

$$0 < d_0 = d(M_2, M_1) = d(p_2, p_1) = \sup\{\, d(p,q) : (p,q) \in M_2 \times M_1 \,\},$$

where d is the Lorentzian distance function. Let φ be a maximal geodesic from M_2 to M_1 realizing this distance with endpoints p_2 and p_1, and parametrized by arc length.

Denote by $\bar{d}$ the Lorentzian distance function to M_2, i.e., for $p \in I^+(M_2)$

$$\bar{d}(p) = \sup_{q \in M_2} d(q, p).$$

Since φ is maximal, $\Gamma = \{\, \varphi(t) : 0 \le t < d_0 \,\}$ contains no focal points of M_2 [O'Neill 1983, Theorem 34, p. 285]. Hence there exists an open neighborhood

$\mathcal{V} = \mathcal{V}(\Gamma)$ such that $\bar{d}$ is smooth in $\mathcal{V}$ [O'Neill 1983, Proposition 30], because $\bar{d}$ is a component of the inverse of the normal exponential map of M_2.

Now, M_2 is the level set $\{\bar{d} = 0\}$, and the level sets

$$M(t) = \{ p \in \mathcal{V} : \bar{d}(p) = t \}$$

are smooth hypersurfaces; $x^0 = \bar{d}$ is a time function in $\mathcal{V}$ and generates a normal Gaussian coordinate system, since $\langle D\bar{d}, D\bar{d} \rangle = -1$. Hence, by Equation (3–3) on page 302, the mean curvature $\bar{H}(t)$ of $M(t)$ satisfies

$$\dot{\bar{H}} = |\bar{A}|^2 + \bar{R}_{\alpha\beta} \nu^\alpha \nu^\beta,$$

and therefore we have

$$\dot{\bar{H}} \geq \frac{1}{n} |\bar{H}|^2 - \Lambda > 0, \tag{3–7}$$

in view of the assumption $\sqrt{n\Lambda} < \tau_2$.

Next, consider a tubular neighborhood $\mathcal{U}$ of M_1 with corresponding normal Gaussian coordinates (x^α). The level sets

$$\tilde{M}(s) = \{x^0 = s\}, \qquad -\epsilon < s < 0,$$

lie in the past of $M_1 = \tilde{M}(0)$ and are smooth for small ϵ.

Since the geodesic φ is normal to M_1, it is also normal to $\tilde{M}(s)$ and the length of the geodesic segment of φ from $\tilde{M}(s)$ to M_1 is exactly $-s$, i.e., equal to the distance from $\tilde{M}(s)$ to M_1. Hence we deduce

$$d(M_2, \tilde{M}(s)) = d_0 + s;$$

that is, $\{ \varphi(t) : 0 \leq t \leq d_0 + s \}$ is also a maximal geodesic from M_2 to $\tilde{M}(s)$. We conclude further that, for fixed s, the hypersurface $\tilde{M}(s) \cap \mathcal{V}$ is contained in the past of $M(d_0 + s)$ and touches $M(d_0 + s)$ in $p_s = \varphi(d_0 + s)$. The maximum principle then implies

$$H|_{\tilde{M}(s)}(p_s) \geq H|_{M(d_0+s)}(p_s) > \tau_2,$$

in view of (3–7) above.

On the other hand, the mean curvature of $\tilde{M}(s)$ converges to the mean curvature of M_1 if s tends to zero; hence

$$H_1(\varphi(d_0)) \geq \tau_2,$$

contradicting the assumption that $H_1 < H_2$. □

Corollary 3.2. *The CMC hypersurfaces with mean curvature*

$$\tau > \sqrt{n\Lambda}$$

are uniquely determined.

Proof. Let $M_1 = \operatorname{graph} u_1$ and $M_2 = \operatorname{graph} u_2$ be hypersurfaces with mean curvature τ and suppose that, say,

$$\{\, x \in \mathcal{S}_0 : u_1(x) < u_2(x) \,\} \neq \varnothing.$$

Consider a tubular neighborhood of M_1 with a corresponding future-oriented normal Gaussian coordinate system (x^α). Then the evolution of the mean curvature of the coordinate slices satisfies

$$\dot{\bar{H}} = |\bar{A}|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \geq \frac{1}{n}|\bar{H}|^2 - \Lambda > 0$$

in a neighborhood of M_1; i.e., the coordinate slices $M(t) = \{x^0 = t\}$ with $t > 0$ all have mean curvature $\bar{H}(t) > \tau$. Using now M_1 and $M(t)$, $t > 0$, as barriers, we infer that for any $\tau' \in \mathbb{R}$, $\tau < \tau' < \bar{H}(t)$, there exists a spacelike hypersurface $M_{\tau'}$ with mean curvature τ' such that $M_{\tau'}$ can be expressed as $\operatorname{graph} u$ over M_1, where

$$0 < u < t.$$

For a proof see [Gerhardt 1983, Section 6]; a different more transparent proof of this result has been given in [Gerhardt 2000b].

Writing $M_{\tau'}$ as graph over $\mathcal{S}_0$ in the original coordinate system without changing the notation for u, we obtain

$$u_1 < u,$$

and by choosing t small enough, we may also conclude that

$$E(u) = \{\, x \in \mathcal{S}_0 : u(x) < u_2(x) \,\} \neq \varnothing,$$

which is impossible, in view of the preceding result. □

Lemma 3.3. *Under the assumptions of Theorem 1.1, if $M_{\tau_0} = \operatorname{graph} u_{\tau_0}$ is a CMC hypersurface with mean curvature $\tau_0 > \sqrt{n\Lambda}$, then the future of M_{τ_0} can be foliated by CMC hypersurfaces*

$$I^+(M_{\tau_0}) = \bigcup_{\tau_0 < \tau < \infty} M_\tau. \tag{3–8}$$

Each set M_τ can be written over $\mathcal{S}_0$ as

$$M_\tau = \operatorname{graph} u(\tau, \cdot\,),$$

such that u is strictly monotone increasing with respect to τ and continuous in $[\tau_0, \infty) \times \mathcal{S}_0$.

Proof. The monotonicity and continuity of u follow from Lemma 3.1 and Corollary 3.2, in view of the a priori estimates.

It remains to verify the relation (3–8). Letting $p = (t, y^i) \in I^+(M_{\tau_0})$, we have to show $p \in M_\tau$ for some $\tau > \tau_0$.

In [Gerhardt 1983, Theorem 6.3] it is proved that there exists a family

$$\{ M_\tau : \tau_0 \le \tau < \infty \},$$

of CMC hypersurfaces M_τ if there is a future mean curvature barrier.

Define $u(\tau, \cdot)$ by

$$M_\tau = \operatorname{graph} u(\tau, \cdot).$$

Then $u(\tau_0, y) < t < u(\tau^*, y)$ for some large τ^*, because the mean curvature barrier condition together with Lemma 3.1 implies that the CMC hypersurfaces run into the future singularity, if τ goes to infinity.

In view of the continuity of $u(\cdot, y)$ we conclude that there exists τ_1 such that $\tau_0 < \tau_1 < \tau^*$ and

$$u(\tau_1, y) = t.$$

Hence $p \in M_{\tau_1}$. □

Remark 3.4. *The continuity and monotonicity of u holds in any coordinate system (x^α), even in those that do not cover the future completely like the normal Gaussian coordinates associated with a spacelike hypersurface, which are defined in a tubular neighborhood.*

The proof of Theorem 1.1 is now almost finished. The remaining arguments are identical to those in [Gerhardt 2003, Section 2], but for the convenience of the reader, we shall briefly summarize the main steps.

We have to show that the mean curvature parameter τ can be used as a time function in $\{\tau_0 < \tau < \infty\}$, i.e., τ should be smooth with a nonvanishing gradient. Both properties are local.

First step: Fix an arbitrary $\tau' \in (\tau_0, \infty)$, and consider a tubular neighborhood $\mathcal{U}$ of $M' = M_{\tau'}$. Each set $M_\tau \subset \mathcal{U}$ can then be written as $\operatorname{graph} u(\tau, \cdot)$ over M'. For small $\epsilon > 0$ we have

$$M_\tau \subset \mathcal{U} \qquad \text{for all } \tau \in (\tau' - \epsilon, \tau' + \epsilon),$$

and with the help of the implicit function theorem we now show that u is smooth. Define the operator G by

$$G(\tau, \varphi) = H(\varphi) - \tau,$$

where $H(\varphi)$ is an abbreviation for the mean curvature of $\operatorname{graph} \varphi|_{M'}$. Then G is smooth and from (3–5) (page 302) we deduce that $D_2 G(\tau', 0)\varphi$ equals

$$-\Delta\varphi + (\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\varphi,$$

where the Laplacian, the second fundamental form and the normal correspond to M'. Hence $D_2 G(\tau', 0)$ is an isomorphism and the implicit function theorem implies that u is smooth.

Second step: Still in the tubular neighborhood of M', define the coordinate transformation

$$\Phi(\tau, x^i) = (u(\tau, x^i), x^i);$$

note that $x^0 = u(\tau, x^i)$. Then

$$\det D\Phi = \frac{\partial u}{\partial \tau} = \dot{u}.$$

We know that $\dot{u}$ is nonnegative. If it is strictly positive, Φ is a diffeomorphism, and hence τ is smooth with nonvanishing gradient. A proof that $\dot{u} > 0$ is given in [Gerhardt 2003, Lemma 2.2], but we give a simpler one: The CMC hypersurfaces in $\mathcal{U}$ satisfy an equation

$$H(u) = \tau,$$

where the left hand-side can be expressed as in Equation (3–4), page 302. Differentiating both sides with respect to τ and evaluating for $\tau = \tau'$, i.e., on M', where $u(\tau', \cdot\,) = 0$, we get

$$-\Delta\dot{u} + (|A|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\dot{u} = 1.$$

In a point where $\dot{u}$ attains its minimum, the maximum principle implies

$$(|A|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta)\dot{u} \geq 1.$$

Hence $\dot{u} \neq 0$ and $\dot{u}$ is therefore strictly positive.

4. Proof of Theorem 1.2

Let x^0 be a time function satisfying the assumptions of Theorem 1.2. In other words, $N_+ = \{a_0 < x^0 < b\}$, the mean curvature of the slices $M(t) = \{x^0 = t\}$ is nonnegative, and

$$\lim_{t \to b} |M(t)| = 0.$$

Also let M_k be a sequence of spacelike achronal hypersurfaces such that

$$\liminf_{M_k} x^0 = b.$$

Write M_k as graph u_k over $\mathcal{S}_0$. Then

$$g_{ij} = e^{2\psi}(u_i u_j + \sigma_{ij}(u, x))$$

is the induced metric, where we dropped the index k for better readability, and the volume element of M_k has the form

$$d\mu = v\sqrt{\det(\bar{g}_{ij}(u, x))}\, dx,$$

where

$$v^2 = 1 - \sigma^{ij} u_i u_j < 1, \tag{4–1}$$

and $(\bar{g}_{ij}(t,\cdot))$ is the metric of the slices $M(t)$.

From (3–2) we deduce

$$\frac{d}{dt}\sqrt{\det(\bar{g}_{ij}(t,\cdot))} = -e^{\psi}\bar{H}\sqrt{\det(\bar{g}_{ij})} \le 0. \tag{4–2}$$

Now, let $a_0 < t < b$ be fixed. Then for almost every k we have

$$t < u_k \tag{4–3}$$

and hence

$$\begin{aligned}|M_k| &= \int_{\mathcal{S}_0} v\sqrt{\det(\bar{g}_{ij}(u_k,x))}\,dx \\ &\le \int_{\mathcal{S}_0} \sqrt{\det(\bar{g}_{ij}(t,x))}\,dx = |M(t)|,\end{aligned}$$

in view of (4–1), (4–2) and (4–3). We conclude that $\limsup |M_k| \le |M(t)|$ for all $a_0 < t < b$, and thus $\lim|M_k| = 0$.

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