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Dedicated to Professor George Lusztig on his sixtieth birthday.

Let Γ be any canonical left cell of the affine Weyl group W_a of type $\tilde{A}_{n-1}$ for $n > 1$. We describe the lower boundary hyperplanes for Γ, answering two questions of Humphreys.

Let W_a be an affine Weyl group and let Φ be the root system of the corresponding Weyl group. Fix a positive root system Φ^+ of Φ. There is a bijection from W_a to the set of alcoves in the euclidean space E spanned by Φ. We identify the elements of W_a with the alcoves (also with the topological closure of the alcoves) of E. According to a result of Lusztig and Xi [1988], we know that the intersection of any two-sided cell of W_a with the dominant chamber of E is exactly a single left cell of W_a, called a canonical left cell. When W_a is of type $\tilde{A}_{n-1}$, with $n > 1$, there is a bijection ϕ from the set of two-sided cells of W_a to the set of partitions of n; see Remark 2.1 and subsequent paragraphs, as well as [Shi 1986].

From now on, unless otherwise specified, we always assume that W_a is an affine Weyl group of type $\tilde{A}_{n-1}$, where $n > 1$. This article answers two questions posed recently by J. E. Humphreys (private communication):

(1) Can one find the set $B(L)$ of all the lower boundary hyperplanes for any canonical left cell L of W_a?

(2) How does the partition $\phi(L)$ determine the set $B(L)$, and in which case does the set $B(L)$ determine the partition $\phi(L)$ also?

In the first two sections, we collect some concepts and known results for later use. In Section 3, we give criteria for a hyperplane to be the lower boundary of a canonical left cell of W_a. Then we prove our main results in Section 4.

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1. Sign types

Let $\boldsymbol{n} = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. An $\boldsymbol{n}$*-sign type* (or just a sign type) is by definition a matrix $X = (X_{ij})_{i,j\in \boldsymbol{n}}$ over the symbol set $\{+, \bigcirc, -\}$, with

$$\{X_{ij}, X_{ji}\} \in \{\{+, -\}, \{\bigcirc, \bigcirc\}\} \qquad \text{for} \quad i, j \in \boldsymbol{n}.$$

X is determined entirely by its "upper unitriangular" part $X^{\Delta} = (X_{ij})_{i<j}$. We identify X with X^{Δ}. X is *dominant*, if $X_{ij} \in \{+, \bigcirc\}$ for any $i < j$ in $\boldsymbol{n}$, and is *admissible*, if

$$\begin{aligned} - \in \{X_{ij}, X_{jk}\} &\Longrightarrow X_{ik} \leqslant \max\{X_{ij}, X_{jk}\}, \\ - \notin \{X_{ij}, X_{jk}\} &\Longrightarrow X_{ik} \geqslant \max\{X_{ij}, X_{jk}\} \end{aligned} \tag{1–1}$$

for any $i < j < k$ in $\boldsymbol{n}$, where we set a total ordering: $- < \bigcirc < +$.

Lemma 1.1 ([Shi 1987b, Lemma 3.1; Shi 1999, Corollary 2.8]). (1) *A dominant sign type $X = (X_{ij})$ is admissible if and only if for any $i \leqslant h < k \leqslant j$, condition $X_{ij} = \bigcirc$ implies $X_{hk} = \bigcirc$.*

(2) *If an admissible sign type $X = (X_{ij})$ is not dominant, then there exists at least one k with $1 \leqslant k < n$ and $X_{k,k+1} = -$.*

Proof. This is an easy consequence of conditions (1–1). □

Let $E = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^n a_i = 0\}$. This is a euclidean space of dimension $n-1$ with inner product $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n)\rangle = \sum_{i=1}^n a_i b_i$. For $i \neq j$ in $\boldsymbol{n}$, let $\alpha_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$, with 1 and -1 at the i-th and j-th positions, respectively. Then $\Phi = \{\alpha_{ij} \mid 1 \leqslant i \neq j \leqslant n\}$ is the root system of type A_{n-1}, which spans E. $\Phi^+ = \{\alpha_{ij} \in \Phi \mid i < j\}$ is a positive root system of Φ with corresponding simple root system $\Pi = \{\alpha_{i,i+1} \mid 1 \leqslant i < n\}$. For any $\epsilon \in \mathbb{Z}$ and $i < j$ in $\boldsymbol{n}$, define a hyperplane

$$H_{ij;\epsilon} = \{(a_1, \ldots, a_n) \in E \mid a_i - a_j = \epsilon\}. \tag{1–2}$$

Encode a connected component C of $E \setminus \bigcup_{1\leqslant i<j\leqslant n,\, \epsilon\in\{0,1\}} H_{ij;\epsilon}$ by a sign type $X = (X_{ij})_{i<j}$ as follows. Take any $v = (a_1, \ldots, a_n) \in C$ and, for $i < j$ in $\boldsymbol{n}$, set

$$X_{ij} = \begin{cases} + & \text{if } a_i - a_j > 1, \\ - & \text{if } a_i - a_j < 0, \\ \bigcirc & \text{if } 0 < a_i - a_j < 1. \end{cases}$$

X only depends on C, but not on the choice of v; see [Shi 1986, Chapter 5]. Note that not all sign types can be obtained in this way.

Proposition 1.2 ([Shi 1986, Proposition 7.1.1 and §2]). *A sign type $X = (X_{ij})$ can be obtained in the above way if and only if it is admissible.*

Lemma 1.3. *Let $X = (X_{ij})$ be a dominant admissible sign type with $X_{p,p+1} = \bigcirc$ for some p with $1 \leqslant p < n$. Let $X' = (X'_{ij})$ be the sign type given by*

$$X'_{ij} = \begin{cases} X_{ij} & \text{if } (i, j) \neq (p, p+1), \\ - & \text{if } (i, j) = (p, p+1) \end{cases}$$

for $i < j$ in $\boldsymbol{n}$. Then X' is admissible if and only if $X_{ph} = X_{p+1,h}$ for all $h \in \boldsymbol{n}$.

Proof. This is an easy consequence of (1–1). $\square$

For $\alpha \in \Phi$, let s_α be the reflection in α:

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Let T_α be the translation by α: $T(v) = v + \alpha$. Define $s_i = s_{\alpha_{i,i+1}}$ for $1 \leqslant i < n$, and $s_0 = T_{\alpha_{1n}} s_{\alpha_{1n}}$. Then $S = \{s_i \mid 0 \leqslant i < n\}$ forms a distinguished generator set of the affine Weyl group W_a of type $\tilde{A}_{n-1}$.

A connected component in

$$E \setminus \bigcup_{\substack{1 \leqslant i < j \leqslant n \\ k \in \mathbb{Z}}} H_{ij;k}$$

is called an *alcove*. The (right) action of W_a on E induces a simply transitive permutation on the set $\mathfrak{A}$ of alcoves in E. There exists a bijection $w \mapsto A_w$ from W_a to $\mathfrak{A}$ such that A_1 (where 1 is the identity element of W_a) is the unique alcove in the dominant chamber of E whose closure contains the origin and such that $(A_y)w = A_{yw}$ for $y, w \in W_a$; see [Shi 1987a, Proposition 4.2]. To each $w \in W_a$ we associate an admissible sign type $X(w)$ that contains the alcove A_w. An admissible sign type X can be identified with the set $\{w \in W_a \mid X(w) = X\}$.

2. Partitions and Kazhdan–Luzstig cells

Let $(P, \preceq)$ be a finite poset. By a *chain* of P, we mean a totally ordered subset of P (allow to be an empty set). Also, a *cochain* of P is a subset K of P whose elements are pairwise incomparable. A k-*(co)chain family* in P ($k \geqslant 1$) is a subset J of P which is a disjoint union of k (co)chains J_i ($1 \leqslant i \leqslant k$). We usually write $J = J_1 \cup \cdots \cup J_k$.

A *partition* of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$. In particular, when $\lambda_1 = \cdots = \lambda_r = a$, we also write $\lambda = (a^r)$, and call it a *rectangular* partition. Let Λ_n be the set of all partitions of n.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ be in Λ_n. Write $\lambda \leqslant \mu$ if the inequalities $\sum_{j=1}^i \lambda_j \leqslant \sum_{j=1}^i \mu_j$ hold for $i \geqslant 1$. We say that μ is *conjugate* to λ

if $\mu_i = |\{j \mid \lambda_j \geqslant i, 1 \leqslant j \leqslant r\}|$ for $1 \leqslant i \leqslant t$, where $|X|$ stands for the cardinality of the set X.

Let d_k be the maximal cardinality of a k-chain family in P for $k \geqslant 1$. Then $d_1 < d_2 < \cdots < d_r = n = |P|$ for some $r \geqslant 1$. Let $\lambda_1 = d_1$ and $\lambda_i = d_i - d_{i-1}$ for $1 < i \leqslant r$. Then $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ by [Greene 1976, Theorem 1.6]. We get $\phi(P) = (\lambda_1, \ldots, \lambda_r) \in \Lambda_n$, called the *partition associated to chains* in P. Replacing the word "k-chain family" by "k-cochain family", we can also define $\psi(P) = (\mu_1, \ldots, \mu_t) \in \Lambda_n$, again by [Greene 1976, Theorem 1.6], called the *partition associated to cochains* in P. Moreover, $\psi(P)$ is conjugate to $\phi(P)$.

Remark 2.1. Let $(P, \preceq)$ be a finite poset with $\psi(P) = (\mu_1, \ldots, \mu_t)$. For $1 \leqslant k \leqslant t$, let $P^{(k)} = P_1 \cup \cdots \cup P_k$ be a k-cochain family of P with $|P^{(k)}| = \sum_{h=1}^{k} \mu_h$. Then $\mu_1 \geqslant |P_i| \geqslant \mu_k$ for $1 \leqslant i \leqslant k$. In particular, when $\psi(P) = (a^t)$ is rectangular, we have $|P_1| = \cdots = |P_k| = a$. This fact will be used in the proof of Lemma 4.2.

For each admissible sign type $X = (X_{ij})$, we write $i \leqslant_X j$ in $\boldsymbol{n}$ if either $i = j$ or $X_{ij} = +$. By [Shi 1999, Lemma 2.2], the order $\leqslant_X$ is a partial order on $\boldsymbol{n}$. We associate to X two partitions $\phi(X)$ and $\psi(X)$ of n defined above.

Kazhdan and Lusztig [1979] defined certain equivalence classes in a Coxeter system (W, S), called a *left cell*, a *right cell* and a *two-sided cell*.

Let W_a be the affine Weyl group of type $\tilde{A}_{n-1}$ for $n > 1$. Each element w of W_a determines a sign type $X(w)$, and hence it in turn determines two partitions $\phi(w) := \phi(X(w))$ and $\psi(w) := \psi(X(w))$. This defines two maps $\phi, \psi : W_a \longrightarrow \Lambda_n$, each of which induces, by [Shi 1986, Theorem 17.4], a bijection from the set of two-sided cells of W_a to the set Λ_n.

To each $w \in W_a$, we associate a set $\mathcal{R}(w) = \{s \in S \mid ws < w\}$, where $\leqslant$ is the Bruhat order in the Coxeter system (W_a, S). Define

$$Y_0 = \{w \in W_a \mid \mathcal{R}(w) \subseteq \{s_0\}\}.$$

By [Lusztig and Xi 1988, Theorem 1.2], the intersection of Y_0 with any two-sided cell $\phi^{-1}(\lambda)$ ($\lambda \in \Lambda_n$) is a single left cell of W_a, written Γ_λ and called a *canonical left cell*.

3. Lower boundary of a canonical left cell

We now define a lower boundary hyperplane for any $F \subset W_a$, and give criteria for a hyperplane of E to be lower boundary for a canonical left cell of W_a.

For $i < j$ in $\boldsymbol{n}$ and $k \in \mathbb{Z}$, the hyperplane $H_{ij;k}$ divides the space E into three parts: $H^+_{ij;k} = \{v \in E \mid \langle v, \alpha_{ij}\rangle > k\}$, $H^-_{ij;k} = \{v \in E \mid \langle v, \alpha_{ij}\rangle < k\}$, and $H_{ij;k}$. For any set F of alcoves in E, call $H_{ij;k}$ a *lower boundary hyperplane* of F if $\bigcup_{A\in F} A \subset H^+_{ij;k}$ and if there exists some alcove C in F such that $\bar{C} \cap H_{ij;k}$ is a facet of C of dimension $n-2$, where $\bar{C}$ stands for the closure of C in E under the usual topology.

Let Γ be a canonical left cell of W_a. As a subset in W_a, Γ is a union of some dominant sign types, by [Shi 1986, Proposition 18.2.2]; denote by $S(\Gamma)$ the set of these sign types. Regarded as a union of alcoves, the topological closure of Γ in E is connected [Shi 1986, Theorem18.2.1] and is bounded by a certain set of hyperplanes in E of the form $H_{ij;\epsilon}$, for $1 \leqslant i < j \leqslant n$ and $\epsilon = 0, 1$, defined in (1–2). Then a lower boundary hyperplane of Γ must be one of such hyperplanes. Let $B(\Gamma)$ be the set of all the lower boundary hyperplanes of Γ. Given a hyperplane $H_{ij;\epsilon}$ with $1 \leqslant i < j \leqslant n$ and $\epsilon = 0, 1$, we see that $H_{ij;\epsilon} \in B(\Gamma)$ if and only if one of the following conditions holds.

Condition 3.1. $\epsilon = 1$, $X_{ij} = +$ for all $X = (X_{ab}) \in S(\Gamma)$, and there exists some $Y = (Y_{ab}) \in S(\Gamma)$ such that the sign type $Y' = (Y'_{ab})$ defined below is admissible:

$$Y'_{ab} = \begin{cases} Y_{ab} & \text{if } (a, b) \neq (i, j), \\ \bigcirc & \text{if } (a, b) = (i, j). \end{cases}$$

Condition 3.2. $\epsilon = 0$, and there exists some $X = (X_{ab}) \in S(\Gamma)$ with $X_{ij} = \bigcirc$ such that the sign type $X' = (X'_{ab})$ defined by

$$X'_{ab} = \begin{cases} X_{ab} & \text{if } (a, b) \neq (i, j), \\ - & \text{if } (a, b) = (i, j) \end{cases}$$

is admissible.

Remark 3.3. By Lemma 1.1(2), Condition 3.2 happens only if $j = i + 1$.

Proposition 3.4. (1) *$H_{i,i+1;0} \in B(\Gamma)$ if and only if there exists some $X = (X_{ab}) \in S(\Gamma)$ such that $X_{i,h} = X_{i+1,h}$ for all $h \in \mathbf{n}$. In particular, when these conditions hold, we have $X_{i,i+1} = \bigcirc$.*

(2) *If $H_{ij;1} \in B(\Gamma)$ and if either $i \leqslant k < l \leqslant j$ or $k \leqslant i < j \leqslant l$, then $H_{kl;1} \in B(\Gamma)$ if and only if $(i, j) = (k, l)$.*

Proof. Part (1) follows from Condition 3.2 and Lemma 1.3. Then part (2) is a direct consequence of Condition 3.1 and Lemma 1.1(1). □

4. Description of the sets $B_0(\Gamma_\lambda)$ and $B_1(\Gamma_\lambda)$

We now answer the two questions of Humphreys.

Let Γ_λ be the canonical left cell of W_a corresponding to $\lambda \in \Lambda_n$. Let $B_\epsilon(\Gamma_\lambda) = \{H_{ij;\epsilon} \mid H_{ij;\epsilon} \in B(\Gamma_\lambda)\}$ for $\epsilon = 0, 1$.

Lemma 4.1. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda_n$ contains at least two different parts. Then $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$.*

Proof. Let $\mu = (\mu_1, \ldots, \mu_t)$ be the conjugate partition of λ. Then μ also contains at least two different parts. Given any p with $1 \leqslant p < n$, there exists a permutation $a_1, a_2, \ldots, a_t$ of $1, 2, \ldots, t$ such that $m_s < p$ and $m_{s+1} > p$ for some s with $0 \leqslant s < t$, where $m_u := \sum_{k=1}^{u} \mu_{a_k}$ for $0 \leqslant u \leqslant t$ with the convention that $m_0 = 0$. Define a dominant sign type $X = (X_{ij})$ such that for any i, j with $1 \leqslant i < j \leqslant n$, $X_{ij} = \bigcirc$ if and only if $m_h < i < j \leqslant m_{h+1}$ for some h with $0 \leqslant h < t$. Clearly, X is admissible with $\psi(X) = \mu$. Hence $X \in S(\Gamma_\lambda)$. We see also that $X_{ph} = X_{p+1,h}$ for all h such that $1 \leqslant h \leqslant n$. So we conclude that $H_{p,p+1;0} \in B_0(\Gamma_\lambda)$ by Proposition 3.4(1). Our result follows by Remark 3.3. □

Lemma 4.2. *For a rectangular partition $(k^a) \in \Lambda$ with $a, k \in \mathbb{N}$, we have*

$$B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leqslant p < n, a \nmid p\}.$$

Proof. Let $X = (X_{ij})$ be a dominant admissible sign type. Then a maximal cochain in $\boldsymbol{n}$ with respect to $\leqslant_X$ must consist of consecutive numbers. Now suppose $\psi(X) = (a^k)$. Then by Remark 2.1, we can take a maximal k-cochain family $\boldsymbol{n} = P_1 \cup \cdots \cup P_k$ such that $P_h = \{a(h-1)+1, a(h-1)+2, \ldots, ah\}$ with $1 \leqslant h \leqslant k$ are the maximal cochains in $\boldsymbol{n}$ with respect to $\leqslant_X$. We have $X_{a(h-1)+1,ah} = \bigcirc$ and $X_{a(h-1)+1,ah+1} = +$, which are different. So by the arbitrariness of X and by Proposition 3.4(1), we see that

$$H_{ah,ah+1;0} \notin B_0(\Gamma_{(k^a)}) \quad \text{for } 1 \leqslant h < k. \tag{4–1}$$

On the other hand, let $Y = (Y_{ij})$ be a sign type defined by

$$Y_{ij} = \begin{cases} \bigcirc & \text{if } a(h-1) < i < j \leqslant ah \text{ for some } 1 \leqslant h \leqslant k, \\ + & \text{otherwise} \end{cases}$$

for $1 \leqslant i < j \leqslant n$. Then it is clear that Y is dominant admissible with $\psi(Y) = (a^k)$. Suppose $a(h-1) < p < ah$ for some $h \in [1, k]$. Then $Y_{p,p+1} = \bigcirc$. We see also that $Y_{ph} = Y_{p+1,h}$ for all $h \in [1, n]$. By Proposition 3.4(1), we have

$$H_{p,p+1;0} \in B_0(\Gamma_{(k^a)}) \quad \text{for all } p \text{ with } 1 \leqslant p < n \text{ and } a \nmid p.$$

The result follows from this, (4–1), and Remark 3.3. □

Theorem 4.3. $B_0(\Gamma_\lambda) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$ *for all $\lambda \in \Lambda_n$ unless λ is a rectangular partition. In the latter case, say $\lambda = (k^a)$ for $k, a \in \mathbb{N}$, we have* $B_0(\Gamma_{(k^a)}) = \{H_{p,p+1;0} \mid 1 \leqslant p < n, a \nmid p\}$.

Proof. We see that a partition is nonrectangular if and only if it contains at least two different parts. So our result follows immediately from Lemmas 4.1 and 4.2. □

Theorem 4.4. $B_1(\Gamma_\lambda) = \{H_{i,i+r;1} \mid 1 \leqslant i \leqslant n-r\}$ *for* $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda_n$.

Proof. Let $\mu = (\mu_1, \dots, \mu_t) \in \Lambda_n$ be conjugate to λ. First we claim that, for any $X = (X_{ij}) \in S(\Gamma_\lambda)$,

$$X_{i,i+r} = + \quad \text{for } i = 1, \dots, n-r. \tag{4–2}$$

Otherwise, there would exist some $X = (X_{ij}) \in S(\Gamma_\lambda)$ with $X_{i,i+r} = \bigcirc$ for some i, $1 \leqslant i \leqslant n-r$. By Lemma 1.1(1), we would have $X_{hk} = \bigcirc$ for all h, k such that $i \leqslant h < k \leqslant i+r$. Then $\{i, i+1, \dots, i+r\}$ would be a cochain in $\boldsymbol{n}$ with respect to the partial order $\leqslant_X$, whose cardinality is $r+1 > \mu_1 = r$, contradicting the assumption $\psi(\Gamma_\lambda) = (\mu_1, \mu_2, \dots, \mu_t)$.

Next we want to find, for any p with $1 \leqslant p \leqslant n-r$, some $Y = (Y_{ij}) \in S(\Gamma_\lambda)$ such that the sign type $Y' = (Y'_{ij})$ defined by

$$Y'_{ij} = \begin{cases} Y_{ij} & \text{if } (i,j) \neq (p, p+r), \\ \bigcirc & \text{if } (i,j) = (p, p+r) \end{cases} \tag{4–3}$$

for $1 \leqslant i < j \leqslant n$, is admissible. If this happens, we automatically have $\psi(Y') \gneqq \mu$ by the proof of (4–2).

Take a permutation $a_1, a_2, \dots, a_t$ of $1, 2, \dots, t$ satisfying two conditions:

(1) Let $m_u = \sum_{k=1}^{u} \mu_{a_k}$ for $0 \leqslant u \leqslant t$ with the convention that $m_0 = 0$. Then there exists some $s \in [0, t)$ such that $a_{s+1} = 1$, $m_s < p$ and $m_{s+1} \geqslant p$.

(2) s is the largest possible number with the property (1) when $a_1, a_2, \dots, a_t$ ranges over all the permutations of $1, 2, \dots, t$.

Then we have $t - s \geqslant 2$, $p \leqslant m_{s+1} < p+r$ and $m_{s+2} \geqslant p+r$. Define a dominant sign type $Y = (Y_{ij})$ such that $Y_{ij} = \bigcirc$ if and only if either

$$m_u < i < j \leqslant m_{u+1} \text{ for } 0 \leqslant u < t, \quad \text{or} \quad p \leqslant i < j \leqslant p+r \text{ with } (i,j) \neq (p, p+r).$$

By Lemma 1.1(1), Y is admissible with $\psi(Y) = \mu$, i.e., $Y \in S(\Gamma_\lambda)$. Clearly, the sign type Y' obtained from Y as in (4–3) is also dominant admissible by Lemma 1.1(1). This implies by Condition 3.1 that $H_{p,p+r;1}$ belongs to $B_1(\Gamma_\lambda)$ for any $p = 1, \dots, n-r$. The result follows by Proposition 3.4(2). $\square$

Remark 4.5. Theorems 4.3 and 4.4 answer the two questions of Humphreys. In particular, the canonical left cells of W_a associated to the rectangular partitions are determined entirely by the corresponding B_1-set of hyperplanes. From the above description of B_0-sets of hyperplanes, we see that compared with the other canonical left cells of W_a, the positions of the canonical left cells associated to rectangular partitions are farther from the walls of the dominant chamber.

Remark 4.6. When $\lambda = (n)$, we have $B_0(\Gamma_\lambda) = \varnothing$ and

$$B_1(\Gamma_\lambda) = \{H_{i,i+1;1} \mid 1 \leqslant i < n\}.$$

Actually, this is the unique canonical left cell whose B_1-set contains a hyperplane of the form $H_{i,i+1;1}$. Also, this is the unique canonical left cell whose B_0-set is empty. On the other hand, $B_0(\Gamma_{(1^n)}) = \{H_{i,i+1;0} \mid 1 \leqslant i < n\}$ and $B_1(\Gamma_{(1^n)}) = \varnothing$. $\Gamma_{(1^n)}$ is the unique canonical left cell whose B_1-set is empty.

Remark 4.7. When $n \in \mathbb{N}$ is a prime number, the B_0-sets of all the canonical left cells Γ_λ of W_a are $\{H_{i,i+1;0} \mid 1 \leqslant i < n\}$, except for the case where $\lambda = (n)$.

Remark 4.8. Now assume that (W_a, S) is an irreducible affine Weyl group of arbitrary type with Δ a choice of simple roots system of Φ. We are unable to describe the lower boundary hyperplanes for a canonical left cell L of W_a in general. This is because L is not always a union of some sign types (as in the case of type $\tilde{B}_2$). But we know that L is a single sign type when L is in either the lowest or the highest two-sided cell of W_a (see [Shi 1987c; Shi 1988]) for which we can describe its lower boundary hyperplanes: if L is in the lowest two-sided cell of W_a, then $B_1(L) = \{H_{\alpha;1} \mid \alpha \in \Delta\}$ and $B_0(L) = \varnothing$, where $H_{\alpha;1} := \{v \in E \mid \langle v, \alpha^\vee\rangle = 1\}$ and $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$; if L is in the highest two-sided cell of W_a, then $B_1(L) = \varnothing$ and $B_0(L) = \{H_{\alpha;0} \mid \alpha \in \Delta\}$. This extends the result in Remark 4.6. We conjecture that any canonical left cell of W_a is a union of some sign types whenever W_a has a simply-laced type, namely $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$. If this is true, one would be able to describe the lower boundary hyperplanes for the canonical left cells of these groups.

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