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We introduce a noncommutative version of Schur multipliers relative to an operator ideal. In this setting the functions of two variables are replaced by elements from a tensor product of C*-algebras, and the measures (or spectral measures) by representations. For commutative C*-algebras this approach agrees with Birman and Solomyak's theory of double operator integrals. We study the dependence of the spaces of multipliers on the choice of representations and find that the question is closely related to Voiculescu and Arveson's theory of approximately equivalent representations. The space of multipliers universal with respect to the chosen measures is related to the Haagerup tensor product of the algebras.

1. Introduction

Let H and K be Hilbert spaces, let $B(H, K)$ be the Banach space of all bounded linear operators from H into K, and let $\mathfrak{S}_2(H, K)$ be the Hilbert space of Hilbert–Schmidt operators. Each symmetrically normed ideal I induces the norm $|\cdot|_I$ on $\mathscr{X}_I = I(H, K) \cap \mathfrak{S}_2(H, K)$. Let $\Phi : \varphi \to \Phi_\varphi$ be a map from a set G into the algebra $B(\mathfrak{S}_2(H, K))$ of all bounded operators on $\mathfrak{S}_2(H, K)$. If for some $\varphi \in G$, the operator Φ_φ preserves $\mathscr{X}_I$ and is bounded in $|\cdot|_I$, so that $|\Phi_\varphi(R)|_I \le C|R|_I$, for all $R \in \mathscr{X}_I$, then φ is called a (Φ, I)-*multiplier*. Below we consider some examples of (Φ, I)-multipliers with increasing generality.

Let X, Y be arbitrary sets, let $H = l_2(X)$, $K = l_2(Y)$, and let $B(X \times Y)$ be the set of all bounded complex-valued functions on $X \times Y$. Identify each T in $\mathfrak{S}_2(H, K)$ with the corresponding matrix $(t(x, y))$. For $\varphi \in B(X \times Y)$, set $S_\varphi(T) = (\varphi(x, y)t(x, y))$. Then $S : \varphi \mapsto S_\varphi$ is a map from $B(X \times Y)$ into $B(\mathfrak{S}_2(H, K))$, and we call (S, I)-multipliers *Schur I-multipliers*. It is not difficult to check that, at least for separable ideals I, they coincide with Schur I-multipliers as defined in [Bennett 1977] and, for $I = B(H, K)$, in [Pisier 2001].

More generally, for arbitrary measures μ on X and ν on Y, let $H = L_2(X, \mu)$ and $K = L_2(Y, \nu)$. Then $\mathfrak{S}_2(H, K)$ consists of integral operators R with kernels

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$r(x, y)$ on $X \times Y$ such that

$$|R|_2 = \left(\int_{X\times Y} |r(x, y)|^2 \, d\mu(x) \, d\nu(y) \right)^{1/2} < \infty. \tag{1–1}$$

Each $\varphi \in L_\infty(X \times Y, \mu \times \nu)$ defines a bounded linear map Φ_φ on $\mathfrak{S}_2(H, K)$, where $\Phi_\varphi(R)$ is the integral operator with kernel $\varphi(x, y)r(x, y)$. The (Φ, I)-multipliers in this case are called *(μ, ν, I)-multipliers.*

Birman and Solomyak [1967; 1973; 1989] developed a powerful machinery of *double operator integrals* (DOI) in their study of multipliers related to various problems arising in mathematical physics. Starting with two spectral measures $\mathscr{E}$ and $\mathscr{F}$ on sets X and Y, respectively, they define, for each bounded measurable function φ, a map Φ_φ on $\mathfrak{S}_2(H, K)$ by

$$\Phi_\varphi(R) = \int_X \int_Y \varphi(x, y) \, d\mathscr{F}(y) R \, d\mathscr{E}(x).$$

The corresponding (Φ, I)-multipliers are called "functions that define bounded DOI on I"; we will call them $(\mathscr{E}, \mathscr{F}, I)$-multipliers or DOI I-multipliers. For multiplicity-free spectral measures, they coincide with (μ, ν, I)-multipliers.

We consider now a noncommutative version of the example above. In this setting the functions of two variables are replaced by elements of the tensor product $\mathscr{A} \otimes \mathscr{B}$ of C*-algebras, and the spectral measures by representations π, ρ of these algebras. For $\varphi \in \mathscr{A} \otimes \mathscr{B}$, the operator $(\pi \otimes \rho)(\varphi)$ acts on the tensor product $\mathscr{H} = H_\pi \otimes H_\rho$. Identifying $\mathscr{H}$ with $\mathfrak{S}_2(H_\pi^d, H_\rho)$, where H_π^d is the dual of H_π, we may consider this operator as an operator Φ_φ on $\mathfrak{S}_2(H_\pi^d, H_\rho)$ and, in the sense above, speak about I-multipliers. We call them $(\pi \otimes \rho, I)$-multipliers. For commutative C*-algebras $\mathscr{A} = C_0(X)$ and $\mathscr{B} = C_0(Y)$, the $(\pi \otimes \rho, I)$-multipliers coincide with $(\mathscr{E}, \mathscr{F}, I)$-multipliers, where $\mathscr{E}$ and $\mathscr{F}$ are the spectral measures corresponding to the representations π and ρ. Even for commutative algebras the precise description of the spaces of multipliers is known only for $I = B(H)$; for Schur multipliers it was obtained in [Grothendieck 1953], for DOI $B(H)$-multipliers in [Peller 1985].

In this paper we mainly study the dependence of the spaces of multipliers on the choice of the representations and, in the commutative case, on the choice of spectral or scalar measures. Our initial aim was to prove that a continuous function $\varphi(x, y)$ is a $(\mu, \nu, B(H))$-multiplier if and only if it is a Schur multiplier on the product of the supports of μ and ν. In other words, we were going to prove that the space of continuous $(\mu, \nu, B(H))$-multipliers depends only on the supports of the measures. This was conjectured by B. E. Johnson in a discussion with the second author and previously proved in [Kissin and Shulman 1996] for functions of the form $(f(x)-f(y))/(x-y)$. Here this result will be deduced from a result of much more general nature: the space of all (π, ρ, I)-multipliers does not change if the representations π and ρ are replaced by approximately equivalent representations. For its proof we use Voiculescu's noncommutative Weyl–von Neumann theorem. As far as we know, this is the first application of a deep result of the theory of C*-algebras to multipliers and, in particular, to Schur multipliers. In fact, the desire to understand the relation between these branches of the operator theory was our main motivation during this work.

The restriction to the C*-tensor products of C*-algebras reflects our interest in continuous multipliers. However, in the last sections we go further and study "non-continuous" multipliers. More precisely, we consider (μ, ν, I)-multipliers continuous in a pseudotopology instead of a topology. It was shown in [Erdos et al. 1998] that each pair μ and ν of standard measures on X and Y defines a pseudotopology ω on $X \times Y$, and we study (μ, ν, I)-multipliers that are ω-continuous functions. It should be noted that the space of such multipliers is much wider than $C_0(X \times Y)$ and, moreover, all $(\mu, \nu, B(H))$-multipliers are necessarily ω-continuous. The main result here states that an ω-continuous function is a (μ, ν, I)-multiplier if and only if it becomes a Schur multiplier after deleting from X and Y suitable null subsets. As a consequence we show how one can deduce Peller's theorem on DOI $B(H)$-multipliers from Grothendieck's description of Schur multipliers. We also prove that the space of all ω-continuous (μ, ν, I)-multipliers does not change if the measures μ and ν are replaced by equivalent measures.

2. Preliminaries

We need some notions and results from the theory of symmetrically normed (s.n.) ideals. The general reference for this topic is Gohberg and Kreĭn [1965]. We denote the dual space of a Banach space X by X^d and the conjugate operator of $A \in B(X)$ by A^d. Let $\mathcal{F}$ and $C(H)$ be the ideals of finite rank operators and of compact operators in the Banach algebra $(B(H), \|\cdot\|)$ of all bounded operators on a Hilbert space H. A two-sided ideal I of $B(H)$ is *symmetrically normed* if it is a Banach space with respect to a norm $|\cdot|_I$, and

$$|AXB|_I \le \|A\| |X|_I \|B\|, \qquad \text{for } A, B \in B(H) \text{ and } X \in I.$$

Such an ideal I is selfadjoint and, by the Calkin theorem, $\mathcal{F} \subseteq I \subseteq C(H)$.

There is a one-to-one correspondence between the set of symmetrically normed functions (see [Gohberg and Kreĭn 1965]) on the space c_0 of all sequences of real numbers converging to 0 and the set $\mathfrak{J}$ of all pairs (J_0, J) of s.n. ideals, where J_0 is a separable ideal that coincides with the closure of $\mathcal{F}$ in $|\cdot|_{J_0}$, and J is the largest s.n. ideal such that $J_0 \subseteq J$ and the norms $|\cdot|_{J_0}$ and $|\cdot|_J$ coincide on J_0. We call J *coseparable* because there is another, "dual" pair $(\widehat{J_0}, \widehat{J})$ in $\mathfrak{J}$ such that J is isometrically isomorphic to the dual space of $\widehat{J_0}$ via the following correspondence:

every bounded linear functional on $\widehat{J_0}$ has the form

$$F_T(X) = (X, T)_2 = \mathrm{Tr}(T^*X), \qquad \text{with } T \in J \text{ and } \|F_T\| = |T|_J. \tag{2–1}$$

In turn, the ideal $\widehat{J}$ is isometrically isomorphic to the dual space of J_0.

Many ideals are separable and coseparable simultaneously. An important class of such ideals consists of the Schatten ideals $\mathfrak{S}_p$, with $1 \le p < \infty$. We will denote $C(H)$ by $\mathfrak{S}_\infty$ and $B(H)$ by $\mathfrak{S}_b$. The dual ideal $\widehat{\mathfrak{S}}_p$ of the Schatten ideal $\mathfrak{S}_p$ is $\mathfrak{S}_{p'}$, where

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ if } 1 < p < \infty; \qquad p' = 1 \text{ if } p = \infty; \qquad p' = b \text{ if } p = 1.$$

For each s.n. ideal I, there is a unique pair (J_0, J) in $\mathfrak{J}$ such that $J_0 \subseteq I \subseteq J$ and the norms $|\cdot|_{J_0}$, $|\cdot|_I$ and $|\cdot|_J$ coincide on J_0.

If I, J are s.n. ideals and $J \subseteq I$, Proposition 2.1 of [Kissin and Shulman 2005b] tells us that there is $c > 0$ such that

$$|A|_I \le c|A|_J, \qquad \text{for } A \in J. \tag{2–2}$$

For their dual spaces I^d, J^d, we have $I^d \subseteq J^d$ and

$$\|F\|_{J^d} \le c\|F\|_{I^d}, \qquad \text{for } F \in I^d. \tag{2–3}$$

Lemma 2.1. (i) *For coseparable ideals I, J, the following conditions are equivalent*:

$$1)\ I \subseteq J; \qquad 2)\ I_0 \subseteq J; \qquad 3)\ I_0 \subseteq J_0; \qquad 4)\ \widehat{J} \subseteq \widehat{I}.$$

In particular, the following conditions are equivalent:

$$1)\ \mathfrak{S}_2 \subseteq J_0; \qquad 2)\ \mathfrak{S}_2 \subseteq J; \qquad 3)\ \widehat{J} \subseteq \mathfrak{S}_2; \qquad 4)\ \widehat{J_0} \subseteq \mathfrak{S}_2.$$

(ii) *Let $J \subseteq I$ be s.n. ideals. If a bounded map M: $I \to I$ preserves J, its restriction to J is bounded.*

Proof. Part (i) follows from (2–1)–(2–3).

Let $A_n \to A$ and $M(A_n) \to B$ in $(J, |\cdot|_J)$. By (2–2), $|A_n - A|_I \to 0$, so $|M(A_n) - M(A)|_I \to 0$. Therefore,

$$\begin{aligned}|M(A) - B|_I &\le |M(A) - M(A_n)|_I + |M(A_n) - B|_I \\ &\le |M(A) - M(A_n)|_I + c|M(A_n) - B|_J \longrightarrow 0.\end{aligned}$$

Thus $M(A) = B$. Hence M is closed on J and, therefore, bounded. □

Let H, K be Hilbert spaces and I be an s.n. ideal of $B(H)$. Then

$$I(H, K) = \{A \in B(H, K) : (A^*A)^{1/2} \in I\}$$

is a closed left $B(K)$- and right $B(H)$-module supplied with norm

$$|A|_I = \left|(A^*A)^{1/2}\right|_I.$$

If U is an isometry from H onto K, then $I(H, K) = UI$. For $R \in B(H_1, H)$, $S \in B(K, K_1)$, and $A \in I(H, K)$,

$$SAR \in I(H_1, K_1) \quad \text{and} \quad |SAR|_I \le \|S\|\,|A|_I\,\|R\|.$$

If S and R are isometries, then $|SAR|_I = |A|_I$.

The dual space H^d of H is a Hilbert space; there is an antiisometry map ∂ from H onto H^d, where $x^d := \partial(x)$ is given by $x^d(y) = (y, x) = (x^d, y^d)$ [Wegge-Olsen 1993]. The space $\mathfrak{S}_2(H^d, K)$, being a Hilbert space with respect to the scalar product $(T, R) = Tr(R^*T)$, is isometrically isomorphic to the tensor product space $H \otimes K$. More precisely, the linear map θ from the algebraic tensor product $H \odot K$ into the set of all finite rank operators in $B(H^d, K)$ defined by

$$\theta(h \otimes k)x^d = x^d(h)k = (h, x)k \quad \text{for } x \in H,$$

extends to an isometric isomorphism from $H \otimes K$ to $\mathfrak{S}_2(H^d, K)$:

$$(\theta(\xi), \theta(\eta))_2 = \mathrm{Tr}(\theta(\eta)^*\theta(\xi)) = (\xi, \eta) \quad \text{for } \xi, \eta \in H \otimes K.$$

Let θ_1 be the isomorphism from $H_1 \otimes K_1$ on $\mathfrak{S}_2(H_1^d, K_1)$. For $R \in B(H, H_1)$, denote by R^* its adjoint, acting from H_1 to H, and by R^d its conjugate from H_1^d to H^d. Then

$$\|R^d\| = \|R\|, \quad R^d x^d = (R^*x)^d \quad \text{for } x \in H_1,$$

and

$$(RT)^d = T^d R^d, \quad (R^*)^d = (R^d)^*, \quad (\lambda R)^d = \lambda R^d \ \text{ for } \lambda \in C. \tag{2–4}$$

The second of these equalities can be written in the form

$$R^d = \partial R^* \partial_1^{-1}. \tag{2–5}$$

Let $S \in B(K, K_1)$. We have $\theta_1\big((R \otimes S)(h \otimes k)\big) = S\theta(h \otimes k)R^d$ for $h \in H$ and $k \in K$, so

$$\theta_1((R \otimes S)\xi) = S\theta(\xi)R^d, \quad \text{for } \xi \in H \otimes K. \tag{2–6}$$

3. Multipliers and approximate equivalence

A *normed subspace* X of a Hilbert space H is a linear subspace supplied with its own norm $\|\cdot\|_X$. By $b_1(X)$ we denote the closed unit ball of $(X, \|\cdot\|_X)$. As important classes of normed subspaces we mention full and normal subspaces:

(1) X is *full* if $X = H$, while $\|\cdot\|_X$ need not coincide with the norm of H;

(2) X is *normal* if $b_1(X)$ is closed in H.

We define the dual normed subspace $X^\natural$ of a normed subspace X in H by setting

$$X^\natural = \{y \in H\colon \|y\|_{X^\natural} < \infty\}, \qquad \text{where } \|y\|_{X^\natural} = \sup_{x\in X} \frac{|(x, y)|}{\|x\|_X}. \tag{3–1}$$

Then $X \subseteq X^{\natural\natural}$ and $\|x\|_{X^{\natural\natural}} \le \|x\|_X$, for $x \in X$. Thus $b_1(X) \subseteq b_1(X^{\natural\natural})$.

Proposition 3.1. (i) *For a normed subspace X of H the following conditions are equivalent*:

(1) *X is normal*;

(2) *X is a dual of some normed subspace*;

(3) $b_1(X) = b_1(X^{\natural\natural})$.

(ii) *Let X be full. It is normal if and only if*

$$\|x\|_X \le C\|x\|, \qquad \textit{for all } x \in H \textit{ and some } C > 0. \tag{3–2}$$

Proof. (3) ⇒ (2). Set $Y = X^\natural$. 2) ⇒ 1) follows from (3–1).

(1) ⇒ (3). Let $z \in b_1(X^{\natural\natural}) \setminus b_1(X)$. Since $b_1(X)$ is closed in H, then, by the Hahn–Banach theorem, there is $y \in H$ such that $|(z, y)| > 1$ and $|(x, y)| \le 1$, for all $x \in b_1(X)$. Therefore, by (3–1), $y \in X^\natural$ and $\|y\|_{X^\natural} \le 1$, so $\|z\|_{X^{\natural\natural}} > 1$. This contradiction shows that $b_1(X) = b_1(X^{\natural\natural})$. Part (i) is proved.

Let X be full: $H = X = \bigcup_{n=1}^{\infty} n\, b_1(X)$. If X is normal, then $b_1(X)$ is closed. By Baire's theorem, $b_1(X)$ contains an open subset of H and this implies (3–2). The converse is evident. □

Let X be a normed subspace of H. An operator $M \in B(H)$ is called *bounded on the pair* (X, H), if it preserves X and is bounded on X in $\|\cdot\|_X$. The proof of the following result is straightforward.

Lemma 3.2. *Let X be a normed subspace of H, and let M be bounded on* (X, H).

(i) M^* *is bounded on* $(X^\natural, H)$ *and* $\|M^*\|_{B(X^\natural)} \le \|M\|_{B(X)}$.

(ii) *M is bounded on* $(X^{\natural\natural}, H)$.

(iii) *If X is normal, then* $\|M^*\|_{B(X^\natural)} = \|M\|_{B(X)}$.

Let π be a representation of a C*-algebra $\mathcal{A}$ on H and X be a normed subspace of H. An element $a \in \mathcal{A}$ is a (π, X)-*multiplier* if $\pi(a)$ is bounded on (X, H). Set

$$|a|_X^\pi = \|\pi(a)\|_{B(X)} = \sup_{x\in X} \frac{\|\pi(a)x\|_X}{\|x\|_X}. \tag{3–3}$$

We now recall the notions of approximate equivalence and approximate subordination for representations of C*-algebras, introduced in [Voiculescu 1976] (see also [Arveson 1974]) and [Hadwin 1981], respectively. Among the various possible definitions, we use the one given in this last reference.

Definition 3.3. (i) Let π and π' be *-representations of a C*-algebra $\mathcal{A}$ on Hilbert spaces H and H'. The representation π' is approximately subordinate to π (we write $\pi' \ll_a \pi$) if there is a net $\{U_\lambda\}$ of isometries from H' into H such that

$$\|\pi(a)U_\lambda - U_\lambda \pi'(a)\| \to 0, \qquad \text{for all } a \in \mathcal{A}. \tag{3–4}$$

(ii) If the operators U_λ are unitary, then π and π' are approximately equivalent, and we write $\pi \sim_a \pi'$.

Let X' and X be normed subspaces of H' and H, respectively. We say that an approximate subordination $\pi' \ll_a \pi$ or approximate equivalence $\pi' \sim_a \pi$) is (X', X)-consistent if the operators U_λ in Equation (3–3) can be chosen in such a way that

$$U_\lambda X' \subseteq X, \quad U_\lambda^* X \subseteq X', \quad \|U_\lambda x'\|_X \le C\|x'\|_{X'}, \quad \|U_\lambda^* x\|_{X'} \le C\|x\|_X, \tag{3–5}$$

for some $C > 0$ and all $x' \in X'$, $x \in X$.

Proposition 3.4. *Let π' and π be *-representations of $\mathcal{A}$ on H' and H, let X' and X be normed subspaces of H' and H, and let there exist an (X', X)-consistent approximate subordination $\pi' \ll_a \pi$. Suppose that X' is normal. Then any (π, X)-multiplier a in $\mathcal{A}$ is also a (π', X')-multiplier, and*

$$|a|_{X'}^{\pi'} \le C^2 |a|_X^\pi . \tag{3–6}$$

Proof. Set $F_\lambda = \pi(a)U_\lambda - U_\lambda \pi'(a)$. Given $x' \in X'$, we have $U_\lambda^* \pi(a) U_\lambda x' \in X'$,

$$\|U_\lambda^* \pi(a) U_\lambda x'\|_{X'} \le C\|\pi(a)U_\lambda x'\|_X \le C|a|_X^\pi \|U_\lambda x'\|_X \le C^2 |a|_X^\pi \|x'\|_{X'},$$

and $\pi'(a)x' = U_\lambda^* U_\lambda \pi'(a)x' = U_\lambda^* \pi(a) U_\lambda x' - U_\lambda^* F_\lambda x'$.

Set $C_1 = C^2 |a|_X^\pi \|x'\|_{X'}$. Then all $U_\lambda^* \pi(a) U_\lambda x'$ belong to $C_1 b_1(X')$. Since X' is normal, the ball $C_1 b_1(X')$ is closed in H. Since $\|U_\lambda^* F_\lambda x'\|_{H'} \to 0$, we obtain that $\pi'(a)x' \in C_1 b_1(X')$. Thus $\pi'(a)$ preserves X', and

$$\|\pi'(a)x'\|_{X'} \le C_1 = C^2 |a|_X^\pi \|x'\|_{X'},$$

which gives (3–6). □

4. Operators bounded on normed subspaces of $\mathfrak{S}_2$

For an s.n. ideal I, define a normed subspace $\mathcal{X}_I$ of $\mathfrak{S}_2(H, K)$ by setting

$$\mathcal{X}_I = I(H, K) \cap \mathfrak{S}_2(H, K), \qquad \text{with } \|X\|_{\mathcal{X}_I} = |X|_I \text{ for } X \in \mathcal{X}_I.$$

As in general (see Section 2), set

$$\mathscr{X}_I^{\sim} = \{T \in \mathfrak{S}_2(H, K)\colon \text{the map } X \to (X, T)_2 = \mathrm{Tr}(T^* X) \text{ is bounded on } \mathscr{X}_I\}.$$

Let a pair (J_0, J) in $\mathfrak{J}$ be such that $J_0 \subseteq I \subseteq J$ and the norms $|\cdot|_{J_0}$ and $|\cdot|_I$ coincide on J_0. If $\mathfrak{S}_2 \subseteq I$, then, by Lemma 2.1(i), $\mathfrak{S}_2 \subseteq J_0$, so $\mathscr{X}_I = \mathscr{X}_{J_0}$. Let $(\widehat{J_0}, \widehat{J})$ be the corresponding "dual" pair.

Lemma 4.1. (i) $(\mathscr{X}_{J_0})^\natural = \mathscr{X}_{\widehat{J}}$.

(ii) *If* $\mathfrak{S}_2 \subseteq I$ *or if* I *is coseparable* ($I = J$), *then the normed space* $\mathscr{X}_I$ *is normal.*

(iii) *If* $J \subseteq \mathfrak{S}_2$, *then* $(\mathscr{X}_J)^\natural = (\mathscr{X}_I)^\natural = (\mathscr{X}_{J_0})^\natural = \mathscr{X}_{\widehat{J}} = \mathscr{X}_{\widehat{J_0}}$ *and* $(\mathscr{X}_I)^{\natural\natural} = \mathscr{X}_J$.

Proof. Since J_0 is separable and the space $\mathscr{F}(H, K)$ of all finite rank operators from H into K lies in $\mathscr{X}_{J_0}$, we see that $\mathscr{X}_{J_0}$ is dense in $J_0(H, K)$. From this, from (2–1) and (3–1) we obtain $(\mathscr{X}_{J_0})^\natural = \mathscr{X}_{\widehat{J}}$. Part (i) is proved.

If $\mathfrak{S}^2 \subseteq I$, then $\mathscr{X}_I$ is full. By (2–2) and Proposition 3.1, $\mathscr{X}_I$ is normal.

Let $I = J$. By (i), $(\mathscr{X}_{\widehat{J_0}})^\natural = \mathscr{X}_J$. Thus, by Proposition 3.1, $\mathscr{X}_I$ is normal. Part (ii) is proved.

Let $I \subseteq \mathfrak{S}_2$. By Lemma 2.1, $J \subseteq \mathfrak{S}_2$, so that $\mathscr{X}_{J_0} \subseteq \mathscr{X}_I \subseteq \mathscr{X}_J$. It follows from (2–2) that

$$(\mathscr{X}_J)^\natural \subseteq (\mathscr{X}_I)^\natural \subseteq (\mathscr{X}_{J_0})^\natural.$$

By (i), $(\mathscr{X}_{J_0})^\natural = \mathscr{X}_{\widehat{J}}$. By Lemma 2.1, $\mathfrak{S}_2 \subseteq \widehat{J_0} \subseteq \widehat{J}$, so $\mathscr{X}_{\widehat{J}} = \mathscr{X}_{\widehat{J_0}}$. From (2–1) we have $\mathscr{X}_{\widehat{J_0}} \subseteq (\mathscr{X}_J)^\natural$. Combining all, we obtain

$$(\mathscr{X}_J)^\natural = (\mathscr{X}_I)^\natural = (\mathscr{X}_{J_0})^\natural = \mathscr{X}_{\widehat{J}} = \mathscr{X}_{\widehat{J_0}}.$$

Applying (i) again, we complete the proof. □

Denote by $\mathscr{L}(I)$ the algebra of all operators bounded on $(\mathscr{X}_I, \mathfrak{S}_2(H, K))$. Recall that this means that they are bounded operators on $\mathfrak{S}_2(H, K)$, preserve $\mathscr{X}_I$ and are bounded on $\mathscr{X}_I$ in $\|\cdot\|_{\mathscr{X}_I}$. Set

(4–1) $$\mathscr{L}(I)^* = \{M^*\colon M \in \mathscr{L}(I)\} \quad \text{and} \quad \|M\|_I = \|M\|_{B(\mathscr{X}_I)}.$$

If there is an s.n. ideal J such that $\mathscr{X}_J = (\mathscr{X}_I)^\natural$, then it follows from Lemma 3.2 that

(4–2) $$\mathscr{L}(I)^* \subseteq \mathscr{L}(J) \quad \text{and} \quad \|M^*\|_J \le \|M\|_I, \quad \text{for } M \in \mathscr{L}(I).$$

If $\mathscr{X}_I$ is normal, then

(4–3) $$\|M^*\|_J = \|M\|_I.$$

Let $(J_0, J) \in \mathfrak{J}$ and let $(\widehat{J_0}, \widehat{J})$ be the corresponding "dual" pair.

Proposition 4.2. (i) $\mathscr{L}(J_0)^* \subseteq \mathscr{L}(\widehat{J})$ *and* $\|M^*\|_{\widehat{J}} \le \|M\|_{J_0}$, *for all* $M \in \mathscr{L}(J_0)$.

(ii) *If* $J_0 = J$, *then* $\|M^*\|_{\widehat{J}} = \|M\|_J$, *for all* $M \in \mathcal{L}(J)$. *If* J *is reflexive* (*that is*, $J_0 = J$ *and* $\widehat{J_0} = \widehat{J}$), *then also* $\mathcal{L}(J)^* = \mathcal{L}(\widehat{J})$.

(iii) *Let* $J \subseteq \mathfrak{S}_2$ *and let* I *be an s.n. ideal such that* $\widehat{J_0} \subseteq I \subseteq \widehat{J}$ *and the norms* $|\cdot|_{\widehat{J_0}}, |\cdot|_I$, *coincide on* $\widehat{J_0}$. *Then*

$$\mathcal{L}(J_0) \subseteq \mathcal{L}(\widehat{J_0})^* = \mathcal{L}(I)^* = \mathcal{L}(\widehat{J})^* = \mathcal{L}(J),$$

and the inclusion and the equalities are isometric.

In particular, $\mathcal{L}(\mathfrak{S}_p)^* = \mathcal{L}(\mathfrak{S}_{p'})$, *if* $1 < p < \infty$, *where* $p' = p/(p-1)$; $\mathcal{L}(\mathfrak{S}_1)^* = \mathcal{L}(\mathfrak{S}_\infty) = \mathcal{L}(\mathfrak{S}_b)$ *and the norms coincide.*

Proof. Part (i) follows from Lemma 3.2(i) and (4–2).

If $J_0 = J$, then, by Lemma 4.1(ii), the space $\mathcal{X}_{J_0} = \mathcal{X}_J$ is normal, and part (ii) follows from (4–3) and (i).

By Lemma 2.1, $\mathfrak{S}_2 \subseteq \widehat{J_0}$, so that $\mathcal{X}_{\widehat{J_0}} = \mathcal{X}_I = \mathcal{X}_{\widehat{J}}$ $(= \mathfrak{S}_2)$ and the norms coincide. Hence $\mathcal{L}(\widehat{J_0}) = \mathcal{L}(I) = \mathcal{L}(\widehat{J})$ and the norms coincide. By Lemma 4.1, $\mathcal{X}_{\widehat{J}}$ is normal and $(\mathcal{X}_{\widehat{J}})^\natural = (\mathcal{X}_{\widehat{J_0}})^\natural = \mathcal{X}_J$. It follows from (4–2) and (4–3) that

$$\mathcal{L}(\widehat{J})^* \subseteq \mathcal{L}(J) \quad \text{and} \quad \|M\|_{\widehat{J}} = \|M^*\|_J \text{ for } M \in \mathcal{L}(\widehat{J}). \tag{4–4}$$

Combining this with (i), we have

$$\mathcal{L}(J_0) \subseteq \mathcal{L}(\widehat{J_0})^* \subseteq \mathcal{L}(J), \tag{4–5}$$

$$\|M\|_J = \|M^*\|_{\widehat{J_0}} = \|M^*\|_{\widehat{J}} \le \|M\|_{J_0} \quad \text{for } M \in \mathcal{L}(J_0). \tag{4–6}$$

By Lemma 4.1(iii), $(\mathcal{X}_J)^\natural = \mathcal{X}_{\widehat{J_0}}$. Hence, by (4–2),

$$\mathcal{L}(J)^* \subseteq \mathcal{L}(\widehat{J_0}) \quad \text{and} \quad \|M^*\|_{\widehat{J_0}} \le \|M\|_J \text{ for } M \in \mathcal{L}(J).$$

Since $\mathcal{X}_{J_0} \subseteq \mathcal{X}_J$ and the norms $|\cdot|_J$ and $|\cdot|_{J_0}$ coincide on J_0, we have $\|M\|_{J_0} \le \|M\|_J$, for $M \in \mathcal{L}(J_0)$. Combining this with (4–4)–(4–6), we conclude the proof of (iii). □

Let $I \subset R \subset J$ be s.n. ideals. The ideal R is called an *interpolation ideal* for the pair (I, J), if every bounded operator T on J preserving I also preserves R. It follows from Lemma 2.1 that $T|I$ and $T|R$ are bounded operators. All coseparable ideals are interpolation ideals for the pair $(\mathfrak{S}_1, \mathfrak{S}_\infty)$ (see [Mitjagin 1965]).

Using the results of [Boyd 1969], Arazy [1978] associated the Boyd indices (p_{J_0}, q_{J_0}), where $1 \le p_{J_0} \le q_{J_0} \le \infty$, with each separable ideal J_0 and proved that J_0 is an interpolation ideal for a pair $(\mathfrak{S}_p, \mathfrak{S}_q)$ if $p < p_{J_0}$ and $q_{J_0} < q$. For $J_0 = \mathfrak{S}_p$, one has $p_{J_0} = q_{J_0} = p$. In particular, $\mathfrak{S}_r$ is an interpolation ideal for $(\mathfrak{S}_p, \mathfrak{S}_q)$ if $p < r < q$.

Corollary 4.3. *If R is an interpolation ideal for a pair (I, I_1) of separable ideals, then $\mathcal{L}(I) \cap \mathcal{L}(I_1) \subseteq \mathcal{L}(R)$. In particular,*

$$\mathcal{L}(\mathfrak{S}_\infty)^* \cap \mathcal{L}(\mathfrak{S}_\infty) \subseteq \mathcal{L}(J) \quad \textit{for each coseparable ideal } J,$$
$$\mathcal{L}(\mathfrak{S}_r) \subseteq \mathcal{L}(\mathfrak{S}_p) \quad \textit{if } 2 \le p < r \textit{ or } 1 \le r < p \le 2,$$
$$\mathcal{L}(\mathfrak{S}_p) \cap \mathcal{L}(\mathfrak{S}_q) \subseteq \mathcal{L}(J_0) \quad \textit{if } p < p_{J_0} \textit{ and } q_{J_0} < q.$$

5. Multipliers for tensor products of representations

Let $\mathcal{A} \otimes \mathcal{B}$ be the minimal tensor product of C*-algebras $\mathcal{A}$ and $\mathcal{B}$ — the completion of the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ in the minimal C*-norm $\|\cdot\|_{\min}$. If π and ρ are *-representations of $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces H and K, we denote by $\pi \otimes \rho$ their tensor product; it is a *-representation of $\mathcal{A} \otimes \mathcal{B}$ on $H \otimes K$.

Let $\xi \in H \otimes K$. Then $\theta(\xi) \in \mathfrak{S}_2(H^d, K)$. It follows from (2–6) that, for $a \odot b \in \mathcal{A} \odot \mathcal{B}$,

$$\theta((\pi \otimes \rho)(a \odot b)\xi) = \theta((\pi(a) \otimes \rho(b))\xi) = \rho(b)\theta(\xi)\pi(a)^d,$$

where $\pi(a)^d$ is the conjugate of $\pi(a)$ on H^d. Thus the representation $\pi \otimes \rho$ is equivalent to the representation $\sigma_{\pi,\rho}$ of $\mathcal{A} \otimes \mathcal{B}$ on $\mathfrak{S}_2(H^d, K)$ such that

$$\text{(5–1)} \qquad \sigma_{\pi,\rho}(a \odot b)T = \rho(b)T\pi(a)^d, \qquad \text{for } a \in \mathcal{A},\ b \in \mathcal{B},\ T \in \mathfrak{S}_2(H^d, K).$$

We say that $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is a $(\pi \otimes \rho, I)$-multiplier if it is a $(\sigma_{\pi,\rho}, I)$-multiplier, that is, $\sigma_{\pi,\rho}(\varphi) \in \mathcal{L}(I)$. Recall that it means that $\sigma_{\pi,\rho}(\varphi)$ preserves $\mathcal{X}_I = I(H^d, K) \cap \mathfrak{S}_2(H^d, K)$, and its restriction to $\mathcal{X}_I$ is bounded in $|\cdot|_I$.

Denote by $\mathbf{M}_I^{\pi,\rho}(\mathcal{A} \otimes \mathcal{B})$ (or just $\mathbf{M}_I^{\pi,\rho}$) the algebra of all $(\pi \otimes \rho, I)$-multipliers, and by $\|\varphi\|_I^{\pi,\rho}$ the norm of $\sigma_{\pi,\rho}(\varphi)$ on $\mathcal{X}_I$ (see (4–1)):

$$\|\varphi\|_I^{\pi,\rho} = \|\sigma_{\pi,\rho}(\varphi)\|_I.$$

Then $\mathbf{M}_{\mathfrak{S}_2}^{\pi,\rho} = \mathcal{A} \otimes \mathcal{B}$. We have $\mathbf{M}_{\mathfrak{S}_\infty}^{\pi,\rho} = \mathbf{M}_{\mathfrak{S}_b}^{\pi,\rho}$ and, omitting the subscript, write $\mathbf{M}^{\pi,\rho}$ and $\|\varphi\|^{\pi,\rho}$.

Remark. It follows immediately from our definitions that all results of Proposition 4.2 and Corollary 4.3 hold if $\mathcal{L}(I)$ is replaced by $\mathbf{M}_I^{\pi,\rho}$.

Clearly, all algebras $\mathbf{M}_I^{\pi,\rho}$ contain $\mathcal{A} \odot \mathcal{B}$, so they are dense in $\mathcal{A} \otimes \mathcal{B}$. We will see now (and use later on) that the unit ball of $\mathbf{M}_I^{\pi,\rho}$ is norm closed in $\mathcal{A} \otimes \mathcal{B}$. In fact, it is closed in a much stronger sense — with respect to a weaker convergence, which can be considered as the analog of the point convergence in the case of usual Schur multipliers.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Probably the weakest condition for an operator T from $\mathfrak{X}$ into $\mathfrak{Y}$ to be considered as a "limit point" for a set W of operators is the condition that $Tx \in \overline{Wx}$, for any vector $x \in \mathfrak{X}$. In this case one says that T belongs to the *reflexive hull* $\operatorname{Ref}(W)$ of W.

Let τ be a *-representation of a C*-algebra $\mathfrak{D}$ on a Hilbert space $\mathcal{H}$. We say that $\varphi \in \mathfrak{D}$ is a *τ-cluster point* of a convex subset W of $\mathfrak{D}$, if $\tau(\varphi) \in \operatorname{Ref}(\tau(W))$. For $\mathcal{H} = \mathfrak{S}_2(H_1, H_2)$, this means that there are $\varphi_n \in W$ such that

$$\lim_{n\to\infty} \left|\tau(\varphi)(T) - \tau(\varphi_n)(T)\right|_2 = 0 \quad \text{for all } T \in \mathfrak{S}_2(H_1, H_2). \tag{5–2}$$

We say that φ is a *weak τ-cluster point of W* if, for each $T \in \mathfrak{S}_2(H_1, H_2)$, the operator $\tau(\varphi)(T) \in \mathfrak{S}_2(H_1, H_2)$ belongs to $\operatorname{Ref}\{\tau(w)(T)\colon w \in W\}$. By the Hahn–Banach theorem, this means that, for each $x \in H_1$ and $y \in H_2$, there are $\varphi_n \in W$ such that

$$(\tau(\varphi_n)(T)x, y) \to (\tau(\varphi)(T)x, y). \tag{5–3}$$

Recall that, for any normed space $(\mathfrak{X}, \|\cdot\|)$, we denote by $b_r(\mathfrak{X})$ the closed ball $\{x \in \mathfrak{X}\colon \|x\| \le r\}$.

Proposition 5.1. (i) *If $\mathfrak{S}_2 \subseteq I$, then $b_r(\mathbf{M}_I^{\pi,\rho})$ contains all its $\sigma_{\pi,\rho}$-cluster points.*
(ii) *$b_r(\mathbf{M}^{\pi,\rho})$ contains all its weak $\sigma_{\pi,\rho}$-cluster points.*

Proof. Let φ be a $\sigma_{\pi,\rho}$-cluster point of $b_r(\mathbf{M}_I^{\pi,\rho})$. Then, for $T \in \mathfrak{S}_2(H^d, K)$, there are $\varphi_n \in b_r(\mathbf{M}_I^{\pi,\rho})$ such that (5–2) holds. Hence, by (2–2),

$$\begin{aligned}|\sigma_{\pi,\rho}(\varphi)(T)|_I &\le |\sigma_{\pi,\rho}(\varphi-\varphi_n)(T)|_I + |\sigma_{\pi,\rho}(\varphi_n)(T)|_I \\ &\le c|\sigma_{\pi,\rho}(\varphi-\varphi_n)(T)|_2 + \|\varphi_n\|_I^{\pi,\rho}|T|_I \le c|\sigma_{\pi,\rho}(\varphi-\varphi_n)(T)|_2 + r|T|_I,\end{aligned}$$

for some $c > 0$. Thus $|\sigma_{\pi,\rho}(\varphi)(T)|_I \le r|T|_I$, so $\varphi \in b_r(\mathbf{M}_I^{\pi,\rho})$. Part (i) is proved.

Let $I = \mathfrak{S}_\infty$ and let φ be a weak $\sigma_{\pi,\rho}$-cluster of $b_r(\mathbf{M}^{\pi,\rho})$. For $T \in \mathfrak{S}_2(H^d, K)$, $x \in H^d$, $y \in K$, choose $\varphi_n \in b_r(\mathbf{M}^{\pi,\rho})$ satisfying (5–3). Then a similar argument gives

$$|(\sigma_{\pi,\rho}(\varphi)(T)x, y)| \le r\|T\|\|x\|\|y\|.$$

Hence $\varphi \in b(\mathbf{M}^{\pi,\rho})$. □

We consider now how the space of multipliers depends on the choice of representations. The next theorem establishes that $\mathbf{M}_I^{\pi,\rho}$ does not change if π and ρ are replaced by approximately equivalent representations.

Theorem 5.2. *Let $\pi' \ll_a \pi$ and $\rho' \ll_a \rho$. If I is either a coseparable ideal or contains $\mathfrak{S}_2$, then*

$$\mathbf{M}_I^{\pi,\rho} \subseteq \mathbf{M}_I^{\pi',\rho'} \quad \textit{and} \quad \|\varphi\|_I^{\pi',\rho'} \le \|\varphi\|_I^{\pi,\rho} \textit{ for } \varphi \in \mathbf{M}_I^{\pi,\rho}.$$

As a consequence, if $\pi' \sim_a \pi$ and $\rho' \sim_a \rho$, then

$$\mathbf{M}_I^{\pi,\rho} = \mathbf{M}_I^{\pi',\rho'} \quad \textit{and} \quad \|\varphi\|_I^{\pi',\rho'} = \|\varphi\|_I^{\pi,\rho} \textit{ for } \varphi \in \mathbf{M}_I^{\pi,\rho}.$$

Proof. Let isometries $U_\lambda : H' \to H$ and $V_\mu : K' \to K$ satisfy (3–4). Then, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $\|\pi(a)U_\lambda - U_\lambda\pi'(a)\| \to 0$ and $\|\rho(b)V_\mu - V_\mu\rho'(b)\| \to 0$. The operators $W_{\lambda\mu} = U_\lambda \otimes V_\mu$ are isometries from $H' \otimes K'$ into $H \otimes K$, and

$$\begin{aligned}&\big\|(\pi\otimes\rho)(a\otimes b)U_\lambda\otimes V_\mu - U_\lambda\otimes V_\mu(\pi'\otimes\rho')(a\otimes b)\big\|\\ &\qquad\le \big\|(\pi(a)U_\lambda - U_\lambda\pi'(a))\otimes\rho(b)V_\mu\big\| + \big\|U_\lambda\pi'(a)\otimes(\rho(b)V_\mu - V_\mu\rho'(b))\big\|,\end{aligned}$$

which tends to 0. By linearity, $\|(\pi\otimes\rho)(x)W_{\lambda\mu} - W_{\lambda\mu}(\pi'\otimes\rho')(x)\| \to 0$, for all $x \in \mathcal{A} \odot \mathcal{B}$. Since $\|W_{\lambda\mu}\| = 1$, it also holds for all $x \in \mathcal{A} \otimes \mathcal{B}$. Thus

$$\pi'\otimes\rho' \ll_a \pi\otimes\rho.$$

By Lemma 4.1(ii), the normed space $\mathcal{X}_I((H')^d, K')$ is normal. Identifying $H \otimes K$ with $\mathfrak{S}_2(H^d, K)$ and $H' \otimes K'$ with $\mathfrak{S}_2((H')^d, K')$, we have from (2–6) that,

$$W_{\lambda\mu}T = (U_\lambda\otimes V_\mu)T = V_\mu T U_\lambda^d \quad\text{and}\quad W_{\lambda\mu}^* R = (U_\lambda^*\otimes V_\mu^*)R = V_\mu^* R (U_\lambda^*)^d,$$

for $T \in \mathfrak{S}_2((H')^d, K')$ and $R \in \mathfrak{S}_2(H^d, K)$. Since I is an ideal,

$$W_{\lambda\mu}\mathcal{X}_I((H')^d, K') \subseteq \mathcal{X}_I(H^d, K);$$

see (4–6). By (2–4),

$$|W_{\lambda\mu}T|_I \le \|V_\mu\||T|_I\|U_\lambda^d\| \le |T|_I \quad\text{and}\quad |W_{\lambda\mu}^* R|_I \le \|V_\mu^*\||R|_I\|(U_\lambda^*)^d\| \le |R|_I.$$

Hence the approximate subordination $\pi'\otimes\rho' \ll_a \pi\otimes\rho$ satisfies (3–5). Applying Proposition 3.4, we complete the proof. □

Remark. We do not know whether Theorem 5.2 extends to all separable ideals contained in $\mathfrak{S}_2$. Proposition 4.2(i) only gives that $(\mathbf{M}_{J_0}^{\pi,\rho})^* \subseteq \mathbf{M}_J^{\pi',\rho'}$, if $J_0 \subseteq \mathfrak{S}_2$.

Recall that for $T \in B(H)$, $\operatorname{rank}(T) = \dim\overline{(TH)}$. Let π and π' be representations of a C*-algebra $\mathcal{A}$. It was proved in Theorem 5.1 of [Hadwin 1981] that

$$\pi' \ll_a \pi \iff \operatorname{rank}(\pi'(a)) \le \operatorname{rank}(\pi(a)) \quad\text{for each } a \in \mathcal{A}. \tag{5–4}$$

Thus it follows from Theorem 5.2 and (5–4) that, if

$$\operatorname{rank}(\pi'(a)) = \operatorname{rank}(\pi(a)) \quad\text{and}\quad \operatorname{rank}(\rho'(b)) = \operatorname{rank}(\rho(b))$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then $\mathbf{M}_I^{\pi,\rho} = \mathbf{M}_I^{\pi',\rho'}$, and the corresponding norms are equal.

For some applications (see Section 6) it is important that, for representations of separable algebras on the spaces of arbitrary dimension, one need not distinguish infinite values of the rank.

Corollary 5.3. *Let an s.n. ideal I be either coseparable or contain $\mathfrak{S}_2$. Let π, π' and, respectively, ρ, ρ' be representations of separable C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces H, H' and K, K'. If*

$$\min\{\aleph_0, \operatorname{rank}(\pi'(a))\} \le \min\{\aleph_0, \operatorname{rank}(\pi(a))\}$$

and

$$\min\{\aleph_0, \operatorname{rank}(\rho'(b))\} \le \min\{\aleph_0, \operatorname{rank}(\rho(b))\},$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, then $\mathbf{M}_I^{\pi,\rho} \subseteq \mathbf{M}_I^{\pi',\rho'}$ and $\|\varphi\|_I^{\pi',\rho'} \le \|\varphi\|_I^{\pi,\rho}$, for $\varphi \in \mathbf{M}_I^{\pi,\rho}$.

Proof. Let $\varphi \in \mathbf{M}_I^{\pi,\rho}$ and $\|\varphi\|_I^{\pi,\rho} = C$. We have to prove that for every $T \in I(H'^d, K')$,

$$\|\sigma_{\pi',\rho'}(\varphi)(T)\|_I \le C\|T\|_I.$$

Since T is compact, there are separable subspaces $K_0 \subset K'$ and $G_0 \subset H'^d$ such that $T = P_{K_0} T P_{G_0}$, where P_{K_0}, P_{G_0} are the corresponding projections. The subspaces $K_1 = \overline{\rho'(\mathcal{B})K_0}$ of K' and $G_1 = \overline{\pi'^d(\mathcal{A})G_0}$ of H'^d are also separable because $\mathcal{A}$ and $\mathcal{B}$ are separable. We denote by H_1 the orthogonal complement in H' of the annihilator of G_1.

Since H_1 and K_1 are invariant for π' and ρ', respectively, define new representations π'_1 and ρ'_1 of $\mathcal{A}$ and $\mathcal{B}$ by

$$\pi'_1(a) = \pi'(a)|_{H_1} \oplus \mathbf{0} \quad \text{and} \quad \rho'_1(b) = \rho'(b)|_{K_1} \oplus \mathbf{0} \quad \text{for } a \in \mathcal{A},\ b \in \mathcal{B}.$$

Since H_1 and K_1 are separable, it follows from our assumptions that $\operatorname{rank}(\pi'_1(a)) \le \operatorname{rank}(\pi(a))$ and $\operatorname{rank}(\rho'_1(b)) \le \operatorname{rank}(\rho(b))$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. By (5–4),

$$\pi'_1 \ll_a \pi \quad \text{and} \quad \rho'_1 \ll_a \rho.$$

Hence, by Theorem 5.2,

$$\varphi \in \mathbf{M}_I^{\pi'_1,\rho'_1} \quad \text{and} \quad \|\varphi\|_I^{\pi'_1,\rho'_1} \le \|\varphi\|_I^{\pi,\rho}.$$

Thus $\|\sigma_{\pi'_1,\rho'_1}(\varphi)(T)\|_I \le C\|T\|_I$. But by the construction of π'_1 and ρ'_1, we have

$$\sigma_{\pi'_1,\rho'_1}(\varphi)(T) = \sigma_{\pi',\rho'}(\varphi)(T),$$

whence $\|\sigma_{\pi',\rho'}(\varphi)(T)\|_I \le C\|T\|_I$. □

Corollary 5.4. *Let an ideal I be either coseparable or contain $\mathfrak{S}_2$, let π, π' be representations of $\mathcal{A}$, and ρ, ρ' be representations of $\mathcal{B}$. Suppose that $\pi(\mathcal{A})$ and $\rho(\mathcal{B})$ contain no nonzero finite rank operators, and that π' and ρ' are separable and satisfy the condition*

$$\text{(5–5)} \qquad \operatorname{Ker}(\pi) \subseteq \operatorname{Ker}(\pi') \quad \textit{and} \quad \operatorname{Ker}(\rho) \subseteq \operatorname{Ker}(\rho').$$

Then $\mathbf{M}_I^{\pi,\rho} \subseteq \mathbf{M}_I^{\pi',\rho'}$ and $\|\varphi\|_I^{\pi',\rho'} \le \|\varphi\|_I^{\pi,\rho}$, for $\varphi \in \mathbf{M}_I^{\pi,\rho}$.

Proof. It follows from (5–5) that $\operatorname{rank}(\pi'(a)) \le \operatorname{rank}(\pi(a))$, for $a \in \mathcal{A}$, and $\operatorname{rank}(\rho'(b)) \le \operatorname{rank}(\rho(b))$, for $b \in \mathcal{B}$. Hence, by (5–4),

$$\pi' \ll_a \pi \quad \text{and} \quad \rho' \ll_a \rho,$$

and it remains now only to apply Theorem 5.2. □

Remark 5.5. (1) The first condition in Corollary 5.4 can be replaced by the conditions

$$\operatorname{rank}(\pi'(a)) \le \operatorname{rank}(\pi(a)) \quad \text{and} \quad \operatorname{rank}(\rho'(b)) \le \operatorname{rank}(\rho(b)),$$

whenever $\pi(a)$ and $\rho(b)$ are nonzero finite rank operators.

(2) If $\mathcal{A}$ and $\mathcal{B}$ are separable, the condition in Corollary 5.4 that π' and ρ' are separable can be omitted.

Applying Corollary 5.4 to simple C*-algebras we get the following result.

Corollary 5.6. *Let I be either a coseparable ideal, or $\mathfrak{S}_2 \subseteq I$. If $\mathcal{A}$ and $\mathcal{B}$ are simple C^*-algebras different from $\mathfrak{S}_\infty$, then $\mathbf{M}_I^{\pi,\rho}$ is the same for all separable representations π of $\mathcal{A}$ and ρ of $\mathcal{B}$.*

For $I = \mathfrak{S}_\infty$ or $\mathfrak{S}_b$, the conditions in Corollary 5.4 can be further simplified if the representations π, ρ have separating vectors. This simplification is based on the results of Smith [1991].

Recall that a vector $x \in H$ is *separating* for a representation π of $\mathcal{A}$ if the map $T \to Tx$ is injective on the second commutant $\pi(\mathcal{A})''$. This is equivalent to the existence of a *cyclic vector* for the commutant $\pi(\mathcal{A})'$. In the lemma below, $\mathbf{1}$ is the identity operator on a fixed Hilbert space $\mathcal{H}$. The representations $\pi \otimes \mathbf{1}$ and $\rho \otimes \mathbf{1}$ act on $H \otimes \mathcal{H}$ and $K \otimes \mathcal{H}$, respectively.

Lemma 5.7. *Let *-representations π and ρ of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ on H and K have separating vectors. Then $\mathbf{M}^{\pi,\rho} = \mathbf{M}^{\pi\otimes\mathbf{1},\rho\otimes\mathbf{1}}$ and the norms coincide.*

The proof of the lemma follows along the lines of the proof of in [Smith 1991, Theorem 2.1] and we omit it.

Corollary 5.8. *Let representations π of $\mathcal{A}$ and ρ of $\mathcal{B}$ have separating vectors. Then $\mathbf{M}^{\pi,\rho} \subseteq \mathbf{M}^{\pi',\rho'}$ and $\|\varphi\|^{\pi',\rho'} \le \|\varphi\|^{\pi,\rho}$, for $\varphi \in \mathbf{M}^{\pi,\rho}$, if representations π' and ρ' satisfy* (5–5).

Proof. If $\dim \mathcal{H}$ is sufficiently large, then condition (5–5) implies

$$\operatorname{rank}(\pi'(a)) \le \operatorname{rank}(\pi(a) \otimes \mathbf{1}) \quad \text{and} \quad \operatorname{rank}(\rho'(b)) \le \operatorname{rank}(\rho(b) \otimes \mathbf{1}),$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Hence, by (5–4), $\pi' \ll_a \pi \otimes \mathbf{1}$ and $\rho' \ll_a \rho \otimes \mathbf{1}$, and to complete the proof it remains only to apply Theorem 5.2 and Lemma 5.7. □

6. Universal multipliers

In this section we consider only the case $I = \mathfrak{S}_b$. We saw in Corollary 5.8 that in this case all multipliers for a pair of faithful representations with separating vectors are multipliers for all pairs of representations. Let us denote by $\mathbf{M}(\mathscr{A} \otimes \mathscr{B})$ the algebra of "universal" multipliers; it consists of all elements of $\mathscr{A} \otimes \mathscr{B}$ that are $(\pi \otimes \rho, \mathfrak{S}_b)$-multipliers for all pairs (π, ρ). Clearly $\mathscr{A} \odot \mathscr{B} \subseteq \mathbf{M}(\mathscr{A} \otimes \mathscr{B}) \subseteq \mathscr{A} \otimes \mathscr{B}$. For $\varphi \in \mathbf{M}(\mathscr{A} \otimes \mathscr{B})$, set

$$\|\varphi\|_r = \sup_{\pi,\rho} \|\varphi\|^{\pi,\rho}. \tag{6–1}$$

It is not difficult to see that $\|\varphi\|_r < \infty$. Indeed, if $\|\varphi\|^{\pi_n,\rho_n} \to \infty$, consider the representations $\pi = \bigoplus \pi_n$ and $\rho = \bigoplus \rho_n$. Then $\|\varphi\|^{\pi_n,\rho_n} \le \|\varphi\|^{\pi,\rho}$ for all n, a contradiction.

As usual, we denote by $\mathscr{A}^{\mathrm{op}}$ the C*-algebra that consists of all elements of $\mathscr{A}$ and has the same norm and involution, but the reverse multiplication: $a \circ b = ba$. If π is a *-representation of $\mathscr{A}$ on H, the map $\pi^{\mathrm{op}} : a \to \pi(a)^d$ is a *-representation of $\mathscr{A}^{\mathrm{op}}$ on H^d.

Recall that the Haagerup norm on $\mathscr{A} \odot \mathscr{B}$ is defined by

$$\|w\|_h = \inf\Big\{ \big\| \textstyle\sum a_i a_i^* \big\|^{1/2} \big\| \sum b_i^* b_i \big\|^{1/2} : w = \sum a_i \otimes b_i \Big\}.$$

It is known that

$$\|w\|_{\min} \le \|w\|_h, \qquad \text{for } w \in \mathscr{A} \odot \mathscr{B}.$$

Define a "pseudo-Haagerup" norm on $\mathscr{A} \odot \mathscr{B}$ by setting

$$\|w\|_{ph} = \inf\Big\{ \big\| \textstyle\sum a_i a_i^* \big\|^{1/2} \big\| \sum b_i b_i^*) \big\|^{1/2} : w = \sum a_i \otimes b_i \Big\}. \tag{6–2}$$

It is a norm, because $\|w\|_{ph} = \|\Gamma w\|_h$, where Γ: $\mathscr{A} \odot \mathscr{B} \to \mathscr{B} \odot \mathscr{A}^{\mathrm{op}}$ is a linear bijection defined by $\Gamma(a \otimes b) = b \otimes a$.

Theorem 6.1. *The Haagerup and pseudo-Haagerup norms satisfy* $\|w\|_r = \|w\|_{ph}$ *for* $w \in \mathscr{A} \odot \mathscr{B}$.

Proof. Let π and ρ be representations of $\mathscr{A}$ and $\mathscr{B}$ on H and K. For $w = \sum a_i \otimes b_i$ in $\mathscr{A} \odot \mathscr{B}$, set $A_i = \pi(a_i)$, $B_i = \rho(b_i)$. Take $T \in \mathfrak{S}_2(H^d, K)$, $x \in H$, $y \in K$. By (5–1),

$$\begin{aligned} |(\sigma_{\pi,\rho}(w)Tx, y)| &= \Big|\sum_i (B_i T A_i^d x, y)\Big| \le \sum_i \big|(T A_i^d x, B_i^* y)\big| \\ &\le \sum_i \|T A_i^d x\| \|B_i^* y\| \le \|T\| \Big(\sum_i \|A_i^d x\|^2\Big)^{1/2} \Big(\sum_i \|B_i^* y\|^2\Big)^{1/2}. \end{aligned}$$

We have

$$\sum_i \|B_i^* y\|^2 = \sum_i (B_i^* y, B_i^* y) = \Big(y, \Big(\sum_i B_i B_i^*\Big) y\Big)$$
$$\leq \Big\|\rho\Big(\sum_i b_i b_i^*\Big)\Big\| \, \|y\|^2 \leq \Big\|\sum_i b_i b_i^*\Big\| \, \|y\|^2.$$

We see from (2–4) that $(\pi(a)^d)^* \pi(a)^d = \pi(aa^*)^d$ for $a \in \mathcal{A}$. From this, and using (2–4) again, we get $\sum_i \|A_i^d x\|^2 \leq \big\|\sum_i a_i a_i^*\big\| \, \|x\|^2$. Therefore

$$\big|(\sigma_{\pi,\rho}(w) T x, y)\big| \leq \|T\| \, \Big\|\sum_i a_i a_i^*\Big\|^{1/2} \Big\|\sum_i b_i b_i^*\Big\|^{1/2} \|x\| \, \|y\|.$$

Hence

$$\|\sigma_{\pi,\rho}(w) T\| \leq \|w\|_{ph} \|T\|,$$

so $\|w\|^{\pi,\rho} \leq \|w\|_{ph}$. Thus

(6–3) $$\|w\|_r \leq \|w\|_{ph} \quad \text{for } w \in \mathcal{A} \odot \mathcal{B}.$$

To prove the converse inequality, denote by $\mathcal{G}$ the space of all linear functionals g on $\mathcal{A} \odot \mathcal{B}$ such that

$$|g(w)| \leq \|w\|_{ph}, \quad \text{for } w \in \mathcal{A} \odot \mathcal{B}.$$

For $g \in \mathcal{G}$, let $\hat{g}$ be the linear functional on $\mathcal{B} \odot \mathcal{A}^{\mathrm{op}}$ acting by the rule $\hat{g}(w) = g(\Gamma w)$ for $w \in \mathcal{B} \odot \mathcal{A}^{\mathrm{op}}$. By (6–2), $|\hat{g}(w)| \leq \|w\|_h$. Hence $\hat{g}$ extends to a bounded functional on the Haagerup tensor product $\mathcal{B} \otimes_h \mathcal{A}^{\mathrm{op}}$ and $\|\hat{g}\| \leq 1$. Consider now the bilinear map on $\mathcal{B} \times \mathcal{A}^{\mathrm{op}}$ defined by the formula: $G(b, a) = \hat{g}(b \otimes a)$, for $b \in \mathcal{B}$ and $a \in \mathcal{A}^{\mathrm{op}}$. It follows from Theorems 1.5.2 and 1.5.4 of [Sinclair and Smith 1995] that there exist *-representations ρ of $\mathcal{B}$ on K and τ of $\mathcal{A}^{\mathrm{op}}$ on L, a bounded operator $T \colon L \to K$, and elements $x \in L$ and $y \in K$ with $\|x\| = \|y\| = 1$, such that

$$G(b, a) = (\rho(b) T \tau(a) x, y), \quad \text{for } b \in \mathcal{B}, \ a \in \mathcal{A}^{\mathrm{op}},$$

and $\|G\|_{cb} = \|\hat{g}\|_h = \|T\| \leq 1$.

Set $H = L^d$ and $\pi(a) = \tau(a)^d$. Then π is a *-representation of $\mathcal{A}$ on H and

(6–4) $$g(a \otimes b) = \hat{g}(b \otimes a) = G(b, a) = (\rho(b) T \pi(a)^d x, y),$$

for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. For $w = \sum_i a_i \otimes b_i$, denote by $\sigma^{\infty}_{\pi,\rho}(w)$ the extension of $\sigma_{\pi,\rho}(w)$ from $\mathfrak{S}_2(H^d, K)$ to $\mathfrak{S}_\infty(H^d, K)$. Let $\lambda_{\pi,\rho}(w)$ be the second adjoint of $\sigma^{\infty}_{\pi,\rho}(w)$ acting on the second dual space $B(H^d, K)$. By (6–1),

(6–5) $$\|\lambda_{\pi,\rho}(w)\| = \|\sigma^{\infty}_{\pi,\rho}(w)\| = \|w\|^{\pi,\rho} \leq \|w\|_r.$$

For $T \in \mathfrak{S}_\infty(H^d, K)$, we have $\sigma^{\infty}_{\pi,\rho}(w) T = \sum \rho(b_i) T \pi(a_i)^d$. This implies that $\lambda_{\pi,\rho}(w) T = \sum \rho(b_i) T \pi(a_i)^d$ for all $T \in B(H^d, K)$. Hence, by (6–4), $g(w) =$

$(\lambda_{\pi,\rho}(w)Tx, y)$. Using (6–5), we obtain $|g(w)| \leq \|\lambda_{\pi,\rho}(w)T\| \leq \|w\|_r$. Thus

$$|g(w)| \leq \|w\|_r \quad \text{for } g \in \mathcal{G} \text{ and } w \in \mathcal{A} \odot \mathcal{B}.$$

Making use of the Hahn–Banach theorem, we have $\|w\|_{ph} = \sup_{g\in\mathcal{G}} |g(w)| \leq \|w\|_r$, for $w \in \mathcal{A} \odot \mathcal{B}$. Combining this with (6–3), we complete the proof. □

We say that a net $\{d_\nu\}$ of elements of a C*-algebra $\mathcal{D}$ *point-weakly converges* to $d \in \mathcal{D}$, and write

$$d_\nu \overset{\text{pw}}{\to} d,$$

if for each irreducible representation τ of $\mathcal{D}$, $\tau(d_\nu) \to \tau(d)$ in the weak operator topology. Denote by $(\mathcal{A} \odot \mathcal{B})^\sim$ the linear space of all $\varphi \in \mathcal{A} \otimes \mathcal{B}$ for which there is a net $\{w_\nu\}$ in $\mathcal{A} \odot \mathcal{B}$ point-weakly converging to φ such that $\sup \|w_\nu\|_{ph} < \infty$.

Theorem 6.2. $(\mathcal{A} \odot \mathcal{B})^\sim \subseteq \mathbf{M}(\mathcal{A} \otimes \mathcal{B})$.

Proof. Let $w_\nu \in \mathcal{A} \odot \mathcal{B}$, $w_\nu \overset{\text{pw}}{\to} \varphi \in \mathcal{A} \otimes \mathcal{B}$ and $D = \sup \|w\|_{ph} < \infty$. To prove that $\varphi \in \mathbf{M}(\mathcal{A} \otimes \mathcal{B})$, we have to check that $\|\varphi\|^{\pi,\rho} \leq D$ for all representations π, ρ.

Let firstly π and ρ be direct sums of irreducible representations: $\pi = \bigoplus_{\lambda\in\Lambda} \pi_\lambda$ and $\rho = \bigoplus_{\gamma\in\Gamma} \rho_\gamma$ act on Hilbert spaces $H = \bigoplus H_\lambda$ and $K = \bigoplus K_\lambda$, respectively. By Theorem 6.1, $\|w_\nu\|^{\pi,\rho} \leq D$, for each ν, so $\|\sigma_{\pi,\rho}(w_\nu)(T)\| \leq \|w_\nu\|^{\pi,\rho}\|T\| \leq D\|T\|$, for any operator $T \in \mathfrak{S}_2(H^d, K)$. To prove that $\|\sigma_{\pi,\rho}(\varphi)(T)\| \leq D\|T\|$, it suffices to show that the operators $\sigma_{\pi,\rho}(w_\nu)(T)$ tend to $\sigma_{\pi,\rho}(\varphi)(T)$ in the weak operator topology. Moreover, the standard boundedness arguments show that it suffices to prove that

$$(\sigma_{\pi,\rho}(w_\nu)(T)x, y) \to (\sigma_{\pi,\rho}(\varphi)(T)x, y), \tag{6–6}$$

for each $x \in U = \bigcup H_\lambda^d$ and $y \in V = \bigcup K_\gamma$, since U, V are generating subsets in H^d and K, respectively.

For $x \in H_\lambda^d$ and $y \in K_\gamma$, set $R = x \otimes y$. We have from (5–1) that, for each $\psi \in \mathcal{A} \otimes \mathcal{B}$, $\sigma_{\pi,\rho}(\psi)(R) = \sigma_{\pi_\lambda,\rho_\gamma}(\psi)(R)$. Hence, we obtain from (2–1) that

$$\begin{aligned}(\sigma_{\pi,\rho}(\psi)(T)x, y) &= \operatorname{Tr}(y \otimes \sigma_{\pi,\rho}(\psi)(T)x) = \operatorname{Tr}(\sigma_{\pi,\rho}(\psi)(T)(x \otimes y)^*)\\ &= \overline{(x \otimes y, \sigma_{\pi,\rho}(\psi)(T))_2} = \overline{(\sigma_{\pi,\rho}(\psi^*)(R), T)_2}\\ &= \overline{(\sigma_{\pi_\lambda,\rho_\gamma}(\psi^*)(R), T)_2}\,.\end{aligned}$$

Since $\sigma_{\pi_\lambda,\rho_\gamma}$ is an irreducible representation $\mathcal{A} \otimes \mathcal{B}$ and $w_\nu^* \overset{\text{pw}}{\to} \varphi^*$, it follows that (6–6) holds.

Now let π and ρ be arbitrary. Consider the representation τ of $\mathcal{A}$, which is the direct sum of all irreducible representations of $\mathcal{A}$ repeated $\dim(\mathcal{H}_\pi)$ times. Then, for each $a \in \mathcal{A}$, we have $\operatorname{rank}(\pi(a)) \leq \operatorname{rank}(\tau(a))$, whence $\pi \ll_a \tau$; see (5–4). Similarly, there is a representation χ of $\mathcal{B}$, which is a direct sum of irreducible representations, such that $\rho \ll_a \chi$. By Theorem 5.2, $\mathbf{M}^{\tau,\chi} \subseteq \mathbf{M}^{\pi,\rho}$, so $\varphi \in \mathbf{M}^{\pi,\rho}$. Thus $\varphi \in \mathbf{M}(\mathcal{A} \otimes \mathcal{B})$ and $(\mathcal{A} \odot \mathcal{B})^\sim \subseteq \mathbf{M}(\mathcal{A} \otimes \mathcal{B})$. □

Problem 6.3. Does $(\mathcal{A} \odot \mathcal{B})^\sim$ coincide with $\mathbf{M}(\mathcal{A} \otimes \mathcal{B})$?

We will see further that for commutative C*-algebras the answer is positive.

7. Multipliers of commutative algebras; (μ, ν)-multipliers

The commutativity of the C*-algebras $\mathcal{A}$ and $\mathcal{B}$ implies significant simplifications to the previous results and constructions. To begin with, each representation of a commutative algebra has a separating vector. Hence, by Corollary 5.8, the algebra $\mathbf{M}^{\pi,\rho}(\mathcal{A} \otimes \mathcal{B})$ depends only on the kernels of the representations π, ρ. In particular, if π, ρ are faithful, then $\mathbf{M}^{\pi,\rho}(\mathcal{A} \otimes \mathcal{B}) = \mathbf{M}(\mathcal{A} \otimes \mathcal{B})$ and $\|\varphi\|^{\pi,\rho} = \|\varphi\|_r$. Since $\|\varphi\|_{ph} = \|\varphi\|_h$ for commutative algebras, Theorem 6.1 shows that, for faithful π and ρ,

$$\|\varphi\|^{\pi,\rho} = \|\varphi\|_h, \qquad \text{for } \varphi \in \mathcal{A} \odot \mathcal{B}.$$

It was proved in [1953] that the norm $\|\cdot\|_h$ on $\mathcal{A} \otimes \mathcal{B}$ is equivalent to the projective tensor norm $\|\cdot\|_\gamma$. Thus in the case of commutative algebras our Theorem 6.2 implies that the Varopoulos tensor algebra $V(X, Y) = C(X) \hat{\otimes} C(Y)$ and its "tilde-algebra" (see [Graham and McGehee 1979]) are topologically included into $\mathbf{M}(C(X) \otimes C(Y))$. In fact, this theorem deals with a wider "tilde-extension" consisting of pointwise limits of $\|\cdot\|_\gamma$-bounded nets. We will return to this topic later.

Let $\mathcal{U}$ be a commutative operator C*-algebra on H with the space Λ of all maximal ideals. Then

$$H = \bigoplus_{\gamma \in \Gamma} H_\gamma,$$

where all $H_\gamma \approx L_2(\Lambda, \mu_\gamma)$ are invariant for $\mathcal{U}$, and each $f \in \mathcal{U}$ acts on H_γ as a multiplication operator. The antiisometric involution $j : \{g_\gamma(\lambda)\} \mapsto \{\overline{g_\gamma(\lambda)}\}$ on H induces an involution on $\mathcal{U}$ given by $jAj = A^*$, for $A \in \mathcal{U}$. Taking into account (2–5), which here becomes $\partial A^* \partial^{-1} = A^d$, we see that the unitary operator $V = \partial j$ from H to H^d establishes a unitary equivalence of A and A^d: $A^d = VAV^{-1}$. For representations π of $\mathcal{A}$ on H and ρ of $\mathcal{B}$ on K, we identify $\mathfrak{S}_2(H^d, K)$ with $\mathfrak{S}_2(H, K)$ by the formula $U(T) = TV$, for $T \in \mathfrak{S}_2(H^d, K)$. Using this and (5–1), we will assume that $\sigma_{\pi,\rho}$ acts on $\mathfrak{S}_2(H, K)$ by the formula

$$\sigma_{\pi,\rho}(a \otimes b)R = \rho(b)R\pi(a).$$

We now prove that, for commutative $\mathcal{A}$, $\mathcal{B}$, the subalgebras $M_I^{\pi,\rho}$ in $\mathcal{A} \otimes \mathcal{B}$ are involutive.

Proposition 7.1. *If $\mathcal{A}$ and $\mathcal{B}$ are commutative, then $\mathbf{M}_I^{\pi,\rho}(\mathcal{A}\otimes\mathcal{B})$ is a *-subalgebra of $\mathcal{A}\otimes\mathcal{B}$ for each pair of representations π, ρ and each s.n. ideal I. Moreover,*

$$\|\varphi^*\|_I^{\pi,\rho} = \|\varphi\|_I^{\pi,\rho}, \qquad \textit{for } \varphi \in \mathbf{M}_I^{\pi,\rho}(\mathcal{A}\otimes\mathcal{B}).$$

Proof. Consider the antiisometric involutions j on H_π and i on K_ρ such that $\pi(a^*) = j\pi(a)j$ and $\rho(b^*) = i\rho(b)i$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then, for $T \in \mathfrak{S}_2(H, K)$,

$$\sigma_{\pi,\rho}(a^*\otimes b^*)(T) = \rho(b^*)T\pi(a^*) = i\{\sigma_{\pi,\rho}(a\otimes b)(iTj)\}j.$$

Hence $\sigma_{\pi,\rho}(\varphi^*)(T) = i\{\sigma_{\pi,\rho}(\varphi)(iTj)\}j$, for all $\varphi \in \mathcal{A}\otimes\mathcal{B}$. For any s.n. ideal I and any operator $T \in I(H, K)$, $iTj \in I(H, K)$ and $|iTj|_I = |T|_I$. Thus it follows that $\varphi \in \mathbf{M}_I^{\pi,\rho}(\mathcal{A}\otimes\mathcal{B})$ implies $\varphi^* \in \mathbf{M}_I^{\pi,\rho}(\mathcal{A}\otimes\mathcal{B})$, and $\|\varphi^*\|_I^{\pi,\rho} = \|\sigma_{\pi,\rho}(\varphi^*)\|_I = \|\sigma_{\pi,\rho}(\varphi)\|_I = \|\varphi\|_I^{\pi,\rho}$. □

Let X be the space of all maximal ideals of a commutative C*-algebra $\mathcal{A}$. Then $\mathcal{A} = C_0(X)$ and each representation π of $\mathcal{A}$ corresponds to a spectral measure $\mathcal{E}_\pi$ on X, that is, a σ-additive map from the σ-algebra of all Borel subsets of X to the lattice of projections in H_π. An isolated point x in the support, $\operatorname{supp}(\mathcal{E}_\pi)$, of $\mathcal{E}_\pi$ must be an atom: $\mathcal{E}_\pi(\{x\}) \neq 0$. To apply the results of the previous sections we need to express $\operatorname{rank}(\pi(f))$ in terms of the spectral measure. Set

$$\mathcal{S}(f, \mathcal{E}_\pi) = \{x \in \operatorname{supp}(\mathcal{E}_\pi) : f(x) \neq 0\}.$$

Lemma 7.2. *For $f \in C_0(X)$, $\operatorname{rank}(\pi(f)) < \infty$ if and only if $\mathcal{S}(f, \mathcal{E}_\pi)$ consists of a finite number of points of finite multiplicity* $(\dim(\mathcal{E}_\pi(\{x\})) < \infty)$. *In this case*

$$\operatorname{rank}(\pi(f)) = \sum_{x\in S(f,\mathcal{E}_\pi)} \dim(\mathcal{E}_\pi(\{x\})).$$

Proof. If $\mathcal{S}(f, \mathcal{E}_\pi)$ is infinite, it contains a countable set of points with disjoint neighbourhoods. Hence $\operatorname{rank}(\pi(f))$ is infinite. Let $\mathcal{S}(f, \mathcal{E}_\pi) = \{x_1, \dots, x_n\}$. Since f is continuous, $\mathcal{E}_\pi(\{x_i\}) \neq 0$ and $\pi(f) = \sum_i f(x_i)\mathcal{E}_\pi(\{x_i\})$. □

It follows from Lemma 7.2 that the kernel of a representation depends only on the support of the corresponding spectral measure.

Corollary 7.3. *Let π, π' and ρ, ρ' be, respectively, representations of commutative C*-algebras $\mathcal{A} = C_0(X)$ and $\mathcal{B} = C_0(Y)$. Let*

$$\operatorname{supp}(\mathcal{E}_{\pi'}) \subset \operatorname{supp}(\mathcal{E}_\pi) \quad \textit{and} \quad \operatorname{supp}(\mathcal{E}_{\rho'}) \subset \operatorname{supp}(\mathcal{E}_\rho). \tag{7–1}$$

Then

(i) $\mathbf{M}^{\pi,\rho}(\mathcal{A}\otimes\mathcal{B}) \subseteq \mathbf{M}^{\pi',\rho'}(\mathcal{A}\otimes\mathcal{B})$, *and the inclusion is contractive.*

(ii) *Suppose that I is either a coseparable s.n. ideal or $\mathfrak{S}_2 \subseteq I$. Let the representations π' and ρ' be separable, and let, for all isolated points $x \in \operatorname{supp}(\mathscr{E}_\pi)$ and $y \in \operatorname{supp}(\mathscr{E}_\rho)$,*

$$\dim(\mathscr{E}_{\pi'}(\{x\})) \le \dim(\mathscr{E}_\pi(\{x\})) \quad \textit{and} \quad \dim(\mathscr{E}_{\rho'}(\{y\})) \le \dim(\mathscr{E}_\rho(\{y\})).$$

Then $\mathbf{M}_I^{\pi,\rho}(\mathscr{A} \otimes \mathscr{B}) \subseteq \mathbf{M}_I^{\pi',\rho'}(\mathscr{A} \otimes \mathscr{B})$, and the inclusion is contractive.

Proof. The inclusions (7–1) imply (5–4). Since all representations of commutative C*-algebras have separating vectors, (i) follows from Corollary 5.8.

Applying Lemma 7.2, Corollary 5.4 and Remark 5.5, we get (ii). □

Let μ and ν be measures on X and Y, let $H = L_2(X, \mu)$ and $K = L_2(Y, \nu)$. Then $\mathfrak{S}_2(H, K)$ consists of integral Hilbert–Schmidt operators R with kernels $r(x, y)$ on $X \times Y$ satisfying (1–1). Each $\varphi \in L_\infty(X \times Y, \mu \times \nu)$ defines a bounded linear map Φ_φ on $\mathfrak{S}_2(H, K)$: $\Phi_\varphi(R)$ is the integral operator with kernel $\varphi(x, y)r(x, y)$. Recall from the Introduction that if Φ_φ preserves $\mathscr{X}_I$ and is bounded in $|\cdot|_I$, then φ is called a (μ, ν, I)-multiplier. We denote by $\|\Phi_\varphi\|_I$ the norm of the operator Φ_φ acting on $\mathscr{X}_I$, and by $\mathfrak{M}_{\mu,\nu}(I)$ the set of all (μ, ν, I)-multipliers.

Every multiplicity-free representation of $C_0(X)$ is defined by a regular σ-finite Borel measure μ on X, and acts on $L_2(X, \mu)$ by multiplication operators:

$$\pi_\mu(f)h(x) = f(x)h(x).$$

Let ρ_ν be a multiplicity-free representation of $C_0(Y)$ defined by a regular σ-finite Borel measure ν on Y. Then $C_0(X) \otimes C_0(Y) = C_0(X \times Y)$ and, for each φ in $C_0(X \times Y)$, $\sigma_{\pi_\mu,\rho_\nu}(\varphi)$, acts on $\mathfrak{S}_2(H, K)$ by multiplying the integral kernels of operators $R \in \mathfrak{S}_2(H, K)$ by φ. Thus

$$\sigma_{\pi_\mu,\rho_\nu}(\varphi) = \Phi_\varphi \quad \text{for } \varphi \in C_0(X \times Y).$$

Therefore $(\pi_\mu \otimes \pi_\nu, I)$-multipliers are *continuous* (μ, ν, I)-multipliers, and

$$\mathbf{M}_I^{\pi_\mu,\rho_\nu}(C_0(X) \otimes C_0(Y)) = \mathfrak{M}_{\mu,\nu}(I) \cap C_0(X \times Y).$$

We will use the simplified notations and write

$$\mathbf{M}_I^{\mu,\nu} \text{ instead of } \mathbf{M}_I^{\pi_\mu,\rho_\nu} \quad \text{and} \quad \|\varphi\|_I^{\mu,\nu} \text{ instead of } \|\varphi\|_I^{\pi_\mu,\rho_\nu}.$$

Thus

$$\|\varphi\|_I^{\mu,\nu} \overset{\text{def}}{=} \|\varphi\|_I^{\pi_\mu,\rho_\nu} \overset{\text{def}}{=} \|\sigma_{\pi_\mu,\rho_\nu}(\varphi)\|_I = \|\Phi_\varphi\|_I \quad \text{for } \varphi \in C_0(X \times Y).$$

Corollary 7.4. *Let X, Y be locally compact spaces with countable bases. Let μ, μ' and ν, ν' be σ-finite Borel measure on X and Y, respectively. Let I be either a coseparable s.n. ideal or $\mathfrak{S}_2 \subseteq I$. If*

$$\operatorname{supp}(\mu') \subseteq \operatorname{supp}(\mu) \quad \textit{and} \quad \operatorname{supp}(\nu') \subseteq \operatorname{supp}(\nu),$$

then $\mathbf{M}_I^{\mu,\nu}(C_0(X)\otimes C_0(Y)) \subseteq \mathbf{M}_I^{\mu',\nu'}(C_0(X)\otimes C_0(Y))$, *and the inclusion is contractive.*

Proof. Since X, Y have countable bases and all measures are σ-finite, the corresponding $L_2(\cdot,\cdot)$ spaces are separable. For any $A\subset X$, $\mathscr{E}_{\pi_\mu}(A)$ is the multiplication operator by the characteristic function of A. Hence $\operatorname{supp}(\mu)$ coincides with $\operatorname{supp}(\mathscr{E}_{\pi_\mu})$. Since $\dim(\mathscr{E}_{\pi_\mu}(\{x\}))=1$ for each isolated point $x\in\operatorname{supp}(\mu)$, our result follows from Corollary 7.3. □

Our next aim is to relate continuous (μ,ν,I)-multipliers to Schur I-multipliers. Let $H=l_2(X)$ be the Hilbert space of all complex-valued functions g on X such that $\sum_{x\in X}|g(x)|^2<\infty$. Denote by τ_X the representation of $C_0(X)$ on $l_2(X)$ by diagonal operators

$$(\tau_X(h)g)(x)=h(x)g(x)\qquad \text{for } h\in C_0(X),\ g\in l_2(X).$$

Let $K=l_2(Y)$. Each $T\in\mathfrak{S}_2(H,K)$ corresponds to a matrix $(t(x,y))$ with $\sum|t(x,y)|^2<\infty$. For a bounded complex-valued function φ on $X\times Y$, the operator $S_\varphi(T)=(\varphi(x,y)t(x,y))$ is bounded on $\mathfrak{S}_2(H,K)$. If S_φ preserves $\mathscr{X}_I=I(H,K)\cap\mathfrak{S}_2(H,K)$ and is bounded in $|\cdot|_I$, then φ is called a Schur I-multiplier and $\|S_\varphi\|_I$ denotes the norm of the operator S_φ acting on $\mathscr{X}_I$. Clearly, Schur I-multipliers on $X\times Y$ are exactly (τ_X,τ_Y,I)-multipliers.

Theorem 7.5. *Let X, Y be locally compact spaces with countable bases and let μ,ν be Borel σ-finite measures on X and Y, with* $\operatorname{supp}(\mu)=X$, $\operatorname{supp}(\nu)=Y$. *Suppose that an s.n. ideal I is either coseparable or $\mathfrak{S}_2\subseteq I$. A function $\varphi\in C_0(X\times Y)$ is a (μ,ν,I)-multiplier on $X\times Y$ if and only if it is a Schur I-multiplier on $X\times Y$. In this case $\|S_\varphi\|_I=\|\varphi\|_I^{\mu,\nu}$.*

Proof. Since $L_2(X,\mu)$ is a separable space, $\operatorname{rank}(\pi_\mu(f))\le\aleph_0$, for $f\in C_0(X)$. Let us show that

$$\operatorname{rank}(\pi_\mu(f))=\min\{\aleph_0,\operatorname{rank}(\tau_X(f))\}. \tag{7–2}$$

If $\operatorname{rank}(\tau_X(f))\ge\aleph_0$, then, by Lemma 7.2, $\operatorname{rank}(\pi_\mu(f))$ can not be finite, so (7–2) holds. If $\operatorname{rank}\tau_X(f)=n<\infty$, then the set $\mathscr{S}(f,\mathscr{E}_\tau)$ consists of n points. By the continuity of f, these points must be isolated in X. Hence, by Lemma 7.2, $\operatorname{rank}(\pi_\mu(f))=n$, and (7–2) holds. Since the same equality holds for π_ν and τ_Y, and the C*-algebras $C_0(X)$, $C_0(Y)$ are separable, our result follows from Corollary 5.3. □

Problem 7.6. Let X and Y be locally compact spaces with countable bases and $I=\mathfrak{S}_p$. Is each Schur I-multiplier $\varphi\in C_0(X\times Y)$ a $(\pi\otimes\rho,I)$-multiplier for all separable representations π of $C_0(X)$ and ρ of $C_0(Y)$?

The positive answer to this problem follows from the previous results in two cases: if X and Y have no isolated points, and if $I = \mathfrak{S}_\infty$.

(1) *Assume X and Y have no isolated points.* Let μ, ν be Borel σ-finite measures without atoms, with $\operatorname{supp}(\mu) = X$, $\operatorname{supp}(\nu) = Y$. Then $\pi_\mu(C_0(X))$ and $\rho_\nu(C_0(Y))$ have no nonzero finite rank operators, and $\operatorname{Ker}(\pi_\mu) = \operatorname{Ker}(\rho_\nu) = \{0\}$. By Theorem 7.5, φ is a (μ, ν, I)-multiplier on $X \times Y$. By Corollary 5.4, it is a $(\pi \otimes \rho, I)$-multiplier for all separable representations π of $C_0(X)$ and ρ of $C_0(Y)$.

(2) *Assume $I = \mathfrak{S}_\infty$.* Let $I = \mathfrak{S}_\infty$. Every cyclic representation of $C_0(X)$ is of the form π_μ. Each separable representation π of $C_0(X)$ is equivalent to a subrepresentation of $\pi_\mu \otimes \mathbf{1}_{\mathcal{H}}$, for separable $\mathcal{H}$ and some cyclic representation π_μ. By Theorem 7.5, φ is a (μ, ν, I)-multiplier on $X \times Y$. By Lemma 5.7, it is a $((\pi_\mu \otimes \mathbf{1}_{\mathcal{H}}) \otimes (\rho_\nu \otimes \mathbf{1}_{\mathcal{H}}), I)$-multiplier. Hence it is a $(\pi \otimes \rho, I)$-multiplier.

For $I = \mathfrak{S}_\infty$ (or equivalently $\mathfrak{S}_b$, $\mathfrak{S}_1$), Schur I-multipliers were described by Grothendieck in [1953] (see also Theorems 5.1 and 5.5 in [Pisier 2001]): φ is a Schur $\mathfrak{S}_\infty$-multiplier if and only if there are bounded families $\{u_\lambda\}$ and $\{v_\lambda\}$ of functions on X and Y, such that φ belongs to the pointwise closure of the convex hull of $\{u_\lambda(x)v_\lambda(y)\}$. It can be easily seen from the proof in [Pisier 2001] that if $\varphi \in C_0(X, Y)$, one can choose u_λ, v_λ among Borel functions. Since each Borel function $u(x)$ with $|u(x)| \le 1$ can be pointwise approximated by functions from $b_1(C_0(X))$, the inclusion of Theorem 6.2 is, in fact, an equality for commutative $\mathcal{A}$ and $\mathcal{B}$.

Corollary 7.7. *If $\mathcal{A}$ and $\mathcal{B}$ are commutative, then* $(\mathcal{A} \odot \mathcal{B})^{\sim} = M(\mathcal{A} \otimes \mathcal{B})$.

Recall one of the equivalent definitions (cf. [Birman and Solomyak 1967]) of a double operator integral (DOI). Let $\mathcal{E}$, $\mathcal{F}$ be spectral measures on X and Y with values in the sets $\mathcal{P}(H)$ and $\mathcal{P}(K)$ of all projections in $B(H)$ and $B(K)$, respectively. One defines their direct product $\mathcal{G}$ as a spectral measure on $X \times Y$ with values in $\mathcal{P}(\mathfrak{S}_2(H, K))$ by $\mathcal{G}(A \times B)(T) = \mathcal{F}(B)T\mathcal{E}(A)$, and further extends it from rectangulars to all Borel sets. For a bounded Borel function φ on $X \times Y$, one defines the operator $\mathfrak{I}_\varphi$ on $\mathfrak{S}_2(H, K)$ by

$$\mathfrak{I}_\varphi = \int \varphi(x, y)\, d\mathcal{G}.$$

If $\mathfrak{I}_\varphi$ is bounded on $I \cap \mathfrak{S}_2$ in $|\cdot|_I$, then one says that φ defines DOI on I.

Let now $\varphi \in C_0(X \times Y)$, and let spectral measures $\mathcal{E}_\pi$, $\mathcal{F}_\rho$ correspond to representations π and ρ of $C_0(X)$ and $C_0(Y)$, respectively. Then φ defines DOI on I if and only if $\varphi \in \mathbf{M}_I^{\pi,\rho}(C_0(X) \otimes C_0(Y))$. Thus the DOI theory, restricted to continuous functions, can be considered as a part of the general operator multipliers theory for tensor products of representations of C*-algebras. In particular, Corollary 7.3 states that the space of continuous functions that define bounded DOI depends only on the supports of the spectral measures and (if $I \neq \mathfrak{S}_1, \mathfrak{S}_\infty, \mathfrak{S}_b$) on the multiplicity of their atoms.

Some results in this section are known or can be deduced from the DOI theory. Proposition 7.1 was, in fact, proved by Birman and Solomyak [1967; 1973]. We presented the proof here because it is short and "coordinate free". For functions that define DOI on $\mathfrak{S}_\infty$ (or, equivalently, on $\mathfrak{S}_1$, $\mathfrak{S}_b$), a precise description was obtained by Peller [1985], completing previous results of Birman and Solomyak [1967; 1973] (a transparent proof of Peller's theorem can be found in the recent book [Hiai and Kosaki 2003]). Without stating this directly, Peller's theorem shows that only supports of the spectral measures are essential in the description of $\mathbf{M}^{\pi,\rho}$. No definitive description of $\mathbf{M}_I^{\pi,\rho}$ is known for other I. We will discuss Peller's theorem at the end of Section 8.

8. The notion of ω-continuity and an analog of Luzin's theorem

Our goal now is to remove the restriction of continuity on (μ, ν, I)-multiplier in the main results of Section 6. Moreover, we are going to extend these results to functions on the product of measure spaces (X, μ) and (Y, ν) without distinguished topologies. On the other hand, even in this case, in order to be a (μ, ν, I)-multiplier (at least if $I = \mathfrak{S}_\infty$; see Proposition 9.1), a function still has to be "continuous" in some natural *pseudotopology*, called ω-pseudotopology, associated in [Erdos et al. 1998] with the product of measure spaces. In this section we establish some auxiliary results on ω-continuous functions.

Recall that a pseudotopology on a set is defined by a family of its subsets (called *pseudoopen*), which is closed under finite intersections and countable unions. The complements of pseudoopen sets are called *pseudoclosed*. A complex-valued function is *pseudocontinuous* if the preimages of open sets are pseudoopen.

The ω-pseudotopology on the product of measure spaces is defined as follows. A subset N of $X \times Y$ is called *marginally null* if there are subsets $F \subseteq X$ and $S \subseteq Y$ of zero measure such that $N \subseteq (F \times Y) \cup (X \times S)$. A set E is *ω-open* if there is a countable family of measurable rectangles $A_n \times B_n$ such that the symmetric difference of $\bigcup (A_n \times B_n)$ and E is marginally null. The space of all ω-continuous functions on $X \times Y$ is denoted by $C_{\mu,\nu}(X \times Y)$.

A measure space (X, μ) is called *standard* if there is a topology on X (called *admissible*) with respect to which μ is a σ-finite Radon measure, that is, for each measurable set A of finite measure and each $\varepsilon > 0$, there is a compact set F such that $F \subseteq A$ and $\mu(A \setminus F) < \varepsilon$. A standard space (X, μ) is *separable* if there is an admissible topology in which X has a countable base.

Lemma 8.1. *Let $Z \times W \subseteq \bigcup_{i=1}^n (A_i \times B_i)$ for $A_i \subseteq X$, $B_i \subseteq Y$ and $n < \infty$. Then there are finite families of disjoint sets $\{X_p\}_{p=1}^m$ in Z and $\{Y_j\}_{j=1}^k$ in W such that* *each $X_p \times Y_j$ is contained in at least one of $A_i \times B_i$ and*

$$Z = \bigcup_{p=1}^{m} X_p, \quad W = \bigcup_{j=1}^{k} Y_j.$$

Proof. When z spans Z, there is only a finite number of different sets

$$A_z = Z \cap \Big(\bigcap_{z \in A_i} A_i \Big) \cap \Big(\bigcap_{z \notin A_i} (Z \setminus A_i) \Big).$$

Denote them by $X_1, \dots, X_m$. Choosing, similarly, sets $Y_1, \dots, Y_k$ in W, we obtain the sets $\{X_p\}$, $\{Y_j\}$ satisfying all conditions of the lemma. □

Denote by χ_E the characteristic function of a set E. We say that a function g on $X \times Y$ is *simple* if there are measurable, disjoint sets $\{X_i\}_{i=1}^n$, $\{Y_j\}_{j=1}^m$, with $n, m < \infty$, such that

$$X = \bigcup_{i=1}^{n} X_i, \quad Y = \bigcup_{j=1}^{m} Y_j, \quad \text{and} \quad g = \sum_{i,j} \alpha_{ij} \chi_{X_i} \chi_{Y_j} \text{ with } \alpha_{ij} \in \mathbb{C}.$$

Let φ be a function on $X \times Y$ and let $Z \subseteq X$, $W \subseteq Y$ be measurable. Set

$$\lambda(\varphi, Z \times W) = \sup\big\{ |\varphi(x, y) - \varphi(x', y')| : x, x' \in Z,\ y, y' \in W \big\}.$$

For $\varepsilon > 0$, a function φ is called *ε-decomposable* on $Z \times W$ if there are measurable sets $\{X_i\}_{i=1}^n$, $\{Y_j\}_{j=1}^m$, with $n, m < \infty$, such that

$$Z \subseteq \bigcup_{i=1}^{n} X_i, \quad W \subseteq \bigcup_{j=1}^{m} Y_j, \quad \text{and} \quad \lambda(\varphi, X_i \times Y_j) < \varepsilon \text{ for all } i, j.$$

Theorem 8.2. *Let (X, μ) and (Y, ν) be standard finite measure spaces. For a function φ on $X \times Y$, the following conditions are equivalent.*

(i) *φ is ω-continuous.*

(ii) *For each $\varepsilon > 0$, there are measurable sets X_ε and Y_ε such that $\mu(X \setminus X_\varepsilon) < \varepsilon$, $\nu(Y \setminus Y_\varepsilon) < \varepsilon$ and φ is ε-decomposable on $X_\varepsilon \times Y_\varepsilon$.*

(iii) *For each $\varepsilon > 0$, there are measurable sets X_ε and Y_ε such that $\mu(X \setminus X_\varepsilon) < \varepsilon$, $\nu(Y \setminus Y_\varepsilon) < \varepsilon$ and $\varphi | X_\varepsilon \times Y_\varepsilon$ is a uniform limit of simple functions.*

Proof. (i) $\Longrightarrow$ (ii). Choosing admissible topologies on X, Y and compacts $Q \subset X$ and $K \subset Y$ such that $\mu(X \setminus Q) < \varepsilon/2$ and $\nu(Y \setminus K) < \varepsilon/2$, we only need to prove the implication for Q, K and $\varepsilon/2$. Thus we may assume that X and Y are compacts in these topologies.

Cover the range of φ by open disks D^k of radius $\varepsilon/2$. Since φ is continuous, the sets $\varphi^{-1}(D^k)$ are ω-open. Hence, for each k, there are marginally null sets N_k

in $X \times Y$ and measurable sets A_i^k in X and B_i^k in Y such that

$$\varphi^{-1}(D^k) = N_k \cup \bigcup_{i=1}^{\infty} (A_i^k \times B_i^k)$$

Thus

$$X \times Y = N \cup \bigcup_{i=k=1}^{\infty} (A_i^k \times B_i^k),$$

where N is a marginally null set, and $\lambda(\varphi, A_i^k \times B_i^k) < \varepsilon$, for all i, k.

Choose $U \subset X$, $V \subset Y$ such that

$$N \subseteq (X \times V) \cup (U \times Y), \quad \mu(U) \leq \varepsilon/2, \quad \nu(V) \leq \varepsilon/2.$$

The set $\kappa = (X \setminus U) \times (Y \setminus V)$ is ω-closed and

$$\kappa \subseteq \bigcup_{i=k=1}^{\infty} (A_i^k \times B_i^k).$$

By [Erdos et al. 1998, Lemma 3.4], there are sets $R_\varepsilon \subset X$ and $T_\varepsilon \subset Y$, with $\mu(X \setminus R_\varepsilon) < \varepsilon/2$ and $\nu(Y \setminus T_\varepsilon) < \varepsilon/2$, such that the set

$$\kappa \cap (R_\varepsilon \times T_\varepsilon) = (R_\varepsilon \setminus U) \times (T_\varepsilon \setminus V)$$

is covered by a finite number of the rectangles $A_i^k \times B_i^k$. Setting $X_\varepsilon = R_\varepsilon \setminus U$, $Y_\varepsilon = T_\varepsilon \setminus V$, we obtain what we need.

(ii) $\Longrightarrow$ (iii). For $\varepsilon_n = 2^{-n}\varepsilon$, choose X_{ε_n}, Y_{ε_n} as in (ii): $\mu(X \setminus X_{\varepsilon_n}) < \varepsilon_n$, $\nu(Y \setminus Y_{\varepsilon_n}) < \varepsilon_n$ and $X_{\varepsilon_n} \times Y_{\varepsilon_n}$ is covered by a finite family of rectangles

$$\{A_j^n \times B_j^n\}_{j=1}^{p(n)}, \quad \text{with } \lambda(\varphi, A_j^n \times B_j^n) < \varepsilon_n.$$

Set

$$X_\varepsilon = \bigcap_{n=1}^{\infty} X_{\varepsilon_n}, \quad Y_\varepsilon = \bigcap_{n=1}^{\infty} Y_{\varepsilon_n}.$$

Then $\mu(X \setminus X_\varepsilon) < \varepsilon$, $\nu(Y \setminus Y_\varepsilon) < \varepsilon$, and, for each n,

$$X_\varepsilon \times Y_\varepsilon \subseteq X_{\varepsilon_n} \times Y_{\varepsilon_n} \subseteq \bigcup_{j=1}^{p(n)} (A_j^n \times B_j^n).$$

It follows from Lemma 8.1 that there is a simple function φ_n on $X \times Y$ such that $\sup\{|\varphi(x, y) - \varphi_n(x, y)| : (x, y) \in X_\varepsilon \times Y_\varepsilon\} < \varepsilon_n$.

(iii) $\Longrightarrow$ (i). Every simple function is ω-continuous. By [Erdos et al. 1998, Lemma 3.3], the uniform limit of ω-continuous functions is ω-continuous. Hence, for each $\varepsilon > 0$, the function $\varphi|X_\varepsilon \times Y_\varepsilon$ is ω-continuous. The set

$$N = (X \times Y) \setminus \bigcup_{n=1}^{\infty} \big(X_{1/n} \times Y_{1/n}\big)$$

is marginally null. Hence, for any open set $G \subset \mathbb{C}$,

$$\varphi^{-1}(G) \setminus \bigcup_{n=1}^{\infty} (\varphi^{-1}(G) \cap \big(X_{1/n} \times Y_{1/n}\big)) = \varphi^{-1}(G) \cap N$$

is a marginally null set. Since all $\varphi^{-1}(G) \bigcap \big(X_{1/n} \times Y_{1/n}\big)$ are ω-open, $\varphi^{-1}(G)$ is ω-open and φ is ω-continuous. □

A sequence $\{X_n\}$ of measurable sets in (X, μ) is *exhaustive* if

$$X_n \subseteq X_{n+1} \quad \text{and} \quad \mu\Big(X - \bigcup_{n=1}^{\infty} X_n\Big) = 0.$$

Fix an admissible topology on a standard measure space (X, μ). Then X has an exhaustive sequence $\{X_n\}$ of compact sets. For each n, there are disjoint compacts $\{K_i(n)\}$ in X_{n+1} such that $K_1(n) = X_n$ and $\mu\big(X_{n+1} - \bigcup_i K_i(n)\big) = 0$. Hence there are disjoint compact sets $\{K_n\}$ in X such that $\mu\big(X - \bigcup_n K_n\big) = 0$.

The following result can be considered as an ω-version of Luzin's theorem.

Theorem 8.3. *Let μ and ν be Radon σ-finite measures on topological spaces X and Y. For a function φ on $X \times Y$ the following conditions are equivalent.*

(i) *φ is ω-continuous.*

(ii) *For any $\varepsilon > 0$, there are measurable sets $X_\varepsilon \subseteq X$ and $Y_\varepsilon \subseteq Y$ such that $\mu(X - X_\varepsilon) < \varepsilon$, $\nu(Y - Y_\varepsilon) < \varepsilon$ and φ is continuous on $X \times Y$.*

(iii) *There are exhaustive sequences $\{X_n\}$ and $\{Y_n\}$ of compacts in X and Y such that φ is continuous on each $X_n \times Y_n$.*

Proof. Step 1. First we will prove the theorem for compact X and Y.

(i) $\Rightarrow$ (ii). Let E be a measurable subset of X. By Luzin's theorem, for $\delta > 0$, there is a compact subset K of X such that $\mu(X \setminus K) < \delta$ and χ_E is continuous on K. Hence if g is a simple function on $X \times Y$, there are compacts $K \subseteq X$, $R \subseteq Y$ such that $\mu(X \setminus K) < \delta$, $\nu(Y \setminus R) < \delta$ and g is continuous on $K \times R$.

Let φ be ω-continuous. For $\varepsilon > 0$, let sets X_ε and Y_ε be chosen as in Theorem 8.2 (iii) and let simple functions φ_n uniformly converge to $\varphi|X_\varepsilon \times Y_\varepsilon$. Set $\varepsilon_n = 2^{-n}\varepsilon$. By the argument above, there are compacts $K_n \subseteq X$, $R_n \subseteq Y$ such that $\mu(X \setminus K_n) < \varepsilon_n$, $\nu(Y \setminus R_n) < \varepsilon_n$ and the functions φ_n are continuous on $K_n \times R_n$. Set

$$L(\varepsilon) = X_\varepsilon \cap \bigcap_{n=1}^{\infty} K_n \quad \text{and} \quad M(\varepsilon) = Y_\varepsilon \cap \bigcap_{n=1}^{\infty} R_n.$$

Then $\mu(X \setminus L(\varepsilon)) \le 2\varepsilon$ and $\nu(Y \setminus M(\varepsilon)) \le 2\varepsilon$. All φ_n are continuous on $L(\varepsilon) \times M(\varepsilon)$ and uniformly converge to $\varphi|L(\varepsilon) \times M(\varepsilon)$. Hence φ is continuous on $L(\varepsilon) \times M(\varepsilon)$.

(ii) ⇒ (iii). Set

$$L_n = \bigcap_{k=n}^{\infty} L(\varepsilon_k) \quad \text{and} \quad M_n = \bigcap_{k=n}^{\infty} M(\varepsilon_k).$$

Then $L_n \subseteq L_{n+1}$, $M_n \subseteq M_{n+1}$, and φ is continuous on $L_n \times M_n$. Furthermore,

$$\mu(X \setminus L_n) \le \sum_{k=n}^{\infty} \mu(X \setminus L(\varepsilon_k)) < \varepsilon 2^{2-n},$$

$$\nu(Y \setminus M_n) \le \sum_{k=n}^{\infty} \nu(Y \setminus M(\varepsilon_k)) < \varepsilon 2^{2-n}.$$

Thus $\{L_n\}$, $\{M_n\}$ are exhaustive sequences. Since μ, ν are Radon measures, there are compacts $E_n \subseteq L_n$ and $F_n \subseteq M_n$ such that $\mu(L_n \setminus E_n) < 1/n$, $\nu(M_n \setminus F_n) < 1/n$. Hence

$$X_n = \bigcup_{k=1}^{n} E_k \quad \text{and} \quad Y_n = \bigcup_{k=1}^{n} F_k$$

form exhaustive sequences of compacts in X and Y.

Step II. Now assume that X and Y are not compact spaces. Let $\{F_n\}$ and $\{G_n\}$ be disjoint compact sets in X and Y such that $\mu(X \setminus \bigcup_n F_n) = \nu(Y \setminus \bigcup_n G_n) = 0$. For $\varepsilon > 0$, set $\varepsilon_n = \varepsilon 2^{-n}$.

(i) ⇒ (ii). It follows from step I that, for each pair (n, m), there are sets $R_{n,m}(\varepsilon) \subset F_n$ and $T_{n,m}(\varepsilon) \subset G_m$ such that φ is continuous on $R_{n,m}(\varepsilon) \times T_{n,m}(\varepsilon)$, $\mu(F_n \setminus R_{n,m}(\varepsilon)) \le \varepsilon_n \varepsilon_m$, and $\nu(G_m \setminus T_{n,m}(\varepsilon)) \le \varepsilon_n \varepsilon_m$. Set

$$R_n(\varepsilon) = \bigcap_{m=1}^{\infty} R_{n,m}(\varepsilon) \quad \text{and} \quad T_m(\varepsilon) = \bigcap_{n=1}^{\infty} T_{n,m}(\varepsilon). \tag{8–1}$$

Then $\mu(F_n \setminus R_n(\varepsilon)) \le \varepsilon_n \varepsilon$, $\mu(G_m \setminus T_m(\varepsilon)) \le \varepsilon_m \varepsilon$, and the map φ is continuous on $R_n(\varepsilon) \times T_m(\varepsilon)$ for each pair (n, m). Set

$$X_\varepsilon = \bigcup_{n=1}^{\infty} R_n(\varepsilon) \quad \text{and} \quad Y_\epsilon = \bigcup_{m=1}^{\infty} T_m(\varepsilon). \tag{8–2}$$

These are the sets we need.

(ii) ⇒ (iii). We preserve the notations above. Let $\varepsilon_k = 2^{-k}$. It follows from step I that, for each pair (n, m), there are increasing sequences of compact sets $\{R_{n,m}(\varepsilon_k)\}_{k=1}^{\infty}$ in F_n and $\{T_{n,m}(\varepsilon_k)\}_{k=1}^{\infty}$ in G_m such that

$$\mu(F_n \setminus R_{n,m}(\varepsilon_k)) \le \varepsilon_n \varepsilon_m \varepsilon_k, \qquad \nu(G_m \setminus T_{n,m}(\varepsilon_k)) \le \varepsilon_n \varepsilon_m \varepsilon_k,$$

and φ is continuous on $R_{n,m}(\varepsilon_k) \times T_{n,m}(\varepsilon_k)$. The compact sets $R_n(\varepsilon_k) \subseteq F_n$ and $T_m(\varepsilon_k) \subseteq G_m$ (see (8–1)) increase with k, $\mu(F_n \setminus R_n(\varepsilon_k)) \le \varepsilon_n \varepsilon_k$, $\mu(G_m \setminus T_m(\varepsilon_k)) \le \varepsilon_m \varepsilon_k$, and φ is continuous on $R_n(\varepsilon_k) \times T_m(\varepsilon_k)$. The compact sets $X_k = X_{\varepsilon_k}$ and

$Y_k = Y_{\varepsilon_k}$ (see (8–2)) form exhaustive sequences in X and Y, and φ is continuous on each $X_k \times Y_k$.

The proof of (iii) $\Rightarrow$ (i) is the same as in Theorem 8.2. □

9. Schur multipliers and discontinuous (μ, ν)-multipliers

In contrast to Schur multipliers, (μ, ν, I)-multipliers are not sensitive to changes on null sets. Therefore one cannot expect that the classes of noncontinuous (μ, ν, I)-multipliers and of Schur I-multipliers coincide. In this section we will show that for any I, ω-continuous (μ, ν, I)-multipliers coincide marginally a.e. with Schur I-multipliers. More precisely, an ω-continuous function is a (μ, ν, I)-multiplier if and only if it becomes a Schur I-multiplier after deleting a marginally null subset.

Remark. Two ω-continuous functions φ, φ' coincide a.e. if and only if they coincide marginally a.e. Indeed, set $\psi = \varphi - \varphi'$. If $\psi \equiv 0$ marginally a.e., then $\psi \equiv 0$ a.e. Suppose that ψ vanishes a.e. The set $L = \{z \in \mathbb{C} : \psi(z) \neq 0\}$ is ω-open and $(\mu \otimes \nu)(L) = 0$. Therefore it coincides with some union of rectangles $A_n \times B_n$ up to a marginally null set. Hence $\mu(A_n)\nu(B_n) = 0$, so all $A_n \times B_n$ are marginally null. Thus L is a marginally null set.

Our restriction to ω-continuous functions is strongly motivated by the following result.

Proposition 9.1. *If* (X, μ) *and* (Y, ν) *are standard measure spaces, then*

$$\mathfrak{M}_{\mu,\nu}(\mathfrak{S}_\infty) \subseteq C_{\mu,\nu}(X \times Y).$$

Proof. Choose admissible topologies on X and Y, so that $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$, with $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. Let $\varphi \in \mathfrak{M}_{\mu,\nu}(\mathfrak{S}_\infty)$ and let G be an open set in $\mathbb{C}$. Since

$$\varphi^{-1}(G) = \bigcup_n \left(\varphi^{-1}(G) \cap (X_n \times Y_n)\right),$$

we only need to show that each set $\varphi^{-1}(G) \cap (X_n \times Y_n)$ is ω-open. Hence we may assume that $\mu(X) < \infty$ and $\nu(Y) < \infty$.

Set $H = L_2(X, \mu)$ and $K = L_2(Y, \nu)$. All results of Proposition 4.2 hold if $\mathcal{L}(I)$ is replaced by $\mathfrak{M}_{\mu,\nu}(I)$. Hence $\varphi \in \mathfrak{M}_{\mu,\nu}(\mathfrak{S}_1)$. The operator A with kernel $a(x, y) \equiv 1$ is a rank one operator. Hence the operator $\Phi_\varphi(A)$ with kernel $\varphi(x, y)$ belongs to $\mathfrak{S}_1(H, K)$. Hence $\varphi(x, y)$ belongs to the projective tensor product $H \hat{\otimes} K$ and is ω-continuous by Theorem 6.5 of [Erdos et al. 1998]. □

Let $\{X_n\}$, $\{Y_n\}$ be exhaustive sequences of measurable subsets of (X, μ) and (Y, ν), and let χ_n and χ'_n be the characteristic functions of X_n and Y_n. Let μ_n and

ν_n be the restrictions of μ and ν to X_n and Y_n. For $\varphi \in L_\infty(X \times Y, \mu \otimes \nu)$, set

$$\varphi_n = \chi_n \chi_n' \varphi \quad \text{and} \quad \widehat{\varphi}_n = \varphi | X_n \times Y_n.$$

Lemma 9.2. (i) *Let an s.n. ideal I be either coseparable, or contain $\mathfrak{S}_2$. Then φ is a (μ, ν, I)-multiplier if and only if all φ_n are (μ, ν, I)- multipliers and* $\sup_n \|\Phi_{\varphi_n}\|_I < \infty$. *In this case*

$$\|\Phi_\varphi\|_I = \sup_n \|\Phi_{\varphi_n}\|_I.$$

(ii) *Let I be either a separable or coseparable s.n. ideal, or $\mathfrak{S}_b$. Let $X = \bigcup_n X_n$ and $Y = \bigcup_n Y_n$. Then φ is a Schur I-multiplier if and only if all φ_n are Schur I-multipliers and* $\sup_n \|S_{\varphi_n}\|_I < \infty$. *In this case* $\|S_\varphi\|_I = \sup_n \|S_{\varphi_n}\|_I$.

(iii) *φ_n is a (μ, ν, I)-multiplier on $X \times Y$ if and only if $\widehat{\varphi}_n$ is a (μ_n, ν_n, I)-multiplier on $X_n \times Y_n$. Moreover, $\|\Phi_{\varphi_n}\|_I = \|\Phi_{\widehat{\varphi}_n}\|_I$.*

(iv) *φ_n is a Schur I-multiplier on $X \times Y$ if and only if $\widehat{\varphi}_n$ is a Schur I-multiplier on $X_n \times Y_n$. Moreover, $\|S_{\varphi_n}\|_I = \|S_{\widehat{\varphi}_n}\|_I$.*

Proof. Set $\Phi = \Phi_\varphi$ and $\Phi_n = \Phi_{\varphi_n}$. The operators P_n on $H = L_2(X, \mu)$ (identified with H^d as usual) and Q_n on $K = L_2(Y, \nu)$, acting by the multiplication by χ_n and χ_n', respectively, are projections, and

$$\Phi_n(R) = Q_n \Phi(R) P_n, \qquad \text{for } R \in \mathfrak{S}_2(H, K). \tag{9–1}$$

Since P_n and Q_n strongly converge to the identity operators, $\Phi_n(R)$ strongly converge to $\Phi(R)$.

Let I be coseparable. If φ_n are (μ, ν, I)-multipliers, then $\Phi_n(R) \in I$, for $R \in \mathscr{X}_I$. If $\sup_n \|\Phi_n\|_I < \infty$, then $\sup_n |\Phi_n(R)|_I < \infty$, and it follows from Theorem III.5.1 of [Gohberg and Kreĭn 1965] that $\Phi(R) \in I$ and $|\Phi(R)|_I \le \sup_n |\Phi_n(R)|_I$. On the other hand, by (9–1), all

$$|\Phi_n(R)|_I \le \|Q_n\| |\Phi(R)|_I \|P_n\| \le |\Phi(R)|_I.$$

Hence $\|\Phi\|_I = \sup_n \|\Phi_n\|_I$. The proof of the converse statement follows from (9–1) immediately.

If $\mathfrak{S}_2 \subseteq I$, then $\mathfrak{S}_2 \subseteq I \subseteq J$, for some coseparable ideal J. Since all results of Proposition 4.2 hold, if $\mathscr{L}(I)$ is replaced by $\mathfrak{M}_{\mu,\nu}(I)$, the sets of (μ, ν, I)- and (μ, ν, J)-multipliers coincide. Part (i) is proved.

Let I be a coseparable ideal, let I_0 be the corresponding separable ideal and $(\widehat{I_0}, \widehat{I})$ be the pair of the dual ideals. It is well known and follows from duality (see, for example, [Kissin and Shulman 2005a, Lemma 5.1]) that the sets of Schur I-, I_0-, $\widehat{I}$- and $\widehat{I_0}$-multipliers (in particular, $\mathfrak{S}_b$-, $\mathfrak{S}_1$- and $\mathfrak{S}_\infty$-multipliers) coincide and the norms of the multipliers are equal. Hence we only need to prove (ii) for coseparable ideals. This proof is identical to the proof of (i) with Φ replaced by S, $L_2(X,\mu)$ by $l_2(X)$ and $L_2(Y,\nu)$ by $l_2(Y)$.

Set $H_n = L_2(X_n,\mu_n)$, $K_n = L_2(Y_n,\nu_n)$. For $R \in \mathfrak{S}_2(H,K)$ with kernel r, let $\delta_n(R)$ be the integral operator from H_n into K_n with kernel $\widehat{r} = r|X_n \times Y_n$. Then δ_n maps $\mathscr{X}_I(H,K)$ onto $\mathscr{X}_I(H_n,K_n)$, and it is an isometry from $Q_n\mathscr{X}_I(H,K)P_n$ onto $\mathscr{X}_I(H_n,K_n)$. Conversely, for $\widehat{R} \in \mathfrak{S}_2(H_n,K_n)$ with kernel $\widehat{r}$, let $\Delta_n(\widehat{R})$ be the integral operator from H into K with kernel r that vanishes outside $X_n \times Y_n$ and $r|X_n \times Y_n = \widehat{r}$. Then Δ_n is an isometry from $\mathscr{X}_I(H_n,K_n)$ onto $Q_n\mathscr{X}_I(H,K)P_n$, and

$$\delta_n(\Delta_n(\widehat{R})) = \widehat{R} \quad \text{for } \widehat{R} \in \mathscr{X}_I(H_n,K_n),$$

$$\Delta_n(\delta_n(R)) = R \quad \text{for } R \in Q_n\mathscr{X}_I(H,K)P_n.$$

We also have $\Phi_{\varphi_n}(R) \in Q_n\mathscr{X}_I(H,K)P_n$ for $R \in \mathscr{X}_I(H,K)$, $\delta_n\Phi_{\varphi_n} = \Phi_{\widehat{\varphi}_n}\delta_n$, $\Phi_{\varphi_n}\Delta_n = \Delta_n\Phi_{\widehat{\varphi}_n}$, and

$$\|\Phi_{\varphi_n}\|_I = \sup\{\left|\Phi_{\varphi_n}(R)\right|_I : R \in Q_n\mathscr{X}_I(H,K)P_n,\ |R|_I = 1\}.$$

Making use of these formulae, one obtains a proof of (iii). Part (iv) is evident. □

We will prove now an analogue of Theorem 7.5 for ω-continuous functions.

Theorem 9.3. *Let I be either a coseparable ideal, or a separable ideal containing $\mathfrak{S}_2$. Let (X,μ) and (Y,ν) be standard measure spaces with countable bases. An ω-continuous function φ on $X \times Y$ is a (μ,ν,I)-multiplier if and only if there are null sets $X_0 \subset X$, $Y_0 \subset Y$ such that φ is a Schur I-multiplier on $(X \setminus X_0) \times (Y \setminus Y_0)$. In this case the sets X_0, Y_0 can be chosen in such a way that $\|\varphi\|_I^{\mu,\nu} = \|S_{\tilde{\varphi}}\|_I$, where $\tilde{\varphi} = \varphi|(X \setminus X_0) \times (Y \setminus Y_0)$.*

Proof. Choose admissible topologies on X and Y. By Theorem 8.3, there are exhaustive sequences $\{X_n\}$ and $\{Y_n\}$ of compact sets in X and Y such that φ is continuous on each $X_n \times Y_n$. Let μ_n and ν_n be the restrictions of μ and ν to X_n and Y_n. One can assume that $\mathrm{supp}(\mu_n) = X_n$ and $\mathrm{supp}(\nu_n) = Y_n$. Indeed, set $K_n = \mathrm{supp}(\mu_n)$. If $K_n \neq X_n$, replace X_n by K_n. If $x \in K_n$ then, for each neighbourhood U_x of x in X, we have $\mu(X_n \cap U_x) \neq 0$. Hence $\mu(X_{n+1} \cap U_x) \neq 0$, so $x \in K_{n+1}$. Thus $K_n \subseteq K_{n+1}$. Since $X_n = K_n \cup N_n$ and $\mu(N_n) = 0$, the sequence $\{K_n\}$ is exhaustive and $\mathrm{supp}(\mu|K_n) = K_n$.

By Lemma 9.2(i), φ is a (μ,ν,I)-multiplier if and only if all its restrictions φ_n to $X_n \times Y_n$ are (μ,ν,I)-multipliers and the norms are bounded. Moreover, $\|\Phi_\varphi\|_I = \sup_n \|\Phi_{\varphi_n}\|_I$. Since $\widehat{\varphi}_n$ is continuous on $X_n \times Y_n$, it follows from Theorem 7.5 that $\widehat{\varphi}_n$ is a (μ_n,ν_n,I)-multiplier if and only $\widehat{\varphi}_n$ is a Schur I-multiplier on $X_n \times Y_n$; in this case $\|S_{\widehat{\varphi}_n}\|_I = \|\widehat{\varphi}_n\|_I^{\mu_n,\nu_n}$. By Lemma 9.2(ii), the restriction $\tilde{\varphi}$ of φ to $\left(\bigcup_n X_n\right) \times \left(\bigcup_n Y_n\right)$ is a Schur I-multiplier if and only if all φ_n are Schur I-multipliers and the norms are bounded. In this case, $\|S_{\tilde{\varphi}}\|_I = \sup_n \|S_{\varphi_n}\|$. Taking now into account Lemma 9.2(iii) and (iv), we complete the proof. □

Remark. Theorem 9.3 allows us to deduce Peller's [1985] description of double operator integrable functions from Grothendieck's description of Schur $\mathfrak{S}_\infty$-multipliers. Indeed, let $\mathscr{E}$, $\mathscr{F}$ be spectral measures on locally compact spaces X and Y. Denote by π and ρ the representations of $C_0(X)$ and $C_0(Y)$ corresponding to $\mathscr{E}$ and $\mathscr{F}$. Then (see the discussion at the end of Section 6) the set of double operator integrable functions with respect to $\mathscr{E}$, $\mathscr{F}$ coincides with $\mathbf{M}^{\pi,\rho}$. Let μ, ν be scalar measures such that $\operatorname{supp}(\mathscr{E}) = \operatorname{supp}(\mu)$ and $\operatorname{supp}(\mathscr{F}) = \operatorname{supp}(\nu)$. Then it follows from Corollary 7.3 that $\mathbf{M}^{\pi,\rho} = \mathbf{M}^{\mu,\nu}$. By Proposition 9.1, every function $\varphi \in \mathbf{M}^{\mu,\nu}$ is ω-continuous. Hence, by Theorem 9.3, φ becomes a Schur $\mathfrak{S}_\infty$-multiplier after deleting some null subsets from X and Y. Applying [Pisier 2001, Theorem 5.1], we get $\varphi(x, y) = (a(x), b(y))$, where a, b are bounded Hilbert space valued functions. This is the first part of Peller's theorem. Furthermore, by the proof of Theorem 5.5 of [Pisier 2001], there are a probability space (T, τ) and bounded functions $a(x, t)$, $b(y, t)$ on $X \times T$ and $Y \times T$ such that $\varphi(x, y) = \int_T a(x, t)b(y, t)\, d\tau$. This is the second (much stronger) statement in Peller's result.

We now relate ω-continuous (μ, ν, I)-multipliers for different pairs of measures, just as we did for continuous (μ, ν, I)-multipliers in Corollary 7.4. Let μ be a σ-finite Radon measure on a topological space X and let Σ be the σ-algebra of all μ-measurable sets in (X, μ). Let a measure μ' on Σ be *absolutely continuous with respect to* μ, that is, $\mu(E) = 0$ implies $\mu'(E) = 0$ for $E \in \Sigma$. Then, for every μ-measurable subset Z of X, $\operatorname{supp}(\mu'|Z) \subseteq \operatorname{supp}(\mu|Z)$.

Theorem 9.4. *Let I be either a coseparable ideal, or $\mathfrak{S}_2 \subseteq I$. Let μ and ν be σ-finite Radon measures on topological spaces X and Y with countable bases. Let σ-finite measures μ' and ν' on X and Y be absolutely continuous with respect to μ and ν, respectively. Then every (μ, ν, I)-multiplier φ is also a (μ', ν', I)-multiplier and $\|\Phi_{\varphi,\mu',\nu'}\|_I \le \|\Phi_{\varphi,\mu,\nu}\|_I$.*

Proof. By Theorem 8.3, there are exhaustive (with respect to μ and ν) sequences $\{X_n\}$ and $\{Y_n\}$ of compact sets in X and Y such that the functions

$$\widehat{\varphi}_n = \varphi | X_n \times Y_n$$

are continuous. Then $\{X_n\}$ and $\{Y_n\}$ are also exhaustive sequences with respect to μ' and ν'. Let μ_n and μ_n' be the restrictions of μ and μ' to X_n, and let ν_n and ν_n' be the restrictions of ν and ν' to Y_n. By Lemma 9.2(i) and (iii), the functions $\widehat{\varphi}_n$ are (μ_n, ν_n, I)-multipliers and $\|\Phi_{\varphi,\mu,\nu}\|_I = \sup_n \|\Phi_{\widehat{\varphi}_n,\mu_n,\nu_n}\|_I$.

Since $\operatorname{supp}(\mu_n') \subseteq \operatorname{supp}(\mu_n)$ and $\operatorname{supp}(\nu_n') \subseteq \operatorname{supp}(\nu_n)$, it follows from Corollary 7.4 that $\widehat{\varphi}_n$ are also (μ_n', ν_n', I)-multipliers and $\|\Phi_{\widehat{\varphi}_n,\mu_n',\nu_n'}\|_I \le \|\Phi_{\widehat{\varphi}_n,\mu_n,\nu_n}\|_I$. Applying again Lemma 9.2(i) and (iii), we conclude that φ is a (μ', ν', I)-multiplier and $\|\Phi_{\varphi,\mu',\nu'}\|_I \le \|\Phi_{\varphi,\mu',\nu'}\|_I$. □

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