AN EICHLER–ZAGIER MAP FOR JACOBI FORMS OF
HALF-INTEGRAL WEIGHT

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We construct an Eichler–Zagier map for Jacobi cusp forms of half-integral weight. As an application, we show there exists no Hecke-equivariant map from index 1 to index $p$ ($p$ prime), when the weight is half-integral.

The aim of this paper is to generalize the Eichler–Zagier map for Jacobi forms of half-integral weight, which is formally defined as

\[
\mathcal{F}_m : \sum_{0 > D, r \in \mathbb{Z}} c(D, r) e\left(\frac{r^2 - D}{4m} \tau + rz\right) \mapsto \sum_{0 > D \in \mathbb{Z}} \left( \sum_{r \equiv D \pmod{4m}} c(D, r) \right) e(|D|\tau).
\]

We prove that it is a Hecke-equivariant map from Jacobi cusp forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4M)$, index $m$ and character $\chi$ ($k$ and $\chi$ are even) into a certain subspace of cusp forms of weight $k$ on $\Gamma_1(16m^2M)$. First we derive this assertion for $m = 1$ by proving that $\mathcal{F}_1$ maps respective Poincaré series. For the general index $m$, we apply certain operator $I_m$ (see (2) for the definition) which changes the index $m$ into index 1 and then apply $\mathcal{F}_1$ to obtain the required mapping property.

In order to give a Maass relation for each prime $p$ for Siegel modular forms of half-integral weight and degree two, Y. Tanigawa [1986] obtained a Hecke-equivariant map from the space of index 1 Jacobi forms of half-integral weight into certain modular forms of integral weight and he constructed the map $V_{p^2}$ from the space of Jacobi forms of index 1 into index $p^2$. As a natural question, he asked the existence of a connection between Jacobi forms of index 1 and index $p$ ($p$ is a prime) in the case of half-integral weight. We show that there is no such Hecke-equivariant map as an application of the nature of the map $\mathcal{F}_m$.

**Notation and background.** Throughout this paper, unless otherwise specified, the letters $k, m, M, N$ will stand for natural numbers and $\tau$ for an element of $\mathcal{H}$, the complex upper half-plane.

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For a complex number \( z \), we write \( \sqrt{z} \) for the square root with argument in \( (-\pi/2, \pi/2) \), and we set \( z^{k/2} = (\sqrt{z})^k \) for any \( k \in \mathbb{Z} \).

For integers \( a, b \), let \( \left( \frac{a}{b} \right) \) denote the generalized quadratic residue symbol. Let \( S_k(N, \psi) \) denote the space of cusp forms of weight \( k \) and level \( N \) with character \( \psi \).

We write the Fourier expansion of a modular form \( f \) as
\[
 f(\tau) = \sum_{n \geq 1} a_f(n) e^{2\pi i n \tau}.
\]

For \( z \in \mathbb{C} \) and \( c, d \in \mathbb{Z} \), we put \( e^c(z) = e^{2\pi i cz/d} \). We also write \( e^c(z) = e_c(z), e^1(z) = e(z) \). The symbol \( a \equiv \square (b) \) means that \( a \) is a square modulo \( b \). For two forms \( f \) and \( g \) (either in the space of modular forms of integral weight or in the space of Jacobi forms of half-integral weight), \( \langle f, g \rangle \) denotes the Petersson inner product of \( f \) and \( g \). For a Dirichlet character \( \psi \) modulo \( 4m \), the twisting operator on modular forms of integral weight is given by
\[
 (1) \quad R_\psi = \frac{1}{W_\psi} \sum_{u \mod 4m} \overline{\psi}(u) \begin{pmatrix} 4m & u \\ 0 & 4m \end{pmatrix},
\]
where \( W_\psi = \sum_{u \mod (4m)} \psi(u) e(u/4m) \). It follows that \( \langle f \mid R_\psi, g \rangle = \langle f, g \mid R_\psi \rangle \), where \( f, g \in S_k(\Gamma_1(16mM)) \) and
\[
 R_\psi : \sum_{n \geq 1} a_f(n) e(n \tau) \mapsto \sum_{n \geq 1} \overline{\psi}(n) a_f(n) e(n \tau).
\]

For a natural number \( d \), the operators \( U(d) \) and \( B(d) \) are defined on formal power series by
\[
 U(d) : \sum_{n \geq 1} a(n) e(n \tau) \mapsto \sum_{n \geq 1} a(nd) e(n \tau),
\]
\[
 B(d) : \sum_{n \geq 1} a(n) e(n \tau) \mapsto \sum_{n \geq 1} a(n) e(nd \tau).
\]

For \( n \geq 1 \), let \( P_n \) denote the \( n \)-th Poincaré series in \( S_k(N, \psi) \) whose \( \ell \)-th Fourier coefficient is given by
\[
 g_n(\ell) = 2 \pi i^{k-1} n^{(k-1)/2} \sum_{c \geq 1, \ N|c} K_{N, \chi}(n, \ell; c) J_{k-1} \left( \frac{4 \pi \sqrt{n \ell}}{c} \right),
\]
where \( \delta(\ell, n) \) is the Kronecker-delta function, \( J_{k-1}(x) \) is the Bessel function of order \( k-1 \) and \( K_{N, \chi}(n, \ell; c) \) is the Kloosterman sum defined by
\[
 K_{N, \chi}(n, \ell; c) = \frac{1}{c} \sum_{dd^{-1}=1} \overline{\psi}(d) e_c(n d^{-1} + \ell d).
\]
1. A certain space of cusp forms of integral weight

For $m, M \in \mathbb{N}$, let $\chi \mod M$ be a Dirichlet character and $\chi_m(n) = \left(\frac{n}{m}\right)$ be the quadratic character modulo $m$ or $4m$ according as $m \equiv 1$ or $m \equiv 3 \pmod{4}$.

Let

\[
S = \{ \ell \in \mathbb{N} : 1 \leq \ell \leq 4m, \ell \equiv 0 \pmod{4m} \},
\]

\[
S^* = \{ \ell \in S : p^2 \mid 4mM \text{ implies } p \nmid \ell, \text{ with } p \text{ prime} \}.
\]

If $\ell \in S$, define

\[
S_k^{\square, \ell}(16mM, \chi \chi_m) := S_k(16mM, \chi \chi_m) \mid R_{\ell},
\]

where

\[
R_{\ell} : \sum_{n \geq 1} a(n) e(n\tau) \mapsto \sum_{n \geq 1} a(n) e(n\tau). \quad -n \equiv \ell \pmod{4m}
\]

For $\ell \in S$, let $t = (\ell, 4m)$. A formal computation shows that

\[
R_{\ell} = U(t) R(\ell) B(t),
\]

with

\[
R(\ell) = \frac{1}{\varphi(4m/t)} \sum_{\psi \mod 4m/t} \overline{\psi}(-\ell/t) R_{\psi},
\]

where $\varphi(n)$ is the Euler totient function. Using the mapping properties of $U(t)$, $R_{\psi}$ and $B(t)$ in the said order, we verify that $S_k^{\square, \ell}(16mM, \chi \chi_m)$ is a subspace of $S_k(\Gamma(1, 16m^2M))$. Finally we define

\[
S_k^{\square}(16mM, \chi \chi_m) = \sum_{\ell \in S} S_k^{\square, \ell}(16mM, \chi \chi_m).
\]

2. Jacobi forms of half-integral weight

For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, let $\tilde{\alpha} = (\alpha, \phi(\tau))$, where $\phi(\tau)$ is a holomorphic function on $\mathbb{H}$ such that $\phi^2(\tau) = t(c\tau + d)$, with $t \in \{1, -1\}$. Then the set $G = \{ \tilde{\alpha} : \alpha \in \text{SL}_2(\mathbb{R}) \}$ is a group with group law

\[
(\alpha_1, \phi_1(\tau))(\alpha_2, \phi_2(\tau)) = (\alpha_1\alpha_2, \phi_1(\alpha_2\tau)\phi_2(\tau)).
\]

If $\alpha \in \Gamma_0(4)$, set

\[
j(\alpha, \tau) = \left(\frac{\tau}{d}\right) \left(\frac{-4}{d}\right)^{-1/2} \left(c\tau + d\right)^{1/2}.
\]

We set $\alpha^* = (\alpha, j(\alpha, \tau))$; the association $\alpha \mapsto \alpha^*$ is an injective map from $\Gamma_0(4)$ into $G$. Let $G^j$ be the set of all triplets $[\tilde{\alpha}, X, s]$, $\alpha \in \text{SL}_2(\mathbb{R})$, $X \in \mathbb{R}^2$, $s \in \mathbb{C}$, $|s| = 1$. Then $G^j$ is a group, with group law given by

\[
[\tilde{\alpha_1}, X_1, s_1][\tilde{\alpha_2}, X_2, s_2] = \left[\tilde{\alpha_1}\tilde{\alpha_2}, X_1\alpha_2 + X_2, s_1s_2 \cdot \det\left(\begin{pmatrix} X_1 & \alpha_2 \\ & X_2 \end{pmatrix}\right)\right].
\]
The stroke operator \( |_{k+1/2,m} \) is defined on functions \( \phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \) by
\[
\phi |_{k+1/2,m}[\tilde{\alpha},X,s] = \delta^m \phi(\tau) - 2k - 1 e^m \left( \frac{-e(z + \lambda \tau + \mu)^2}{\epsilon \tau + \delta} + 2\lambda^2 \tau + 2\lambda z + \lambda \mu \right) \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z + \lambda \tau + \mu}{c\tau + d} \right),
\]
where \([\tilde{\alpha},X,s] \in G^J\).

The Jacobi group for \( \Gamma_0(4N) \) is a subgroup \( \Gamma_0^J(4N)^* \) of \( G^J \), given by
\[
\Gamma_0^J(4N)^* = \{[\alpha^*,X] : \alpha \in \Gamma_0(4N), X \in \mathbb{Z}^2\}.
\]
A Jacobi form \( \phi(\tau, z) \) of weight \( k + \frac{1}{2} \) and index \( m \) for the group \( \Gamma_0(4M) \), with character \( \chi \), is a holomorphic function \( \phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \) satisfying the following conditions:

(i) \( \phi |_{k+1/2,m}[\gamma^*,X](\tau, z) = \chi(d) \phi(\tau, z) \), where \( \chi \) is a Dirichlet character mod \( 4M \) and \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(4M) \).

(ii) For every \( \alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \), the image \( \phi |_{k+1/2,m}[\tilde{\alpha},(0,0)](\tau, z) \) has a Fourier development of the form
\[
\sum_{n,r \in \mathbb{Q}, \frac{r^2}{4} \leq 4nm} c_{\alpha}(n,r) e(n\tau + rz),
\]
where the sum ranges over rational numbers \( n, r \) with bounded denominators subject to the condition \( r^2 \leq 4nm \).

Further, if \( r^2 < 4nm \) whenever \( c_{\alpha}(n,r) \neq 0 \), then \( \phi \) is called a Jacobi cusp form. We denote by \( J_{k+1/2,m}(4M,\chi) \) the space of Jacobi forms of weight \( k + \frac{1}{2} \), index \( m \) for \( \Gamma_0(4M) \) with character \( \chi \), and by \( J_{k+1/2,m}^{\text{cusp}}(4M,\chi) \) the subspace of \( J_{k+1/2,m}(4M,\chi) \) consisting of Jacobi cusp forms. A Jacobi form \( \phi \) has a Fourier expansion of the form
\[
\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}, \frac{r^2}{4} \leq 4nm} c(n,r) e(n\tau + rz).
\]

Since \( c(n,r) = c(n',r') \) if \( r'^2 - 4n'm = r^2 - 4nm \) and \( r' \equiv r \pmod{2m} \), we write the Fourier expansion of \( \phi \) as
\[
\phi(\tau, z) = \sum_{D > 0, d \in \mathbb{Z}, D \equiv r^2 \pmod{4m}} c_{\phi}(D,r) e \left( \frac{r^2 - D}{4m} \tau + rz \right).
\]
Let $D < 0$ be a discriminant and $r$ an integer modulo $2m$ with $D \equiv r^2 \pmod{4m}$. Then the $(D, r)$-th Poincaré series, denoted by $P_{(D, r)}$, is defined by

$$P_{(D, r)}(\tau, z) = \sum_{\gamma \in \Gamma_0(4M) \setminus \Gamma_0(4M)} \bar{\chi}(\gamma) e(n\tau + rz) \bigg|_{k+1/2,m} \gamma.$$

We state the following proposition without proof.

**Proposition 2.1.** The Poincaré series $P_{(D, r)}$ lies in $J_{k+1/2,m}^{\text{cusp}}(4M, \chi)$ and satisfies

$$\langle \phi, P_{(D, r)} \rangle = \alpha_{k, m} |D|^{-k+1} c_\phi(D, r),$$

for each $\phi \in J_{k+1/2,m}^{\text{cusp}}(4M, \chi)$, where $\alpha_{k, m} = \Gamma(k - 1) m^{k-3/2} / (2\pi^{k-1})$. It has a Fourier development of the form

$$P_{(D, r)}(\tau, z) = \sum_{0 \leq r', r' \in \mathbb{Z}, \frac{D'}{r'} = r \pmod{4m}} \left(g_{D, r}(D', r') + \bar{\chi}(-1) g_{D, r}(D', -r')\right) e\left(\frac{r'^2 - D'}{4m} \tau + r'z\right),$$

where $D = r^2 - 4mn$, $D' = r'^2 - 4mn'$, and $g_{D, r}(D', r')$ is given by

$$\delta_m(D, r, D', r') + i^{-k-3/2} \pi \sqrt{\frac{2}{m}} \left(\frac{D'}{D}\right)^{k/2} \sum_{c > 1 \pmod{4Mc}} H_{m, c, \chi}(D, r, D', r') J_k\left(\frac{\pi \sqrt{DD'}}{mc}\right),$$

with

$$\delta_m(D, r, D', r') = \begin{cases} 1 & \text{if } D' = D \text{ and } r' \equiv r \pmod{2m}, \\ 0 & \text{otherwise}. \end{cases}$$

and

$$H_{m, c, \chi}(D, r, D', r') = c^{-3/2} e^{-r'r'/(2mc)} \times \sum_{d, \lambda(c) \pmod{dd^{-1} \equiv 1(c)}} \bar{\chi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{1/2} e_c \left(d^{-1} (m\lambda^2 + r\lambda + n) + dn' - \lambda r'\right).$$

### 3. The Eichler–Zagier map

First we consider the space $J_{k+1/2,1}^{\text{cusp}}(4M, \chi)$. Put $D = D_0 \ell^2$, $r = r_0 \ell$ in Proposition 2.1. In the Fourier coefficient of $P_{(D_0 \ell^2, r_0 \ell)}$, the Kloosterman-type sum is periodic as a function of $\ell$ of period $2c$. Hence, for any $h \pmod{2c}$, its Fourier transform (after replacing $\ell$ by $\ell d$ and $\lambda$ by $\lambda d$) becomes

$$\frac{1}{2c^{5/2}} \sum_{\ell / (2c), \lambda(c)^*} \bar{\chi}(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{1/2} \times e_{2c} \left(d(2\lambda^2 + 2r_0 \ell \lambda + 2n_0 \ell^2 + 2n - 2r \lambda - r_0 \ell r - h \ell)\right).$$
Since $4|c$, the sum over $\lambda$ is nonzero only if $r_0 \ell \equiv r \pmod{2}$. Hence, the sum over $\lambda$ becomes

$$\sum_{\lambda(c)} e_c(d\lambda^2) e_c(-d \left( \frac{r_0 \ell - r}{2} \right)^2).$$

Again, the fact that $4|c$ and $\gcd(c, d) = 1$ gives the identity

$$\frac{1}{\sqrt{2c}} \sum_{\lambda(c)} e_c(d\lambda^2) = \left( \frac{c}{d} \right) \left( -\frac{4}{d} \right)^{-1/2}.$$

Thus, the Fourier transform simplifies to

$$\frac{\sqrt{i}}{\sqrt{2c^2}} \sum_{(\ell(2c), d(c))^r} \overline{\chi}(d) e_{4c}(d(D_0 \ell^2 + D - 2h\ell)) = \frac{\sqrt{i}}{4\sqrt{2c^2}} \sum_{(\ell(2c), d(4c))^r} \overline{\chi}(d) e_{4c}(d(D_0 \ell^2 + D - 2h\ell)).$$

which is the Fourier transform of the corresponding Kloosterman sum of integral weight.

More precisely:

**Theorem 3.1.** The Eichler–Zagier map $\mathcal{H}_1$ maps $J_{k+1/2,1}^{\text{cusp}}(4M, \chi)$ into $S_k^\square(16M, \chi)$.

**Proof.** We shall prove that the $(D, r)$-th Fourier coefficient of $P(D_0 \ell^2, r_0 \ell)$ is equal (up to constant) $|D|$-th Fourier coefficient of $P(D_0 \ell^2)$. It is easy to see that

$$\delta_1(D_0 \ell^2, r_0 \ell, D, r) = \delta_{|D_0|\ell^2, |D|}.$$

We consider both the Kloosterman sums as periodic functions of period $2c$. The arguments put forth above shows that for each $c \geq 1$, with $4M|c$, the Fourier transform of $H_{1,c,\chi}(D_0 \ell^2, r_0 \ell, D, r)$ is equal to (up to the required constants) the Fourier transform of the Kloosterman sum (corresponding to integral weight) $K_{16M,\chi}(D_0 \ell^2, |D|; 4c)$. This proves the theorem. □

**The index-changing operator $I_m$.** If $\phi \in J_{k+1/2,m}^{\text{cusp}}(4M, \chi)$, define $I_m$ by

$$ (2) \quad \phi \mid I_m(\tau, z) = \sum_{\lambda \pmod{m}} e(\lambda^2 \tau + 2\lambda z) \phi(m\tau, z + \lambda \tau). $$

**Proposition 3.2.** $I_m$ maps $J_{k+1/2,m}^{\text{cusp}}(4M, \chi)$ into $J_{k+1/2,1}^{\text{cusp}}(4mM, \chi \chi_m)$. The Fourier development of $\phi \mid I_m$ is of the form

$$ \phi \mid I_m(\tau, z) = \sum_{0 < D, r \in \mathbb{Z}} \left( \sum_{D \equiv r^2 \pmod{4}} \sum_{c \in \mathbb{Z}/2m} D \equiv r^2 \pmod{2} \chi(D, s) \right) e \left( \frac{r^2 - D}{4} \tau + rz \right). $$
Proof. It is easy to see that
\[
\phi \mid I_m(\tau, z) = m^{-k/2-1/4} \sum_{\lambda \bmod m} \phi_1(\tau) \Delta_m(\lambda, 0)(\tau, z),
\]
where \(\phi_1(\tau) = \phi(\tau, z / \sqrt{m})\) and \(\Delta_m\) is the diagonal matrix \(\text{diag}(\sqrt{m}, 1/\sqrt{m})\).

The proposition now follows directly from the preceding expression. \(\square\)

Using the equality \(\mathcal{D}_m = I_m \mathcal{D}_1\), together with Theorem 3.1 and Proposition 3.2, we have:

**Theorem 3.3.** The map \(\mathcal{D}_m\) takes \(J^{\text{cusp}}_{k+1/2, m}(4M, \chi)\) into \(S_k(16mM, \chi\chi_m)\).

### 4. Half-integral weight Jacobi forms of index 1 and index \(p\)

In the case of integral weight Jacobi forms, the well-known map \(V_p\) is a Hecke-equivariant map from \(J_{k,1}\) into \(J_{k,p}\) (\(p\) is a prime). If we replace \(k\) by \(k+\frac{1}{2}\), then we have a Hecke-equivariant map \(V_p\) from \(J_{k+1/2,1}(4M)\) into \(J_{k+1/2,p}(4M)\), which was given by Tanigawa [1986]. Therefore, existence of a Hecke-equivariant map from index 1 into \(p\) in the case of half-integral weight Jacobi forms seems to be a natural question.

As an application of the map \(\mathcal{D}_m\), we show that there does not exist a Hecke-equivariant map from \(J^{\text{cusp}}_{k+1/2,1}(4)\) into \(J^{\text{cusp}}_{k+1/2,p}(4)\).

Let
\[
N = \begin{cases} 
 p & \text{if } p \equiv 1 \pmod{4}, \\
 p^2 & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

Let \(\psi \pmod{N}\) be a primitive Dirichlet character such that \(\psi^2 = \chi_p\). Let \(R_\psi\) be the twisting operator defined as in (1). Then, \(R_\psi\) maps \(S_k(16N^2, \chi_p)\) into \(S_k(16N^2)\) and commutes with Hecke operators \(T_n\), \((n, p) = 1\). Further, if \(f \in S_k(16N^2, \chi_p)\), we have
\[
(f \mid R_\psi) \mid W_p = f \mid R_\psi,
\]
where \(W_p\) is the \(W\)-operator on \(S_k(16N^2)\) for the prime \(p\).

**Case 1:** \(p \equiv 3 \pmod{4}\). Let \(f \in S_k(4p, \chi_p)\) be a normalized Hecke eigenform. Since \(f \mid R_\psi \in S_k(4p^4)\) and it is an eigenform for all the Hecke operators and the \(W\) operators, it is a newform in \(S_k^{\text{new}}(4p^4)\). Hence, by the theory of newforms, it is not equivalent to a level-1 Hecke eigenform.

**Case 2:** \(p \equiv 1 \pmod{4}\). Let \(f \in S_k^{\text{new}}(4p, \chi_p)\) be a normalized Hecke eigenform. Then, \(f \mid R_\psi \in S_k^{\text{new}}(4p^2)\). Since \(f \mid R_\psi = S_k^{\text{new}}(4p^2)\), and \(\psi^2 = \overline{\psi}\) (as \(\psi^2\) is quadratic), we get \(f \mid R_\psi\) and \(f \mid R_{R_\psi} = R_{\psi'}\) are newforms in \(S_k^{\text{new}}(4p^2)\). Thus, the form \(f\) is not equivalent to a level-1 Hecke eigenform. Now, we let \(f \in S_k(p, \chi_p)\).
Arguments as above again show that \( f \) is not equivalent to a level-1 Hecke eigenform.

Thus, we conclude that a normalized Hecke eigenform in \( S_k(4p, \chi_p) \) is not equivalent to a normalized Hecke eigenform in \( S_k(4) \). In view of the mapping property proved in Theorem 3.3, we have proved:

**Theorem 4.1.** There is no Hecke-equivariant map from the space \( J_{k+1/2,1}^{\text{cusp}}(4) \) into the space \( J_{k+1/2,1}^{\text{cusp}}(4) \).

In this connection the following question seems natural.

What contribution do half-integral weight Jacobi forms of square-free index make to the construction of a “Maass space” (if one exists) for degree-2 Siegel modular forms of half-integral weight?

**References**


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