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We construct an Eichler–Zagier map for Jacobi cusp forms of half-integral weight. As an application, we show there exists no Hecke-equivariant map from index 1 to index $p$ ($p$ prime), when the weight is half-integral.

The aim of this paper is to generalize the Eichler–Zagier map for Jacobi forms of half-integral weight, which is formally defined as

$$\mathcal{F}_m : \sum_{0 > D, r \in \mathbb{Z}} c(D, r) e\left(\frac{r^2 - D}{4m} \tau + rz\right) \mapsto \sum_{0 > D \in \mathbb{Z}} \left( \sum_{r \equiv D (4m)} c(D, r) \right) e(|D|\tau).$$

We prove that it is a Hecke-equivariant map from Jacobi cusp forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4M)$, index $m$ and character $\chi$ ($k$ and $\chi$ are even) into a certain subspace of cusp forms of weight $k$ on $\Gamma_1(16m^2 M)$. First we derive this assertion for $m = 1$ by proving that $\mathcal{F}_1$ maps respective Poincaré series. For the general index $m$, we apply certain operator $I_m$ (see (2) for the definition) which changes the index $m$ into index 1 and then apply $\mathcal{F}_1$ to obtain the required mapping property.

In order to give a Maass relation for each prime $p$ for Siegel modular forms of half-integral weight and degree two, Y. Tanigawa [1986] obtained a Hecke-equivariant map from the space of index 1 Jacobi forms of half-integral weight into certain modular forms of integral weight and he constructed the map $V_p^2$ from the space of Jacobi forms of index 1 into index $p^2$. As a natural question, he asked the existence of a connection between Jacobi forms of index 1 and index $p$ ($p$ is a prime) in the case of half-integral weight. We show that there is no such Hecke-equivariant map as an application of the nature of the map $\mathcal{F}_m$.

Notation and background. Throughout this paper, unless otherwise specified, the letters $k, m, M, N$ will stand for natural numbers and $\tau$ for an element of $\mathcal{H}$, the complex upper half-plane.

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For a complex number $z$, we write $\sqrt{z}$ for the square root with argument in $(-\pi/2, \pi/2)$, and we set $z^{k/2} = (\sqrt{z})^k$ for any $k \in \mathbb{Z}$.

For integers $a, b$, let $(\frac{a}{b})$ denote the generalized quadratic residue symbol. Let $S_k(N, \psi)$ denote the space of cusp forms of weight $k$ and level $N$ with character $\psi$.

We write the Fourier expansion of a modular form $f$ as

$$f(\tau) = \sum_{n \geq 1} a_f(n)e^{2\pi in\tau}. $$

For $z \in \mathbb{C}$ and $c, d \in \mathbb{Z}$, we put $e_c^z(z) = e^{2\pi i cz/d}$. We also write $e_c^1(z) = e^z$, $e_1^1(z) = e^z$, and $e_1^1(z) = e^z$. The symbol $a \equiv \square (b)$ means that $a$ is a square modulo $b$. For two forms $f$ and $g$ (either in the space of modular forms of integral weight or in the space of Jacobi forms of half-integral weight), $\langle f, g \rangle$ denotes the Petersson inner product of $f$ and $g$. For a Dirichlet character $\psi$ modulo 4$m$, the twisting operator on modular forms of integral weight is given by

$$R_\psi = \frac{1}{W_\psi} \sum_{u \mod 4m} \bar{\psi}(u) \begin{pmatrix} 4m & u \\ 0 & 4m \end{pmatrix},$$

where $W_\psi = \sum_{u \mod (4m)} \psi(u)e(u/4m)$. It follows that $\langle f \mid R_\psi, g \rangle = \langle f, g \mid R_\psi \rangle$, where $f, g \in S_k(\Gamma_1(16M))$ and

$$R_\psi : \sum_{n \geq 1} a_f(n)e(n\tau) \mapsto \sum_{n \geq 1} \psi(n)a_f(n)e(n\tau).$$

For a natural number $d$, the operators $U(d)$ and $B(d)$ are defined on formal power series by

$$U(d) : \sum_{n \geq 1} a(n)e(n\tau) \mapsto \sum_{n \geq 1} a(nd)e(n\tau),$$

$$B(d) : \sum_{n \geq 1} a(n)e(n\tau) \mapsto \sum_{n \geq 1} a(n)e(n\tau).$$

For $n \geq 1$, let $P_n$ denote the $n$-th Poincaré series in $S_k(N, \psi)$ whose $\ell$-th Fourier coefficient is given by

$$g_n(\ell) = \delta(\ell, n) + 2\pi i^{-k}(\ell/n)^{(k-1)/2} \sum_{c \geq 1, N \mid |c|} K_{N, \chi}(n, \ell; c)J_{k-1} \left( \frac{4\pi \sqrt{n\ell}}{c} \right),$$

where $\delta(\ell, n)$ is the Kronecker-delta function, $J_{k-1}(x)$ is the Bessel function of order $k - 1$ and $K_{N, \chi}(n, \ell; c)$ is the Kloosterman sum defined by

$$K_{N, \chi}(n, \ell; c) = \frac{1}{c} \sum_{d(c) \equiv \ell (\mod c)} \bar{\psi}(d)n(d^{-1} + \ell d).$$
1. A certain space of cusp forms of integral weight

For $m, M \in \mathbb{N}$, let $\chi \mod M$ be a Dirichlet character and $\chi_m(n) = \left( \frac{m}{n} \right)$ be the quadratic character modulo $m$ or $4m$ according as $m \equiv 1$ or $m \equiv 3 \pmod{4}$.

Let

$$S = \{ \ell \in \mathbb{N} : 1 \leq \ell \leq 4m, \ell \equiv \square (4m) \},$$

$$S^* = \{ \ell \in S : p^2 | 4mM \text{ implies } p \nmid \ell, \text{ with } p \text{ prime} \}.$$

If $\ell \in S$, define

$$S_k^{\square, \ell}(16mM, \chi \chi_m) := S_k(16mM, \chi \chi_m) \mid R_\ell,$$

where

$$R_\ell : \sum_{n \geq 1} a(n)e(n\tau) \mapsto \sum_{n \geq 1} a(n)e(n\tau).$$

For $\ell \in S$, let $t = (\ell, 4m)$. A formal computation shows that

$$R_\ell = U(t)R(\ell)B(t),$$

with

$$R(\ell) = \frac{1}{\varphi(4m/t)} \sum_{\psi \mod 4m/t} \overline{\psi(-\ell/t)}R_\psi,$$

where $\varphi(n)$ is the Euler totient function. Using the mapping properties of $U(t)$, $R_\psi$ and $B(t)$ in the said order, we verify that $S_k^{\square, \ell}(16mM, \chi \chi_m)$ is a subspace of $S_k(\Gamma_1(16m^2M))$. Finally we define

$$S_k^{\square}(16mM, \chi \chi_m) = \sum_{\ell \in S} S_k^{\square, \ell}(16mM, \chi \chi_m).$$

2. Jacobi forms of half-integral weight

For $\alpha = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R})$, let $\tilde{\alpha} = (\alpha, \phi(\tau))$, where $\phi(\tau)$ is a holomorphic function on $\mathcal{H}$ such that $\phi^2(\tau) = t(c\tau + d)$, with $t \in \{1, -1\}$. Then the set $G = \{ \tilde{\alpha} : \alpha \in \text{SL}_2(\mathbb{R}) \}$ is a group with group law

$$(\alpha_1, \phi_1(\tau))(\alpha_2, \phi_2(\tau)) = (\alpha_1\alpha_2, \phi_1(\alpha_2\tau)\phi_2(\tau)).$$

If $\alpha \in \Gamma_0(4)$, set

$$j(\alpha, \tau) = \left( \frac{c}{d} \right) \left( -\frac{d}{c} \right)^{-1/2} (c\tau + d)^{1/2}.$$ 

We set $\alpha^* = (\alpha, j(\alpha, \tau))$; the association $\alpha \mapsto \alpha^*$ is an injective map from $\Gamma_0(4)$ into $G$. Let $G^J$ be the set of all triplets $[\tilde{\alpha}, X, s]$, $\alpha \in \text{SL}_2(\mathbb{R})$, $X \in \mathbb{R}^2$, $s \in \mathbb{C}$, $|s| = 1$. Then $G^J$ is a group, with group law given by

$$[\tilde{\alpha_1}, X_1, s_1][\tilde{\alpha_2}, X_2, s_2] = \left[ \tilde{\alpha_1}\tilde{\alpha_2}, X_1\alpha_2 + X_2, s_1s_2 \cdot \left( \det \left( \begin{array}{cc} X_1 & \alpha_2 \\ X_2 & \end{array} \right) \right) \right].$$
The stroke operator \( |_{k+1/2,m} \) is defined on functions \( \phi : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C} \) by

\[
\phi |_{k+1/2,m}[\tilde{\alpha}, X, s] = s^m \phi(\tau)^{-2k-1} e^m \left( \frac{-c(z+\lambda \tau + \mu)}{c\tau + d} + 2\lambda^2 \tau + 2\lambda z + \lambda \mu \right) \phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z+\lambda \tau + \mu}{c\tau + d} \right),
\]

where \( [\tilde{\alpha}, X, s] \in G^J \).

The Jacobi group for \( \Gamma_0(4N) \) is a subgroup \( \Gamma_0^J(4N)^* \) of \( G^J \), given by

\[
\Gamma_0^J(4N)^* = \{ [\alpha^*, X] : \alpha \in \Gamma_0(4N), X \in \mathbb{Z}^2 \}.
\]

A Jacobi form \( \phi(\tau, z) \) of weight \( k + \frac{1}{2} \) and index \( m \) for the group \( \Gamma_0(4M) \), with character \( \chi \), is a holomorphic function \( \phi : \mathcal{H} \times \mathbb{C} \rightarrow \mathbb{C} \) satisfying the following conditions:

(i) \( \phi |_{k+1/2,m}[\gamma^*, X](\tau, z) = \chi(d) \phi(\tau, z) \), where \( \chi \) is a Dirichlet character mod \( 4M \) and \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(4M) \).

(ii) For every \( \alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}_2(\mathbb{Z}) \), the image \( \phi |_{k+1/2,m}[\tilde{\alpha}, (0, 0)](\tau, z) \) has a Fourier development of the form

\[
\sum_{n, r \in \mathbb{Q}, r^2 \leq 4nm} c_\alpha(n, r) e(n \tau + rz),
\]

where the sum ranges over rational numbers \( n, r \) with bounded denominators subject to the condition \( r^2 \leq 4nm \).

Further, if \( r^2 < 4nm \) whenever \( c_\alpha(n, r) \neq 0 \), then \( \phi \) is called a Jacobi cusp form. We denote by \( J_{k+1/2,m}(4M, \chi) \) the space of Jacobi forms of weight \( k + \frac{1}{2} \), index \( m \) for \( \Gamma_0(4M) \) with character \( \chi \), and by \( J_{k+1/2,m}^{\text{cusp}}(4M, \chi) \) the subspace of \( J_{k+1/2,m}(4M, \chi) \) consisting of Jacobi cusp forms. A Jacobi form \( \phi \) has a Fourier expansion of the form

\[
\phi(\tau, z) = \sum_{n, r \in \mathbb{Z}, r^2 \leq 4nm} c(n, r) e(n \tau + rz).
\]

Since \( c(n, r) = c(n', r') \) if \( r'^2 - 4nm = r^2 - 4nm \) and \( r' \equiv r \pmod{2m} \), we write the Fourier expansion of \( \phi \) as

\[
\phi(\tau, z) = \sum_{0 \geq D, r, \tilde{D} \in \mathbb{Z}, D \equiv r^2 \pmod{4m}} c_\phi(D, r) e \left( \frac{r^2 - D}{4m} \tau + rz \right).
\]
Let $D < 0$ be a discriminant and $r$ an integer modulo $2m$ with $D ≡ r^2 (4m)$. Then the $(D, r)$-th Poincaré series, denoted by $P_{(D, r)}$, is defined by

$$P_{(D, r)}(τ, z) = \sum_{γ \in Γ_0(4M)c \setminus Γ_0(4M)} \bar{χ}(γ)e(nτ + rz)\big|_{k+1/2, m}.$$ 

We state the following proposition without proof.

**Proposition 2.1.** The Poincaré series $P_{(D, r)}$ lies in $J_{k+1/2, m}^{\text{cusp}}(4M, χ)$ and satisfies

$$\langle φ, P_{(D, r)} \rangle = α_{k,m}|D|^{-k+1}c_φ(D, r),$$

for each $φ ∈ J_{k+1/2, m}^{\text{cusp}}(4M, χ)$, where $α_{k,m} = Γ(k - 1)m^{k-3/2}/(2π^{k-1})$. It has a Fourier development of the form

$$P_{(D, r)}(τ, z) = \sum_{0 > D', r' \in \mathbb{Z}} \left( g_{D,r}(D', r') + \chi(-1)g_{D,r}(D', -r') \right)e\left(\frac{r'^2 - D'}{4m}τ + r'z\right),$$

where $D = r^2 - 4mn$, $D' = r'^2 - 4mn'$, and $g_{D,r}(D', r')$ is given by

$$δ_m(D, r, D', r') + i^{-k-3/2}π\sqrt{\frac{2}{m}\left(\frac{D'}{D}\right)}^{k/2} \sum_{c \geq 1 \atop c \mid 4M|c} H_{m,c,χ}(D, r, D', r') J_k\left(\frac{π\sqrt{DD'}}{mc}\right),$$

with

$$δ_m(D, r, D', r') = \begin{cases} 1 & \text{if } D' = D \text{ and } r' \equiv r \pmod{2m}, \\ 0 & \text{otherwise}. \end{cases}$$

and

$$H_{m,c,χ}(D, r, D', r') = e^{-3/2}e^{-rr'/(2mc)}$$

$$\times \sum_{\frac{d, λ(c)}{dd^{-1} = 1}} \bar{χ}(d)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{1/2}e_2\left(d^{-1}(mλ^2 + rλ + n) + dn' - λr'\right).$$

### 3. The Eichler–Zagier map

First we consider the space $J_{k+1/2, 1}^{\text{cusp}}(4M, χ)$. Put $D = D_0ℓ^2, r = r_0ℓ$ in Proposition 2.1. In the Fourier coefficient of $P_{(D_0ℓ^2, r_0ℓ)}$, the Kloosterman-type sum is periodic as a function of $ℓ$ of period $2c$. Hence, for any $h \pmod{2c}$, its Fourier transform (after replacing $ℓ$ by $ℓd$ and $λ$ by $λd$) becomes

$$\frac{1}{2c^{3/2}} \sum_{\frac{d, λ(c)}{(2c)_||d(c)} ≡ \lambda(c)} \bar{χ}(d)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{1/2}$$

$$\times e_{2c}\left(d(2λ^2 + 2r_0λ + 2n_0ℓ^2 + 2n - 2rλ - r_0ℓr - hℓ)\right).$$
Since $4|c$, the sum over $\lambda$ is nonzero only if $r_0 \ell \equiv r \pmod{2}$. Hence, the sum over $\lambda$ becomes

$$\sum_{\lambda \pmod{c}} e_c(d\lambda^2) e_c\left(-d\left(\frac{r_0 \ell - r}{2}\right)^2\right).$$

Again, the fact that $4|c$ and $\gcd(c, d) = 1$ gives the identity

$$\frac{1}{\sqrt{2c}} \sum_{\lambda \pmod{c}} e_c(d\lambda^2) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}.$$

Thus, the Fourier transform simplifies to

$$\frac{\sqrt{i}}{\sqrt{2c^2}} \sum_{(2c), d(c)^*} \overline{\chi}(d) e_{4c}(d(D_0 \ell^2 + D - 2h \ell)) = \frac{\sqrt{i}}{4\sqrt{2c^2}} \sum_{(2c), d(4c)^*} \overline{\chi}(d) e_{4c}(d(D_0 \ell^2 + D - 2h \ell)),$$

which is the Fourier transform of the corresponding Kloosterman sum of integral weight.

More precisely:

**Theorem 3.1.** The Eichler–Zagier map $\mathcal{E}_1$ maps $J_{k+1/2,1}^{\text{cusp}}(4M, \chi)$ into $S_k^{\square}(16M, \chi)$.

**Proof.** We shall prove that the $(D, r)$-th Fourier coefficient of $P_{(D_0 \ell^2, r_0 \ell)}$ is equal (up to constant) to $|D|$-th Fourier coefficient of $P_{|D_0| \ell^2}$. It is easy to see that

$$\delta_1(D_0 \ell^2, r_0 \ell, D, r) = \delta_{|D_0| \ell^2, |D|}.$$

We consider both the Kloosterman sums as periodic functions of period $2c$. The arguments put forth above shows that for each $c \geq 1$, with $4M|c$, the Fourier transform of $H_{1,c,\chi}(D_0 \ell^2, r_0 \ell, D, r)$ is equal to (up to the required constants) the Fourier transform of the Kloosterman sum (corresponding to integral weight) $K_{16M,\chi}(|D_0| \ell^2, |D|; 4c)$. This proves the theorem. \qed

**The index-changing operator $I_m$.** If $\phi \in J_{k+1/2,m}^{\text{cusp}}(4M, \chi)$, define $I_m$ by

$$\phi | I_m(\tau, z) = \sum_{\lambda \pmod{m}} e(\lambda^2 \tau + 2\lambda z) \phi(m \tau, z + \lambda \tau).$$

**Proposition 3.2.** $I_m$ maps $J_{k+1/2,m}^{\text{cusp}}(4M, \chi)$ into $J_{k+1/2,1}^{\text{cusp}}(4mM, \chi\chi_m)$. The Fourier development of $\phi | I_m$ is of the form

$$\phi | I_m(\tau, z) = \sum_{0 < D, r \in \mathbb{Z}} \left(\sum_{s \pmod{2m}} c_\phi(D, s) e\left(\frac{r^2 - D}{4} \tau + rz\right)\right).$$
Proof. It is easy to see that

\[\phi \mid I_m(\tau, z) = m^{-k/2-1/4} \sum_{\lambda \equiv 0 \pmod{m}} \phi_{1/\sqrt{m}} \mid_{k, 1} \left[ \Delta_m, (\lambda, 0) \right] (\tau, z),\]

where \(\phi_{1/\sqrt{m}}(\tau, z) = \phi(\tau, z/\sqrt{m})\) and \(\Delta_m\) is the diagonal matrix diag\((\sqrt{m}, 1/\sqrt{m})\). The proposition now follows directly from the preceding expression. \(\square\)

Using the equality \(\mathcal{D}_m = I_m \mathcal{D}_1\), together with Theorem 3.1 and Proposition 3.2, we have:

**Theorem 3.3.** The map \(\mathcal{D}_m\) takes \(J_{k+1/2, m}^{cusp}(4M, \chi)\) into \(S_k(16mM, \chi\chi_m)\).

4. Half-integral weight Jacobi forms of index 1 and index \(p\)

In the case of integral weight Jacobi forms, the well-known map \(V_p\) is a Hecke-equivariant map from \(J_{k, 1}\) into \(J_{k, p}\) (\(p\) is a prime). If we replace \(k\) by \(k + \frac{1}{2}\), then we have a Hecke-equivariant map \(V_p^2\) from \(J_{k+1/2, 1}(4M)\) into \(J_{k+1/2, p^2}(4M)\), which was given by Tanigawa [1986]. Therefore, existence of a Hecke-equivariant map from index 1 into \(p\) in the case of half-integral weight Jacobi forms seems to be a natural question.

As an application of the map \(\mathcal{D}_m\), we show that there does not exist a Hecke-equivariant map from \(J_{k+1/2, 1}^{cusp}(4)\) into \(J_{k+1/2, p}(4)\).

Let

\[N = \begin{cases} p & \text{if } p \equiv 1 \pmod{4}, \\ p^2 & \text{if } p \equiv 3 \pmod{4}. \end{cases}\]

Let \(\psi \pmod{N}\) be a primitive Dirichlet character such that \(\psi^2 = \chi_p\). Let \(R_\psi\) be the twisting operator defined as in (1). Then, \(R_\psi\) maps \(S_k(16N^2, \chi_p)\) into \(S_k(16N^2)\) and commutes with Hecke operators \(T_n\), \((n, p) = 1\). Further, if \(f \in S_k(16N^2, \chi_p)\), we have

\[\left( f \mid R_\psi \right) \mid W_p = f \mid R_\psi,\]

where \(W_p\) is the \(W\)-operator on \(S_k(16N^2)\) for the prime \(p\).

**Case 1:** \(p \equiv 3 \pmod{4}\). Let \(f \in S_k(4p, \chi_p)\) be a normalized Hecke eigenform. Since \(f \mid R_\psi \in S_k(4p^4)\) and it is an eigenform for all the Hecke operators and the \(W\) operators, it is a newform in \(S_k^{\text{new}}(4p^4)\). Hence, by the theory of newforms, it is not equivalent to a level-1 Hecke eigenform.

**Case 2:** \(p \equiv 1 \pmod{4}\). Let \(f \in S_k^{\text{new}}(4p, \chi_p)\) be a normalized Hecke eigenform. Then, \(f \mid R_\psi \in S_k^{\text{new}}(4p^2)\). Since \(f \mid R_\psi \in S_k^{\text{new}}(4p^2)\), and \(\psi^3 = \overline{\psi}\) (as \(\psi^2\) is quadratic), we get \(f \mid R_\psi\) and \(f \mid R_\psi \mid R_\chi_p\) are newforms in \(S_k^{\text{new}}(4p^2)\). Thus, the form \(f\) is not equivalent to a level-1 Hecke eigenform. Now, we let \(f \in S_k(p, \chi_p)\).
Arguments as above again show that $f$ is not equivalent to a level-1 Hecke eigenform.

Thus, we conclude that a normalized Hecke eigenform in $S_k(4p, \chi_p)$ is not equivalent to a normalized Hecke eigenform in $S_k(4)$. In view of the mapping property proved in Theorem 3.3, we have proved:

**Theorem 4.1.** There is no Hecke-equivariant map from the space $J_{k+1/2,1}(4)$ into the space $J_{k+1/2,p}(4)$.

In this connection the following question seems natural.

What contribution do half-integral weight Jacobi forms of square-free index make to the construction of a “Maass space” (if one exists) for degree-2 Siegel modular forms of half-integral weight?

**References**


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