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Let (M, g) be a compact smooth Riemannian surface with boundary. In this paper, we use blowing-up analysis to prove that some Moser–Trudinger trace inequalities hold on certain function spaces, and that the extremal functions exist in those function spaces without any additional hypothesis on (M, g).

1. Introduction and main results

Let (M, g) be a compact smooth Riemannian surface, and $H^{1,2}(M)$ the completion of $C^\infty(M)$ under the norm

$$\|u\|_{H^{1,2}(M)} = \left(\int_M (|\nabla u|^2 + |u|^2)\, dV_g \right)^{1/2}.$$

A result of N. Trudinger [1967] implies that there exists a constant α such that

$$\sup_{\|u\|_{H^{1,2}(M)}=1} \int_M e^{\alpha u^2} dV_g < +\infty.$$

J. Moser proved the following theorems:

Theorem A [Moser 1970/71]. *Let Ω be an open domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant C which depends only on n such that if u is smooth, has compact support contained in Ω and its gradient ∇u satisfies $\int_M |\nabla u|^n dx \leq 1$, then*

$$\int_\Omega e^{\alpha_n |u|^{n/(n-1)}} dx \leq C|\Omega|,$$

where $\alpha_n = n(\omega_{n-1})^{1/n-1}$ and ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^n$. If α_n is replaced by any $\alpha > \alpha_n$, the integral on the left-hand is still finite, but can be made arbitrarily large by an appropriate choice of u.

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Theorem B [Moser 1970/71]. *There exists an absolute constant c_0 such that if u is a smooth function on S^2 with $\int_{S^2} |\nabla u|^2\, dS = 1$ and $\int_{S^2} u\, dV_g = 0$, then*

$$\int_{S^2} e^{4\pi u^2} dS \le c_0.$$

The constant 4π is the best possible in the same sense as α_n in Theorem A.

Recall that Sobolev's theorems, see e.g. [10], assert existence of imbedding $W_0^{1,p}(\Omega) \to L^q(\Omega)$ for $1 < p < n$ and $W_0^{1,p}(\Omega) \to C^0(\Omega)$ for $p > n$, where $1/q = 1/p - 1/n$. Thus Theorem A represents a sharp way to fill in the gap at the critical exponent $p = n$. Theorem B plays the same role for the Sobolev theorems on S^2.

Moser's work was extended in [Adams 1988; Fontana 1993; Nolasco and Tarantello 1998; Chang and Yang 1988; Ding et al. 1997]. Generally, the inequalities obtained by those mathematicians are also called Moser–Trudinger inequalities.

It is well known that Moses–Trudinger inequalities play an important role in the study of partial differential equations, especially those that arise in geometry and physics. There has been much work on such inequalities and their applications; see, for example, [Trudinger 1967; Cohn and Lu 2002; Carleson and Chang 1986; Chang 1996; Flucher 1992; Lin 1996; Jost and Wang 2001] and the references therein.

Li and Zhu [1997] established some sharp Sobolev trace inequalities on n-dimensional compact Riemannian manifolds with smooth boundaries. Recently, Liu generalized a result of Osgood, Phillips and Sarnak [Osgood et al. 1988]:

Theorem C [Liu 2002]. *Let (M, g) be a compact Riemannian surface with boundary ∂M, then there exists a constant C, which depends only on the geometry of M, such that for all $u \in H^{1,2}(M)$*

$$\log \int_{\partial M} e^u ds_g \le \frac{1}{4\pi} \int_M |\nabla u|^2 dV_g + \int_{\partial M} u\, ds_g + C. \tag{1–1}$$

The value $\dfrac{1}{4\pi}$ is sharp.

A strong version of (1–1) has also been obtained:

Theorem D [Li and Liu 2005]. *Let (M, g) be a compact Riemannian surface with boundary ∂M. Then*

$$\sup_{\substack{\int_M |\nabla u|^2\, dV_g = 1 \\ \int_{\partial M} u\, dS_g = 0}} \int_{\partial M} e^{\pi u^2} dS_g < +\infty, \tag{1–2}$$

and

$$\sup_{\substack{\int_M |\nabla u|^2 dV_g = 1 \\ \int_{\partial M} u\, dS_g = 0}} \int_{\partial M} e^{\alpha u^2} dS_g = +\infty$$

for any $\alpha > \pi$. *Moreover, there is a function* $u \in C^\infty(\bar{M})$ *which satisfies that* $\int_M |\nabla u|^2 dV_g = 1$, $\int_{\partial M} u = 0$, *and*

$$\int_{\partial M} e^{\pi u^2} dS_g = \sup_{\substack{\int_M |\nabla v|^2 dV_g = 1 \\ \int_{\partial M} v\, dS_g = 0}} \int_{\partial M} e^{\pi v^2} dS_g.$$

Theorems C and D are proved by blowing-up analysis, a method closely related to those used by Schoen [1984] in his solution of the Yamabe problem, Escobar and Schoen [1986] for finding conformal metrics with prescribed curvatures in higher dimensions, and Ding, Jost, Li and Wang [Ding et al. 1997] in their solution of the differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemannian surface.

In this paper we study some trace inequalities similar to (1–2). Let

$$\mathcal{H}_1 = \left\{ u \in H^{1,2}(M) : \int_M |\nabla u|^2 dV_g = 1, \int_M u\, dV_g = 0 \right\},$$
$$\mathcal{H}_2 = \left\{ u \in H^{1,2}(M) : \int_M (|\nabla u|^2 + u^2)\, dV_g = 1 \right\}.$$

Theorem 1.1. *Let* (M, g) *be a compact Riemannian surface with boundary* ∂M. *Then*

$$\sup_{u \in \mathcal{H}_1} \int_{\partial M} e^{\pi u^2} dS_g < +\infty$$

and $\sup_{u \in \mathcal{H}_1} \int_{\partial M} e^{\alpha u^2} dS_g = +\infty$ *for any* $\alpha > \pi$. *Moreover, there is a function* $u \in C^\infty(\bar{M}) \cap \mathcal{H}_1$ *such that*

$$\int_{\partial M} e^{\pi u^2} dS_g = \sup_{v \in \mathcal{H}_1} \int_{\partial M} e^{\pi v^2} dS_g. \tag{1–3}$$

Our method to prove Theorem 1.1 is similar to that of [Li and Liu 2005]. Precisely speaking, we divide the proof into two steps. Firstly, for any $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{H}_1$ be a maximizer of the functional

$$J_{\pi-\varepsilon}(u) = \int_{\partial M} e^{(\pi-\varepsilon)u^2} dS_g$$

on the space $\mathcal{H}_1$. Let G be a Green's function on M. Then G takes the form

$$G(x, p) = -\frac{1}{\pi} \log r(x) + A_p + O(r)$$

in a normal coordinate system around p, where $r(x) = \mathrm{dist}(x, p)$ and A_p is a constant. If the sequence $\{u_\varepsilon\}$ blows up, i.e.,

$$|u_\varepsilon|(x_\varepsilon) = \sup_{x \in M} |u_\varepsilon|(x) \to +\infty \quad \text{as} \quad \varepsilon \to 0,$$

we obtain

$$\sup_{u\in\mathcal{H}_1}\int_{\partial M} e^{\pi u^2}dS_g \le \operatorname{Vol}\partial M + 2\pi e^{\pi A_p}. \tag{1–4}$$

In the second step, we construct a blowing up sequence $\phi_\varepsilon \in \mathcal{H}_1$ such that

$$J_\pi(\phi_\varepsilon) = \int_{\partial M} e^{\pi\phi_\varepsilon^2}dS_g > \operatorname{Vol}\partial M + 2\pi e^{\pi A_p}$$

for sufficiently small ε. This contradicts step 1, and implies that blowing up cannot occur. The weak compactness of $L^p(M)$ $(p > 1)$ gives the existence of the extremal function, i.e., (1–3) holds.

It should be mentioned that x_ε lies on ∂M naturally in [Li and Liu 2005] because u_ε is a harmonic function there. But in our case, passing to any subsequence, we cannot assume $x_\varepsilon \in \partial M$ and $u_\varepsilon(x_\varepsilon) \to +\infty$ simultaneously. Also, in the second step, the blowing up sequence we constructed (see Section 5) is different from that of [Li and Liu 2005].

Using the same idea described above, we also obtain:

Theorem 1.2. *Let (M, g) be as in Theorem 1.1. Then*

$$\sup_{u\in\mathcal{H}_2}\int_{\partial M} e^{\pi u^2}dS_g < +\infty$$

and $\sup_{u\in\mathcal{H}_2}\int_{\partial M} e^{\alpha u^2}dS_g = +\infty$ for any $\alpha > \pi$. Moreover, there is a function $u \in C^\infty(\overline{M}) \cap \mathcal{H}_2$ such that

$$\int_{\partial M} e^{\pi u^2}dS_g = \sup_{u\in\mathcal{H}_2}\int_{\partial M} e^{\pi v^2}dS_g.$$

Clearly, Theorem C is a corollary of Theorem D. Similar results can also be derived from Theorems 1.1 and 1.2; for instance, we can substitute $\int_M u\,dV_g$ for $\int_{\partial M} u\,ds_g$, or $(1/4\pi)\,\|u\|_{H^{1,2}(M)}$ for $(1/4\pi)\int_M |\nabla u|^2 dV_g + \int_{\partial M} u\,ds_g$ in the right side of inequality (1–1). Theorems 1.1 and 1.2 are independent of Theorems C and D. They are more interesting than Theorem C because we obtain boundary estimates without direct boundary conditions.

For simplicity, we often omit the volume elements dV_g and dS_g when we write the integrals on M and ∂M respectively, and sometimes denote different constants by the same c. The reader can distinguish them easily from the context.

Most of the remainder of this paper is devoted to the proof of Theorem 1.1. In Section 2, we establish two regularity lemmas for use later. In Section 3, we prove that π is the best constant. And we derive an upper bound of $J_\pi(u)$ under the assumption that u_ε blows up in Section 4. A blowing up sequence ϕ_ε is constructed to reach a contradiction in Section 5, and this completes the proof of Theorem 1.1. In Section 6 we outline the proof of Theorem 1.2.

2. Regularity lemmas

Lemma 2.1. *Suppose $f \in L^q(M)$, $h \in H^{1,q}(M)$, $1<q<2$, and $2<p<2q/(2-q)$. let $u \in H^{1,2}(M)$ be a solution of the equation*

$$\begin{cases} \Delta u = f & \text{in } \mathring{M} \\ \dfrac{\partial u}{\partial n} = h & \text{on } \partial M, \end{cases}$$

where $\mathring{M}$ denotes the interior of M. Then u lies in $L^\infty(M)$ and we have

$$\|u\|_{L^\infty(M)} \le c\big(\|f\|_{L^q(M)} + \|h\|_{L^p(M)} + \|\nabla h\|_{L^q(M)} + \|u\|_{L^2(M)}\big),$$

where c is a constant depending only on M.

Proof. We use De Giorgi iteration. Choose a C^∞ vector field ζ whose restriction on ∂M is the outward unit normal vector field. By Stokes' theorem we have, for any $\varphi \in C^\infty(M)$,

$$\tag{2–1} \begin{aligned} -\int_M \nabla u \nabla \varphi &= \int_M f\varphi - \int_{\partial M} \varphi \frac{\partial u}{\partial n} = \int_M f\varphi - \int_M \operatorname{div}(\varphi h \zeta) \\ &= \int_M \big(f - h \operatorname{div}\zeta - \langle \zeta, \nabla h\rangle_g\big)\varphi - \int_M h\langle \zeta, \nabla\varphi\rangle_g \\ &\equiv \int_M f^0 \varphi - \int_M \langle \vec{h}, \nabla\varphi\rangle_g, \end{aligned}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the Riemannian inner product, $f^0 = f - h\operatorname{div}\zeta - \langle \zeta, \nabla h\rangle_g$, and $\vec{h} = h\zeta$. Clearly, $f^0 \in L^q(M)$ and $\vec{h} \in L^p(M)$.

For $0<k<+\infty$, define $v_k = (u-k)^+$, $M_k = \{x \in M : v_k(x) > 0\}$. By Hölder's inequality,

$$\tag{2–2} |M_k| \le \frac{\|u\|_{L^1(M)}}{k} \le \frac{|M|^{1/2}\|u\|_{L^2(M)}}{k},$$

where $|M_k|$ and $|M|$ represent the 2-dimensional measure of M_k and M respectively. Inserting $\varphi = v_k$ into (2–1), one has

$$\tag{2–3} \begin{aligned} \int_M |\nabla v_k|^2 &= \int_M \nabla u \nabla v_k = -\int_M f^0 v_k + \int_M \langle \vec{h}, \nabla v_k\rangle_g \\ &\le \Big(\int_M (f^0)^q\Big)^{1/q}\Big(\int_M v_k^{q'}\Big)^{1/q'} + \Big(\int_{M_k} |\vec{h}|^2\Big)^{1/2}\Big(\int_M |\nabla v_k|^2\Big)^{1/2}, \end{aligned}$$

where $|\cdot| = \sqrt{\langle\,\cdot\,,\,\cdot\,\rangle}$ and $1/q' + 1/q = 1$. Since $1 < q < 2$, we have $q' > 2$. Choose r sufficiently large that $1/q' - 1/r > 1/2 - 1/p$ and $r(1/2 - 1/p) > 1$. By the Sobolev imbedding theorem,

$$\|v_k\|_{L^r(M)} \le c(\|v_k\|_{L^2(M)} + \|\nabla v_k\|_{L^2(M)}) \le c(\|v_k\|_{L^r(M)}|M_k|^{1/2-1/r} + \|\nabla v_k\|_{L^2(M)}),$$

where c is a constant depending only on M.

Without loss of generality we assume that $\|u\|_{L^2(M)} = 1$. According to (2–2), there exists a large integer number k_0 such that $c|M_k|^{1/2-1/r} < 1$ for $k > k_0$. Hence

$$\|v_k\|_{L^r(M)} \le c\|\nabla v_k\|_{L^2(M)} \quad \text{for} \quad k \ge k_0. \tag{2–4}$$

By (2–3), we have

$$\begin{aligned}\|\nabla v_k\|^2_{L^2(M)} &\le \|f^0\|_{L^q(M)}\|v_k\|_{L^r(M)}|M_k|^{1/q'-1/r} + \|\vec{h}\|_{L^2(M_k)}\|\nabla v_k\|_{L^2(M)} \\ &\le c\|f^0\|_{L^q(M)}|M_k|^{1/q'-1/r}\|\nabla v_k\|_{L^2(M)} + \|\vec{h}\|_{L^2(M_k)}\|\nabla v_k\|_{L^2(M)},\end{aligned}$$

which gives

$$\|\nabla v_k\|_{L^2(M)} \le c\|f^0\|_{L^q(M)}|M_k|^{1/q'-1/r} + \|\vec{h}\|_{L^p(M)}|M_k|^{1/2-1/p}.$$

Note that $1/q' - 1/r > 1/2 - 1/p$. We have

$$\|\nabla v_k\|_{L^2(M)} \le c\tau|M_k|^{1/2-1/p}, \tag{2–5}$$

where $\tau = \|f^0\|_{L^q(M)} + \|\vec{h}\|_{L^p(M)}$.

On the other hand, for $h > k$, we have

$$\int_M v_k^r \ge \int_{M_h} (u-k)^r \ge |M_h|(h-k)^r.$$

Combining this with (2–4) and (2–5), we get $|M_h| \le K(h-k)^{-r}|M_k|^\beta$, with $K \equiv \tilde{c}\tau^r$ for some constant $\tilde{c}$, $\beta \equiv (1/2 - 1/p)r > 1$, and $k_0 < k < h < h_1 < +\infty$ for any sufficiently large h_1. By [Troianiello 1987, Lemma 2.9], $|M_{k_0+\hat{k}}| = 0$ for some $\hat{k} > 0$; that is, $u \le k_0 + \hat{k}$ in M. With the same argument, one can deduce that $-u \le k_0 + \hat{k}$ in M. □

Theorem 3.17 of [Troianiello 1987] yields an immediate consequence:

Lemma 2.2. *Suppose that $f \in L^p(M)$ and $h \in H^{1,p}(M)$ for some $p \ge 2$, and that $u \in H^{1,2}(M)$ is a solution of*

$$\begin{cases} \Delta u = f & \text{in } \mathring{M} \\ \dfrac{\partial u}{\partial n} = h & \text{on } \partial M. \end{cases}$$

Then $u \in H^{2,p}(M)$.

3. The best constants

We now prove that the best constant in Theorem 1.1 is π. Here best means that

$$\sup_{u\in\mathcal{H}_1}\int_{\partial M} e^{\alpha u^2} < +\infty \quad \text{for } \alpha < \pi,$$
$$\sup_{u\in\mathcal{H}_1}\int_{\partial M} e^{\alpha u^2} = +\infty \quad \text{for } \alpha > \pi.$$

The following lemma is well known:

Lemma 3.1. *Let M be a compact Riemannian surface with boundary. Then there exists a positive number α such that* $\sup_{u\in\mathcal{H}_1}\int_M e^{\alpha u^2} < \infty$.

Lemma 3.2. *Set* $\alpha_2 = \sup\{\alpha : \sup_{u\in\mathcal{H}_1}\int_M e^{\alpha u^2} < +\infty\}$. *Then* $\alpha_2 = 2\pi$.

Proof. Step 1. We first prove that $\alpha_2 \ge 2\pi$.

Suppose $\alpha_2 < 2\pi$. There exists a sequence $u_\varepsilon \in \mathcal{H}_1$ such that $\int_M e^{(\alpha_2+\varepsilon)u_\varepsilon^2} \to +\infty$ as $\varepsilon \to 0$. One can see that there exists a $p \in M$ such that for any $r > 0$,

$$\int_{B_r(p)} e^{(\alpha_2+\varepsilon)u_\varepsilon^2} \to +\infty \quad \text{as} \quad \varepsilon \to 0, \tag{3–1}$$

where $B_r(p)$ is a geodesic ball centered at p with radius r. For otherwise, using a covering argument, one has $\int_M e^{(\alpha_2+\varepsilon)u_\varepsilon^2} \le c$ for ε small enough, which contradicts the definition of u_ε. By the Poincaré inequality, $\{u_\varepsilon\}$ is bounded in $H^{1,2}(M)$, and so is $\{|u_\varepsilon|\}$. Hence there is $u \in H^{1,2}(M)$ such that $|u_\varepsilon| \rightharpoonup u$ (weak convergence) in $H^{1,2}(M)$ and $|u_\varepsilon| \to u$ (strong convergence) in $L^2(M)$ as $\varepsilon \to 0$. For any $\eta > 0$, we claim that

$$\lim_{\varepsilon\to 0}\int_M |\nabla(|u_\varepsilon| - \eta)^+|^2 = 1, \tag{3–2}$$

where $(|u_\varepsilon| - \eta)^+$ is the positive part of $|u_\varepsilon| - \eta$. Suppose (3–2) does not hold. Clearly, $\liminf_{\varepsilon\to 0}\int_M |\nabla(|u_\varepsilon| - \eta)^+|^2 < 1$. By the definition of α_2, passing to a subsequence, we can choose $\alpha' > \alpha_2$ such that

$$\int_M \exp\left(\alpha'\left((|u_\varepsilon| - \eta)^+ - \frac{1}{\operatorname{Vol} M}\int_M (|u_\varepsilon| - \eta)^+\right)^2\right) \le c$$

for sufficiently small ε. Using the Poincaré inequality and the inequality $ab \le \delta a^2 + b^2/(4\delta)$ for any $\delta > 0$, we can choose some $\varepsilon' > 0$ such that $\alpha'/(1+\varepsilon') > \alpha_2$ and $\int_M e^{\alpha' u_\varepsilon^2/(1+\varepsilon')} \le c$, which contradicts (3–1) for ε small enough, and implies (3–2).

Let $v_\varepsilon = \min\{|u_\varepsilon|, \eta\}$. Then v_ε is bounded in $H^{1,2}(M)$. So there exists $v \in H^{1,2}(M)$ such that $v_\varepsilon \rightharpoonup v$ weakly in $H^{1,2}(M)$ and $v_\varepsilon \to v$ strongly in $L^2(M)$. Obviously,

$$|u_\varepsilon| = v_\varepsilon + (|u_\varepsilon| - \eta)^+. \tag{3–3}$$

Note that

$$1=\int_M |\nabla u_\varepsilon|^2 \ge \int_M \big|\nabla |u_\varepsilon|\big|^2 = \int_M |\nabla v_\varepsilon|^2 + \int_M \big|\nabla (|u_\varepsilon|-\eta)^+\big|^2.$$

By (3–2), we have $\int_M |\nabla v_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$. By the Poincaré lemma, $\int_M |v_\varepsilon - \bar v_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$, where $\bar v_\varepsilon = (\mathrm{Vol}\, M)^{-1}\int_M v_\varepsilon$. Note that $v_\varepsilon \to v$ strongly in $L^2(M)$. One has $v = \bar v$ almost everywhere in M. From (3–3), we know that

$$u = v + (u-\eta)^+ \quad \text{a.e. in } M.$$

By an appropriate choice of η, one easily derives that $v = 0$ and $u = 0$ a.e. in M.

Recall that $|u_\varepsilon| \to u$ strongly in $L^2(M)$. One has

$$|u_\varepsilon| \to 0 \quad \text{strongly in } L^2(M) \quad \text{as } \varepsilon \to 0. \tag{3–4}$$

Now we turn to (3–1). Take $p \in \partial M$. Choose an isothermal coordinate system (U, ψ) around p such that $\psi : U \to \mathbb{B}^+_{2r}$. Choose a cut-off function $\varphi \in C^\infty(M)$ such that $\varphi \equiv 1$ on $B_r(p)$ and $\varphi \equiv 0$ outside $B_{4r/3}(p)$. By (3–4), we have

$$\int_{B_r(p)} |\nabla(\eta u_\varepsilon)|^2 \le \int_M |\nabla(\eta u_\varepsilon)|^2 \le 1+\varepsilon''$$

for some $\varepsilon'' > 0$ with $2\pi/(1+\varepsilon'') > \alpha_2$, provided that ε is sufficiently small. Define

$$\tilde u_\varepsilon(s,t) = \begin{cases} (\eta u_\varepsilon)(s,t) & \text{for } t \ge 0,\\ (\eta u_\varepsilon)(s,-t) & \text{for } t < 0.\end{cases}$$

Then $\int_{\mathbb{B}_{2r}} |\nabla \tilde u_\varepsilon|^2\, ds\, dt \le 2+2\varepsilon''$. By Moser's inequality, we then obtain the bound $\int_{\mathbb{B}_{2r}} e^{4\pi \tilde u_\varepsilon^2/(2+2\varepsilon')}\, ds\, dt \le c$. Hence

$$\int_{B_r(p)} e^{2\pi u_\varepsilon^2/(1+\varepsilon')} \le 2\int_{\mathbb{B}_{2r}} e^{4\pi \tilde u_\varepsilon^2/(2+2\varepsilon')}\, ds\, dt \le 2c \tag{3–5}$$

for sufficiently small r. This contradicts (3–1).

When p is an interior point in M, one can get a contradiction as above without any difficulty. In this case, $\tilde u_\varepsilon$ is not needed any more; one need only consider u_ε itself. This completes the proof of step 1.

Step 2. To prove the opposite inequality, $\alpha_2 \le 2\pi$, take any $p \in \partial M$ and choose an isothermal coordinate system around p. Set

$$u_\varepsilon = \begin{cases} -\sqrt{\dfrac{1}{2\pi}\log\dfrac{1}{\varepsilon}} & \text{in } B_{\delta\sqrt{\varepsilon}}(p),\\[2mm] \dfrac{\sqrt{2}}{\sqrt{-\pi\log\varepsilon}}\log\dfrac{r}{\delta} & \text{in } B_\delta(p)\setminus B_{\delta\sqrt{\varepsilon}}(p),\\[2mm] C_\varepsilon \varphi & \text{in } M\setminus B_\delta(p),\end{cases} \tag{3–6}$$

where $\varphi \in C_0^\infty(M\backslash B_\delta(p))$, $0 \le \varphi \le 1$, and C_ε is chosen to satisfy $\int_M u_\varepsilon = 0$. It is easy to check that

$$\int_M |\nabla u_\varepsilon|^2 \to 1 \quad \text{as} \quad \varepsilon \to 0$$

and that

$$\int_M \exp\biggl(\alpha\Bigl(\frac{u_\varepsilon}{\|u_\varepsilon\|_{L^2}}\Bigr)^2\biggr) \ge \exp\biggl(\frac{\alpha}{2\pi\|\nabla u_\varepsilon\|_{L^2}^2}\log\frac{1}{\varepsilon}\biggr)\operatorname{Vol} B_{\delta\sqrt{\varepsilon}} \ge C\varepsilon^{1-\alpha/(2\pi\|\nabla u_\varepsilon\|_{L^2}^2)}$$

for any $\alpha > 2\pi$; the latter lower bound approaches $+\infty$ as $\varepsilon \to 0$. Therefore $\alpha_2 \le 2\pi$. □

Lemma 3.3. *Set $J_\alpha(u) = \int_{\partial M} e^{\alpha u^2}$. Then*

$$\sup_{u\in\mathcal{H}_1} J_\alpha(u) < +\infty \quad \textit{for } \alpha < \pi \quad \textit{and} \quad \sup_{u\in\mathcal{H}_1} J_\alpha(u) = +\infty \quad \textit{for } \alpha > \pi.$$

Proof. Take a smooth vector field ζ whose restriction on ∂M is the outward unit normal vector field. Using the divergence theorem and Lemma 3.2, one has

$$\begin{aligned}\int_{\partial M} e^{(\pi-\varepsilon)u^2} &= \int_M \operatorname{div}(\zeta e^{(\pi-\varepsilon)u^2}) = \int_M \bigl(\operatorname{div}(\zeta) + 2(\pi-\varepsilon)u\langle\zeta, \nabla u\rangle_g\bigr)e^{(\pi-\varepsilon)u^2}\\ &\le C\biggl(1+\int_M |\nabla u||u|e^{(\pi-\varepsilon)u^2}\biggr)\\ &\le C\Bigl(1+\|\nabla u\|_{L^2(M)}\|u\|_{L^p(M)}\|e^{(\pi-\varepsilon)u^2}\|_{L^{(2\pi-\varepsilon)/(\pi-\varepsilon)}(M)}\Bigr)\end{aligned}$$

for all $u \in \mathcal{H}_1$, where $1/p + 1/2 + (\pi-\varepsilon)/(2\pi-\varepsilon) = 1$. Combining this estimate with the Sobolev imbedding theorem, one has $\sup_{u\in\mathcal{H}_1} J_{\pi-\varepsilon}(u) < +\infty$ for any $\varepsilon > 0$, which implies that $\sup_{u\in\mathcal{H}_1} J_\alpha(u) < +\infty$ for any $\alpha < \pi$.

To complete the proof of the lemma, we employ (3–6) to check that for any $\alpha > \pi$, $J_\alpha(u_\varepsilon)$ diverges to $+\infty$ as $\varepsilon \to 0$. □

4. Blowing up analysis

We now use the method of blowing up to prove (1–4). The same method has also been used in [Li 2001; Li 2005].

The proof consists of several lemmas.

Lemma 4.1. *The functional $J_{\pi-\varepsilon}(u)$ defined in the space $\mathcal{H}_1$ admits a smooth maximizer $u_\varepsilon \in \mathcal{H}_1$.*

Proof. It is obvious that there exists $u_\varepsilon \in \mathcal{H}_1$ such that

$$J_{\pi-\varepsilon}(u_\varepsilon) = \sup_{u\in\mathcal{H}_1} J_{\pi-\varepsilon}(u).$$

The function u_ε satisfies the Euler–Lagrange equation

$$\text{(4–1)}\qquad \begin{cases} \Delta u_\varepsilon = \dfrac{\mu_\varepsilon}{2\lambda_\varepsilon} & \text{in } \mathring{M}, \\ \dfrac{\partial u_\varepsilon}{\partial n} = \dfrac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} & \text{on } \partial M, \end{cases}$$

where

$$\text{(4–2)}\qquad \lambda_\varepsilon = (\pi-\varepsilon)\int_{\partial M} u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2} \quad \text{and} \quad \mu_\varepsilon = \frac{2(\pi-\varepsilon)}{\operatorname{Vol} M}\int_{\partial M} u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2}.$$

Write $h(u_\varepsilon) = (\pi-\varepsilon)/\lambda_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2}$. By the Orlicz space imbedding (see [Struwe 1988]), $e^{u_\varepsilon^2} \in L^p(M)$ for any $p > 0$. Hence $h(u_\varepsilon) \in H^{1,q}(M)$ for any $1 < q < 2$. By Lemma 2.1 we have $u_\varepsilon \in L^\infty(M)$, hence $h(u_\varepsilon) \in H^{1,2}(M)$. By Lemma 2.2, $u_\varepsilon \in H^{2,2}(M)$. The Sobolev imbedding theorem then implies that $h(u_\varepsilon) \in H^{1,p}(M)$ for some $p > 2$. Again, by Lemma 2.2, $u_\varepsilon \in H^{2,p}(M)$. The Sobolev imbedding theorem gives $u_\varepsilon \in C^1(M)$. Using Lemma 2.2 repeatedly, we conclude that $u_\varepsilon \in C^\infty(M)$. □

Lemma 4.2. $\liminf_{\varepsilon\to 0} \lambda_\varepsilon > 0$.

Proof. The following estimate is elementary

$$\operatorname{Vol}\partial M < \sup_{u\in\mathcal{H}_1}\int_{\partial M} e^{\pi u^2} = \lim_{\varepsilon\to 0}\int_{\partial M} e^{(\pi-\varepsilon)u_\varepsilon^2} \le \operatorname{Vol}\partial M + \liminf_{\varepsilon\to 0}\lambda_\varepsilon,$$

which gives $\liminf_{\varepsilon\to 0}\lambda_\varepsilon > 0$. □

Lemma 4.3. *$\mu_\varepsilon/\lambda_\varepsilon$ is bounded with respect to ε.*

Proof. By (4–2) and Lemma 4.2, we have

$$\frac{|\mu_\varepsilon|}{\lambda_\varepsilon} \le \frac{2(\pi-\varepsilon)}{\operatorname{Vol} M}\int_{\partial M}\frac{|u_\varepsilon|}{\lambda_\varepsilon} e^{(\pi-\varepsilon)u_\varepsilon^2} \le \frac{2(\pi-\varepsilon)}{\operatorname{Vol} M}\left(\frac{e^{\pi-\varepsilon}}{\lambda_\varepsilon} + \frac{1}{\pi-\varepsilon}\right) \le C. \qquad \square$$

Write $c_\varepsilon = |u_\varepsilon|(x_\varepsilon) = \max_{x\in M}(x)$. If $\{c_\varepsilon\}$ is bounded, then by the standard elliptic estimate with respect to Equation (4–1), there exists $u \in \mathcal{H}_1 \cap C^\infty(M)$ such that $u_\varepsilon \to u$ in $C^\infty(M)$ as $\varepsilon \to 0$, and Theorem 1.1 follows immediately. Henceforth we assume $c_\varepsilon \to +\infty$ as $\varepsilon \to 0$.

Passing to a subsequence, we may assume that $\mu_\varepsilon \ge 0$ for all $\varepsilon > 0$, for otherwise we consider $-u_\varepsilon$ instead of u_ε in (4–1)–(4–2). We consider separately the possibilities that $\{u_\varepsilon(x_\varepsilon)\}$ approaches $+\infty$ or $-\infty$ or as $\varepsilon \to 0$.

Take first the case $u_\varepsilon(x_\varepsilon) \to +\infty$. Applying the maximum principle to (4–1), we see that $x_\varepsilon \in \partial M$. Passing to a subsequence, we may assume $x_\varepsilon \to p$ for some $p \in \partial M$.

Lemma 4.4. *Define*

$$r_\varepsilon = \frac{1}{\pi-\varepsilon}\frac{\lambda_\varepsilon}{c_\varepsilon^2} e^{-(\pi-\varepsilon)c_\varepsilon^2}. \tag{4–3}$$

Then $r_\varepsilon c_\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof. By the first equality in (4–2), we have

$$1 = \frac{\pi-\varepsilon}{\lambda_\varepsilon}\int_{\partial M} u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2} \le \frac{\pi-\varepsilon}{\lambda_\varepsilon} e^{\pi c_\varepsilon^2}\int_{\partial M} u_\varepsilon^2 \le c\frac{\pi-\varepsilon}{\lambda_\varepsilon} e^{\pi c_\varepsilon^2}$$

for some constant c, where we have used the Sobolev trace imbedding theorem. This implies that $r_\varepsilon c_\varepsilon \to 0$ as $\varepsilon \to 0$. □

Choose an isothermal coordinate system (U, ϕ) near p such that $\phi(p) = 0$, ϕ maps U to $\mathbb{R}^2_+ := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ and $\phi(U \cap \partial M) \subset \partial\mathbb{R}^2_+$.

Set

$$\psi_\varepsilon(x) = u_\varepsilon(x_\varepsilon + r_\varepsilon x)/c_\varepsilon, \qquad \varphi_\varepsilon(x) = c_\varepsilon\big(u_\varepsilon(x_\varepsilon + r_\varepsilon x) - c_\varepsilon\big). \tag{4–4}$$

Lemma 4.5. $\psi_\varepsilon \to 1$ *in* $C^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ *as* $\varepsilon \to 0$.

Proof. By (4–1), for ε is sufficiently small we have

$$\begin{cases} \Delta\psi_\varepsilon = \dfrac{r_\varepsilon^2}{c_\varepsilon}\dfrac{\mu_\varepsilon}{2\lambda_\varepsilon} & \text{in } B_R^+(0), \\[2ex] \dfrac{\partial\psi_\varepsilon}{\partial n} = \dfrac{r_\varepsilon}{c_\varepsilon}\dfrac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} & \text{on } B_R(0)\cap\partial\mathbb{R}^2_+, \end{cases}$$

for any $R > 0$. As in the proof of Lemma 4.1, it is not hard to see that $\psi_\varepsilon \to 1$ in $C^2(\overline{B^+_{R/2}(0)})$ as $\varepsilon \to 0$. □

Lemma 4.6. *The functions* φ_ε *converge in* $C^2_{\mathrm{loc}}(\overline{\mathbb{R}^2_+})$ *as* $\varepsilon \to 0$ *to some* φ *satisfying*

$$\begin{cases} -\Delta_{\mathbb{R}^2}\varphi = 0 & \text{in } \mathbb{R}^2_+ \\[1ex] \dfrac{\partial\varphi}{\partial n} = e^{2\pi\varphi} & \text{on } \partial\mathbb{R}^2_+, \\[1ex] \varphi(0) = \sup\varphi = 0. \end{cases}$$

Proof. By (4–1), we have

$$\begin{cases} \Delta\varphi_\varepsilon(x) = c_\varepsilon r_\varepsilon^2\dfrac{\mu_\varepsilon}{2\lambda_\varepsilon} & \text{in } B_R^+(0), \\[2ex] \dfrac{\partial\varphi_\varepsilon}{\partial n} = \dfrac{u_\varepsilon}{c_\varepsilon}\exp\Big((\pi-\varepsilon)\varphi_\varepsilon\Big(1+\dfrac{u_\varepsilon}{c_\varepsilon}\Big)\Big) & \text{on } \partial\mathbb{R}^2_+\cap B_R(0) \end{cases}$$

for any $R > 0$. Using Lemma 2.2, we have $\varphi_\varepsilon \to \varphi$ in $C^2(\overline{B^+_{R/2}(0)})$ as $\varepsilon \to 0$ for some $u \in C^2(\overline{B^+_{R/2}(0)})$. Clearly u satisfies the required conditions. □

It is not difficult to see that

$$\int_{B_R(0)\cap\partial\mathbb{R}^2_+} e^{2\pi\varphi} \le \liminf_{\varepsilon\to 0} \int_{B_{Rr_\varepsilon}(x_\varepsilon)\cap\partial M} \frac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2} \le 1,$$

which gives

$$\int_{\partial\mathbb{R}^2_+} e^{2\pi\varphi} \le 1.$$

By a result in [Li and Zhu 1995], we have

$$\varphi(x) = -\frac{1}{2\pi}\log\bigl(\pi^2 x_1^2 + (1+\pi x_2)^2\bigr).$$

A direct calculation gives

$$\int_{\partial\mathbb{R}^2_+} e^{2\pi\varphi} = 1.$$

Following [Li and Liu 2005], we define $u_\varepsilon^c = \min\{\frac{c_\varepsilon}{c}, u_\varepsilon\}$.

Lemma 4.7. *For any $c > 1$, we have* $\lim_{\varepsilon\to 0}\int_M |\nabla u_\varepsilon^c|^2 = \frac{1}{c}$.

Proof. Using Stokes' formula, (4–1) and Lemma 4.5, we have

$$\begin{aligned}
\int_M \Bigl|\nabla\Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+\Bigr|^2 &= \int_M \nabla u_\varepsilon \nabla\Bigl(\Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+\Bigr)\\
&= \int_{\partial M}\Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+ \frac{\partial u_\varepsilon}{\partial n} - \int_M \Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+ \Delta u_\varepsilon\\
&= \int_{\partial M}\Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+ \frac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} - \int_M \Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+ \frac{\mu_\varepsilon}{2\lambda_\varepsilon}\\
&\ge \int_{\partial M\cap B_{Rr_\varepsilon(x_\varepsilon)}} \bigl(u_\varepsilon - c_\varepsilon/c\bigr)^+ \frac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} + o_\varepsilon(1)\\
&= \frac{c-1}{c}\int_{\partial\mathbb{R}^{2+}\cap B_R(0)} e^{2\pi\varphi} + o_\varepsilon(R) + o_\varepsilon(1),
\end{aligned}$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$, and $o_\varepsilon(R)\to 0$ for any fixed R as $\varepsilon\to 0$. Letting $\varepsilon\to 0$ first, and then $R\to+\infty$, we obtain

$$\liminf_{\varepsilon\to 0}\int_M \Bigl|\nabla\Bigl(u_\varepsilon - \frac{c_\varepsilon}{c}\Bigr)^+\Bigr|^2 \ge \frac{c-1}{c}.$$

With the same argument, we get

$$\liminf_{\varepsilon\to 0}\int_M |\nabla u_\varepsilon^c|^2 \ge \frac{1}{c}.$$

Note that since
$$\int_M \left|\nabla\left(u_\varepsilon - \frac{c_\varepsilon}{c}\right)^+\right|^2 + \int_M |\nabla u_\varepsilon^c|^2 = 1,$$
we have $\liminf_{\varepsilon\to 0}\int_M |\nabla u_\varepsilon^c|^2 = c^{-1}$. □

Lemma 4.8. *Under the assumption that $c_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we have the estimate*
$$\sup_{u\in\mathcal{H}_1} J_\pi(u) \le \operatorname{Vol}\partial M + \frac{1}{\pi}\limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}.$$

Proof. For any $c > 1$, we have
$$\begin{aligned}\int_{\partial M} e^{(\pi-\varepsilon)u_\varepsilon^2} &= \int_{\partial M\cap\{u_\varepsilon\le c_\varepsilon/c\}} e^{(\pi-\varepsilon)u_\varepsilon^2} + \int_{\partial M\cap\{u_\varepsilon> c_\varepsilon/c\}} e^{(\pi-\varepsilon)u_\varepsilon^2}\\ &\le \int_{\partial M} e^{(\pi-\varepsilon)(u_\varepsilon^c)^2} + c^2\frac{\lambda_\varepsilon}{c_\varepsilon^2}\int_{\partial M}\frac{u_\varepsilon^2}{\lambda_\varepsilon}e^{(\pi-\varepsilon)u_\varepsilon^2}.\end{aligned}$$
By Lemma 4.7, according to step 1 in the proof of Lemma 3.2, one can see that $u_\varepsilon^c \to 0$ a.e. in M as $\varepsilon \to 0$. Substituting u_ε^c for u in (3–5), one immediately has
$$\int_{\partial M} e^{(\pi-\varepsilon)(u_\varepsilon^c)^2} \to \operatorname{Vol}\partial M \quad\text{as}\quad \varepsilon\to 0.$$
Hence
$$\sup_{u\in\mathcal{H}_1} J_\pi(u) = \lim_{\varepsilon\to 0}\int_{\partial M} e^{(\pi-\varepsilon)u_\varepsilon^2} \le \operatorname{Vol}\partial M + \frac{c^2}{\pi}\limsup_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon}.$$
Letting $c \to 1$, the conclusion of the lemma follows. □

The next result is an immediate consequence of Lemma 4.8:

Corollary 4.9. $\lambda_\varepsilon/c_\varepsilon \to +\infty$ *as* $\varepsilon \to 0$.

Lemma 4.10. *For any $\phi \in C^\infty(\partial M)$, we have*

$$\lim_{\varepsilon\to 0}\int_{\partial M}\phi\,\frac{\pi-\varepsilon}{\lambda_\varepsilon}c_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} = \phi(p). \tag{4–5}$$

Proof. For any fixed $c > 1$, we partition ∂M into its intersections with
$$D_1 = \left(\left\{u_\varepsilon > \frac{c_\varepsilon}{c}\right\}\setminus B_{Rr_\varepsilon}(x_\varepsilon)\right),\quad D_2 = \left(\left\{u_\varepsilon \le \frac{c_\varepsilon}{c}\right\}\setminus B_{Rr_\varepsilon}(x_\varepsilon)\right),\quad D_3 = B_{Rr_\varepsilon}(x_\varepsilon).$$
Denote by I_1, I_2, I_3 the partial integrals in (4–5) taken over D_1, D_2, D_3. Then
$$\begin{aligned}|I_1| &\le c\sup_{\partial M}|\varphi|\int_{\partial M\cap(\{u_\varepsilon>\frac{c_\varepsilon}{c}\}\setminus B_{Rr_\varepsilon}(x_\varepsilon))}\frac{\pi-\varepsilon}{\lambda_\varepsilon}u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2}\\ &\le c\sup_{\partial M}|\varphi|\left(1-\int_{\partial M\cap B_{Rr_\varepsilon}(x_\varepsilon)}\frac{\pi-\varepsilon}{\lambda_\varepsilon}u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2}\right)\\ &\le c\sup_{\partial M}|\varphi|\left(1-\int_{\partial B_R^+(0)\cap\partial\mathbb{R}^2_+} e^{2\pi\varphi} + o_\varepsilon(R)\right),\end{aligned}$$

where $o_\varepsilon(R)\to 0$ as $\varepsilon\to 0$ for any fixed R. Letting $\varepsilon\to 0$ first, and then $R\to+\infty$, one has $I_1\to 0$. Next,

$$\begin{aligned}|I_2| &\le (\pi-\varepsilon)\sup_{\partial M}|\varphi|\frac{c_\varepsilon}{\lambda_\varepsilon}\int_{\partial M}|u_\varepsilon|e^{(\pi-\varepsilon)(u_\varepsilon^c)^2}\\ &\le \pi\sup_{\partial M}|\varphi|\frac{c_\varepsilon}{\lambda_\varepsilon}\|u_\varepsilon\|_{L^{(c+1)/(c-1)}(\partial M)}\|e^{(\pi-\varepsilon)(u_\varepsilon^c)^2}\|_{L^{(c+1)/2}(\partial M)}\\ &\le \tilde{C}\sup_{\partial M}|\varphi|\frac{c_\varepsilon}{\lambda_\varepsilon},\end{aligned}$$

where $\tilde{C}$ is a constant depending on M and c, here we have used Hölder's inequality and Sobolev imbedding theorem. By Corollary 4.9, we get $I_2\to 0$ as $\varepsilon\to 0$. Finally,

$$\begin{aligned}I_3 &= \int_{\partial M\cap B_{Rr_\varepsilon}(x_\varepsilon)}\varphi\,\frac{\pi-\varepsilon}{\lambda_\varepsilon}\,c_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} = \int_{\partial\mathbb{R}^2_+\cap\partial B_R^+(0)}\varphi\,\frac{u_\varepsilon}{c_\varepsilon}\,e^{(\pi-\varepsilon)\varphi_\varepsilon(1+u_\varepsilon/c_\varepsilon)}\\ &= \varphi(p)\Big(\int_{\partial B_R^+(0)\cap\partial\mathbb{R}^2_+}e^{2\pi\varphi}+o_\varepsilon(R)\Big).\end{aligned}$$

As before, letting $\varepsilon\to 0$ first, then $R\to+\infty$, we get $I_3\to\varphi(p)$. Combining all three estimates, we get the conclusion of the lemma. □

Lemma 4.11. $|\nabla u_\varepsilon|^2\rightharpoonup\delta_p$ *weakly in the sense of measure.*

Proof. Set

$$A=\Big\{q\in M:\lim_{r\to 0}\liminf_{\varepsilon\to 0}\int_{B_r(q)}|\nabla u_\varepsilon|^2>0\Big\}.$$

We claim that A contains only one point.

Suppose not. Then, for any $q\in M$, we have $\lim_{r\to 0}\liminf_{\varepsilon\to 0}\int_{B_r(q)}|\nabla u_\varepsilon|^2<1$. There exist positive numbers r and δ such that

$$\int_{B_r(q)}|\nabla u_\varepsilon|^2\le\delta(q)<1.$$

With the assumption $c_\varepsilon\to+\infty$ as $\varepsilon\to 0$, step 1 in the proof of Lemma 3.2 implies that $u_\varepsilon\to 0$ in $L^2(M)$, and hence $\int_{B_r(q)}u_\varepsilon\to 0$ as $\varepsilon\to 0$. It is not difficult to see that there exists a constant $\alpha(q)>\pi$ such that

$$\int_{\partial M\cap\partial B_r(q)}e^{\alpha(q)u_\varepsilon^2}\le C_q$$

for some constant C_q depending on q. By a covering argument, there exists an $\alpha>\pi$ such that

$$\int_{\partial M}e^{\alpha u_\varepsilon^2}\le C$$

for some constant C. This contradicts the choice of u_ε, and our claim follows.

Next we claim that $A = \{p\}$. Let q be the unique point in A, and suppose $q \neq p$. Choose a smooth function ψ such that $\psi(p) \neq \psi(q)$. By Stokes' theorem and Equation (4–1), we have

$$\int_M \psi |\nabla u_\varepsilon|^2 = \int_{\partial M} \psi \frac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2} - \int_M u_\varepsilon \frac{\mu_\varepsilon}{2\lambda_\varepsilon} - \int_M u_\varepsilon \nabla\psi \nabla u_\varepsilon.$$

Clearly the last two terms here tend to 0 as $\varepsilon \to 0$. As in the proof of Lemma 4.10, we can show that

$$\lim_{\varepsilon\to 0} \int_{\partial M} \psi \frac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2} = \psi(p).$$

On the other hand, $\lim_{\varepsilon\to 0} \int_M \psi|\nabla u_\varepsilon|^2 = \psi(q)$. Hence $\psi(p) = \psi(q)$, which contradicts the choice of ψ. This completes the proof of the lemma. □

Lemma 4.12. *$c_\varepsilon u_\varepsilon \rightharpoonup G$ weakly in $H^{1,q}(M)$ for any $q : 1 < q < 2$. For any $\Omega \Subset M \setminus \{p\}$, we have $c_\varepsilon u_\varepsilon \to G$ in $C^\infty(\overline{\Omega})$, where G satisfies*

$$\begin{cases} -\Delta G = \delta_p - \dfrac{1}{\operatorname{Vol} M} & \text{in } M, \\ \displaystyle\int_M G = 0, \quad \dfrac{\partial G}{\partial n}\Big|_{\partial M\setminus\{p\}} = 0. \end{cases} \tag{4–6}$$

Proof. By Equation (4–1), we have

$$\begin{cases} \Delta(c_\varepsilon u_\varepsilon) = c_\varepsilon \dfrac{\mu_\varepsilon}{2\lambda_\varepsilon} & \text{in } M, \\ \dfrac{\partial(c_\varepsilon u_\varepsilon)}{\partial n} = \dfrac{\pi-\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} & \text{on } \partial M. \end{cases}$$

Integrating both sides on M, one has

$$\int_M c_\varepsilon \frac{\mu_\varepsilon}{2\lambda_\varepsilon} = \int_M \Delta(c_\varepsilon u_\varepsilon) = \int_{\partial M} \frac{\pi-\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2}.$$

By Lemma 4.10, we immediately get $c_\varepsilon \mu_\varepsilon/(2\lambda_\varepsilon) \to 1/\operatorname{Vol} M$ as $\varepsilon \to 0$.

For any q in the range $1 < q < 2$, denote its conjugate by q', so $1/q + 1/q' = 1$. It is well known that

$$\int_M |\nabla(c_\varepsilon u_\varepsilon)|^q \leq \sup\left\{ \int_M \nabla\phi \nabla(c_\varepsilon u_\varepsilon)\, dV_g : \|\phi\|_{H^{1,q'}} = 1 \right\}.$$

The Sobolev embedding theorem yields $\|\phi\|_{C^0(M)} \le C$, where C is a constant depending only on M. Using the divergence theorem and (4–1), we have

$$\begin{aligned}\int_M \nabla\phi\,\nabla(c_\varepsilon u_\varepsilon) &= \int_{\partial M} \phi\,\frac{\partial(c_\varepsilon u_\varepsilon)}{\partial n} - \int_M \phi\Delta(c_\varepsilon u_\varepsilon)\\ &= \int_{\partial M} \phi\frac{\pi-\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} - c_\varepsilon\frac{\mu_\varepsilon}{2\lambda_\varepsilon}\int_M \phi.\end{aligned}$$

By Lemma 4.10 again, we obtain

$$\int_M |\nabla(c_\varepsilon u_\varepsilon)|^q \le C\|\phi\|_{C^0(M)} \le C.$$

This, together with Poincaré's inequality, implies that $c_\varepsilon u_\varepsilon$ is bounded in $H^{1,q}(M)$. Hence there exists $G \in H^{1,q}(M)$ such that $c_\varepsilon u_\varepsilon \rightharpoonup G$ weakly in $H^{1,q}(M)$ as $\varepsilon \to 0$. For any $\phi \in C^\infty(M)$, we have

$$\begin{aligned}\int_M \nabla\phi\nabla(c_\varepsilon u_\varepsilon) &= \int_{\partial M} \phi\frac{\partial(c_\varepsilon u_\varepsilon)}{\partial n} - \int_M \phi\Delta(c_\varepsilon u_\varepsilon)\\ &= \int_{\partial M} \phi\frac{\pi-\varepsilon}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} - c_\varepsilon\frac{\mu_\varepsilon}{2\lambda_\varepsilon}\int_M \phi\\ &\longrightarrow \phi(p) - \frac{1}{\mathrm{Vol}\,M}\int_M \phi \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Hence

$$\int_M \nabla G\,\nabla\phi = \phi(p) - \frac{1}{\mathrm{Vol}\,M}\int_M \phi,$$

and Equation (4–6) holds.

For any $\Omega \Subset M \setminus \{p\}$, we choose a smooth function η on M such that $\eta \equiv 1$ on Ω, and $\eta \equiv 0$ near p. By Lemma 4.11, $\eta u_\varepsilon \to 0$ in $L^2(M)$ as $\varepsilon \to 0$. This, together with the convergence $u_\varepsilon \to 0$ in $L^2(M)$ as $\varepsilon \to 0$, implies that $e^{(\pi-\varepsilon)u_\varepsilon^2}$ is uniformly bounded in $L^r(\overline{\Omega})$ with respect to ε for any $r > 1$. Standard elliptic estimates imply that $c_\varepsilon u_\varepsilon \to G$ in $C^k(\overline{\Omega})$ for any positive integer k. This completes the proof of the lemma. □

In the following, we use the capacity technique to derive the upper bound of $J_\pi(u)$. Take an isothermal coordinate system (U, ϕ) near p such that $\phi(p) = 0$ and ϕ maps U inside $\mathbb{R}^2_+$ and $U \cap \partial M$ inside $\partial\mathbb{R}^2_+$. In this coordinate system we can write $g = e^{2f}(dx_1^2 + dx_2^2)$, with $f(0) = 0$. Set $\phi(x_\varepsilon) = (x_\varepsilon^1, 0)$. Let $\mathbb{B}_r = \mathbb{B}_r(x_\varepsilon^1, 0) \subset \mathbb{R}^2$ be the standard ball centered at $(x_\varepsilon^1, 0)$ with radius r. Define

$$i_\varepsilon = \inf_{\partial\mathbb{B}^+_{Rr_\varepsilon}\setminus\partial\mathbb{R}^2_+} u_\varepsilon\circ\phi^{-1}, \quad s_\varepsilon = \sup_{\partial\mathbb{B}^+_{\delta}\setminus\partial\mathbb{R}^2_+} u_\varepsilon\circ\phi^{-1}, \quad \tilde{u}_\varepsilon = \max\{s_\varepsilon, \min\{u_\varepsilon\circ\phi^{-1}, i_\varepsilon\}\}.$$

Clearly,

$$\int_{\mathbb{B}_\delta^+\setminus\mathbb{B}_{Rr_\varepsilon}^+}|\nabla\tilde u_\varepsilon|^2 \le \int_{\phi^{-1}(B_\delta^+)\setminus\phi^{-1}(B_{Rr_\varepsilon}^+)}|\nabla u_\varepsilon|^2 \le 1-\int_{\phi^{-1}(B_\delta^+)}|\nabla u_\varepsilon|^2-\int_{\phi^{-1}(B_{Rr_\varepsilon}^+)}|\nabla u_\varepsilon|^2. \tag{4–7}$$

Define a function space

$$\Lambda_\varepsilon=\Big\{u\in H^{1,2}(\mathbb{B}_\delta^+\setminus\mathbb{B}_{Rr_\varepsilon}^+)\,:\,u|_{\partial\mathbb{B}_\delta^+\setminus\partial\mathbb{R}_+^2}=s_\varepsilon,\ u|_{\partial\mathbb{B}_{Rr_\varepsilon}^+\setminus\partial\mathbb{R}_+^2}=i_\varepsilon,\ \frac{\partial u}{\partial n}\Big|_{\partial\mathbb{R}_+^2}=0\Big\}.$$

It is easy to see that $\inf_{u\in\Lambda_\varepsilon}\int_{\mathbb{B}_\delta^+\setminus\mathbb{B}_{Rr_\varepsilon}^+}|\nabla_{\mathbb{R}^2}u|^2$ is attained by the unique solution of the equation

$$\begin{cases}\Delta\Phi=0 & \text{in}\quad \mathbb{B}_\delta^+\setminus\mathbb{B}_{Rr_\varepsilon}^+\\ \Phi\in\Lambda_\varepsilon.\end{cases}$$

One can check that

$$\Phi=\frac{s_\varepsilon(\log r-\log(Rr_\varepsilon))+i_\varepsilon(\log\delta-\log r)}{\log\delta-\log(Rr_\varepsilon)},$$

whence

$$\int_{\mathbb{B}_\delta^+\setminus\mathbb{B}_{Rr_\varepsilon}^+}|\nabla\Phi|^2=\pi\frac{(s_\varepsilon-i_\varepsilon)^2}{\log\delta-\log Rr_\varepsilon}. \tag{4–8}$$

By Lemma 4.7, we have

$$\int_{\phi^{-1}(\mathbb{B}_{Rr_\varepsilon}^+)}|\nabla u_\varepsilon|^2=\frac{1}{c_\varepsilon^2}\Big(\frac{1}{\pi}\log R+\frac{1}{\pi}\log\frac{\pi}{2}+O\Big(\frac{\log R}{R}\Big)+o_\varepsilon(1)\Big).$$

Lemma 4.12 then yields

$$\int_{M\setminus\phi^{-1}(B_\delta^+)}|\nabla u_\varepsilon|^2=\frac{1}{c_\varepsilon^2}\Big(-\frac{1}{\pi}\log\delta+A_p+O(\delta\log\delta)+o_\varepsilon(1)\Big). \tag{4–9}$$

By (4–7) and (4–8), we have

$$\frac{\pi s_\varepsilon^2-2\pi s_\varepsilon i_\varepsilon+\pi i_\varepsilon^2}{\log\delta-\log(Rr_\varepsilon)}<1-\frac{1}{c_\varepsilon^2}\Big(-\frac{1}{\pi}\log\delta+A_p+O(\delta\log\delta)+o_\varepsilon(1)\Big)-\frac{1}{c_\varepsilon^2}\Big(\frac{1}{\pi}\log R+\frac{1}{\pi}\log\frac{\pi}{2}+O\Big(\frac{\log R}{R}\Big)+o_\varepsilon(1)\Big). \tag{4–10}$$

From Lemma 4.7 and Lemma 4.12, one can see that

$$i_\varepsilon=c_\varepsilon-\frac{\log(1+\pi^2R^2)+o_\varepsilon(R)}{2\pi c_\varepsilon},\qquad s_\varepsilon=\frac{-\log\delta+\pi A_p+O(\delta)+o_\varepsilon(R)}{\pi c_\varepsilon}.$$

Adding this and (4–3) to (4–10), we have

$$\log \frac{\lambda_\varepsilon}{c_\varepsilon^2} \le -\varepsilon c_\varepsilon^2 + \log(2\pi^2) + \pi A_p + o_\varepsilon(\delta) + o_\varepsilon(R) + o_\varepsilon(1) + o_\delta(1) + o_R(1).$$

Letting $\varepsilon \to 0$ first, then $\delta \to 0$ and $R \to +\infty$, we obtain

$$\limsup_{\varepsilon\to 0} \frac{\lambda_\varepsilon}{c_\varepsilon^2} \le 2\pi^2 e^{\pi A_p}.$$

Together with Lemma 4.8, this estimate yields $\sup_{u\in\mathcal{H}_1} J_\pi(u) \le \operatorname{Vol}\partial M + 2\pi e^{\pi A_p}$. In fact, we have proved the following:

Proposition 4.13. *Under the assumption that $\mu_\varepsilon \ge 0$ and $u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$, we obtain*

$$\sup_{u\in\mathcal{H}_1} J_\pi(u) \le \operatorname{Vol}\partial M + 2\pi e^{\pi \max_{p\in\partial M} A_p}.$$

For the other case, $\mu_\varepsilon \ge 0$ and $u_\varepsilon(x_\varepsilon) \to -\infty$, we only need to replace (4–4) by $\varphi_\varepsilon(x) = -c_\varepsilon\big(u_\varepsilon(x_\varepsilon + r_\varepsilon x) + c_\varepsilon\big)$. Using the same arguments we have used from Lemma 4.5 to Proposition 4.13, we also get:

Proposition 4.14. *Under the assumption that $\mu_\varepsilon \ge 0$ and $u_\varepsilon(x_\varepsilon) \to -\infty$ as $\varepsilon \to 0$, we obtain*

$$\sup_{u\in\mathcal{H}_1} J_\pi(u) \le \operatorname{Vol}\partial M + 2\pi e^{\pi \max_{p\in\partial M} A_p}.$$

5. Existence results

Assume $A_p = \max_{p\in\partial M} A_p$ for some $p \in \partial M$. In this section, we will construct a blowing up sequence ϕ_ε with $\int_M |\nabla\phi_\varepsilon|^2 = 1$, and

$$\int_{\partial M} e^{\pi(\phi_\varepsilon - \bar\phi_\varepsilon)^2} > \operatorname{Vol}\partial M + 2\pi e^{\pi A_p}, \quad \text{where } \bar\phi_\varepsilon = \frac{1}{\operatorname{Vol} M}\int_{\partial M} \phi_\varepsilon.$$

Take an isothermal coordinate system (U, ψ) around p such that $\psi(p) = (0, 0)$, ψ maps $\partial M \cap U)$ inside $\partial\mathbb{R}^2_+$, and $g = e^{2f}(ds^2 + dt^2)$ with $f(0) = 0$. Let R be a function of ε such that $R \to +\infty$ and $R\varepsilon \to 0$ as $\varepsilon \to 0$. For sufficiently small $r > 0$, write $\mathbb{B}_r^+ = \mathbb{B}_r^+(0, -\varepsilon/\pi) = \mathbb{B}_r(0, -\varepsilon/\pi) \cap \overline{\mathbb{R}^2_+}$, $B_r^+ = \psi^{-1}(\mathbb{B}_r^+)$ and

$$\tilde\phi_\varepsilon(s, t) = c + \frac{-(1/2\pi)\log(\pi^2 s^2/\varepsilon^2 + (\pi t/\varepsilon + 1)^2) + B}{c} \quad \text{in } \mathbb{B}^+_{R\varepsilon},$$

for some constants B, c.

Set

$$\phi_\varepsilon = \begin{cases} \tilde\phi_\varepsilon \circ \psi(x) & \text{if } x \in B^+_{R\varepsilon}, \\ (G - \eta\beta)/c & \text{if } x \in B^+_{2R\varepsilon} \setminus B^+_{R\varepsilon}, \\ G/c & \text{if } x \in M \setminus B^+_{2R\varepsilon}, \end{cases}$$

where B, c are constants to be defined later, $\beta = G + (1/\pi)\log r - A_p = O(r)$, $\eta \in C_0^\infty(B_{2R\varepsilon})$ with $\eta \equiv 1$ on $B_{R\varepsilon}$, and $\max|\nabla\eta| = O(1/(R\varepsilon))$.

To ensure that $\phi_\varepsilon \in H^{1,2}(M)$, we assume

$$c + \frac{-(1/2\pi)\log(\pi^2R^2) + B}{c} = \frac{-(1/\pi)\log(R\varepsilon) + A_p}{c},$$

which gives

$$2\pi c^2 = 2\log\pi - 2\pi B - 2\log\varepsilon + 2\pi A_p. \tag{5–1}$$

By (4–9), we have

$$\int_{B^+_{R\varepsilon}} |\nabla\phi_\varepsilon|^2 = \frac{1}{\pi c^2}\log\frac{\pi}{2} + \frac{1}{\pi c^2}\log R + O\Big(\frac{\log R}{R}\Big),$$

$$\int_{M\setminus B^+_{R\varepsilon}} |\nabla\phi_\varepsilon|^2 = \int_{M\setminus B^+_{R\varepsilon}} \frac{|\nabla G|^2}{c^2} + \int_{B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}} \frac{|\nabla(\eta\beta)|^2}{c^2} - \frac{2}{c^2}\int_{B^+_{2R\varepsilon}\setminus B^+_{R\varepsilon}} \nabla G\nabla(\eta\beta).$$

Let I_1, I_2, I_3 be the three summands on the right-hand side of the last equation. Clearly, $I_2 = c^{-2}O(R\varepsilon)$ and $I_3 = c^{-2}O(R\varepsilon)$. Next,

$$\begin{aligned} I_1 &= \frac{1}{c^2}\int_{\partial(M\setminus B^+_{R\varepsilon})} G\frac{\partial G}{\partial n} - \frac{1}{c^2}\int_{M\setminus B^+_{R\varepsilon}} G\Delta G \\ &= \frac{1}{c^2}\int_{\partial M\setminus\partial B^+_{R\varepsilon}} G\frac{\partial G}{\partial n} - \frac{1}{c^2}\int_{\partial B^+_{R\varepsilon}\setminus\partial M} G\frac{\partial G}{\partial n} + \frac{1}{c^2}\frac{1}{\operatorname{Vol} M}\Big(\int_{B^+_{R\varepsilon}} G\Big) + \frac{1}{c^2}O(\frac{1}{R}) \\ &= \frac{1}{c^2}\Big(-\frac{1}{\pi}\log(R\varepsilon) + A_p + O(R\varepsilon\log(R_\varepsilon)) + O\Big(\frac{1}{R}\Big)\Big), \end{aligned}$$

whence

$$\int_{M\setminus B^+_{R\varepsilon}} |\nabla\phi_\varepsilon|^2 = \frac{1}{c^2}\Big(-\frac{1}{\pi}\log(R\varepsilon) + A_p + O(R\varepsilon\log(R_\varepsilon)) + O\Big(\frac{1}{R}\Big)\Big).$$

Combining the two estimates above, one has

$$\int_M |\nabla\phi_\varepsilon|^2 = \frac{1}{c^2}\Big(-\frac{1}{\pi}\log\varepsilon + \frac{1}{\pi}\log\frac{\pi}{2} + A_p + O(R\varepsilon\log(R\varepsilon)) + O\Big(\frac{\log R}{R}\Big)\Big).$$

To ensure that $\int_M |\nabla\phi_\varepsilon|^2 = 1$, we set

$$c^2 = -\frac{1}{\pi}\log\varepsilon + \frac{1}{\pi}\log\frac{\pi}{2} + A_p + O(R\varepsilon\log(R\varepsilon)) + O\Big(\frac{\log R}{R}\Big).$$

By (5–1), one can determine B as

$$B = \frac{1}{\pi}\log 2 + O(R\varepsilon\log(R\varepsilon)) + O\Big(\frac{\log R}{R}\Big).$$

A straightforward computation gives

$$\bar{\phi}_\varepsilon = \frac{1}{\operatorname{Vol} M}\int_M \phi_\varepsilon = \frac{1}{c}\Big(O\big((R\varepsilon)^2\log R\big) + O\big((R\varepsilon)^2\log\varepsilon\big) + O\big((R\varepsilon)^2\log(R\varepsilon)\big)\Big).$$

Then

$$\begin{aligned}
&\int_{\partial B^+_{R\varepsilon}\cap\partial M} \exp\big(\pi(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2\big)\\
&= \int_{\partial\mathbb{B}^+_{R\varepsilon}\cap\partial\mathbb{R}^2_+} \exp\left(\pi\left(c-\frac{\log\left(\pi^2 s^2/\varepsilon^2+(1+\pi t/\varepsilon)^2\right)+c\bar{\phi}_\varepsilon-2\pi B}{2\pi c}\right)^2+O(R\varepsilon)\right)ds\\
&\geq \int_{\partial\mathbb{B}^+_{R\varepsilon}\cap\partial\mathbb{R}^2_+} \exp\left(\pi c^2-\log\left(\pi^2\frac{s^2}{\varepsilon^2}+1\right)+2\pi B-c\bar{\phi}_\varepsilon\right)e^{O(R\varepsilon)}ds\\
&= 2\pi e^{\pi A_p}\Big(\frac{2}{\pi}\arctan(\pi R)\Big)\exp\left(O(R\varepsilon\log R\varepsilon)+O\Big(\frac{\log R}{R}\Big)\right)\\
&= 2\pi e^{\pi A_p}\left(1+O(R\varepsilon\log(R\varepsilon))+O\Big(\frac{\log R}{R}\Big)\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\partial M\setminus\partial B^+_{R\varepsilon}} \exp\big(\pi(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2\big) &\geq \int_{\partial M\setminus\partial B^+_{R\varepsilon}} \big(1+\pi^2(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2\big)\\
&\geq \operatorname{Vol}\partial M-\operatorname{Vol}(\partial M\cap\partial B^+_{R\varepsilon})+\pi^2\int_{\partial M\setminus B^+_{2R\varepsilon}}\frac{(G-c\bar{\phi}_\varepsilon)^2}{c^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\partial M} e^{\pi(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2} \geq \operatorname{Vol}\partial M+\frac{\pi^2}{c^2}\int_{\partial M\setminus B^+_{2R\varepsilon}} &\big(G-O(R\varepsilon\log R\varepsilon)\big)^2\\
&+O\big(R\varepsilon\log(R\varepsilon)\big)+O\Big(\frac{\log R}{R}\Big).
\end{aligned}$$

Set $R=\log^2\varepsilon$. Then $R\to+\infty$, $R\varepsilon\to 0$, $c^2(\log R)/R\to 0$, $c^2R\varepsilon\log R\varepsilon\to 0$. Hence

$$\int_{\partial M} e^{\pi(\phi_\varepsilon-\bar{\phi}_\varepsilon)^2} > \operatorname{Vol}\partial M+2\pi e^{\pi A_p}$$

when ε is sufficiently small. This contradicts Proposition 4.13 or Proposition 4.14. Hence Theorem 1.1 holds. □

6. Proof of Theorem 1.2

Lemma 6.1. *Set* $\tilde{\alpha}_2=\sup\left\{\alpha;\ \sup_{u\in\mathcal{H}_2}\int_M e^{\alpha u^2}<+\infty\right\}$. *Then* $\tilde{\alpha}_2=2\pi$.

Proof. We first show that $\tilde{\alpha}_2 \geq 2\pi$. For any $\alpha < 2\pi$ and $u \in \mathcal{H}_2$, we set

$$\tilde{u} = u - \frac{1}{\operatorname{Vol} M} \int_M u.$$

Then $\tilde{u} \in \mathcal{H}_1$. So, by Lemma 3.2,

$$\int_M e^{\alpha \tilde{u}^2} \leq \sup_{v \in \mathcal{H}_1} \int_M e^{\alpha v^2} < +\infty.$$

For any $u \in \mathcal{H}_2$, we have

$$\int_M e^{\alpha u^2} \leq e^{c(\varepsilon')} \int_M e^{\alpha(1+\varepsilon')\tilde{u}^2}$$

for some $\varepsilon' > 0$. One can choose ε' such that $\alpha(1+\varepsilon') < 2\pi$, which gives

$$\int_M e^{\alpha u^2} \leq \sup_{v \in \mathcal{H}_1} \int_M e^{\alpha(1+\varepsilon')v^2} < +\infty.$$

Hence

$$\sup_{u \in \mathcal{H}_2} \int_M e^{\alpha u^2} < +\infty.$$

Next we prove that $\tilde{\alpha}_2$ cannot be greater than 2π. To do this, the example in the proof of Lemma 3.2 still works here. For $p \in \partial M$, we set

$$u_\varepsilon = \begin{cases} -\sqrt{\dfrac{1}{2\pi} \log \dfrac{1}{\varepsilon}} & \text{in } B_{\delta\sqrt{\varepsilon}}(p), \\ \dfrac{\sqrt{2}}{\sqrt{-\pi \log \varepsilon}} \log \dfrac{r}{\delta} & \text{in } B_\delta(p) \setminus B_{\delta\sqrt{\varepsilon}}(p) \\ C_\varepsilon \varphi & \text{in } M \setminus B_\delta(p), \end{cases}$$

where $\varphi \in C_0^\infty(M \setminus B_\delta(p))$, $0 \leq \varphi \leq 1$, and C_ε is chosen to satisfy $\int_M u_\varepsilon = 0$. It is easy to check that

$$\|u_\varepsilon\|^2_{H^{1,2}(M)} = \int_M \left(|\nabla u_\varepsilon|^2 + u_\varepsilon^2\right) \to 1 \quad \text{as} \quad \varepsilon \to 0,$$

and for any $\alpha > 2\pi$

$$\begin{aligned} \int_M \exp\left(\alpha \left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{H^{1,2}(M)}}\right)^2\right) &\geq \exp\left(\frac{\alpha}{2\pi \|u_\varepsilon\|^2_{H^{1,2}(M)}} \log \frac{1}{\varepsilon}\right) \operatorname{Vol} B_{\delta\sqrt{\varepsilon}} \\ &\geq C\varepsilon^{1 - \frac{\alpha}{2\pi \|u_\varepsilon\|^2_{H^{1,2}(M)}}}, \end{aligned}$$

which approaches $+\infty$ as $\varepsilon \to 0$. Therefore $\tilde{\alpha}_2 \leq 2\pi$, as needed. ☐

Using the same argument as in the proof of Lemma 3.3, we obtain:

Lemma 6.2.

$$\sup_{u\in\mathscr{H}_2} J_\alpha(u) < +\infty \quad \textit{for } \alpha < \pi \quad \textit{and} \quad \sup_{u\in\mathscr{H}_2} J_\alpha(u) = +\infty \quad \textit{for } \alpha > \pi.$$

Similarly to Lemma 4.1, one has:

Lemma 6.3. *The functional $J_{\pi-\varepsilon}(u)$ defined in the space $\mathscr{H}_2$ admits a smooth maximizer $u_\varepsilon \in \mathscr{H}_2$.*

Proof. The proof of the existence of u_ε is the same as that of Lemma 4.1. The Euler–Lagrange equation of u_ε is

$$\begin{cases} \Delta u_\varepsilon = u_\varepsilon & \text{in } \mathring{M}, \\ \dfrac{\partial u_\varepsilon}{\partial n} = \dfrac{\pi-\varepsilon}{\lambda_\varepsilon} u_\varepsilon e^{(\pi-\varepsilon)u_\varepsilon^2} & \text{on } \partial M, \end{cases} \tag{6–1}$$

where

$$\lambda_\varepsilon = (\pi-\varepsilon)\int_M u_\varepsilon^2 e^{(\pi-\varepsilon)u_\varepsilon^2}.$$

Using Lemmas 2.1 and 2.2 repeatedly, we get $u_\varepsilon \in C^\infty(M)$. □

The rest of the proof of Theorem 1.2 is almost the same as that of Theorem 1.1; we only give its outline. Without loss of generality, we may assume $u_\varepsilon \geq 0$ in M. Set $c_\varepsilon = u_\varepsilon(x_\varepsilon) = \max_{x\in M} u_\varepsilon(x)$. If $\{c_\varepsilon\}$ is bounded, it is not difficult to see that Theorem 1.2 holds. Hence we assume that $c_\varepsilon \to +\infty$ as $\varepsilon \to 0$. This is equivalent to saying $\int_M e^{\alpha u_\varepsilon^2} \to +\infty$ for any $\alpha > 2\pi$, which implies that $u_\varepsilon \to 0$ strongly in $L^2(M)$ (see the first step in the proof of Lemma 3.2). Applying the maximum principle to (6–1), we find that $x_\varepsilon \in \partial M$. Assume that x_ε converges to p, so $p \in \partial M$. Let r_ε, $\varphi_\varepsilon(x)$ and $\psi_\varepsilon(x)$ be as in Section 4. Then

$$\varphi_\varepsilon \to \varphi = -\frac{1}{2\pi}\log\bigl(\pi^2 x_1^2 + (1+\pi x_2)^2\bigr) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^2_+).$$

Moreover $c_\varepsilon u_\varepsilon \rightharpoonup G$ weakly in $H^{1,q}(M)$ for any q such that $1 < q < 2$. The function $G \in C^\infty(M \setminus \{p\})$ satisfies

$$\begin{cases} -\Delta G + G = \delta_p & \text{in } M, \\ \int_M G = 1. \end{cases}$$

In a normal coordinate system around p, the Green's function G has the representation $G = -(1/\pi)\log r + A_p + O(r)$, where $r(x) = \operatorname{dist}(p, x)$ is the distance function, A_p is a constant depending only on p, and $A_p + O(r)$ is called the regular part. Repeating the other steps taken in Section 4, we obtain

$$\sup_{u\in\mathscr{H}_2} (u) \leq \operatorname{Vol}\partial M + 2\pi e^{\pi A_p}. \tag{6–2}$$

The blowing up sequence we constructed in Section 5 still works here; one can check that

$$J_\pi\left(\frac{\phi_\varepsilon}{\|\phi_\varepsilon\|_{H^{1,2}(M)}}\right) > \operatorname{Vol}\partial M + 2\pi e^{\pi A_p}$$

for sufficiently small ε, which contradicts (6–2) and so proves Theorem 1.2. $\square$

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