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We obtain a Green current in the sense of Gillet–Soulé on an arithmetic quotient of a complex hyperball for the modular cycle stemming from a complex subhyperball of codimension greater than one, generalizing the classical construction of the automorphic Green function for the modular curves.

Introduction and basic notation

Let X be a complex manifold and Y its analytic subvariety of codimension r. The Green current for Y is defined to be a current $\mathcal{G}$ of $(r-1, r-1)$-type on X such that $\mathrm{dd}^{\mathrm{c}}\mathcal{G} + \delta_Y$ is represented by a C^∞-form of (r, r)-type on X. In the arithmetic intersection theory developed by Gillet and Soulé, the role played by the algebraic cycles in the conventional intersection theory is replaced with the arithmetic cycles. In a heuristic sense, Green currents are regarded as the "archimedean" ingredient of such arithmetic cycles [Gillet and Soulé 1990].

Consider the case when X is the quotient of a Hermitian symmetric domain G/K by an arithmetic lattice Γ in the semisimple Lie group G, and Y is a modular cycle stemming from a modular imbedding $H/H \cap K \hookrightarrow G/K$, where H is a reductive subgroup of G such that $H \cap K$ is maximally compact in H. Inspired by the classical works on the resolvent kernel functions of the Laplacian on Riemannian surfaces [Hejhal 1983] and also by a series of works [Miatello and Wallach 1989; 1992], T. Oda posed a plan to construct a Green current for Y making use of a secondary spherical function on $H\backslash G$, giving evidence for the divisorial case with some conjectures. Among many possible choices of the Green currents for a modular cycle Y, this construction may provide a way to fix a natural one. If $r = 1$, namely Y is a modular divisor, we already obtained a satisfactory result by properly introducing the secondary spherical functions [Oda and Tsuzuki 2003]. So it is quite natural to ask whether the same method works for the higher codimensional case. Here we focus on the case when G/K is an n-dimensional complex hyperball and $H/H \cap K$ is also a complex hyperball of codimension $r > 1$.

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After two preliminary sections, in the third section we introduce the vector-valued secondary spherical function ϕ_s together with its simple characterization in Theorem 3.2. Then we form the Poincaré series $G_s(g) = \sum_{\gamma \in H\cap\Gamma\backslash\Gamma} \phi_s(\gamma g)$ which is shown in Theorem 4.2 to be L^1-convergent to yield an $(r-1, r-1)$-current on $\Gamma\backslash G/K$ when the parameter s lies in some right-half plane. In order to analyze G_s further, in Section 5 we study its derivatives $(-\frac{1}{2s}\frac{d}{ds})^j G_s$ with sufficiently large j by the same technique used in [Gon and Tsuzuki 2002]. Through the inductive argument of Theorem 6.2, we show in Theorem 6.3 that the function $s \mapsto G_s$ has a meromorphic continuation to the whole s-plane with a functional equation and has a simple pole at the point $s = n - 2r + 2$. We put $\mathcal{G}$ to be the constant term of the Laurent expansion of G_s at $s = n - 2r + 2$ and prove in Theorem 6.5 that the $(r-1, r-1)$-current $\mathcal{G}$ is a Green current for our Y. For that purpose we introduce another (r, r)-current Ψ_s as a suitable Poincaré series and study its properties. Among other things we show that the value of Ψ_s at its regular point $s = n - 2r + 2$ is square-integrable harmonic form representing the current $\mathrm{dd^c}\mathcal{G} + \delta_D$ up to a constant multiple; see Theorem 7.5.

This paper is a continuation of the joint work of Professor Takayuki Oda and the author [Oda and Tsuzuki 2003]. Thanks are due to Professor Takayuki Oda for his interest in this work and fruitful discussions.

1. Preliminaries

Let n and r be integers such that $2 \leqslant r < n/2$. For a matrix $X = (x_{ij}) \in \mathrm{M}_n(\mathbb{C})$, X^* denotes its conjugate transpose $(\bar{x}_{ji})$. Consider the two involutions σ and θ of the Lie group $G = \mathsf{U}(n, 1) := \{g \in \mathrm{GL}_{n+1}(\mathbb{C}) \mid g^* \mathtt{I}_{n,1} g = \mathtt{I}_{n,1}\}$ defined by $\theta(g) = \mathtt{I}_{n,1}\, g\, \mathtt{I}_{n,1}$ and $\sigma(g) = \mathtt{S}\, g\, \mathtt{S}$ respectively. Here $\mathtt{I}_{n,1} := \mathrm{diag}(\mathtt{I}_n, -1)$ and $\mathtt{S} = \mathrm{diag}(\mathtt{I}_{n-r}, -\mathtt{I}_r, 1)$. Then $K := \{g \in G \mid \theta(g) = g\} \cong \mathsf{U}(n) \times \mathsf{U}(1)$ is a maximal compact subgroup in G and $H := \{g \in G \mid \sigma(g) = g\} \cong \mathsf{U}(n-r, 1) \times \mathsf{U}(r)$ is a symmetric subgroup of G such that $K_H := H \cap K \cong \mathsf{U}(n-r) \times \mathsf{U}(r) \times \mathsf{U}(1)$ is maximally compact in H.

The group G acts on the hyperball $\mathfrak{D} = \{z = {}^t(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 < 1\}$ transitively by the fractional linear transformation

$$g \cdot z = \frac{g_{11} z + g_{12}}{g_{21} z + g_{22}}, \quad g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in G,\ z \in \mathbb{C}^n.$$

Under the identification $G/K \cong \mathfrak{D}$ of G-manifolds sending gK to $g \cdot 0$, the quotient H/K_H corresponds to the H-orbit of 0, i.e., $\mathfrak{D}^H := \{z \in \mathfrak{D} \mid z_{n-r+1} = \cdots = z_n = 0\}$. In particular the real codimension of H/K_H in G/K is $2r$.

The Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ is realized in its complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_{n+1}(\mathbb{C})$ as an $\mathbb{R}$-subalgebra of all $X \in \mathfrak{gl}_{n+1}(\mathbb{C})$ such that $X^* \mathtt{I}_{n,1} + \mathtt{I}_{n,1} X = \mathtt{O}_{n+1}$. Let $\mathfrak{p}$

be the orthogonal complement of $\mathfrak{k} := \mathrm{Lie}(K)$ in $\mathfrak{g}$ with respect to the G-invariant, nondegenerate $\mathbb{R}$-bilinear form $\langle X, Y\rangle = 2^{-1}\mathrm{tr}(XY)$ on $\mathfrak{g}$. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We put G/K a unique G-invariant complex structure such that the map $G/K \cong \mathfrak{D}$ is biholomorphic.

For $1 \leqslant i, j \leqslant n+1$, let $\mathrm{E}_{i,j} := (\delta_{ui}\delta_{vj})_{uv} \in \mathfrak{gl}_{n+1}(\mathbb{C})$ be the matrix unit. Put

$$X_i := \mathrm{E}_{i,n+1} \ \text{ and } \ \bar{X}_i = \mathrm{E}_{n+1,i}, \quad 1 \leqslant i \leqslant n-1;$$
$$X_0 := \mathrm{E}_{n,n+1} \ \text{ and } \ \bar{X}_0 = \mathrm{E}_{n+1,n}.$$

Then $\mathfrak{p}_+ = \sum_{i=0}^{n-1} \mathbb{C}X_i$ corresponds to the holomorphic tangent space of G/K at K, and $\mathfrak{p}_- = \sum_{i=0}^{n-1} \mathbb{C}\bar{X}_i$ to the antiholomorphic tangent space.

The exterior algebra $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ is decomposed to the direct sum of subspaces

$$\textstyle\bigwedge^{p,q} \mathfrak{p}_{\mathbb{C}}^* := \big(\bigwedge^p \mathfrak{p}_+^*\big) \otimes \big(\bigwedge^q \mathfrak{p}_-^*\big), \quad p, q \in \mathbb{N}.$$

With $\{\omega_i\}$ the basis of $\mathfrak{p}_+^*$ dual to $\{X_i\}$ and $\{\bar{\omega}_i\}$ the basis of $\mathfrak{p}_-^*$ dual to $\{\bar{X}_i\}$, put

$$\omega := \tfrac{\sqrt{-1}}{2} \sum_{i=0}^{n-1} \omega_i \wedge \bar{\omega}_i \in \textstyle\bigwedge^{1,1}\mathfrak{p}_{\mathbb{C}}^* \cap \bigwedge \mathfrak{p}^* \quad \text{and} \quad \mathrm{vol} := \tfrac{1}{n!}\omega^n \in \textstyle\bigwedge^{n,n}\mathfrak{p}_{\mathbb{C}}^* \cap \bigwedge \mathfrak{p}^*.$$

The dual inner product on $\mathfrak{p}^*$ naturally extends to the Hermitian inner product $(\cdot\,|\,\cdot)$ of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$. The Hodge star operator $*$ is the $\mathbb{C}$-linear automorphism of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ such that $*\bar{\alpha} = \overline{*\alpha}$ and $(\alpha|\beta)\,\mathrm{vol} = \alpha \wedge *\bar{\beta}$, for $\alpha, \beta \in \bigwedge \mathfrak{p}_{\mathbb{C}}^*$. We remark that

$$\{X_j + \bar{X}_j,\ \sqrt{-1}(X_j - \bar{X}_j)\}_{j=0}^{n-1}$$

is an orthonormal basis of $\mathfrak{p}$, dual to $\{2^{-1}(\omega_j + \bar{\omega}_j), -2^{-1}\sqrt{-1}(\omega_j - \bar{\omega}_j)\}_{j=0}^{n-1}$. For $\alpha \in \bigwedge \mathfrak{p}_{\mathbb{C}}^*$, define the endomorphism $e(\alpha) : \bigwedge \mathfrak{p}_{\mathbb{C}}^* \to \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ by $e(\alpha)\beta = \alpha \wedge \beta$. As usual, we have the Lefschetz operator $L := e(\omega)$ and its adjoint operator $\Lambda = e^*(\omega)$ acting on the finite dimensional Hilbert space $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ [Wells 1980, Chapter V].

Put $\mathfrak{h} = \mathrm{Lie}(H)$. Then θ restricts to a Cartan involution of $\mathfrak{h}$ giving the decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. The complex structure of $\mathfrak{p}$ induces that of $\mathfrak{h} \cap \mathfrak{p}$ giving the decomposition $(\mathfrak{h} \cap \mathfrak{p})_{\mathbb{C}} = (\mathfrak{h} \cap \mathfrak{p})_+ \oplus (\mathfrak{h} \cap \mathfrak{p})_-$ with

$$(\mathfrak{h} \cap \mathfrak{p})_+ = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_+ = \sum_{i=1}^{n-r} \mathbb{C}X_i \quad \text{and} \quad (\mathfrak{h} \cap \mathfrak{p})_- = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{p}_- = \sum_{i=1}^{n-r} \mathbb{C}\bar{X}_i.$$

We introduce two tensors ω_H and η as

$$\omega_H := \tfrac{\sqrt{-1}}{2} \sum_{i=1}^{n-r} \omega_i \wedge \bar{\omega}_i \quad \text{and} \quad \eta := \tfrac{\sqrt{-1}}{2} \sum_{j=n-r+1}^{n-1} \omega_i \wedge \bar{\omega}_i = \omega - \omega_H - \tfrac{\sqrt{-1}}{2}\omega_0 \wedge \bar{\omega}_0.$$

For $0 \leqslant p \leqslant r-1$ let S_p denote the set of all subsets $J \subset \{n-r+1, \dots, n-1\}$ such that $\sharp(J) = p$. Then a computation shows that

$$\eta^p = (\tfrac{\sqrt{-1}}{2})^p \, p! \sum_{J \in S_p} \prod_{j \in J} \omega_j \wedge \bar{\omega}_j, \quad 0 \leqslant p \leqslant r-1.$$

From [Wells 1980, (1.5), p. 163], we have

$$\Lambda(\eta^p) = p(r-p)\,\eta^{p-1}, \quad 0 \leqslant p \leqslant r-1. \tag{1}$$

The coadjoint representation of K on $\mathfrak{p}^*$ is extended to the unitary representation $\tau : K \to \mathrm{GL}(\bigwedge \mathfrak{p}^*_{\mathbb{C}})$ in such a way that $\tau(k)(\alpha \wedge \beta) = \tau(k)\alpha \wedge \tau(k)\beta$ holds for all $\alpha, \beta \in \bigwedge \mathfrak{p}^*_{\mathbb{C}}$ and $k \in K$. The differential of τ is also denoted by τ. Then we have

$$\begin{aligned} &\tau(Z)(\alpha \wedge \beta) = (\tau(Z)\alpha) \wedge \beta + \alpha \wedge (\tau(Z)\beta), \\ &(\tau(Z)\alpha | \beta) = -(\alpha | \tau(\bar{Z})\beta) \end{aligned} \tag{2}$$

for $\alpha,\ \beta \in \bigwedge \mathfrak{p}^*_{\mathbb{C}},\ Z \in \mathfrak{k}_{\mathbb{C}}$.

The irreducible decomposition of the K-invariant subspaces $\bigwedge^{p,q} \mathfrak{p}^*_{\mathbb{C}}$ is well-known.

Lemma 1.1. *Let p, q be nonnegative integers such that $p+q \leqslant n$. Put*

$$F_{p,q} := \big\{\alpha \in \textstyle\bigwedge^{p,q} \mathfrak{p}^*_{\mathbb{C}} \mid \Lambda(\alpha) = 0\big\}.$$

Then $F_{p,q}$ is an irreducible K-invariant subspace of $\bigwedge \mathfrak{p}^*_{\mathbb{C}}$. The K-homomorphism L induces a linear injection $\bigwedge^{p-1,q-1} \mathfrak{p}^*_{\mathbb{C}} \to \bigwedge^{p,q} \mathfrak{p}^*_{\mathbb{C}}$ whose image is the orthogonal complement of $F_{p,q}$ in $\bigwedge^{p,q} \mathfrak{p}^*_{\mathbb{C}}$. In other words,

$$\textstyle\bigwedge^{p,q} \mathfrak{p}^*_{\mathbb{C}} = F_{p,q} \oplus L\big(\bigwedge^{p-1,q-1} \mathfrak{p}^*_{\mathbb{C}}\big).$$

Proof. Use [Borel and Wallach 1980, Lemma 4.9, p. 199]. □

The one dimensional $\mathbb{R}$-subspace $\mathfrak{a} = \mathbb{R}Y_0$ with $Y_0 := X_0 + \bar{X}_0 \in \mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{q} \cap \mathfrak{p}$ with $\mathfrak{q}$ the (-1)-eigenspace of $d\sigma$, the differential of σ. Since (G, H) is a symmetric pair, by the general theory [Heckman and Schlichtkrull 1994, Theorem 2.4, p. 108], the group G is a union of double cosets Ha_tK $(t \geqslant 0)$ with

$$a_t := \exp(tY_0) = \mathrm{diag}\left(\mathrm{I}_{n-1}, \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\right), \quad t \in \mathbb{R}.$$

Put $A = \{a_t \mid t \in \mathbb{R}\}$. Let M_0 be the group of all the elements $k \in K$ such that $\mathrm{Ad}(k)Y_0 = Y_0$ and set $M = M_0 \cap H$. Then

$$\begin{aligned} M_0 &= \{\mathrm{diag}(u, u_0, u_0) \mid u \in \mathsf{U}(n-1),\ u_0 \in \mathsf{U}(1)\}, \\ M &= \{\mathrm{diag}(u_1, u_2, u_0, u_0) \mid u_1 \in \mathsf{U}(n-r),\ u_2 \in \mathsf{U}(r-1),\ u_0 \in \mathsf{U}(1)\}. \end{aligned}$$

It is important to know the structure of the M-fixed part of $F_{p,p}$.

Proposition 1.2. *Let p be an integer such that $0 < p \leqslant r$. Put*

$$\mathsf{v}_0^{(p)} = \frac{1}{n-p+1} \sum_{j=0}^{p} c_j^{(p)} L^{p-j} \big((n-p-j+1)\eta^j + \tfrac{\sqrt{-1}}{2} j(r-j)\, \omega_0 \wedge \bar{\omega}_0 \wedge \eta^{j-1}\big),$$

$$\mathsf{v}_1^{(p)} = \frac{-1}{p(n-2p+1)} \sum_{j=0}^{p} c_j^{(p)} L^{p-j} \big((p-j)\eta^j + \tfrac{\sqrt{-1}}{2} j(r-j)\, \omega_0 \wedge \bar{\omega}_0 \wedge \eta^{j-1}\big),$$

where

$$c_j^{(p)} = (-1)^j \binom{p}{j}\binom{n-p+1}{j}\binom{r-1}{j}^{-1}, 0 \leqslant j \leqslant \inf(p, r-1), \quad \text{and} \quad c_r^{(r)} = 1.$$

Then $F_{p,p}^M$ is a two-dimensional space generated by $\mathsf{v}_0^{(p)}$ and $\mathsf{v}_1^{(p)}$.

Proof. Let W be the $\mathbb{C}$-span of elements $\omega_i, \bar{\omega}_i$ $(1 \leqslant i \leqslant n-1)$. Then for each p, q, $\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^*$ is an orthogonal direct sum of four subspaces $\bigwedge^{p,q} W$, $e(\omega_0)(\bigwedge^{p-1,q} W)$, $e(\bar{\omega}_0)(\bigwedge^{p,q-1} W)$ and $e(\omega_0 \wedge \bar{\omega}_0)(\bigwedge^{p-1,q-1} W)$. The space $\bigwedge W$ has the Lefschetz operator L_0 and its adjoint Λ_0. These are also given as $L_0 = e(\omega_H + \eta)|\bigwedge W$ and $\Lambda_0 = \Lambda|\bigwedge W$. Let p, q be natural numbers such that $p + q < n$. Then the subspace $E_{p,q} = \{\beta \in \bigwedge^{p,q} W \mid \Lambda_0(\beta) = 0\}$ is an irreducible M_0-subspace of $\bigwedge W$; see Lemma 1.1. First we show that the linear map $\mathcal{T}^{p,q}$ from the direct sum $E_{p,q} \oplus E_{p-1,q} \oplus E_{p,q-1} \oplus E_{p-1,q-1}$ to $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$ defined by sending $(\beta_1, \beta_2, \beta_3, \xi)$ to

$$\mathcal{T}^{p,q}(\beta_1, \beta_2, \beta_3, \xi) = \beta_1 + (\beta_2 \wedge \omega_0) + (\beta_3 \wedge \bar{\omega}_0) + \tfrac{\sqrt{-1}}{2}\xi \wedge \omega_0 \wedge \bar{\omega}_0 - \frac{L_0(\xi)}{n-p-q+1}$$

is an M_0-isomorphism onto $F_{p,q}$.

An element $\zeta \in \bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^*$ is expressed uniquely using an element $(\zeta_1, \beta_2, \beta_3, \zeta_4)$ in the direct sum $\bigwedge^{p,q} W \oplus \bigwedge^{p-1,q} W \oplus \bigwedge^{p,q-1} W \oplus \bigwedge^{p-1,q-1} W$ as the sum $\zeta = \zeta_1 + (\beta_2 \wedge \omega_0) + (\beta_3 \wedge \bar{\omega}_0) + (\zeta_4 \wedge \omega_0 \wedge \bar{\omega}_0)$. We examine the condition $\Lambda(\zeta) = 0$. Since the four equalities $\Lambda(\zeta_1) = \Lambda_0(\zeta_1)$, $\Lambda(\beta_2 \wedge \omega_0) = \Lambda_0(\beta_2) \wedge \omega_0$, $\Lambda(\beta_3 \wedge \bar{\omega}_0) = \Lambda_0(\beta_3) \wedge \bar{\omega}_0$ and $\Lambda(\zeta_4 \wedge \omega_0 \wedge \bar{\omega}_0) = \Lambda_0(\zeta_4) \wedge \omega_0 \wedge \bar{\omega}_0 + \frac{2}{\sqrt{-1}}\zeta_4$ hold, the condition $\Lambda(\zeta) = 0$ is equivalent to $\Lambda_0(\beta_2) = \Lambda_0(\beta_3) = \Lambda_0(\zeta_4) = 0$ and $\zeta_4 = -\frac{\sqrt{-1}}{2}\Lambda_0(\zeta_1)$. We can write $\zeta_1 = \beta_1 - (n-p-q+1)^{-1} L_0(\xi)$ with $\beta_1 \in E_{p,q}$ and $\xi \in \bigwedge^{p-1,q-1} W$ uniquely; see Lemma 1.1. By [Wells 1980, Proposition 1.1(c), p. 160], we have $\Lambda_0 L_0(\xi) = (n-p-q+1)\xi$ and hence $\zeta_4 = \frac{\sqrt{-1}}{2}\xi$. Since $\Lambda_0(\zeta_4) = 0$, $\xi \in E_{p-1,q-1}$. Consequently we arrive at $\zeta = \mathcal{T}^{p,q}(\beta_1, \beta_2, \beta_3, \xi)$ to know the image of $\mathcal{T}^{p,q}$ is $F_{p,q}$. The injectivity of $\mathcal{T}^{p,q}$ follows from $\operatorname{Im} L_0 \cap E_{p,q} = \{0\}$. Since ω_0, $\bar{\omega}_0$ are M_0-invariant and since L_0 is an M_0-homomorphism, the map $\mathcal{T}^{p,q}$ is also an M_0-homomorphism.

The map $\mathscr{T}^{p,p}$ induces a linear isomorphism

$$E^M_{p,p} \oplus E^M_{p-1,p} \oplus E^M_{p,p-1} \oplus E^M_{p-1,p-1} \cong F^M_{p,p} \tag{3}$$

of the M-invariant part.

Claim: $E^M_{p,q} = \{0\}$ *if* $p \neq q$, $p+q < n$. *For* $0 \leqslant p \leqslant r$, $E^M_{p,p} = \mathbb{C}\,\mathsf{u}^{(p)} \neq \{0\}$, *where*

$$\mathsf{u}^{(p)} = \sum_{j=0}^{p} \frac{n-p-j+1}{n-p+1} c_j^{(p)} L_0^{p-j}(\eta^j).$$

Granting this, we easily prove the proposition easily. Indeed, a direct computation yields $\mathsf{v}_0^{(p)} = \mathscr{T}^{p,p}(\mathsf{u}^{(p)}, 0, 0, 0)$ and $\mathsf{v}_1^{(p)} = \mathscr{T}^{p,p}(0, 0, 0, \mathsf{u}^{(p-1)})$. The claim, together with the decomposition (3) implies these tensors form a basis of $F^M_{p,p}$.

We prove the claim. The element $\mathrm{diag}(\mathtt{I}_{n-1}, u_0, u_0)$ ($u_0 \in \mathsf{U}(1)$) of M acts on $E_{p,q}$ by the scalar u_0^{-p+q}. Hence $E^M_{p,q} = \{0\}$ if $p \neq q$.

From now on, let $0 \leqslant p \leqslant r$. Since M is a symmetric subgroup of M_0 and $E_{p,p}$ is an irreducible M_0-module, we have $\dim_{\mathbb{C}}(E^M_{p,p}) \leqslant 1$. Hence to show that $E^M_{p,p} = \mathbb{C}\,\mathsf{u}^{(p)} \neq \{0\}$, it is sufficient to prove that *the element* $\mathsf{u}^{(p)}$ *is* M*-invariant, nonzero and* $\Lambda_0(\mathsf{u}^{(p)}) = 0$.

Since η is M-invariant and L_0 is an M_0-homomorphism, the M-invariance of $\mathsf{u}^{(p)}$ is clear. We prove that it is nonzero using the expression

$$\mathsf{u}^{(p)} = \sum_{\alpha=0}^{p} \left(\sum_{j=0}^{\alpha} \frac{n-p-j+1}{n-p+1} c_j^{(p)} \binom{p-j}{\alpha-j} \right) \omega_H^{p-\alpha} \wedge \eta^\alpha \tag{4}$$

which we get from the original formula for $\mathsf{u}^{(p)}$ by substituting the binomial expansion of $L_0^{p-j} = e((\omega_H + \eta)^{p-j})$. It is easy to see that the tensors $\omega_H^{p-\alpha} \wedge \eta^\alpha$ are linearly independent. The coefficient of ω_H^p in expression (4) is $c_0^{(p)}$, which is nonzero. Hence we conclude $\mathsf{u}^{(p)} \neq 0$.

There remains to show that $\Lambda_0(\mathsf{u}^{(p)}) = 0$. By [Wells 1980, Proposition 1.1(c), p. 160] and (1), we have

$$\begin{aligned}\Lambda_0 L_0^{p-j}(\eta^j) &= L_0^{p-j}\Lambda_0(\eta^j) + (p-j)(n-j-p)\, L_0^{p-j-1}(\eta^j)\\ &= j(r-j)\, L_0^{p-j}(\eta^{j-1}) + (p-j)(n-j-p)\, L_0^{p-j-1}(\eta^j).\end{aligned}$$

Hence $\Lambda_0(\mathsf{u}^{(p)})$ equals $(n-p+1)^{-1}$ times

$$\sum_{j=0}^{p-1} (n-p-j)\big((j+1)(r-j-1)\, c_{j+1}^{(p)} + (n-p-j+1)(p-j)\, c_j^{(p)}\big)\, L_0^{p-j-1}(\eta^j).$$

We can easily check that the coefficient of each $L_0^{p-j-1}(\eta^j)$ in the expression above is zero. □

For convenience, we put $\mathsf{v}_0^{(0)} = 1$, $\mathsf{v}_1^{(0)} = 0$; these are elements of $F_{0,0} = \mathbb{C}$.

We also need the structure of the $(H \cap K)$-invariant part of $F_{p,p}$ $(0 \leqslant p < r)$.

Proposition 1.3. *For each* $0 < p \leqslant r-1$, *put*

$$\mathsf{w}^{(p)} = \sum_{j=0}^{p} (r-j) c_j^{(p)} L^{p-j}((\omega - \omega_H)^j).$$

For $p = 0$, *we set* $\mathsf{w}^{(0)} = r$. *Then*

$$\mathsf{w}^{(p)} = \frac{(r-p)(n-p+1)}{n-2p+1} \mathsf{v}_0^{(p)} - p(n-p-r+1) \mathsf{v}_1^{(p)}, \tag{5}$$

and $F_{p,p}^{H \cap K} = \mathbb{C}\, \mathsf{w}^{(p)} \neq \{0\}$.

Proof. The $(H \cap K)$-invariance of $\mathsf{w}^{(p)}$ follows from the fact that $\omega - \omega_H$ is $(H \cap K)$-invariant and L is a K-homomorphism. The formula (5) follows from a direct computation. Since $H \cap K = \mathsf{U}(n-r) \times \mathsf{U}(r) \times \mathsf{U}(1)$ is a symmetric subgroup of $K = \mathsf{U}(n) \times \mathsf{U}(1)$, we have $\dim(F_{p,p}^{H \cap K}) \leqslant 1$. Since $\mathsf{w}^{(p)} \neq 0$ belongs to $F_{p,p}^{H \cap K}$, the last assertion follows. □

For convenience, we put $\mu = r-1$ and $l = n-2r+2$. Then our assumption $2 \leqslant r < n/2$ is equivalent to $\mu \geqslant 1$ and $\lambda \geqslant 3$.

Proposition 1.4. *Define* $\mathsf{w} := (\omega - \omega_H)^{\mu}$ *and* $\mathrm{vol}_H = \frac{1}{(n-r)!} \omega_H^{n-r}$. *We have*

$$*\mathrm{vol}_H = \frac{1}{r!} \mathsf{w} \wedge (\omega - \omega_H), \qquad \mathsf{w} \wedge \Big(\eta - \frac{\sqrt{-1}}{2} \mu\, \omega_0 \wedge \bar{\omega}_0\Big) = 0, \tag{6}$$

$$\Lambda(*\mathrm{vol}_H) = \frac{1}{\mu\,!} \mathsf{w}, \qquad \mathsf{w} = \sum_{p=0}^{\mu} \gamma_p\, L^{\mu-p}(\mathsf{w}^{(p)}), \tag{7}$$

where the γ_p, $0 \leqslant p \leqslant \mu$, *are real numbers defined by the recurrence relation*

$$\gamma_\mu = \frac{1}{c_\mu^{(\mu)}}, \quad \gamma_j\, c_j^{(j)} = -\sum_{p=j+1}^{\mu} \gamma_p\, c_j^{(p)}, \quad 0 \leqslant j < \mu.$$

Proof. Using [Wells 1980, Lemma 1.2, p. 161], we easily check the first formula in (6) by a computation. The second follows directly from $\eta^r = 0$ and $\mathsf{w} = \eta^{\mu} + \frac{\sqrt{-1}}{2} \mu\, \omega_0 \wedge \bar{\omega}_0 \wedge \eta^{\mu-1}$. The first formula in (7) is a consequence of (6) and [Wells 1980, (1.5), p. 163]. To justify the last formula, note that its right-hand side equals

$$\sum_{j=0}^{\mu} \Big(\sum_{p=j}^{\mu} \gamma_p\, c_j^{(p)} \Big) (r-j) L^{\mu-j} (\omega - \omega_H)^j;$$

this equals w by the definition of w and γ_j. □

2. Radial parts of several differential operators

We define $2n-1$ elements Z_α, $\bar{Z}_\alpha$ and Z_0 of $\mathfrak{k}_{\mathbb{C}}$ by

$$Z_\alpha = \mathrm{E}_{\alpha,n}, \quad \bar{Z}_\alpha = -\mathrm{E}_{n,\alpha}, \quad 1 \leqslant \alpha \leqslant n-1,$$
$$Z_0 = \sqrt{-1}(\mathrm{E}_{n,n} - \mathrm{E}_{n+1,n+1}).$$

Note that $(\mathfrak{k} \cap \mathfrak{h})_{\mathbb{C}}$ is a direct sum of $\mathrm{Lie}(M)_{\mathbb{C}}$ and the $\mathbb{C}$-span of the $2r-1$ elements $Z_j,\ \bar{Z}_j\ (n-r+1 \leqslant j \leqslant n-1),\ Z_0$.

Lemma 2.1. *For* $1 \leqslant \alpha, \beta \leqslant n-1$, *we have*

$$\tau(Z_\alpha)\bar{\omega}_\beta = 0, \qquad \tau(Z_\alpha)\omega_\beta = -\delta_{\alpha\beta}\,\omega_0,$$
$$\tau(Z_\alpha)\bar{\omega}_0 = \bar{\omega}_\alpha, \qquad \tau(Z_\alpha)\omega_0 = 0,$$
$$\tau(Z_0)\omega_\alpha = -\sqrt{-1}\omega_\alpha, \quad \tau(Z_0)\omega_0 = -2\sqrt{-1}\omega_0. \tag{8}$$

For $n-r+1 \leqslant \alpha \leqslant n-1$, *we have*

$$\tau(Z_\alpha)\eta = -\tfrac{\sqrt{-1}}{2}\omega_0 \wedge \bar{\omega}_\alpha, \quad \tau(\bar{Z}_\alpha)\eta = -\tfrac{\sqrt{-1}}{2}\omega_\alpha \wedge \bar{\omega}_0.$$

Definition. Let C^∞_τ be the space of C^∞-functions $\varphi : G - HK \to \bigwedge \mathfrak{p}^*_{\mathbb{C}}$ having the (H, K)-equivariance property $\varphi(hgk) = \tau(k)^{-1}\varphi(g)$ for $h \in H$, $k \in K$.

Lemma 2.2. *Let* $i, \nu \in \{1, \ldots, n-r\}$ *and* $j, \mu \in \{n-r+1, \ldots, n-1\}$. *Let* $\varphi \in C^\infty_\tau$. *Then*:

$$R_{X_i\bar{X}_\nu}\varphi(a_t) = \Big(\frac{\delta_{i\nu}}{2}\tanh t\frac{d}{dt} + \tau(Z_i\bar{Z}_\nu)\tanh^2 t + \frac{\sqrt{-1}}{4}\delta_{i\nu}(1+\tanh^2 t)\tau(Z_0)\Big)\varphi(a_t) \tag{9}$$

$$R_{X_i\bar{X}_0}\varphi(a_t) = \frac{\tau(Z_i)}{2}\Big(\tanh t\frac{d}{dt} + \frac{\sqrt{-1}}{2}(\tanh^2 t+1)\tau(Z_0)\Big)\varphi(a_t)$$

$$R_{X_0\bar{X}_i}\varphi(a_t) = \frac{\tau(\bar{Z}_i)}{2}\Big(\tanh t\frac{d}{dt} + 2 - \frac{\sqrt{-1}}{2}(\tanh^2 t+1)\tau(Z_0)\Big)\varphi(a_t)$$

$$R_{X_0\bar{X}_0}\varphi(a_t) = \Big(\frac{1}{4}\frac{d^2}{dt^2} + \frac{1}{4}(\tanh t + \coth t)\frac{d}{dt} + \frac{\sqrt{-1}}{2}\tau(Z_0) + \frac{1}{16}(\tanh t + \coth t)^2\tau(Z_0)^2\Big)\varphi(a_t) \tag{10}$$

$$R_{X_i\bar{X}_j}\varphi(a_t) = \tau(Z_i\bar{Z}_j)\varphi(a_t)$$

$$R_{X_j\bar{X}_i}\varphi(a_t) = \tau(Z_j\bar{Z}_i)\varphi(a_t)$$

(11) $$R_{X_j\bar{X}_\mu}\varphi(a_t) = \Big(\frac{\delta_{j\mu}}{2}\coth t\frac{d}{dt} + \tau(Z_j\bar{Z}_\mu)\coth^2 t + \frac{\sqrt{-1}}{4}(1+\coth^2 t)\delta_{j\mu}\tau(Z_0)\Big)\varphi(a_t),$$

$$R_{X_j\bar{X}_0}\varphi(a_t) = \frac{\tau(Z_j)}{2}\Big(\coth t\frac{d}{dt} + \frac{\sqrt{-1}}{2}(1+\coth^2 t)\tau(Z_0)\Big)\varphi(a_t),$$

$$R_{X_0\bar{X}_j}\varphi(a_t) = \frac{\tau(\bar{Z}_j)}{2}\Big(\coth t\frac{d}{dt} + 2 - \frac{\sqrt{-1}}{2}(1+\coth^2 t)\tau(Z_0)\Big)\varphi(a_t).$$

Proof. We prove these formulas through a computation similar to that of [Oda and Tsuzuki 2003, Lemma 7.1.2], using the formulas

$$X_i = -\tanh t\cdot Z_i + \frac{1}{\cosh t}\mathrm{Ad}(a_t)^{-1}X_i, \qquad 1\leqslant i\leqslant n-r,$$

$$X_j = -\coth t\cdot Z_j + \frac{1}{\sinh t}\mathrm{Ad}(a_t)^{-1}Z_j, \qquad n-r+1\leqslant j\leqslant n-1,$$

$$X_0 = \frac{1}{2}Y_0 + \coth(2t)\frac{\sqrt{-1}}{2}Z_0 - \frac{\sqrt{-1}}{2\sinh(2t)}\mathrm{Ad}(a_t)^{-1}Z_0$$

and their complex conjugates. □

Casimir operators. Let Ω, Ω_K and Ω_{M_0} be the Casimir elements of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{m}_0 :=$ Lie(M_0) corresponding to the invariant form $\langle X, Y\rangle = 2^{-1}\mathrm{tr}(XY)$. Since $\{X_\alpha+\bar{X}_\alpha, \sqrt{-1}(X_\alpha-\bar{X}_\alpha)\}_{\alpha=0}^{n-1}$ is an orthonormal basis of $\mathfrak{p}$, we have

(12) $$\Omega = \Omega_K + 2\sum_{\alpha=0}^{n-1}(X_\alpha\bar{X}_\alpha + \bar{X}_\alpha X_\alpha).$$

Since $\{Z_\beta+\bar{Z}_\beta, \sqrt{-1}(Z_\beta-\bar{Z}_\beta)\}_{\beta=1}^{n-1}\cup\{Z_0\}$ is a pseudo-orthonormal basis of the orthogonal complement of $\mathfrak{m}_0$ in $\mathfrak{k}$, we have

(13) $$\Omega_K = \Omega_{M_0} - 2\sum_{\beta=1}^{n-1}(Z_\beta\bar{Z}_\beta + \bar{Z}_\beta Z_\beta) - Z_0^2.$$

Theorem 2.3. *Let $\varphi\in C_\tau^\infty$ be such that its values $\varphi(g)$, $g\in G$, belong to $\bigwedge^{p,p}\mathfrak{p}_{\mathbb{C}}^*$ for $0\leqslant p\leqslant r$. Put $f(z)=\varphi(a_t)$ where $z=\tanh^2 t$. Then $\Omega\varphi(a_t)$ $(t>0)$ equals*

$$4z(1-z)^2\left(\frac{d^2}{dz^2} + \left(\frac{\mu+1}{z} + \frac{n-1}{1-z}\right)\frac{d}{dz} - \frac{\mathcal{S}}{z(1-z)} + \frac{\mathcal{S}'}{z^2(1-z)} + \frac{\tau(\Omega_{M_0})}{4z(1-z)^2}\right) f(z),$$

where $\mathcal{S}$ and $\mathcal{S}'$ are operators on $\bigwedge\mathfrak{p}_{\mathbb{C}}^*$ defined by

$$\mathcal{S} := 2^{-1}\sum_{i=1}^{n-r}\tau(Z_i\bar{Z}_i + \bar{Z}_i Z_i) \quad \textit{and} \quad \mathcal{S}' := 2^{-1}\sum_{j=n-r+1}^{n-1}\tau(Z_j\bar{Z}_j + \bar{Z}_j Z_j).$$

Proof. By (8) and Proposition 1.2, the operator $\tau(Z_0)$ is zero on the space

$$(\textstyle\bigwedge^{p,p}\mathfrak{p}^*_{\mathbb{C}})^M = \langle L^{p-j}\mathsf{v}_0^{(j)}, L^{p-j}\mathsf{v}_1^{(j)} \mid 0 \leqslant j \leqslant p\rangle_{\mathbb{C}},$$

to which the values $\varphi(a_t)$ $(t > 0)$ belong by the (H, K)-equivariance of φ. Hence under the assumption, we have $\tau(Z_0)\varphi(a_t) = 0$ identically. We also note that the center of K acts on $\bigwedge^{p,p}\mathfrak{p}^*_{\mathbb{C}}$ trivially. Noting these remarks and using the fact that $\sum_{\alpha=0}^{n-1}[X_\alpha, \bar{X}_\alpha]$ is in the center of $\mathfrak{k}_{\mathbb{C}}$, we obtain the following expression of $\Omega\varphi(a_t)$ from (12) and (13).

$$\Omega\varphi(a_t) = 4\sum_{\alpha=0}^{n-1} R_{X_\alpha\bar{X}_\alpha}\varphi(a_t) - 2\sum_{\beta=0}^{n-1}\tau(Z_\beta\bar{Z}_\beta + \bar{Z}_\beta Z_\beta)\varphi(a_t) + \tau(\Omega_{M_0})\varphi(a_t).$$

We can compute the first n terms of the right-hand side of this equation using the formulas (9), (10) and (11) to obtain

$$\Omega\varphi(a_t) = \left(\frac{d^2}{dt^2} + \left((2n-2r+1)\tanh t + \frac{2r-1}{\tanh t}\right) - \frac{4\mathscr{S}}{\cosh^2 t}\frac{d}{dt} + \frac{4\mathscr{S}'}{\sinh^2 t} + \tau(\Omega_{M_0})\right)\varphi(a_t). \tag{14}$$

The conclusion follows since

$$\frac{d^2}{dt^2} = 4z(1-z)^2\frac{d^2}{dz^2} + 2(1-z)(1-3z)\frac{d}{dz}, \qquad \frac{d}{dt} = 2z^{1/2}(1-z)\frac{d}{dz}. \tag{15}$$ □

Lemma 2.4. *Let* $0 < p \leqslant \mu$ *and put* $l = n - p - r + 1$ *and* $m = r - p$. *Then*:

$$\frac{1}{mp}\mathscr{S}\mathsf{v}_0^{(p)} = -\frac{l+m}{m(l+p+m)}\mathscr{S}\mathsf{v}_1^{(p)} = \frac{-1}{l+m}\mathsf{v}_0^{(p)} + \frac{l}{m}\mathsf{v}_1^{(p)}, \tag{16}$$

$$-\frac{1}{lp}\mathscr{S}'\mathsf{v}_0^{(p)} = -\frac{l+m}{m(l+p+m)}\mathscr{S}'\mathsf{v}_1^{(p)} = \frac{1}{l+m}\mathsf{v}_0^{(p)} + \mathsf{v}_1^{(p)}, \tag{17}$$

$$\tau(\Omega_{M_0})\mathsf{v}_0^{(p)} = 4p(n-p)\,\mathsf{v}_0^{(p)} \text{ and } \tau(\Omega_{M_0})\mathsf{v}_1^{(p)} = 4(p-1)(n-p+1)\,\mathsf{v}_1^{(p)}. \tag{18}$$

Proof. As shown in the proof of Proposition 1.2, the tensors $\mathsf{v}_0^{(p)}$ and $\mathsf{v}_1^{(p)}$ are obtained as the image of $\mathsf{u}^{(p)} \in E_{p,p}$ and $\mathsf{u}^{(p-1)} \in E_{p-1,p-1}$ respectively by the M_0-homomorphism $\mathscr{T}^{p,p}$. Hence in order to prove (18) we have only to check that the eigenvalue of Ω_{M_0} on $E_{p,p}$ equals $4p(n-p)$. The eigenvalue is easily calculated if we note that the highest weight of an irreducible $M_0 = (\mathsf{U}(n-1)\times\mathsf{U}(1))$-module $E_{p,p}$ is

$$(\overbrace{1,\ldots,1}^{p}, \overbrace{0,\ldots,0}^{n-2p-1}, \overbrace{-1\ldots,-1}^{p}; 0).$$

We prove the formula (17). Since $\mathsf{w}^{(p)}$ is $(H\cap K)$-invariant and since the elements $\bar{Z}_j$, Z_j $(n-r+1 \leqslant j \leqslant n-1)$ belong to $(\mathfrak{h}\cap\mathfrak{k})_{\mathbb{C}}$, we have $\mathscr{S}'\mathsf{w}^{(p)} = 0$. This together with (5) gives the first equality of (17). To obtain the second equality we need some computation. From definition we have the expressions

$$\begin{aligned}(19)\qquad \mathsf{v}_1^{(p)} &= \frac{-1}{n-2p+1} c_p^{(p)} \tfrac{\sqrt{-1}}{2} (r-p)\, \omega_0 \wedge \bar{\omega}_0 \wedge \eta^{p-1} + L(\xi_1),\\ \mathsf{v}_0^{(p)} &= \frac{1}{n-p+1} c_p^{(p)} \left((n-2p+1)\eta^p + \tfrac{\sqrt{-1}}{2} p(r-p)\, \omega_0 \wedge \bar{\omega}_0 \wedge \eta^{p-1}\right) + L(\xi_0),\end{aligned}$$

for some ξ_0 and ξ_1. Using Lemma 2.1 and (2), we compute that

$$\begin{aligned}\tau(\bar{Z}_j Z_j)(\omega_0 \wedge \bar{\omega}_0 \wedge \eta^{p-1}) &= \omega_j \wedge \bar{\omega}_j \wedge \eta^{p-1} + \omega_0 \wedge (-\bar{\omega}_0) \wedge \eta^{p-1}\\ &\quad - \tfrac{\sqrt{-1}}{2} (p-1)\, \omega_0 \wedge \bar{\omega}_j \wedge \omega_j \wedge \bar{\omega}_0 \wedge \eta^{p-2}\end{aligned}$$

for $n-r+1 \leqslant j \leqslant n-1$. Taking the sum over j, we obtain

$$\sum_{j=n-r+1}^{n-1} \tau(\bar{Z}_j Z_j)(\omega_0 \wedge \bar{\omega}_0 \wedge \eta^{p-1}) = -(r-p)\, \omega_0 \wedge \bar{\omega}_0 \wedge \eta^{p-1} + \tfrac{2}{\sqrt{-1}} \eta^p.$$

We use (19) to write the right-hand side in terms of $\mathsf{v}_0^{(p)}$ and $\mathsf{v}_1^{(p)}$:

$$\begin{aligned}\sum_{j=n-r+1}^{n-1} &\tau(\bar{Z}_j Z_j)(\omega_0 \wedge \bar{\omega}_0 \wedge \eta^{p-1})\\ &= \frac{2}{\sqrt{-1}} \frac{n-p+1}{c_p^{(p)}} \left(\frac{1}{n-2p+1} \mathsf{v}_0^{(p)} + \mathsf{v}_1^{(p)}\right) + L(\xi_2)\end{aligned}$$

for some ξ_2. From this and the first equality in (19), we obtain

$$(20)\qquad \sum_{j=n-r+1}^{n-1} \tau(\bar{Z}_j Z_j) \mathsf{v}_1^{(p)} = -\frac{(r-p)(n-p+1)}{n-2p+1} \left(\frac{1}{n-2p+1} \mathsf{v}_0^{(p)} + \mathsf{v}_1^{(p)}\right) + L(\xi_3)$$

for some ξ_3. Since all terms in this identity except $L(\xi_3)$ belong to $F_{p,p}$ and since $F_{p,p} \cap \operatorname{Im} L = \{0\}$, the residual term $L(\xi_3)$ has to be zero. Noting that $\mathsf{v}_0^{(p)}$ and $\mathsf{v}_1^{(p)}$ are real elements of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$, we know that $2\mathcal{S}' \mathsf{v}_1^{(p)}$ is given by the sum of (20) and its complex conjugate. This completes the proof of the second identity of (17). We can deduce (16) from (17) and (18) using the relation

$$\tau(\Omega_K) \mathsf{v}_i^{(p)} = \tau(\Omega_{M_0}) \mathsf{v}_i^{(p)} - 4\mathcal{S} \mathsf{v}_i^{(p)} - 4\mathcal{S}' \mathsf{v}_i^{(p)}$$

obtained from (13), since the eigenvalue of Ω_K on $F_{p,p}$ equals $4p(n+1-p)$. □

Let $0 < p \leqslant \mu$ and $\varphi \in C_\tau^\infty$ be such that its values $\varphi(g)$ belong to $F_{p,p}$. By the (H, K)-equivariance, the vector $\varphi(a_t)$ belongs to the space of M-fixed tensors $F_{p,p}^M = \mathbb{C}\, \mathsf{v}_0^{(p)} \oplus \mathbb{C}\, \mathsf{v}_1^{(p)}$. We can write

$$(21)\qquad \varphi(a_t) = f_0(z)\, \mathsf{v}_0^{(p)} + f_1(z)\, \mathsf{v}_1^{(p)} = [\mathsf{v}_0^{(p)}\ \mathsf{v}_1^{(p)}] \mathsf{F}(z), \quad z = \tanh^2 t \in (0,1),$$

with a $\mathbb{C}^2$-valued C^∞-function

$$\mathsf{F}(z) = \begin{bmatrix} f_0(z) \\ f_1(z) \end{bmatrix}$$

on $0 < z < 1$. Given a complex number s, consider the differential equation

$$\Omega\varphi(g) = (s^2 - \lambda^2)\varphi(g), \quad g \in G - HK. \tag{22}$$

We rewrite this equation in terms of $\mathsf{F}(z)$.

Proposition 2.5. *Let* $\mathsf{Q}^{(p)}(z) := \begin{bmatrix} Q_0^0(z) & Q_0^1(z) \\ Q_1^0(z) & Q_1^1(z) \end{bmatrix}$ *with*

$$Q_0^0(z) = \frac{1}{z^2(1-z)^2}\left(-\frac{pm}{l+m}z^2 + \bigl(p-(m-1)l+\tfrac{1}{4}(n^2-s^2)\bigr)z - \frac{lp}{l+m}\right),$$

$$Q_0^1(z) = -\frac{m(l+p+m)}{(l+m)^2}\frac{1+z}{z^2(1-z)},$$

$$Q_1^0(z) = -lp\frac{1+z}{z^2(1-z)},$$

$$Q_1^1(z) = \frac{1}{z^2(1-z)^2}\left(-\frac{l(l+p+m)}{l+m}z^2 + \bigl(p-(m-1)l+\tfrac{1}{4}(n^2-s^2)\bigr)z - \frac{m(l+p+m)}{l+m}\right).$$

Then $\varphi \in C_\tau^\infty$ with values in $F_{p,p}$ is a solution of (22) *if and only if* $\mathsf{F}(z)$ $(0 < z < 1)$ *satisfies the second order ordinary differential equation*

$$\frac{d^2\mathsf{F}}{dz^2} + \left(\frac{\mu+1}{z} + \frac{n-1}{1-z}\right)\frac{d\mathsf{F}}{dz} + \mathsf{Q}^{(p)}(z)\mathsf{F} = 0. \tag{23}$$

Proof. This follows from Theorem 2.3 and Lemma 2.4. □

The Schmid operator. Let $\nabla_\pm$ be the Schmid operator, which is given by

$$\nabla_+ f(g) = \sum_{i=0}^{n-1} e(\overline{\omega}_i)R_{\overline{X}_i} f(g) \quad \text{and} \quad \nabla_- f(g) = \sum_{i=0}^{n-1} e(\omega_i)R_{X_i} f(g)$$

for a $\bigwedge\mathfrak{p}_\mathbb{C}^*$-valued C^∞-function f on an open subset of G. To describe the radial part of the Schmid operator, we introduce four operators acting on $\bigwedge\mathfrak{p}_\mathbb{C}^*$:

$$\mathcal{P}_+ = \sum_{i=1}^{n-r} e(\bar{\omega}_i)\tau(\bar{Z}_i), \quad \mathcal{P}_- = \sum_{i=1}^{n-r} e(\omega_i)\tau(Z_i),$$
$$\mathcal{R}_+ = \sum_{j=n-r+1}^{n-1} e(\bar{\omega}_j)\tau(\bar{Z}_j), \quad \mathcal{R}_- = \sum_{j=n-r+1}^{n-1} e(\omega_j)\tau(Z_j).$$

Theorem 2.6. *Let $\varphi \in C_\tau^\infty$ and $f(z)$ be the same as in Theorem 2.3. Then for $t > 0$, $\nabla_-\nabla_+\varphi(a_t)$ equals*

$$(24) \quad z(1-z)^2\biggl(e(\omega_0 \wedge \bar{\omega}_0)\,\frac{d^2}{dz^2} - \left(\frac{\mathcal{A}}{z} + \frac{\mathcal{B}}{1-z}\right)\frac{d}{dz} + \frac{\mathcal{P}_-\mathcal{P}_+}{(1-z)^2} + \frac{\mathcal{R}_-\mathcal{R}_+}{z^2(1-z)^2} + \frac{\mathcal{C}}{z(1-z)^2}\biggr) zf(z)$$

where the numerators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given by

$$\mathcal{A} = -\tfrac{2}{\sqrt{-1}} e(\omega - \omega_H) + e(\bar{\omega}_0)\mathcal{R}_- - e(\omega_0)\mathcal{R}_+,$$
$$\mathcal{B} = -\tfrac{\sqrt{-1}}{2} e(\omega) + e(\omega_0 \wedge \bar{\omega}_0) + e(\bar{\omega}_0)(\mathcal{P}_- + \mathcal{R}_-) - e(\omega_0)(\mathcal{P}_+ + \mathcal{R}_+),$$
$$\mathcal{C} = e(\omega_0)(\mathcal{P}_+ + \mathcal{R}_+) + \mathcal{P}_-\mathcal{R}_+ + \mathcal{R}_-\mathcal{P}_+.$$

Proof. Input the formulas in Lemma 2.2 to the right-hand side of the identity

$$\nabla_-\nabla_+\varphi(a_t) = \sum_{\alpha,\beta=0}^{n-1} R_{X_\alpha \bar{X}_\beta}\varphi(a_t) \wedge \omega_\alpha \wedge \bar{\omega}_\beta.$$

Then a direct computation using (15) yields (24). □

3. Secondary spherical functions

Before we state the main theorem of this section, we prove a lemma which is important not only in this section but also in the global theory to be developed in Section 5.

Lemma 3.1. *For each integer p with $1 \leqslant p \leqslant r$, there exists a unique holomorphic function $s \mapsto \nu_s^{(p)}$ on the domain $\mathbb{C} - L_p$ with*

$$(25) \qquad L_p = \bigl\{ s \in \sqrt{-1}\,\mathbb{R} \mid 2\sqrt{(r-p)(n-p-r+2)} \geqslant |\mathrm{Im}\, s| \bigr\},$$

which takes a positive real value for $s > 0$ and such that

$$\{\nu_s^{(p)}\}^2 = s^2 + 4(r-p)(n-p-r+2).$$

The functional equation $\nu_{-s}^{(p)} = -\nu_s^{(p)}$ holds for $s \in \mathbb{C} - L_p$. Also, if $\mathrm{Re}\, s > 0$, *then* $\mathrm{Re}\, \nu_s^{(p)} > \mathrm{Re}\, \nu_s^{(p+1)} > \mathrm{Re}\, s$.

Proof. Put $c_p = 4(r-p)(n-p-r+2)$. By the residue theorem, the value (mod $2\pi\sqrt{-1}\mathbb{Z}$) of the integral

$$I(s) := \int_{C_s} \frac{\zeta}{\zeta^2 + c_p}\, d\zeta$$

is independent of the choice of the path C_s connecting 1 and s inside the region $\mathbb{C} - L_p$, and locally defines a holomorphic function on $\mathbb{C} - L_p$. Since $\nu_s^{(p)} = \exp(I(s))\sqrt{c_p+1}$ for $s \in \mathbb{C} - L_p$, the first assertion follows. Put $\sigma = \mathrm{Re}\, s$ and $t = \mathrm{Im}\, s$ and suppose $\sigma > 0$. Then we have

$$\mathrm{Re}\, \nu_s^{(p)} = 2^{-1/2}\big(x_{p,s} + \sqrt{x_{p,s}^2 + y_s^2}\,\big)^{1/2},$$

with $x_{p,s} = \sigma^2 - t^2 + c_p$, $y_s = 2\sigma t$. Since $c_p > c_{p+1}$ and $x_{p,s} > x_{p+1,s}$, we have $\mathrm{Re}\, \nu_s^{(p)} > \mathrm{Re}\, \nu_s^{(p+1)}$. The formula $\mathrm{Re}\, \nu_s^{(p)} = \big(\sigma^2 + 2^{-1}(\sqrt{T^2 + 4c_p\sigma^2} - T)\big)^{1/2}$ with $T = t^2 + \sigma^2 - c_p$ obviously implies the inequality $\mathrm{Re}\, \nu_s^{(p)} > \sigma$.

To prove $\nu_{-s}^{(p)} = -\nu_s^{(p)}$, take a path $C_{s,-s}$ lying in the domain $\mathbb{C} - L_p$ from s to $-s$ and C' the image of $C_{s,-s}$ by the map $z = \zeta^2 + c_p$. Then C' is a simple loop rounding the point $\zeta = 0$. By definition, $\nu_s^{(p)} = \exp(I(s) - I(-s))\nu_{-s}^{(p)}$. Since

$$I(s) - I(-s) = \int_{C_{s,-s}} \frac{\zeta}{\zeta^2 + c_p}\, d\zeta = \frac{1}{2}\int_{C'} \frac{dz}{z} = \pm\pi\sqrt{-1},$$

we have $\nu_{-s}^{(p)} = -\nu_s^{(p)}$. □

Consider the holomorphic function

$$d(s) := \prod_{p=1}^{r} \Gamma(\nu_s^{(p)})^{-1}\Gamma(2^{-1}(\nu_s^{(p)} - \lambda) + 1)^{-1}, \quad s \in \mathbb{C} - L_1,$$

and define the related sets

$$D = \{s \in \mathbb{C} - L_1 \mid d(s) \neq 0\}, \quad \tilde{D} = \bigcap_{p=1}^{\mu} \{s \in D \mid \mathrm{Re}\, \nu_s^{(p)} + \mathrm{Re}\, \nu_s^{(p+1)} > 4\}.$$

Note that $\{s \in \mathbb{C} \mid \mathrm{Re}\, s > \lambda\} \subset \tilde{D}$.

The aim of this section is to prove the following theorem.

Theorem 3.2. *There is a unique family of C^∞-functions $\phi_s : G - HK \to \bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*$, s varying over $\tilde{D}$, satisfying the following conditions.*

(i) *For each $g \in G - HK$, the function $s \mapsto \phi_s(g)$ is holomorphic.*

(ii) *ϕ_s has the (H, K)-equivariance property*

$$\phi_s(hgk) = \tau(k)^{-1}\phi_s(g), \quad h \in H,\ k \in K,\ g \in G - HK.$$

(iii) ϕ_s *satisfies the differential equation*

$$\Omega\phi_s(g) = (s^2 - \lambda^2)\phi_s(g), \quad g \in G - HK. \tag{26}$$

(iv) (Recall the $(H \cap K)$-invariant tensor $\mathsf{w} \in \bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*$ defined in Proposition 1.4.) *We have*

$$\lim_{t \to +0} t^{2\mu}\phi_s(a_t) = \mathsf{w}.$$

(v) *If* Re $s > n$, *then* $\phi_s(a_t)$ *decays exponentially as* $t \to +\infty$.

We call the function ϕ_s *the secondary spherical function*.

Differential equations. In this subsection, we fix an $s \in D$. Lemma 1.1 yields the decomposition

$$\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^* = \sum_{p=0}^{\mu} L^{\mu-p}(F_{p,p}) \tag{27}$$

of K-modules. Hence in order to obtain the function ϕ_s as in Theorem 3.2, we have only to construct a function $\phi_s^{(p)} : G - HK \to F_{p,p}$ for each $0 \leqslant p \leqslant \mu$ with the same properties as ϕ_s listed in Theorem 3.2 except condition (iv). Instead we require

$$\lim_{t \to +0} t^{2\mu}\phi_s^{(p)}(a_t) = \mathsf{w}^{(p)},$$

and then form $\phi_s(g) = \sum_{p=0}^{\mu} \gamma_p\, L^{\mu-p}(\phi_s^{(p)}(g))$.

From now on we fix $0 \leqslant p \leqslant \mu$ and examine the conditions to be satisfied by $\phi_s^{(p)}$. We use the notation $l = n - p - r + 1$, $m = r - p$ as in Lemma 2.4.

The case $p > 0$**.** Since $\phi_s^{(p)}$ should be a solution of (22), we first analyze the differential equation (23) in some detail.

Proposition 3.3. *Consider a* $\mathbb{C}^2$*-valued* C^∞*-function* $\mathsf{F}(z) = \begin{bmatrix} f_0(z) \\ f_1(z) \end{bmatrix}$, $0 < z < 1$. *The following conditions on* $\mathsf{F}(z)$ *are equivalent.*

(a) $\mathsf{F}(z)$ *is a solution of* (23) *and also satisfies*

$$f_0(z) = -\frac{z}{lp}\left(\frac{d}{dz} + \frac{m(l+p+m)}{l+m}\,\frac{1}{z} + \frac{l+p+m}{1-z}\right) f_1(z). \tag{28}$$

(b) $\mathsf{F}(z)$ *is a solution of*

$$\frac{d\mathsf{F}}{dz} = \mathsf{B}^{(p)}(z)\mathsf{F} \tag{29}$$

where $\mathsf{B}^{(p)}(z) = \begin{bmatrix} B_0^0(z) & B_0^1(z) \\ B_1^0(z) & B_1^1(z) \end{bmatrix}$ *with entries*

$$B_0^0(z) = -\frac{lp}{l+m}\,\frac{1}{z} - \frac{p}{1-z}, \qquad B_1^0(z) = -\frac{lp}{z},$$

$$B_0^1(z) = -\frac{m(l+p+m)}{(l+m)^2}\,\frac{1}{z} - \frac{s^2-(l-m+1)^2}{4lp}\,\frac{1}{(1-z)^2},$$

$$B_1^1(z) = -\frac{m(l+p+m)}{l+m}\,\frac{1}{z} - \frac{l+p+m}{1-z}.$$

(c) *The entries* f_0 *and* f_1 *of* $\mathsf{F}(z)$ *satisfy* (28) *and*

$$\left(\frac{d^2}{dz^2} + \left(\frac{\mu+2}{z} + \frac{n+1}{1-z}\right)\frac{d}{dz} + \frac{(n+2)^2 - 4l(m-1) - s^2}{4}\left(\frac{1}{z} + \frac{1}{1-z} + \frac{1}{(1-z)^2}\right)\right) f_1(z) = 0. \tag{30}$$

Proof. This follows by direct computation. □

Remark. When $\mathsf{F}(z)$ is related to some $\varphi \in C_\tau^\infty$ by the relation (21), the equation (28) comes from $\mathrm{Pr}(\nabla_+\varphi(a_t)) = 0$, where $\mathrm{Pr} : F_{p,p} \otimes \mathfrak{p}_-^* \to F_{p,p-1}$ is a certain K-projector.

Lemma 3.4. *Let* $\mathsf{F}_0(z)$ *and* $\mathsf{F}_1(z)$ *be two solutions of* (29) *and form the* 2×2*-matrix valued function* $\Phi(z) := [\mathsf{F}_0(z)\ \mathsf{F}_1(z)]$. *Consider a* $\mathbb{C}^2$*-valued* C^∞*-function* $\mathsf{A}(z)$ *on* $0 < z < 1$. *Then the following two conditions on* $\mathsf{A}(z)$ *are equivalent.*

(a) *The function* $\tilde{\mathsf{F}}(z) = \Phi(z)\,\mathsf{A}(z)$ *is a solution of the differential equation* (23).

(b) *The function* $\mathsf{U}(z) = \Phi(z)\dfrac{d\mathsf{A}}{dz}(z)$ *satisfies the differential equation*

$$\frac{d\mathsf{U}}{dz} = -\left(\mathsf{B}^{(p)}(z) + \left(\frac{\mu+1}{z} + \frac{n-1}{1-z}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\mathsf{U}. \tag{31}$$

Proof. Put $p(z) = (\mu+1)z^{-1} + (n-1)(1-z)^{-1}$. Since $\Phi(z)$ satisfies

$$\frac{d^2\Phi}{dz^2} + p(z)\,\frac{d\Phi}{dz} + \mathsf{Q}^{(p)}(z)\Phi = 0 \quad \text{and} \quad \frac{d\Phi}{dz} = \mathsf{B}^{(p)}(z)\Phi,$$

we have

$$\left(\frac{d^2}{dz^2} + p(z)\,\frac{d}{dz} + \mathsf{Q}^{(p)}(z)\right)(\Phi(z)\mathsf{A}(z)) = \left(\frac{d}{dz} + \left(\mathsf{B}^{(p)}(z) + p(z)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\right)\left(\Phi(z)\frac{d\mathsf{A}(z)}{dz}\right),$$

by a direct computation. This identity proves the equivalence of (a) and (b). □

Lemma 3.5. *Consider a $\mathbb{C}^2$-valued C^∞-function* $\mathsf{U}(z) = \begin{bmatrix} u_0(z) \\ u_1(z) \end{bmatrix}$ *on* $0 < z < 1$. *Then* U *is a solution of* (31) *if and only if it satisfies*

$$\frac{z}{lp}\left(\frac{d}{dz} + \frac{lp}{l+m}\,\frac{1}{z} + \frac{p-2}{1-z}\right) u_1(z) = u_0(z)$$

and

$$\left(\frac{d^2}{dz^2} + \left(\frac{\mu+2}{z} + \frac{n-3}{1-z}\right)\frac{d}{dz} + \frac{n^2-s^2-4l(m-1)-8(m+1)}{4}\left(\frac{1}{z} + \frac{1}{1-z}\right)\right.$$
$$\left. + \frac{n^2-s^2-4l(m-1)-8(n-p)}{4}\,\frac{1}{(1-z)^2}\right) u_1(z) = 0.$$

Proof. This is a direct computation. □

The case $p = 0$. Since the function $\phi_s^{(0)}$ is $F_{0,0} = \mathbb{C}$-valued, we can write, thanks to (H, K)-equivariance,

$$\phi_s^{(0)}(a_t) = f_0(z), \quad z = \tanh^2 t,$$

with a C^∞-function $f_0(z)$ on $0 < z < 1$. From (22) we obtain:

Proposition 3.6. *The function* $f_0(z)$ *satisfies the differential equation*

$$\left(\frac{d^2}{dz^2} + \left(\frac{\mu+1}{z} + \frac{n-1}{1-z}\right)\frac{d}{dz} + \frac{1}{z(1-z)^2}\,\frac{\lambda^2 - s^2}{4}\right) f_0(z) = 0. \tag{32}$$

Construction of solutions. Let N be a nonnegative integer and γ_i $(0 \leqslant i \leqslant N)$ complex numbers. For a function $f(s, z)$ on $\{(s, z) \mid s \in D,\ 0 < z < 1\}$, if there exist a meromorphic function $q(s)$ on $\mathbb{C} - L_1$ holomorphic on D and a family of holomorphic functions $\alpha(s)$, $a_i(s; z)$ $(0 \leqslant i \leqslant N)$ on $\{(s, z) \mid s \in \mathbb{C} - L_1,\ |z| < \varepsilon\}$ with some $\varepsilon > 0$ such that

$$f(s; z) = q(s) z^{\alpha(s)}\left(1 + \sum_{i=0}^{N} z^{\gamma_i} (\log z)^i\, a_i(s; z)\right), \quad 0 < z < \varepsilon,\ s \in D,$$

then we write $f(s; z) = q(s) z^{\alpha(s)}(1 + O(s; z^{\gamma_0}, z^{\gamma_1}\log z, \dots, z^{\gamma_N}(\log z)^N))$. Given a $\mathbb{C}^2$-valued function

$$\mathsf{F}(s; z) = \begin{bmatrix} f_0(s; z) \\ f_1(s; z) \end{bmatrix}$$

on $\{(s, z) \mid s \in D,\ 0 < z < 1\}$, we write

$$\mathsf{F}(s; z) = \begin{bmatrix} q_0(s) z^{\alpha_0(s)} \\ q_1(s) z^{\alpha_1(s)} \end{bmatrix} \left(1 + O(s; z^{\gamma_0}, z^{\gamma_1}\log z, \dots, z^{\gamma_N}(\log z)^N)\right)$$

when $f_j(s; z) = q_j(s) y^{\alpha_j(s)}\left(1 + O(s; z^{\gamma_0}, z^{\gamma_1}\log z, \dots, z^{\gamma_N}(\log z)^N)\right)$ for $j = 0, 1$.

We define three new functions:

$$c(s) := \frac{\Gamma(s+1)\,\Gamma(\mu+2)}{\Gamma((s+n)/2+1)\,\Gamma((s-\lambda)/2+1)},$$

$$h_s(z) := {}_2F_1\left(-\frac{s-n}{2}+1, -\frac{s+\lambda}{2}+1\,;\, \mu+2\,;\, z\right),$$

$$H_s(z) := {}_2F_1\left(\frac{s-n}{2}, \frac{s+\lambda}{2}\,;\, s+1\,;\, 1-z\right).$$

Note that $c(s)$ has no poles nor zeros in the domain D.

Lemma 3.7. *Let $s \in D$. Then*

$$h_s(z) = 1 + O(s\,;z) \quad \textit{and} \quad H_s(z) = \frac{c(s)}{\mu+1}\bigl(1 + O(s\,;z, z^{\mu+1}\log z)\bigr).$$

With $y = 1 - z$, we have

$$h_s(z) = \frac{c(s)}{s}(1 + O(s\,;y)) \;\; \textit{if}\;\; \operatorname{Re} s > 0, \quad \textit{and} \quad H_s(z) = 1 + O(s\,;y).$$

Proof. Use [Magnus et al. 1966, p. 49, line 6 and p. 47, last line]. □

The case $p > 0$.

Proposition 3.8. *For $i = 0, 1$, define* $\mathsf{F}_i^{(p)}(s\,;z) = \begin{bmatrix} f_{0i}^{(p)}(s\,;z) \\ f_{1i}^{(p)}(s\,;z) \end{bmatrix}$, $0 < z < 1$, $s \in D$, *by*

$$f_{10}^{(p)}(s\,;z) = (1-z)^{(-\nu_s^{(p+1)}+n)/2+1}\, h_{\nu_s^{(p+1)}}(z),$$

$$f_{11}^{(p)}(s\,;z) = z^{-(\mu+1)}(1-z)^{(\nu_s^{(p+1)}+n)/2+1}\, H_{\nu_s^{(p+1)}}(z),$$

$$f_{00}^{(p)}(s\,;z) = -\frac{(1-z)^{(-\nu_s^{(p+1)}+n)/2}}{lp}\Bigl(z(1-z)\frac{d}{dz} + \frac{\nu_s^{(p+1)}+l+m-1}{2}z + \frac{m(l+p+m)}{l+m}(1-z)\Bigr)\, h_{\nu_s^{(p+1)}}(z),$$

$$f_{01}^{(p)}(s\,;z) = -\frac{z^{-(\mu+1)}(1-z)^{(\nu_s^{(p+1)}+n)/2}}{lp}\Bigl(z(1-z)\frac{d}{dz} + \frac{-\nu_s^{(p+1)}+l+m-1}{2}z - \frac{lp}{l+m}(1-z)\Bigr)\, H_{\nu_s^{(p+1)}}(z).$$

Then $\mathsf{F}_i^{(p)}(s\,;z)$ $(i = 0, 1)$ is a C^∞-solution of (29) *such that*

$$\mathsf{F}_0^{(p)}(s\,;z) = \begin{bmatrix} -m(l+p+m)/(lp(l+m)) \\ 1 \end{bmatrix}\bigl(1 + O(s\,;z)\bigr), \tag{33}$$

$$\mathsf{F}_1^{(p)}(s\,;z) = \frac{c(\nu_s^{(p+1)})}{\mu+1}\begin{bmatrix} 1/(l+m) \\ 1 \end{bmatrix} z^{-(\mu+1)}\bigl(1 + O(s\,;z, z^{\mu+1}\log z)\bigr),$$

and such that for $\operatorname{Re} s > 0$ *and* $y = 1 - z$,

(34)

$$\mathsf{F}_0^{(p)}(s\,;z) = \frac{c(\nu_s^{(p+1)})}{\nu_s^{(p+1)}}\begin{bmatrix} -(\nu_s^{(p+1)}+l+m-1)/(2lp) \\ y \end{bmatrix} y^{(-\nu_s^{(p+1)}+n)/2}\big(1+O(s\,;y)\big),$$

$$\mathsf{F}_1^{(p)}(s\,;z) = \begin{bmatrix} -(-\nu_s^{(p+1)}+l+m-1)/(2lp) \\ y \end{bmatrix} y^{(\nu_s^{(p+1)}+n)/2}\big(1+O(s\,;y)\big),$$

Proof. By Proposition 3.3, we have only to check that the functions $f_{0i}^{(p)}(s\,;z)$ and $f_{1i}^{(p)}(s\,;z)$ are solutions of the equations (28) and (30) for each $i = 0, 1$. This is done by a direct computation. Using Lemma 3.7, we obtain (33) and (34). □

Remark. The function $\mathsf{F}_0^{(p)}\big(\sqrt{s^2-4l(m-1)}\,;\tanh^2 t\big)$ is the A-radial part of an Eisenstein integral for $H\backslash G$ associated with the principal series $\operatorname{Ind}_{P_0}^{G}(E_{p,p}\,;s)$, to be defined in the proof of Proposition 5.5.

By Propositions 3.3 and 3.8, $\mathsf{F}_0^{(p)}(s\,;z)$ and $\mathsf{F}_1^{(p)}(s\,;z)$ are two linearly independent solutions of (23). We proceed to construct two more solutions using Lemma 3.4.

Proposition 3.9. *For* $i = 0, 1$, *define the function* $\mathsf{U}_i^{(p)}(s\,;z) = \begin{bmatrix} u_{0i}^{(p)}(s\,;z) \\ u_{1i}^{(p)}(s\,;z) \end{bmatrix}$, $s \in D$, $0 < z < 1$, *by*

$$u_{10}^{(p)}(s\,;z) = (1-z)^{(-\nu_s^{(p)}+n)/2-1}\,h_{\nu_s^{(p)}}(z),$$

$$u_{11}^{(p)}(s\,;z) = z^{-(\mu+1)}(1-z)^{(\nu_s^{(p)}+n)/2-1}\,H_{\nu_s^{(p)}}(z),$$

$$u_{00}^{(p)}(s\,;z) = \frac{(1-z)^{(-\nu_s^{(p)}+n)/2-2}}{lp}\Big(z(1-z)\frac{d}{dz} + \frac{\nu_s^{(p)}-l-(m+1)}{2}z + \frac{lp}{l+m}(1-z)\Big)\,h_{\nu_s^{(p)}}(z),$$

$$u_{01}^{(p)}(s\,;z) = \frac{z^{-(\mu+1)}(1-z)^{(\nu_s^{(p)}+n)/2-2}}{lp}\Big(z(1-z)\frac{d}{dz} + \frac{-\nu_s^{(p)}-l-(m+1)}{2}z - \frac{m(l+p+m)}{l+m}(1-z)\Big)\,H_{\nu_s^{(p)}}(z).$$

Then $\mathsf{U}_i^{(p)}(s\,;z)$ *is a* C^∞*-solution of* (31).

Proof. A direct computation shows $u_{0i}^{(p)}(s\,z)$ and $u_{1i}^{(p)}(s\,;z)$ satisfy the two differential equations of Lemma 3.5. The conclusion then follows from Lemma 3.5. □

Lemma 3.10. *Let* $\mathsf{F}_i^{(p)}(s\,;z)$ *be the solutions defined in Proposition 3.8 and form the matrix-valued function* $\Phi^{(p)}(s\,;z) = [\mathsf{F}_0^{(p)}(s\,;z)\ \ \mathsf{F}_1^{(p)}(s\,;z)]$ $(0 < z < 1)$. *Define* *the function*

$$\mathsf{A}_i^{(p)}(s\,;z)=\begin{bmatrix}a_{0i}^{(p)}(s\,;z)\\ a_{1i}^{(p)}(s\,;z)\end{bmatrix},\quad 0<z<1,\ s\in\tilde{D},$$

for $i=0,1$, *by*

$$\begin{aligned}a_{00}^{(p)}(s\,;z)&=-z(1-z)^{(\nu_s^{(p+1)}-\nu_s^{(p)})/2-1}H_{\nu_s^{(p+1)}}(z)h_{\nu_s^{(p)}}(z)\\&\quad+\int_0^z(1-w)^{(\nu_s^{(p+1)}-\nu_s^{(p)})/2-2}(1+w)\,H_{\nu_s^{(p+1)}}(w)h_{\nu_s^{(p)}}(w)\,dw,\\ a_{10}^{(p)}(s\,;z)&=z^{\mu+2}(1-z)^{(-\nu_s^{(p+1)}-\nu_s^{(p)})/2-1}h_{\nu_s^{(p+1)}}(z)h_{\nu_s^{(p)}}(z)\\&\quad-\int_0^z w^{\mu+1}(1-w)^{(-\nu_s^{(p+1)}-\nu_s^{(p)})/2-2}(1+w)\,h_{\nu_s^{(p+1)}}(w)h_{\nu_s^{(p)}}(w)\,dw,\\ a_{01}^{(p)}(s\,;z)&=-z^{-\mu}(1-z)^{(\nu_s^{(p+1)}+\nu_s^{(p)})/2-1}H_{\nu_s^{(p+1)}}(z)H_{\nu_s^{(p)}}(z)\\&\quad+\int_1^z w^{-(\mu+1)}(1-w)^{(\nu_s^{(p+1)}+\nu_s^{(p)})/2-2}(1+w)\,H_{\nu_s^{(p+1)}}(w)H_{\nu_s^{(p)}}(w)\,dw,\\ a_{11}^{(p)}(s\,;z)&=z(1-z)^{(-\nu_s^{(p+1)}+\nu_s^{(p)})/2-1}h_{\nu_s^{(p+1)}}(z)H_{\nu_s^{(p)}}(z)\\&\quad-\int_0^z(1-w)^{(-\nu_s^{(p+1)}+\nu_s^{(p)})/2-2}(1+w)\,h_{\nu_s^{(p+1)}}(w)H_{\nu_s^{(p)}}(w)\,dw.\end{aligned}$$

These integrals are convergent, and for $i=0,1$, *we have*

$$\Phi^{(p)}(s\,;z)\,\frac{d}{dz}\mathsf{A}_i^{(p)}(s\,;z)=c(\nu_s^{(p+1)})\mathsf{U}_i^{(p)}(s\,;z),\quad 0<z<1,\ s\in\tilde{D}. \tag{35}$$

Proof. Lemma 3.7 shows that for a given compact set U in D there exists $\varepsilon>0$ such that $h_s(z)$ and $H_s(z)$ are bounded on $(0,\varepsilon]\cup[1-\varepsilon,1)$ uniformly in $s\in U$. Hence the integrands of the formulas above except $a_{01}^{(p)}(s\,;z)$ are bounded on $(0,\varepsilon]$ uniformly in $s\in U$, which implies the convergence of the integrals except $a_{01}^{(p)}(s\,;z)$. To obtain the convergence of $a_{01}^{(p)}(s\,;z)$ we need to assume $s\in\tilde{D}$.

We prove (35). Since $\Delta(z)=\det\Phi^{(p)}(s\,;z)$ is the Wronskian for the fundamental solutions $\mathsf{F}_i^{(p)}(s\,;z)$ of the differential equation (29) it satisfies the relation $(d/dz)\,\Delta(z)=\operatorname{tr}(\mathsf{B}^{(p)}(z))\,\Delta(z)$. Since

$$-\operatorname{tr}(\mathsf{B}^{(p)}(z))=(n+1)\,(1-z)^{-1}+(\mu+1)\,z^{-1},$$

we easily obtain $\Delta(z)=C_s\,z^{-(\mu+1)}(1-z)^{n+1}$ with some constant C_s. Using (33), we have

$$\Delta(z)=-\frac{c(\nu_s^{p+1})}{lp}z^{-(\mu+1)}\big(1+O(s\,;z,z\log z)\big),$$

which implies that $C_s = -c(\nu_s^{(p+1)})/(lp)$. Hence

$$\Phi^{(p)}(s\,;z)^{-1} = -\frac{lp}{c(\nu_s^{(p+1)})} z^{\mu+1}(1-z)^{-(n+1)} \begin{bmatrix} f_{11}^{(p)} & -f_{01}^{(p)} \\ -f_{10}^{(p)} & f_{00}^{(p)} \end{bmatrix} (s\,;z).$$

Direct computation then gives $\dfrac{d}{dz}\mathsf{A}_i^{(p)}(s\,;z) = c(\nu_s^{(p+1)})\,\Phi^{(p)}(s\,;z)^{-1}\mathsf{U}_i^{(p)}(s\,;z)$. □

Proposition 3.11. *For* $0<z<1$ *and* $s \in \tilde{D}$, *set*

$$\tilde{\mathsf{F}}_i^{(p)}(s\,;z) = \Phi^{(p)}(s\,;z)\mathsf{A}_i^{(p)}(s\,;z) = \mathsf{F}_0^{(p)}(s\,;z)\,a_{0i}^{(p)}(s\,;z) + \mathsf{F}_1^{(p)}(s\,;z)\,a_{1i}^{(p)}(s\,;z).$$

For each $i = 0, 1$, *the function* $\tilde{\mathsf{F}}_i^{(p)}(s\,;z)$ *is a* C^∞*-solution of the differential equation* (23) *satisfying the equalities*

$$(36)\quad \tilde{\mathsf{F}}_0^{(p)}(s\,;z) = \frac{c(\nu_s^{(p+1)})}{\mu+2}\begin{bmatrix} 1/(l+m) \\ 1 \end{bmatrix} z\big(1+O(s\,;z,\, z^{\mu+1}\log z)\big),$$

$$\tilde{\mathsf{F}}_1^{(p)}(s\,;z) = -\frac{c(\nu_s^{(p+1)})\,c(\nu_s^{(p)})}{\mu(\mu+1)}\begin{bmatrix} -m(l+p+m)/(lp(l+m)) \\ 1 \end{bmatrix} z^{-\mu} \times\big(1+O(s\,;z,\, z^{\mu+1}\log z,\, z^{2\mu+2}(\log z)^2)\big).$$

When Re $s>0$, *setting* $y = 1-z$, *we have*

$$(37)\quad \tilde{\mathsf{F}}_0^{(p)}(s\,;z) = \frac{c(\nu_s^{(p+1)})\,c(\nu_s^{(p)})}{\nu_s}\,\frac{-2}{\nu_s^{(p)}+l+m+1}\begin{bmatrix} a_0(s) \\ 1 \end{bmatrix} y^{(-\nu_s^{(p)}+n)/2}\big(1+O(s\,;y)\big),$$

$$\tilde{\mathsf{F}}_1^{(p)}(s\,;z) = \frac{-2c(\nu_s^{(p+1)})}{-\nu_s^{(p)}+l+m+1}\begin{bmatrix} a_1(s) \\ 1 \end{bmatrix} y^{(\nu_s^{(p)}+n)/2}\big(1+O(s\,;y)\big),$$

for holomorphic functions $a_0(s)$, $a_1(s)$ *on* $\mathbb{C}$.

Proof. Lemma 3.4 combined with (35) shows that $\tilde{\mathsf{F}}_i^{(p)}(s\,;z)$ is a solution of (23). The formulas (36) and (37) follow from (33), (34) combined with the asymptotics of $a_{ij}^{(p)}(s\,;z)$ which is deduced from their explicit formula and Lemma 3.7. □

Proposition 3.12. *The functions* $\mathsf{F}_0^{(p)}(s\,;z)$, $\mathsf{F}_1^{(p)}(s\,;z)$, $\tilde{\mathsf{F}}_0^{(p)}(s\,;z)$ *and* $\tilde{\mathsf{F}}_1^{(p)}(s\,;z)$ *constructed above form a fundamental system of solutions of the differential equation* (23) *on* $0<z<1$, *which depend holomorphically on* $s \in \tilde{D}$.

Proof. The solutions of (23) form a 4 dimensional $\mathbb{C}$-vector space. We already know that the functions $\mathsf{F}_i^{(p)}(s\,;z)$ $(i=0,1)$ and $\tilde{\mathsf{F}}_i^{(p)}(s\,;z)$ $(i=0,1)$ are solutions of (23) with linearly independent asymptotic behavior as $z \to +0$. □

The case $p = 0$.

Proposition 3.13. *For* $i = 0, 1$, *define* $\tilde{\mathsf{F}}_i^{(0)}(s\,;z)$, $0 < z < 1$, $s \in \tilde{D}$ *by*

$$\tilde{\mathsf{F}}_0^{(0)}(s\,;z) = (1-z)^{(-\nu_s^{(1)}+n)/2}\,{}_2F_1\Big(-\frac{\nu_s^{(1)}-n}{2}, -\frac{\nu_s^{(1)}+\lambda}{2}+1\,;\mu+1\,;z\Big),$$
$$\tilde{\mathsf{F}}_1^{(0)}(s\,;z) = z^{-\mu}(1-z)^{(\nu_s^{(1)}+n)/2}\,{}_2F_1\Big(\frac{\nu_s^{(1)}-n}{2}+1, \frac{\nu_s^{(1)}+\lambda}{2}\,;\nu_s^{(1)}+1\,;1-z\Big).$$

The $\tilde{\mathsf{F}}_i^{(0)}(s\,;z)$ *form a fundamental system of* C^∞*-solutions of* (32) *such that*

$$\tilde{\mathsf{F}}_0^{(0)}(s\,;z) = 1 + O(s\,;z),$$
$$(38)\qquad \tilde{\mathsf{F}}_1^{(0)}(s\,;z) = \frac{c(\nu_s^{(1)})}{\mu(\mu+1)}\,\frac{\nu_s^{(1)}+n}{2}\,z^{-\mu}\big(1+O(s\,;z, z^\mu \log z)\big),$$

and such that for $\operatorname{Re} s > 0$ *and* $y = 1 - z$

$$\tilde{\mathsf{F}}_0^{(0)}(s\,;z) = \frac{c(\nu_s^{(1)})}{(\mu+1)\nu_s^{(1)}}\,\frac{\nu_s^{(1)}+n}{2}\,y^{(-\nu_s^{(1)}+n)/2}\big(1+O(s\,;y)\big),$$
$$(39)\qquad \tilde{\mathsf{F}}_1^{(0)}(s\,;z) = y^{(\nu_s^{(1)}+n)/2}\big(1+O(s\,;y)\big).$$

Proof of Theorem 3.2. First we construct the family of functions ϕ_s ($s \in \tilde{D}$). By the general theory, the map $\psi : (h, t, k) \mapsto h a_t k$ from $H \times (0, \infty) \times K$ onto $G - HK$ is a submersion whose fibres are given as $\psi^{-1}(h a_t k) = \{(hm, t, m^{-1}k) \mid m \in M\}$. Hence we have a well-defined C^∞-function $\phi_s^{(p)} : G - HK \to F_{p,p}$ for each $0 < p \leqslant \mu$ by setting

$$\phi_s^{(p)}(h a_t k) = \frac{\mu(\mu+1)lp}{c(\nu_s^{(p+1)})\,c(\nu_s^{(p)})}\,\tau(k)^{-1}\big(\tilde{f}_{01}^{(p)}(s\,;\tanh^2 t)\,\mathsf{v}_0^{(p)} + \tilde{f}_{11}^{(p)}(s\,;\tanh^2 t)\,\mathsf{v}_1^{(p)}\big).$$

Here $\tilde{\mathsf{F}}_1^{(p)}(s\,;z)$ is the column-matrix with entries $\tilde{f}_{01}^{(p)}(s\,;z)$ and $\tilde{f}_{11}^{(p)}(s\,;z)$, the solution of (23) constructed in Proposition 3.11. For $p = 0$ we also have a well-defined C^∞-function $\phi_s^{(0)} : G - HK \to \mathbb{C}$ such that

$$\phi_s^{(0)}(h a_t k) = \frac{\mu(\mu+1)}{c(\nu_s^{(1)})}\,\frac{2}{\nu_s^{(1)}+n}\,\tilde{\mathsf{F}}_1^{(0)}(s\,;\tanh^2 t), \quad (h, t, k) \in H \times (0, \infty) \times K,$$

where $\tilde{\mathsf{F}}_1^{(0)}(s\,;z)$ is the function defined in Proposition 3.13. Define the function $\phi_s : G - HK \to \bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*$ by $\phi_s(g) := \sum_{p=0}^{\mu} \gamma_p\, L^{\mu-p}(\phi_s^{(p)}(g))$ with $\{\gamma_p\}$ the family of real numbers in Proposition 1.4. Then, by Propositions 3.11 and 3.13, $\{\phi_s\}_{s \in \tilde{D}}$ has the required properties.

We prove the uniqueness of the family ϕ_s. Assume we are given another family of functions φ_s ($s \in \tilde{D}$) possessing the same properties as ϕ_s. Then we can write $\varphi_s(g) = \sum_{p=0}^{\mu} \gamma_p L^{\mu-p}(\varphi_s^{(p)}(g))$ with C^∞-functions $\varphi_s^{(p)} : G - HK \to F_{p,p}$

uniquely along the decomposition (27). We shall show that for each $0 \leqslant p \leqslant \mu$, $\phi_s^{(p)}(g) = \varphi_s^{(p)}(g)$ for all $g \in G - HK$ and $s \in \tilde{D}$; by condition (i) of Theorem 3.2, we have only to show it assuming $\mathrm{Re}\, s > n$.

First consider the case $p > 0$. Then $\varphi_s^{(p)}(a_t) = g_0(\tanh^2 t)\, \mathsf{v}_0^{(p)} + g_1(\tanh^2 t)\, \mathsf{v}_1^{(p)}$ with a C^∞-function $\mathsf{G}(z) = \left[\begin{smallmatrix} g_0(z) \\ g_1(z) \end{smallmatrix}\right]$ on $0 < z < 1$. Since $\varphi_s^{(p)}$ satisfies (22), the function $\mathsf{G}(z)$ is a solution of (23). Hence by Proposition 3.12, there are complex numbers c_0, c_1, d_0, d_1 such that

$$\mathsf{G}(z) = c_0 \mathsf{F}_0^{(p)}(s\,;z) + c_1 \mathsf{F}_1^{(p)}(s\,;z) + d_0 \tilde{\mathsf{F}}_0^{(p)}(s\,;z) + d_1 \tilde{\mathsf{F}}_1^{(p)}(s\,;z).$$

We examine the behavior as $z \to +0$ on both sides of this identity to show $c_1 = 0$. We have $\lim_{z\to+0} z^{\mu+1}\mathsf{G}(z) = 0$ by condition (iv). By Propositions 3.8 and 3.11 we have

$$\lim_{z\to+0} z^{\mu+1}\mathsf{F}_0^{(p)}(s\,;z) = \lim_{z\to+0} z^{\mu+1}\tilde{\mathsf{F}}_0^{(p)}(s\,;z) = \lim_{z\to+0} z^{\mu+1}\tilde{\mathsf{F}}_1^{(p)}(s\,;z) = 0,$$
$$\lim_{z\to+0} z^{\mu+1}\mathsf{F}_1^{(p)}(s\,;z) \neq 0.$$

Hence $c_1 = 0$ and $c_0\mathsf{F}_0^{(p)} + d_0\tilde{\mathsf{F}}_0^{(p)} + d_1\tilde{\mathsf{F}}_1^{(p)} = \mathsf{G}$. Similarly we can use the behavior as $y = 1 - z \to +0$ to conclude $c_0 = d_0 = 0$. Indeed, since we assume $\mathrm{Re}\, s > n$, we have $\lim_{t\to+\infty} e^{(-\nu_s^{(p)}+n)t}\mathsf{G}(\tanh^2 t) = 0$ by condition (v), and

$$\lim_{t\to+\infty} e^{(-\nu_s^{(p)}+n)t}\mathsf{F}_0^{(p)}(s\,;\tanh^2 t) = \lim_{t\to+\infty} e^{(-\nu_s^{(p)}+n)t}\tilde{\mathsf{F}}_1^{(p)}(s\,;\tanh^2 t) = 0$$

by Propositions 3.8 and 3.11. Hence $d_0 = 0$ and $c_0\mathsf{F}_0^{(p)} + d_1\tilde{\mathsf{F}}_1^{(p)} = \mathsf{G}$. Since $\mathrm{Re}\, \nu_s^{(p)} > \mathrm{Re}\, \nu_s^{(p+1)} > \mathrm{Re}\, s > n$ (see Lemma 3.1), we have

$$\lim_{t\to+\infty} e^{(-\nu_s^{(p+1)}+n)t}\tilde{\mathsf{F}}_1^{(p)}(s\,;\tanh^2 t) = \lim_{t\to+\infty} e^{(-\nu_s^{(p+1)}+n)t}\mathsf{G}(\tanh^2 t) = 0$$

by condition (v) and Proposition 3.11. Hence $c_0 = 0$.

Consequently, $\mathsf{G} = d_1\tilde{\mathsf{F}}_1^{(p)}$. By condition (iv) and Proposition 1.3, we have

$$\lim_{z\to+0} z^{\mu}\mathsf{G}(z) = \begin{bmatrix} m(l+p+m)/(l+m) \\ -lp \end{bmatrix}.$$

At the same time, Proposition 3.11 gives

$$\lim_{z\to+0} z^{\mu}\tilde{\mathsf{F}}_1^{(p)}(s\,;z) = \frac{c(\nu_s^{(p+1)})c(\nu_s^{(p)})}{\mu(\mu+1)lp}\begin{bmatrix} m(l+p+m)/(l+m) \\ -lp \end{bmatrix},$$

so the constant d_1 equals $\mu(\mu+1)lp/\big(c(\nu_s^{(p+1)})c(\nu_s^{(p)})\big)$.

From the definition of $\phi_s^{(p)}$, we have $\varphi_s^{(p)}(a_t) = \phi_s^{(p)}(a_t)$. This is sufficient to conclude $\varphi_s^{(p)}(g) = \phi_s^{(p)}(g)$ for all $g \in G - HK$ by the (H, K)-equivariance condition. The discussion for the case $p = 0$ is quite similar. This completes the proof of uniqueness.

Some properties of the secondary spherical function.

Theorem 3.14. *Let ϕ_s ($s \in \tilde{D}$) be the secondary spherical function constructed in Theorem 3.2.*

(a) *There exist μ polynomial functions $\mathsf{a}_\alpha(s)$ $(0 \leqslant \alpha \leqslant \mu - 1)$ with values in $\left(\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*\right)^M$, positive number ε and $\left(\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*\right)^M$-valued holomorphic functions $\mathsf{b}_{i(s,z)}$ $(i = 0, 1, 2)$ on $\{(s, z) \mid s \in \tilde{D},\ |z| < \varepsilon\}$ such that*

$$\mathsf{a}_0(s) = \mathsf{w}, \quad \mathsf{a}_\alpha(-s) = \mathsf{a}_\alpha(s), \quad \deg(\mathsf{a}_\alpha(s)) \leqslant 2\alpha,$$

and such that for $s \in \tilde{D},\ z = \tanh^2 t \in (0, \varepsilon)$,

$$\phi_s(a_t) = \sum_{\alpha=0}^{\mu-1} \frac{\mathsf{a}_\alpha(s)}{z^{\mu-\alpha}} + \mathsf{b}_0(s\,;z) + \mathsf{b}_1(s\,;z)\ \log z + \mathsf{b}_2(s\,;z)\, z^{\mu+2}(\log z)^2. \tag{40}$$

(b) *There exists a positive number ε' and $\left(\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*\right)^M$-valued holomorphic functions $\mathsf{f}^{(p)}(s\,;y)$ $(0 \leqslant p \leqslant \mu)$ on $\{(s, y) \mid |y| < \varepsilon',\ \mathrm{Re}\, s > n\}$ such that*

$$\phi_s(a_t) = \sum_{p=0}^{\mu} y^{(\nu_s^{(p)}+n)/2}\, \mathsf{f}^{(p)}(s\,;y), \quad \mathrm{Re}\, s > n,\ y = \frac{1}{\cosh^2 t} \in (0, \varepsilon'). \tag{41}$$

(c) *For any differential operator with holomorphic coefficient ∂_s on $\tilde{D}$, the function $\partial_s\phi_s(g)$ on $G - HK$ belongs to C_τ^∞.*

Proof. (a) Recall the construction of ϕ_s. Then (36) and (38) immediately yields the existence of the expression of the form (40) except the nature of the functions $\mathsf{a}_\alpha(s)$. We have $\mathsf{a}_0(s) = \mathsf{w}$ from condition (iv). Hence it remains to show that $\mathsf{a}_\alpha(s)$ is an even polynomial function with degree no more than 2α. For that purpose, we examine the differential equation (22) again. By Theorem 2.3, equation (22) can be written as

$$\left(\frac{d^2}{dz^2} + \left(\frac{\mu+1}{z} + \frac{n-1}{1-z}\right)\frac{d}{dz} + \mathfrak{Q}(s\,;z)\right)\phi_s = 0, \tag{42}$$

with

$$\mathfrak{Q}(s\,;z) = \frac{-\mathcal{S}}{z(1-z)} + \frac{\mathcal{S}'}{z^2(1-z)} + \frac{\tau(\Omega_{M_0}) + \lambda^2 - s^2}{4z(1-z)^2}.$$

We have

$$\mathfrak{Q}(s\,;z) = \frac{\mathcal{S}'}{z^2} + \sum_{\gamma=-1}^{\infty} \mathfrak{Q}_\gamma(s)\, z^\gamma, \quad |z| < 1,$$

with

$$\mathcal{Q}_\gamma(s) = -\mathcal{S} + \mathcal{S}' + \tfrac{1}{4}(\gamma+2)\big(\lambda^2 - s^2 + \tau(\Omega_{M_0})\big), \quad \gamma \geqslant -1.$$

Substitute this expression of $\mathcal{Q}(s\,;z)$ and (40) to the left-hand side of (42) and compute the coefficient of $z^{-\mu+\alpha-2}$. Since it should be zero, we obtain the recurrence relation among the tensors $\mathsf{a}_\alpha(s)$: for $0<\alpha\leqslant\mu-1$,

$$(43) \qquad \big(\mathcal{S}' - \alpha(\mu-\alpha)\big)\; \mathsf{a}_\alpha(s) = -\sum_{\gamma=0}^{\alpha-1} \big(\mathcal{Q}_{\alpha-\gamma-2}(s) - (n-1)(\mu-\gamma)\big)\, \mathsf{a}_\gamma(s).$$

By (17), the operator $\mathcal{S}'$ preserves the subspaces $L^{\mu-p}(F^M_{p,p})$ $(0<p\leqslant\mu)$ and the restriction of $\mathcal{S}'$ is represented with respect to the basis $L^{\mu-p}(\mathsf{v}_i^{(p)})$ $(i=0,1)$ by

$$\mathsf{Q}^{(p)}_{-2} = \begin{bmatrix} -\dfrac{pl}{l+m} & -\dfrac{m(l+p+m)}{(l+m)^2} \\[2ex] -lp & -\dfrac{m(l+p+m)}{l+m} \end{bmatrix}.$$

Since $\det\big(\mathsf{Q}^{(p)}_{-2} - \alpha(\mu-\alpha)\big) = \alpha(\alpha+1)(\alpha-\mu-1)(\alpha-\mu)$ is not zero for $0<\alpha\leqslant\mu-1$, the operator $\mathcal{S}'-\alpha(\mu-\alpha)$ is invertible on $\big(\bigwedge^{\mu,\mu}\mathfrak{p}^*_{\mathbb{C}}\big)^M$. Therefore we can determine $\mathsf{a}_\alpha(s)$ uniquely by the recurrence relation (43), from which we easily know that $\mathsf{a}_\alpha(s)$ is an even polynomial function with degree no more than 2α.

By the construction of ϕ_s, part (c) is obvious. Part (b) follows from (37) and (39). □

The function ψ_s. For $s \in \tilde{D}$, define the function $\psi_s : G - HK \to \bigwedge^{r,r}\mathfrak{p}^*_{\mathbb{C}}$ by

$$\psi_s(g) = \nabla_-\nabla_+\phi_s(g), \quad g \in G - HK.$$

We easily see that ψ_s has the same properties as (2) and (3) of the previous theorem for ϕ_s. It also behaves similarly to (40) for ϕ_s near $t=0$,

$$(44) \qquad \psi_s(a_t) = \sum_{\alpha=0}^{\mu-1} \frac{\mathsf{c}_\alpha(s)}{z^{\mu-\alpha}} + O(s;1,\log z, z^\mu(\log z)^2),$$

except that the degree of the $(\bigwedge^{r,r}\mathfrak{p}^*_{\mathbb{C}})^M$-valued polynomial $\mathsf{c}_\alpha(s)$ is no more than $2(\alpha+1)$. Indeed by applying the expression (24) of the Schmid operator to (40) we can obtain the required expression with one extra term of the form $\mathsf{u}z^{-(\mu+1)}$. The only thing we have to do here is to show $\mathsf{u}=0$. By a direct computation, we have $\mathsf{u} = (\mu(\mu+1)e(\omega_0\wedge\bar{\omega}_0) + \mu\mathcal{A} + \mathcal{R}_-\mathcal{R}_+)\mathsf{w}$. Since w is $(H\cap K)$-invariant and the elements $\bar{Z}_j$, Z_j with $n-r+1\leqslant j\leqslant n-1$ belong to $(\mathfrak{k}\cap\mathfrak{h})_{\mathbb{C}}$, we have $\mathcal{R}_\pm\mathsf{w}=0$. Hence $\mathsf{u} = \mu(2/\sqrt{-1})\,\big(\eta - (\sqrt{-1}/2)\,\mu\,\omega_0\wedge\bar{\omega}_0\big)\wedge\mathsf{w}$. By Proposition 1.4, the right-hand side of this identity is zero.

4. Poincaré series

Let Γ be a discrete subgroup of G. We assume that (G, H, Γ) is arranged as follows. There exists a connected reductive $\mathbb{Q}$-group G, a closed $\mathbb{Q}$-subgroup H of G and an arithmetic subgroup Δ of $\mathsf{G}(\mathbb{Q})$ such that there exists a morphism of Lie groups from $\mathsf{G}(\mathbb{R})$ onto G with compact kernel which maps $\mathsf{H}(\mathbb{R})$ onto H and Δ onto Γ. Set $K_H = H \cap K$, $\Gamma_H = \Gamma \cap H$.

Invariant measures. Let dk and dk_0 be the Haar measures of compact groups K and K_H with total volume 1. There is a unique Haar measure dg on G such that the quotient measure dg/dk corresponds to the measure on the symmetric space G/K determined by the invariant volume form vol. We define dh on H analogously: dh/dk_0 corresponds to the measure on H/K_H determined by vol_H.

Lemma 4.1. *For any measurable functions f on G we have*

$$\int_G f(g)\,dg = \int_H dh \int_K dk \int_0^\infty f(ha_t k)\varrho(t)\,dt$$

with dt the usual Lebesgue measure on $\mathbb{R}$ and

$$\varrho(t) = 2c_r\,(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1}, \quad c_r = \pi^r/\mu!.$$

Proof. For closed subgroups $Q_1 \subset Q_2$ of G with Lie algebras $\mathfrak{q}_i = \mathrm{Lie}(Q_i)$ for $i = 1, 2$, we regard $(\mathfrak{q}_2/\mathfrak{q}_1)^* \subset \mathfrak{g}^*$ by the dual map of the composite of the orthogonal projection $\mathfrak{g} \to \mathfrak{q}_2$ and the canonical surjection $\mathfrak{q}_2 \to \mathfrak{q}_2/\mathfrak{q}_1$. Let $\mathrm{vol}_{\mathfrak{q}_2/\mathfrak{q}_1}$ be the element $\xi_1 \wedge \cdots \wedge \xi_s \in \bigwedge(\mathfrak{q}_2/\mathfrak{q}_1)^*$ with $\{\xi_i\}$ any orthonormal basis of $(\mathfrak{q}_2/\mathfrak{q}_1)^*$. Assume Q_1 is compact. Then there exists a unique left Q_2-invariant s-form Z_{Q_2/Q_1} on Q_2/Q_1 whose value at $o = eQ_2$ is $\mathrm{vol}_{\mathfrak{q}_2/\mathfrak{q}_1}$. Let dZ_{Q_2/Q_1} be the Q_2-invariant measure on Q_2/Q_1 corresponding to Z_{Q_2/Q_1}. For example, $\mathrm{vol}_{\mathfrak{g}/\mathfrak{k}} = \mathrm{vol}$ and $\mathrm{vol}_{\mathfrak{h}/\mathfrak{m}} = \mathrm{vol}_H \wedge \mathrm{vol}_{\mathfrak{h}\cap\mathfrak{k}/\mathfrak{m}}$.

The decomposition $G = HAK$ yields the diffeomorphism j from $H/M \times (0, \infty)$ to $(G - HK)/K$ defined by $j(\dot{h}, t) = ha_t\,K$ ($\dot{h} \in H/M$, $t > 0$); see [Heckman and Schlichtkrull 1994, Theorem 2.4, p. 108]. A simple computation with the aid of the formulas in the proof of Lemma 2.2 proves the identity

$$j^*\mathrm{Z}_{G/K} = 2(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1}\,\mathrm{Z}_{H/M} \wedge dt.$$

Hence under the identification $H/M \times (0, \infty) = (G - HK)/K$ we have

$$dZ_{G/K}(\dot{g}) = 2(\sinh t)^{2r-1}(\cosh t)^{2n-sr+1}\,dZ_{H/M}(\dot{h})\,dt. \tag{45}$$

The measure $dZ_{G/K}(\dot{g})$ is precisely dg/dk in our normalization. Let dm be the Haar measure of M with total volume one. The resulting quotient measure dh/dm

on H/M should be proportional to $dZ_{H/M}(\dot{h})$. We determine the proportionality constant C_0 in such a way that

$$\frac{dh}{dm} = C_0\, dZ_{H/M}(\dot{h}). \tag{46}$$

We have a decomposition

$$dZ_{H/M}(\dot{h}) = \frac{dh}{dk_0}\, dZ_{H\cap K/M}(\dot{k}_0),$$

since $dZ_{H/H\cap K}(\dot{h}) = dh/dk_0$. Using the relation $dh/dm = (dh/dk_0)\,(dk_0/dm)$, it follows that

$$\frac{dk_0}{dm} = C_0\, dZ_{H\cap K/M}(\dot{k}_0).$$

Since the total measure of $H\cap K/M$ with respect to dk_0/dm is 1, we then obtain the equality $C_0^{-1} = \int_{H\cap K/M} dZ_{H\cap K/M}(\dot{k}_0)$. To compute this integral, we use the diffeomorphism $\Pi : H\cap K/M \to \mathbb{S}^{2r-1}$ defined by

$$\Pi(\mathrm{diag}(u_1, u_2, u_0)\, M) = u_0^{-1}u_2(\mathsf{e}), \quad (u_1, u_2, u_0) \in \mathsf{U}(n-r)\times\mathsf{U}(r)\times\mathsf{U}(1),$$

with $\mathbb{S}^{2r-1} := \left\{ z\in\mathbb{C}^r \,\middle|\, \sum_{i=1}^r |z_i|^2 = 1\right\}$ and $\mathsf{e} := {}^t(0,\dots,0,1) \in \mathbb{S}^{2r-1}$. Thus, easily, the pullback of the volume form $\mathsf{Z}_{\mathbb{S}^{2r-1}} = \sum_{i=1}^{2r}(-1)^{i-1}x_i\, dx_1\wedge\cdots\wedge\hat{dx_i}\wedge\dots dx_{2r}$ of the sphere $\mathbb{S}^{2r-1}$ is $2\mathsf{Z}_{H\cap K/M}$. Hence

$$C_0^{-1} = \int_{H\cap K/M} dZ_{H\cap K/M}(\dot{k}_0) = \frac{1}{2}\int_{\mathbb{S}^{2r-1}} dZ_{\mathbb{S}^{2r-1}} = \frac{\pi^r}{\Gamma(r)}.$$

Using (45), (46) and the value of C_0 just obtained, the required integration formula easily follows. □

Currents defined by Poincaré series. Let $\mathfrak{F}$ denote the set of the families of functions $\{\varphi_s\}_{s\in\tilde{D}}$ such that $\varphi_s = \partial_s\phi_s$ $(s\in\tilde{D})$ or $\varphi_s = \partial_s\psi_s$ $(s\in\tilde{D})$ with some differential operator ∂_s with holomorphic coefficient on $\tilde{D}$.

For $\{\varphi_s\}\in\mathfrak{F}$, introduce the Poincaré series

$$\tilde{P}(\varphi_s)(g) = \sum_{\gamma\in\Gamma_H\backslash\Gamma} \varphi_s(\gamma g), \quad g\in G, \tag{47}$$

which is the most basic object in our investigation. First of all, we discuss its convergence in a weak sense. Note that φ_s takes its values in a finite dimensional Hilbert space $\bigwedge\mathfrak{p}_{\mathbb{C}}^*$ with the norm $\|\alpha\| = (\alpha|\alpha)^{1/2}$.

Theorem 4.2. *The function in s defined by the integral*

$$\tilde{P}(\|\varphi_s\|) := \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma_H\backslash\Gamma}\|\varphi_s(\gamma g)\|\right) d\dot{g}$$

is locally bounded on Re $s > n$. *For each s with* Re $s > n$, *the series* (47) *converges absolutely almost everywhere in* $g \in G$ *to define an* L^1*-function on* $\Gamma\backslash G$.

Proof. We assume $\varphi_s = \partial\phi_s$. (The proof for $\partial\psi_s$ is the same.) Let U be a compact subset of Re $s > n$. Then by Theorem 3.14 and Lemma 3.1, there exist positive numbers a, δ and C_0 such that

$$\|\varphi_s(a_t)\| \leqslant C_0\, t^{-2\mu}, \qquad (s,t) \in U \times (0,a],$$
$$\|\varphi_s(a_t)\| \leqslant C_0\, e^{-t(2n+\delta)}, \quad (s,t) \in U \times [a,\infty).$$

From the form of $\varrho(t)$, we can find a positive constant C_1 such that

$$\varrho(t) \leqslant C_1\, t^{2\mu+1} \text{ for } t \in (0,a] \quad \text{and} \varrho(t) \leqslant C_1\, e^{2nt} \text{ for } \quad t \in [a,\infty).$$

First as in the proof of [Oda and Tsuzuki 2003, Proposition 3.1.1] and then by using the estimations above, we obtain the uniform bound of $\tilde{P}(\|\varphi_s\|)$ for $s \in U$. □

If Γ is neat, then the quotient space $\Gamma\backslash G/K$ acquires a structure of complex manifold from the one on $G/K \cong \mathfrak{D}$. Let $\pi : G/K \to \Gamma\backslash G/K$ be the natural projection. Let $A(\Gamma\backslash G/K)$ denote the space of C^∞-differential forms on $\Gamma\backslash G/K$ and $A_c(\Gamma\backslash G/K)$ the subspace of compactly supported forms. Given $\alpha \in A(\Gamma\backslash G/K)$, there is a unique C^∞-function $\tilde{\alpha} : G \to \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ such that $\tilde{\alpha}(\gamma g k) = \tau(k)^{-1}\tilde{\alpha}(g)$ for $\gamma \in \Gamma$ and $k \in K$ and such that

$$(48) \quad \langle(\pi^*\alpha)(gK), dL_g(\xi_o)\rangle = \langle\tilde{\alpha}(g), \xi_o\rangle, \quad g \in G,\ \xi_o \in \bigwedge \mathfrak{p} = \bigwedge \mathrm{T}_o(G/K).$$

Here L_g denotes the left translation on G/K by the element g and we identify $\mathfrak{p}$ with $\mathrm{T}_o(G/K)$, the tangent space of G/K at $o = eK$. Then for any α, $\beta \in A(\Gamma\backslash G/K)$, we have

$$(49) \qquad \int_{\Gamma\backslash G/K} \alpha \wedge *\bar{\beta} = \int_{\Gamma\backslash G} (\tilde{\alpha}(g)|\tilde{\beta}(g))\, d\dot{g}.$$

For any left Γ-invariant function f on G, the integral

$$\mathcal{J}_H(f;g) = \int_{\Gamma_H\backslash H} f(hg)\,dh, \quad g \in G,$$

plays a fundamental role in our further study. We already discussed the convergence problem of this integral in [Oda and Tsuzuki 2003, §3.2]. For convenience we recall the result. If Γ is cocompact, we take a compact fundamental domain $\mathfrak{S}^1$ for Γ in G and $t_{\mathfrak{S}^1}$ the constant function 1. Hence $G = \Gamma\mathfrak{S}^1$ in this case. If Γ is not cocompact, then one can fix a complete set of representatives P^i $(1 \leqslant i \leqslant h)$ of Δ-conjugacy classes of $\mathbb{Q}$-parabolic subgroups in G together with $\mathbb{Q}$-split tori $\mathbb{G}_m \cong \mathsf{A}^i$ in the radical of P^i such that an eigencharacter of $\mathrm{Ad}(t)$ $(t \in \mathbb{G}_m)$ in the Lie algebra of P^i is one of t^j $(j = 0, 1, 2)$. For each i, let T^i and N^i be the images in G of $\mathsf{A}^i(\mathbb{R})$ and the unipotent radical of $\mathsf{P}^i(\mathbb{R})$ respectively. Then we can choose a Siegel domain $\mathfrak{S}^i$ in G with respect to the Iwasawa decomposition $G = N^i T^i K$ for each i such that G is a union of $\Gamma\mathfrak{S}^i$ $(1 \leqslant i \leqslant h)$. Let $t_{\mathfrak{S}^i} : \mathfrak{S}^i \to (0, \infty)$ be the function $t_{\mathfrak{S}^i}(n_i\, \underline{t}_i\, k) = t$ $(n_i\, \underline{t}_i\, k \in \mathfrak{S}^i)$. Here $\underline{t}_i$ denotes the image of $t \in \mathbb{G}_m(\mathbb{R}) \cong \mathsf{A}^i(\mathbb{R})$ in T^i.

Given $\delta \in (2rn^{-1}, 1)$, let $\mathfrak{M}_\delta$ be the space of all left Γ-invariant C^∞-functions $f : G \to \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ with the K-equivariance $f(gk) = \tau(k)^{-1} f(g)$ such that, for any $\varepsilon \in (0, \delta)$ and $D \in U(\mathfrak{g}_{\mathbb{C}})$, we have

$$\|R_D\varphi(g)\| \prec t_{\mathfrak{S}^i}(g)^{(2-\varepsilon)n}, \quad g \in \mathfrak{S}^i,\ 1 \leqslant i \leqslant h.$$

Proposition 4.3. *Let $f \in \mathfrak{M}_\delta$ with $\delta \in (2rn^{-1}, 1)$ and $D \in U(\mathfrak{g}_{\mathbb{C}})$.*

(1) *For any $\varepsilon \in (2rn^{-1}, \delta)$*

$$\mathcal{J}_H(\|R_D f\|; a_t) \prec e^{(2-\varepsilon)nt}, \quad t \geqslant 0.$$

The function $\mathcal{J}_H(f; g)$ is of class C^∞, belongs to C_τ^∞ and

$$\mathcal{J}_H(R_D f; g) = R_D \mathcal{J}_H(f; g), \quad g \in G.$$

(2) *For any $\{\varphi_s\} \in \mathfrak{F}$, the integral*

$$\int_{\Gamma\backslash G} |(\tilde{P}(\varphi_s)(g)|R_D f(g))|\, d\dot{g}$$

is finite if $\operatorname{Re} s > 3n - 2r$, *and*

$$\int_{\Gamma\backslash G} (\tilde{P}(\varphi_s)(g)|R_D f(g))\, d\dot{g} = \int_0^\infty \varrho(t)\, (\varphi_s(a_t)|R_D \mathcal{J}_H(f; a_t))\, dt.$$

Proof. See [Oda and Tsuzuki 2003, Theorem 3.2.1] and its proof. The number $\tau_{\mathbb{Q}}(G, \sigma)$ is $2(1 - rn^{-1})$ in our setting here. □

Proposition 4.4. *There exists a unique current $P(\varphi_s)$ on $\Gamma\backslash G/K$ such that for $\alpha \in A_{\mathrm{c}}(\Gamma\backslash G/K)$,*

$$\langle P(\varphi_s), *\bar{\alpha}\rangle = \int_{\Gamma\backslash G} (\tilde{P}(\varphi_s)(g)|\tilde{\alpha}(g)) d\dot{g} = \int_0^\infty \varrho(t)\, (\varphi_s(a_t)|\mathcal{J}_H(\tilde{\alpha}; a_t))\, dt. \tag{50}$$

Let ∂_s be a holomorphic differential operator on $\tilde{D}$. Then for any $\alpha \in A_{\mathrm{c}}(\Gamma\backslash G/K)$, the function $s \mapsto \langle P(\varphi_s), \alpha\rangle$ is holomorphic on $\tilde{D}$ and $\partial_s\langle P(\varphi_s), \alpha\rangle = \langle P(\partial_s\varphi_s), \alpha\rangle$.

Proof. The L^1-function $\tilde{P}(\varphi_s)$ on $\Gamma\backslash G$ satisfies $\tilde{P}(\varphi_s)(gk) = \tau(k)^{-1}\tilde{P}(\varphi_s)(g)$ for all $k \in K$. Hence it defines an L^1-form $P(\varphi_s)$ on $\Gamma\backslash G/K$ naturally. Then the linear functional $\alpha \mapsto \int_{\Gamma\backslash G/K} P(\varphi_s) \wedge \alpha$ on $A_{\mathrm{c}}(\Gamma\backslash G/K)$ is a current, which we also denote by $P(\varphi_s)$. The first equality of (50) follows from (49). Let $\alpha \in A_{\mathrm{c}}(\Gamma\backslash G/K)$. Since $\tilde{\alpha}$ is left Γ-invariant and has compact support mod Γ, the image $\tilde{\alpha}(\Gamma\backslash G)$ is a compact subset of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$. Hence there is a positive constant C_0

such that $|(\mathsf{u}|\tilde{\alpha}(g))| \leqslant C_0\|\mathsf{u}\|$ for all $\mathsf{u} \in \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ and $g \in G$. Then for any loop C in $\operatorname{Re} s > n$, we have

$$\int_C ds \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma_H\backslash\Gamma} |(\varphi_s(\gamma g)|\tilde{\alpha}(\gamma g))|\, d\dot{g} \leqslant C_0 \int_C ds \int_{\Gamma\backslash G} \Bigg(\sum_{\gamma\in\Gamma_H\backslash\Gamma} \|\varphi_s(\gamma g)\|\Bigg)\, d\dot{g}$$
$$= C_0 \int_C \tilde{P}(\|\varphi_s\|)\, ds.$$

By Theorem 4.2, the last integral in s is finite. Hence by Fubini's theorem and by the holomorphicity of each function $s \mapsto (\varphi_s(g)|\tilde{\alpha}(g))$, we have

$$\int_C ds \int_{\Gamma\backslash G} (\tilde{P}(\varphi_s)(g)|\tilde{\alpha}(g))\, d\dot{g} = 0.$$

Since C is arbitrary, the holomorphicity in s of the integral (50) follows. The second equality of (50) follows by a standard computation. □

Differential equation of functions related to $\tilde{G}_s$. For each $0 \leqslant j \leqslant \mu$, put $\boldsymbol{\delta}_{j,s}\tilde{G}_s = (c_r\,\mu\,\mu!)^{-1}\tilde{P}(\boldsymbol{\delta}_{s,j}\phi_s)$ with

$$\boldsymbol{\delta}_{j,s} = \frac{1}{j!}\left(\frac{1}{2s}\frac{d}{ds}\right)^j, \quad j \in \mathbb{N},$$

which we will regard as a $\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*$-valued L^1-function on $\Gamma\backslash G$.

Theorem 4.5. *Let $0 \leqslant j \leqslant \mu$ be an integer. Suppose* $\operatorname{Re} s > 3n - 2r$. *Then for any $f \in \mathfrak{M}_\delta$ with $\delta \in (2rn^{-1}, 1)$, we have*

$$\int_{\Gamma\backslash G} (\boldsymbol{\delta}_{j,s}\tilde{G}_s(g)|R_{(\Omega+\lambda^2-\bar{s}^2)^{j+1}} f(g))\, d\dot{g} = \frac{4}{\mu!}\,(\mathsf{w}|\mathcal{J}_H(f\,;e)).$$

Proof. For $f \in \mathfrak{M}_\delta$, let $I_j(f)$ denote the integral above. Then we shall prove $I_j(f) = I_{j-1}(f)$ for $1 \leqslant j \leqslant \mu$ and $I_0(f) = \frac{4}{\mu!}(\mathsf{w}|\mathcal{J}_H(f\,;e))$, which obviously imply the equality. Since we already have Proposition 4.3, we can set aside the convergence argument of various integrals in these formulas. Fix $0 \leqslant j \leqslant \mu$ and put

$$\tilde{F}(g) = \mathcal{J}_H(R_{(\Omega-s^2+\lambda^2)^j} f\,; g), \quad g \in G.$$

Since $\tilde{F} \in C_\tau^\infty$, (14) implies

$$R_{(\Omega+\lambda^2-\bar{s}^2)}\tilde{F}(a_t) = (\mathcal{D} + \lambda^2 - \bar{s}^2)F(t)$$

with $\mathcal{D}$ the differential operator in $t > 0$ given by the formula inside the bracket of (14), and $F(t) = \tilde{F}(a_t)$. By Proposition 4.4(2), we have

$$I_j(f) = \frac{1}{c_r\,\mu\,\mu!}\int_0^\infty \varrho(t)\,\big(\boldsymbol{\delta}_{j,s}\phi_s(a_t)|(\mathcal{D} + \lambda^2 - \bar{s}^2)F(t)\big)\, dt. \tag{51}$$

Let $0 < \varepsilon < R$. Then performing integration-by-parts and using the fact that the operators $\mathcal{S}$, $\mathcal{S}'$ and $\tau(\Omega_{M_0})$ are self-adjoint, we obtain

$$\text{(52)} \quad \int_{\varepsilon}^{R} \varrho(t)\,\big(\boldsymbol{\delta}_{j,s}\phi_s(a_t)|(\mathcal{D}+\lambda^2-\bar{s}^2)F(t)\big)\,dt \\ = -\Phi(R)+\Phi(\varepsilon)+\int_{\varepsilon}^{R}\varrho(t)\big((\mathcal{D}+\lambda^2-s^2)\boldsymbol{\delta}_{j,s}\phi_s(a_t)|F(t)\big)\,dt$$

with

$$\begin{aligned}\Phi(t) = \varrho(t)\big(\boldsymbol{\delta}_{j,s}\phi_s(a_t)|F'(t)\big) - \varrho'(t)\big(\boldsymbol{\delta}_{j,s}\phi_s(a_t)|F(t)\big) \\ - \varrho(t)\Big(\frac{d}{dt}\boldsymbol{\delta}_{j,s}\phi_s(a_t)|F(t)\Big) \\ + \varrho(t)\big((2n-2r+1)\tanh t+(2r-1)\coth t\big)\big(\boldsymbol{\delta}_{j,s}\phi_s(a_t)|F(t)\big).\end{aligned}$$

Now we use (40). Since the function $\mathsf{a}_\alpha(s)$ appearing there is an even polynomial function of degree no more than 2α, we have

$$\boldsymbol{\delta}_{j,s}\phi_s(a_t) = \frac{\mathsf{w}_{\mu-j}}{z^{\mu-j}}(1+O(s\,;z,z\log z))$$

with $\mathsf{w}_\alpha = (\boldsymbol{\delta}_{\alpha,s}\mathsf{a}_\alpha)(0)$. Using this, we easily see that

$$\Phi(t) = 4c_r(\mu-j)\big(\mathsf{w}_{\mu-j}|F(0)\big)\,z^j\big(1+O(s\,;z,z\log z)\big)$$

to conclude that

$$\text{(53)} \qquad \lim_{\varepsilon\to+0}\Phi(\varepsilon) = 4c_r\,\mu\delta_{0j}(\mathsf{w}|F(0))$$

with δ_{0j} the Kronecker delta. To compute the limit of $\Phi(R)$ as R goes to infinity, we use (41) and Lemma 3.1. After a computation we obtain

$$\text{(54)} \qquad \lim_{R\to+\infty}\Phi(R) = 0.$$

Here we need the inequality $\operatorname{Re} s > n$ which follows from $\operatorname{Re} s > 3n-2r$. A simple computation (or [Gon and Tsuzuki 2002, Claim 3.1.6, p. 240]) shows the first equality of

$$\text{(55)} \qquad \begin{aligned}(\mathcal{D}+\lambda^2-s^2)\boldsymbol{\delta}_{j,s}\phi_s(a_t) &= \boldsymbol{\delta}_{j,s}(\mathcal{D}+\lambda^2-s^2)\phi_s(a_t)+\boldsymbol{\delta}_{j-1,s}\phi_s(a_t) \\ &= \boldsymbol{\delta}_{j-1,s}\phi_s(a_t),\end{aligned}$$

and the differential equation of (22) gives the second equality. Putting together equations (51)–(55), we obtain the result. □

5. Spectral expansion

In this section we investigate the spectral expansion of the functions $\boldsymbol{\delta}_{j,s}\tilde{G}_s$ to obtain a meromorphic continuation of the current-valued function $s \mapsto G_s$, which is already holomorphic on the half plane $\operatorname{Re} s > n$.

Spectral expansion. In order to describe the spectral decomposition of the function $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s$, we need some preparations.

For positive q, let $\mathscr{L}^q_\Gamma(\tau)$ denote the Banach space of all measurable functions $f : G \to \bigwedge \mathfrak{p}^*_{\mathbb{C}}$ such that for all $\gamma \in \Gamma$ and $k \in K$, $f(\gamma g k) = \tau(k)^{-1} f(g)$ and $\int_{\Gamma\backslash G} \|f(g)\|^q \, d\dot{g} < \infty$. For $0 \leqslant d \leqslant n$, let $\mathscr{L}^q_\Gamma(\tau)^{(d)}$ denote the subspace of those functions $f \in \mathscr{L}^q_\Gamma(\tau)$ with values in $\bigwedge^{d,d}\mathfrak{p}^*_{\mathbb{C}}$. The inner product of two functions f_1 and f_2 in $\mathscr{L}^2_\Gamma(\tau)$ is given as

$$\langle f_1 | f_2 \rangle = \int_{\Gamma\backslash G} (f_1(g) | f_2(g)) \, d\dot{g}.$$

For each $0 \leqslant d \leqslant n$, let $\{\lambda^{(d)}_n\}_{n\in\mathbb{N}}$ be the increasing sequence of the eigenvalues of the bidegree (d,d)-part of the Laplacian $\widetilde{\Delta} = -R_\Omega$ such that each eigenvalue occurs with its multiplicity. Choose an orthonormal system $\{\tilde{\alpha}^{(d)}_n\}_{n\in\mathbb{N}}$ in $\mathscr{L}^2_\Gamma(\tau)^{(d)}$ such that $\widetilde{\Delta}\tilde{\alpha}^{(d)}_n = \lambda^{(d)}_n \tilde{\alpha}^{(d)}_n$ for each n. From now on we assume Γ is *not* cocompact. Recall the parabolic subgroups P^i used to construct the Siegel domains $\mathfrak{S}^i$ (page 338). Let $P^i = M^i_0 T^i N^i$ be its Langlands decomposition with $M^i_0 := Z_K(T^i)$. For each i let $\Gamma_{P^i} = \Gamma \cap P^i$ and $\Gamma_{M^i_0} = M^i_0 \cap (\Gamma_{P^i} N^i)$. Then $\Gamma_{M^i_0}$ is a finite group which is trivial if Γ is torsion free.

For a vector $\mathsf{u} \in \mathsf{V}^{(d)}_i := \big(\bigwedge^{d,d}\mathfrak{p}^*_{\mathbb{C}}\big)^{\Gamma_{M^i_0}}$ and a complex number s, define the function $\varphi^i_s(\mathsf{u}; g)$ on G using the Iwasawa decomposition $G = N^i T^i K$ by

$$\varphi^i_s(\mathsf{u}; n_i \, \underline{t}_i \, k) = t^{s+n} \tau(k)^{-1} \mathsf{u},$$

for $n_i \in N^i$, $t > 0$, $k \in K$. The Eisenstein series associated with u is defined by the absolutely convergent infinite series

$$\mathsf{E}^i(s; \mathsf{u}; g) = \sum_{\gamma \in \Gamma_{P^i}\backslash\Gamma} \varphi^i_s(\mathsf{u}; \gamma g), \quad g \in G, \tag{56}$$

if $\operatorname{Re} s > n$. There exists a family of linear maps $\mathsf{E}^i(s)$ from $\mathsf{V}^{(d)}_i$ to the space of automorphic forms on $\Gamma\backslash G$, which depends meromorphically on $s \in \mathbb{C}$ and is holomorphic on the imaginary axis, such that $(\mathsf{E}^i(s)(\mathsf{u}))(g) = \mathsf{E}^i(s; \mathsf{u}; g)$ coincides with (56) when $\operatorname{Re} s > n$. If $\mathsf{u} \in \mathsf{V}^{(d)}_i$ is an eigenvector of the Casimir operator $\Omega_{M^i_0}$ of M^i_0 with eigenvalue $c \in \mathbb{C}$, then $R_\Omega \mathsf{E}(s; \mathsf{u}) = (s^2 - n^2 + c)\, \mathsf{E}^i(s; \mathsf{u})$ for any $s \in \mathbb{C}$ where $\mathsf{E}^i(s)$ is regular. It is also known that there exists a meromorphic family of operators $\{\mathsf{c}^i_j(s)\}$ with $\mathsf{c}^i_j(s) \in \operatorname{Hom}_{\mathbb{C}}\big(\mathsf{V}^{(d)}_i, \mathsf{V}^{(d)}_j\big)$ satisfying the functional equations

$$\begin{aligned} \mathsf{E}^i(-s) &= \sum_{j=1}^{h} \mathsf{E}^j(s) \circ \mathsf{c}^i_j(s), \quad 1 \leqslant i \leqslant h, \\ \sum_{j=1}^{h} \mathsf{c}^i_j(-\bar{s})^* \circ \mathsf{c}^i_j(s) &= \mathrm{Id}_{\mathsf{V}_i^{(d)}}, \quad 1 \leqslant i \leqslant h. \end{aligned} \tag{57}$$

Lemma 5.1. *For $0 \leqslant p \leqslant d$ and $\varepsilon \in \{0, 1\}$, let $\mathsf{W}_i^{(d)}(p; \varepsilon)$ be the eigenspace of $\tau(\Omega_{M_0^i})$ on $\mathsf{V}_i^{(d)}$ corresponding to the eigenvalue $(2p - \varepsilon)(2n - 2p + \varepsilon)$. Then we have the orthogonal decomposition*

$$\mathsf{V}_i^{(d)} = \bigoplus_{p=0}^{d} \bigoplus_{\varepsilon\in\{0,1\}} \mathsf{W}_i^{(d)}(p; \varepsilon).$$

Proof. First we recall the construction of invariant tensors given in Proposition 1.2. For each p, q, let $E_{p,q}$ be the M_0-module defined in the proof of that proposition and $\mathcal{T}^{p,q}$ be the M_0-homomorphism constructed there. For convenience we set $E_{p,q} = \{0\}$ when p or q is negative. Choose a parabolic subgroup P_0 of G with Levi subgroup M_0A. Then we can find an element $\tilde{k}_i \in K$ such that $P^i = \tilde{k}_i P_0 \tilde{k}_i^{-1}$, $M_0^i = \tilde{k}_i M_0 \tilde{k}_i^{-1}$, and $T^i = \tilde{k}_i A \tilde{k}_i^{-1}$. For $p \in \mathbb{N}$, put $p(+) := p$ and $p(-) := p - 1$. For $\varepsilon, \varepsilon' \in \{+, -\}$, set $E_{p,\varepsilon\varepsilon'} = \mathcal{T}^{p,p}(E_{p(\varepsilon),p(\varepsilon')})$; then $E^i_{p,\varepsilon,\varepsilon'} := \tau(\tilde{k}_i)(E_{p,\varepsilon\varepsilon'})$ is an irreducible sub M_0^i-module of $F_{p,p}$. For each $0 \leqslant p < 2n$, the space $F_{p,p}$, when considered as an M_0^i-module, is decomposed to the orthogonal direct sum of four subspaces $E^i_{p,\varepsilon\varepsilon'}$ with $\varepsilon, \varepsilon' \in \{+, -\}$. The operator $\tau(\Omega_{M_0^i})$ acts on $E^i_{p,\varepsilon\varepsilon'}$ by the scalar $(p(\varepsilon) + p(\varepsilon'))(2n - p(\varepsilon) - p(\varepsilon'))$. Now put

$$\begin{aligned} \mathsf{W}_i^{(d)}(p; 0) &= L^{d-p}(E^i_{p,++})^{\Gamma_{M_0^i}} \oplus L^{d-p-1}(E^i_{p+1,--})^{\Gamma_{M_0^i}}, \\ \mathsf{W}_i^{(d)}(p; 1) &= L^{d-p}(E^i_{p,+-})^{\Gamma_{M_0^i}} \oplus L^{d-p}(E^i_{p,-+})^{\Gamma_{M_0^i}}. \end{aligned} \tag{58}$$

Then the operator $\tau(\Omega_{M_0^i})$ acts on $\mathsf{W}_i^{(d)}(p; \varepsilon)$ for $0 \leqslant p \leqslant d$ and $\varepsilon \in \{0, 1\}$ as the scalar $(2p - \varepsilon)(2n - 2p + \varepsilon)$. By the K-decomposition (27), the space $\mathsf{V}_i^{(d)}$ is decomposed as the orthogonal direct sum of those subspaces $\mathsf{W}_i^{(d)}(p; \varepsilon)$. □

Lemma 5.2. (a) *The eigenfunctions $\tilde{\alpha}_m^{(d)}$ lie in the space $\mathfrak{M}_\delta$ for any $\delta \in (2rn^{-1}, 1)$. For each $0 \leqslant j \leqslant \mu$, we have*

$$\int_{\Gamma\backslash G} (\boldsymbol{\delta}_{j,s} \tilde{G}_s(g) | \tilde{\alpha}_m^{(\mu)}(g))\, d\dot{g} = \frac{4\,(\mathsf{w} | \mathcal{I}_H(\tilde{\alpha}_m^{(\mu)}; e))}{\mu!\,(\lambda^2 - \lambda_m^{(\mu)} - s^2)^{j+1}}. \tag{59}$$

(b) *Let U be a compact subset on which the function $\mathsf{E}^i(s)$ is holomorphic. Then for all $\mathsf{u} \in \mathsf{V}_i^{(d)}$, the image $\mathsf{E}^i(s)(\mathsf{u})$ is in $\mathfrak{M}_\delta$ for any $\delta \in (2rn^{-1}, 1)$ such that* $\sup_{s\in U} |\mathrm{Re}\, s| < 1 - \delta$. *If $\mathsf{u} \in \mathsf{V}_i^{(\mu)}$ satisfies $\tau(\Omega_{M_0^i})\mathsf{u} = c\,\mathsf{u}$, we have, for each*

$0 \leqslant j \leqslant \mu$,

$$\int_{\Gamma\backslash G} (\boldsymbol{\delta}_{j,s}\tilde{G}_s(g)|\mathsf{E}^i(\nu;\mathsf{u},;g))\,d\dot{g} = \frac{4\,(\mathsf{w}|\mathcal{J}_H(\mathsf{E}^i(\nu;\mathsf{u});e))}{\mu!\,(\lambda^2-n^2+c-s^2+\nu^2)^{j+1}} \tag{60}$$

for $s \in \mathbb{C}$ *with* $|\mathrm{Re}\, s| < 1-2rn^{-1}$.

Proof. The estimation in [Oda and Tsuzuki 2003, Lemma 3.3.1], which implies the first assertion of (a), is valid for our $\tilde{\alpha}_m^{(d)}$ without modification. The first assertion of (b) follows from [Miatello and Wallach 1989, Lemma A.2.2]. The argument to prove (59) and (60) by Theorem 4.5 is the same as that in the proof of [Oda and Tsuzuki 2003, Proposition 6.2.2]. □

For each index (d, i, p, ε), fix an orthonormal basis $\mathcal{B}_i^{(d)}(p;\varepsilon)$ of the space $\mathsf{W}_i^{(d)}(p;\varepsilon)$.

Theorem 5.3. *Let* $\mathrm{Re}\, s > 3n-2r$. *Then there exists* $\varepsilon > 0$ *such that the function* $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s(g)$ *belongs to the space* $\mathcal{L}_\Gamma^{2+\varepsilon}(\tau)^{(\mu)}$. *The spectral expansion of* $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s$ *is*

$$\boldsymbol{\delta}_{\mu,s}\tilde{G}_s = \mathcal{G}_{\mathrm{dis}}(s) + \sum_{p=0}^{\mu}\sum_{\varepsilon\in\{0,1\}} \mathcal{G}_{\mathrm{c}}^{(p,\varepsilon)}(s)$$

with

$$\mathcal{G}_{\mathrm{dis}}(s) = \sum_{m=0}^{\infty} \frac{4\left(\mathsf{w}|\mathcal{J}_H(\tilde{\alpha}_m^{(\mu)};e)\right)}{\mu!\,(\lambda^2-\lambda_m^{(\mu)}-s^2)^r}\,\tilde{\alpha}_m^{(\mu)}, \tag{61}$$

$$\mathcal{G}_{\mathrm{c}}^{(p,\varepsilon)}(s) = \frac{1}{4\pi\sqrt{-1}} \times \int_{\sqrt{-1}\,\mathbb{R}} \sum_{i=1}^{h} \sum_{\mathsf{u}\in\mathcal{B}_i^{(\mu)}(p;\varepsilon)} \frac{4\left(\mathsf{w}|\mathcal{J}_H(\mathsf{E}^i(\zeta;\mathsf{u});e)\right)}{\mu!\,(\lambda^2-(n-2p+\varepsilon)^2-s^2+\zeta^2)^r}\,\mathsf{E}^i(\zeta;\mathsf{u})\,d\zeta,$$

where the summations in the right-hand side of this formula are convergent in $\mathcal{L}_\Gamma^2(\tau)^{(\mu)}$.

Proof. Since the differential operator $\boldsymbol{\delta}_{\mu,s}$ annihilates the even polynomial functions $\mathsf{a}_j(s)$ $(0 \leqslant j \leqslant \mu-1)$ appearing in (40), the behavior of $\boldsymbol{\delta}_{\mu,s}\phi_s(a_t)$ near $t=0$ is given as $\boldsymbol{\delta}_{\mu,s}\phi_s(a_t) = O(s; 1, \log z, z^{\mu+2}(\log z)^2)$. By (41), the function $\boldsymbol{\delta}_{\mu,s}\phi_s(a_t)$ has an exponential decay as $t \to +\infty$ when $\mathrm{Re}\, s > n$. Hence we can argue in exactly the same way as [Oda and Tsuzuki 2003, §§5.2, 5.3] to obtain the estimate $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s \in \mathcal{L}_\Gamma^{2+\varepsilon}(\tau)^{(\mu)}$ with some $\varepsilon > 0$. (In this reference, the results in Section 4, on which the arguments in §§5.2 and 5.3 rely, remain valid for our (G, H, Γ) without modification.) Once we establish the $L^{2+\varepsilon}$-estimate of $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s$, we can work out the spectral expansion of $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s$ using (59) and (60) by the same argument as in [Oda and Tsuzuki 2003, 6.2]. □

Some properties of Eisenstein period.

Proposition 5.4. *For $1 \leqslant i \leqslant h$ and $\mathsf{u} \in \mathsf{V}_i^{(d)}$, there exists a unique $\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}^*$-valued meromorphic function $\mathcal{P}^i_H(s\,;\mathsf{u})$ on $\mathbb{C}$ which is regular and has the value given by the absolutely convergent integral $\mathcal{J}_H(\mathsf{E}^i(s\,;\mathsf{u})\,;e)$ at any regular point $s \in \mathbb{C}$ of $\mathsf{E}^i(s\,;\mathsf{u})$ in $|\mathrm{Re}\, s| < 1 - 2rn^{-1}$.*

Proof. This can be proved by the same argument as in [Oda and Tsuzuki 2003, Proposition 6.1.1]. □

Proposition 5.5. *Let $1 \leqslant i \leqslant h$ and $1 \leqslant p \leqslant d$. Then for any $\mathsf{u} \in \mathsf{W}_i^{(d)}(p\,;1)$, we have $\mathcal{P}^i_H(s\,;\mathsf{u}) = 0$ identically.*

Proof. We freely use the notation introduced in the proof of Lemma 5.1. For any unitary representation (E,σ) of M_0 and $s \in \mathbb{C}$, let $\mathrm{Ind}_{P_0}^G(E\,;s)$ be the Fréchet space of all C^∞-functions $f : G \to E$ satisfying the relation $f(ma_tng) = e^{(n+s)t}\sigma(m)f(g)$ for any $m \in M_0$, $a_t \in A$ and $n \in N$. The group G acts by the right translation on the space $\mathrm{Ind}_{P_0}^G(E\,;s)$ smoothly. For $\varepsilon, \varepsilon' \in \{+,-\}$, $1 \leqslant i \leqslant h$, $0 \leqslant p \leqslant d$, put

$$\mathcal{U}^i_{p,\varepsilon\varepsilon'} := \mathrm{Hom}_{\Gamma_{M_0^i}}(E^i_{p,\varepsilon\varepsilon'}, \mathbb{C}).$$

Then by the theory of Eisenstein series there exists a meromorphic family of continuous G-homomorphisms

$$\mathsf{E}^i_{p,\varepsilon\varepsilon'}(s) : \mathrm{Ind}_{P_0}^G(E_{p,\varepsilon\varepsilon'}\,;s) \otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'} \to \mathcal{A}(\Gamma\backslash G), \quad s \in \mathbb{C},$$

which is given by the absolutely convergent sum

$$(\mathsf{E}^i_{p,\varepsilon\varepsilon'}(s)(f \otimes \check{u}))(g) = \sum_{\gamma \in \Gamma_{P^i}\backslash\Gamma} \check{u}(\tau(\tilde{k}_i)f(\tilde{k}_i^{-1}\gamma g)), \quad g \in G,$$

for $f \in \mathrm{Ind}_{P_0}^G(E_{p,\varepsilon\varepsilon'}\,;s)$ and $\check{u} \in \mathcal{U}^i_{p,\varepsilon\varepsilon'}$ when $\mathrm{Re}\, s > n$. Here $\mathcal{A}(\Gamma\backslash G)$ denotes the space of (not necessarily K-finite) automorphic forms on G. By slightly extending the argument in the proof of [Oda and Tsuzuki 2003, Proposition 6.1.1], we can show that the integral $\Phi^i_{p,\varepsilon\varepsilon'}(s)(f \otimes \check{u}) := \mathcal{J}_H(\mathsf{E}^i_{p,\varepsilon\varepsilon'}(s)(f \otimes \check{u})\,;e)$ is convergent in some neighborhood of the imaginary axis and the function $\Phi^i_{p,\varepsilon\varepsilon'}(s)$ has a meromorphic continuation to $\mathbb{C}$. The linear map

$$\Phi^i_{p,\varepsilon\varepsilon'}(s) : \mathrm{Ind}_{P_0}^G(E_{p,\varepsilon\varepsilon'}\,;s) \otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'} \to \mathbb{C}$$

is continuous and H-invariant, that is, $\Phi^i_{p,\varepsilon\varepsilon'}(s)$ is an H-spherical distribution. The maps $\mathsf{E}^i_{p,\varepsilon\varepsilon'}(s)$ and $\Phi^i_{p,\varepsilon\varepsilon'}(s)$ naturally induce the linear maps

$$\mathsf{E}^i_{p,\varepsilon\varepsilon'}(s)_* : \mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}, \mathrm{Ind}_{P_0}^G(E_{p,\varepsilon\varepsilon'};s)\big) \otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'} \to \mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}, \mathcal{A}(\Gamma\backslash G)\big),$$

$$\Phi^i_{p,\varepsilon\varepsilon'}(s)_* : \mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}, \mathrm{Ind}_{P_0}^G(E_{p,\varepsilon\varepsilon'};s)\big) \otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'} \to \textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}^*.$$

On the other hand, we define the linear map

$$J^i_{p,\varepsilon\varepsilon'} : \mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ \mathrm{Ind}^G_{P_0}(E_{p,\varepsilon\varepsilon'};s)\big)\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'} \to \mathsf{V}^{(d)}_i$$

by composing the maps

$$\begin{aligned}&\mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ \mathrm{Ind}^G_{P_0}(E_{p,\varepsilon\varepsilon'};s)\big)\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'}\\ &\quad\xrightarrow{J_1} \mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ \mathrm{Ind}^K_{M_0}(E_{p,\varepsilon\varepsilon'}\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'})\big)\\ &\quad\xrightarrow{J_2} \mathrm{Hom}_K\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ \mathrm{Ind}^K_{M^i_0}(E^i_{p,\varepsilon\varepsilon'}\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'})\big) \xrightarrow{J_3} \mathrm{Hom}_{M^i_0}\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ E_{p,\varepsilon\varepsilon'}\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'}\big)\\ &\quad\xrightarrow{J_4} \mathrm{Hom}_{M^i_0}\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ \mathrm{Ind}^{M^i_0}_{\Gamma_{M^i_0}}(1_{\Gamma_{M^i_0}})\big) \xrightarrow{J_5} \mathrm{Hom}_{\Gamma_{M^i_0}}\big(\textstyle\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ 1_{\Gamma_{M^i_0}}\big)\cong \mathsf{V}^{(d)}_i,\end{aligned}$$

where J_1 is the isomorphism induced by the natural identification of the K-modules $\mathrm{Ind}^G_{P_0}(E_{p,\varepsilon\varepsilon'};s)\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'}\cong \mathrm{Ind}^K_{M_0}(E_{p,\varepsilon\varepsilon'}\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'})$, the map J_2 is induced by the K-isomorphism $\mathrm{Ind}^K_{M_0}(E_{p,\varepsilon\varepsilon'})\cong \mathrm{Ind}^K_{M^i_0}(E^i_{p,\varepsilon\varepsilon'})$ that assigns the function

$$k\mapsto \tau(\tilde{k}_i)f(\tilde{k}_i^{-1}k)\in \mathrm{Ind}^K_{M^i_0}(E^i_{p,\varepsilon\varepsilon'})\quad\text{to}\quad f(k)\in \mathrm{Ind}^K_{M_0}(E_{p,\varepsilon\varepsilon'}),$$

J_3 and J_5 are the isomorphisms giving the Frobenius reciprocity, and J_4 is the inclusion induced by the map $E^i_{p,\varepsilon\varepsilon'}\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'}\hookrightarrow C^\infty(\Gamma_{M^i_0}\backslash M^i_0)$ which identifies the tensor $\mathsf{u}\otimes\check{u}\in E^i_{p,\varepsilon\varepsilon'}\otimes \mathcal{U}^i_{p,\varepsilon\varepsilon'}$ with the function $m\mapsto \langle \check{u},\tau(m)\mathsf{u}\rangle$ on $\Gamma_{M^i_0}\backslash M^i_0$. The map $J^i_{p,\varepsilon\varepsilon'}$ is injective and has the defining formula

$$\langle J^i_{p,\varepsilon\varepsilon'}(\alpha\otimes\check{u}),\xi\rangle = \langle \check{u},\tau(\tilde{k}_i)(\alpha(\xi)(\tilde{k}_i^{-1}))\rangle$$

for $\xi\in\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}$, $\alpha\in \mathrm{Hom}_K\big(\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}},\ \mathrm{Ind}^G_{P_0}(E_{p,\varepsilon\varepsilon'};s)\big)$, and $\check{u}\in \mathcal{U}^i_{p,\varepsilon\varepsilon'}$. From the definition of $J^i_{p,\varepsilon\varepsilon'}$ and (58), we get $\mathsf{W}^{(d)}_i(p;1)=\mathrm{Im}\,J^i_{p,+-}\oplus \mathrm{Im}\,J^i_{p,-+}$. The formula

$$\mathsf{E}^i(s;\mathsf{u}) = (\mathsf{E}^i_{p,+-}(s)_*\oplus \mathsf{E}^i_{p,-+}(s)_*)\circ (J^i_{p,+-}\oplus J^i_{p,-+})^{-1}(\mathsf{u}),\qquad \mathsf{u}\in \mathsf{W}^{(d)}_i(p;1),$$

which follows directly from the definitions when $\mathrm{Re}\,s>n$, remains valid as an identity of meromorphic functions on $\mathbb{C}$. When $|\mathrm{Re}\,s|$ is sufficiently small, the integration on $\Gamma_H\backslash H$ of this formula yields yet another formula: for $\mathsf{u}\in \mathsf{W}^{(d)}_i(p;1)$,

$$(62)\qquad \mathcal{J}_H(\mathsf{E}^i(s;\mathsf{u});e) = (\Phi^i_{p,+-}(s)_*\oplus \Phi^i_{p,-+}(s)_*)\circ (J^i_{p,+-}\oplus J^i_{p,-+})^{-1}(\mathsf{u}).$$

By the general theory, there exists an open dense set U such that for $s\in U$ the space $C^{-\infty}\,\mathrm{Ind}^G_{P_0}(E_{p,q};s)^H$ of H-spherical distributions is isomorphic to $E^{M_0\cap H}_{p,q}=E^M_{p,q}$ [Heckman and Schlichtkrull 1994, Theorem 6.4, p. 151], which is zero when $p\neq q$ by the Claim in the proof of Proposition 1.2. Hence $\Phi^i_{p,+-}(s)=0$ and $\Phi^i_{p,-+}(s)=0$ for $s\in U$. This, combined with (62), implies $\mathcal{J}_H(\mathsf{E}^i(s;\mathsf{u});e)=0$ for generic s. By analytic continuation we obtain $\mathcal{P}^i_H(s;\mathsf{u})=0$ identically. □

Lemma 5.6. *For each $0 \leqslant p \leqslant \mu$, put*

$$(63)\quad \tilde{\mathscr{E}}_p^{(\mu)}(\nu\,;g) = \frac{4}{\mu!}\sum_{i=1}^{h}\sum_{\mathsf{u}\in\mathscr{B}_i^{(\mu)}(p;0)} (\mathsf{w}|\mathscr{P}^i_H(-\bar{\nu}\,;\mathsf{u}))\,\mathsf{E}^i(\nu\,;\mathsf{u}\,;g),\qquad g\in G,\ \nu\in\mathbb{C}.$$

Then $\tilde{\mathscr{E}}_p^{(\mu)}(\nu\,;g)$ is independent of the choice of the orthonormal basis $\mathscr{B}_i^{(\mu)}(p\,;0)$ and satisfies the functional equation $\tilde{\mathscr{E}}_p^{(\mu)}(-s\,;g) = \tilde{\mathscr{E}}_p^{(\mu)}(s\,;g)$.

Proof. The independence of the basis $\mathscr{B}_i^{(\mu)}(p\,;0)$ is clear to see. The functional equation follows from (57). ☐

Meromorphic continuation and functional equations. Let $\mathscr{K}_\Gamma(\tau)$ be the space of C^∞-functions $\tilde{\beta}: G \to \bigwedge \mathfrak{p}^*_{\mathbb{C}}$ with compact support modulo Γ such that $\tilde{\beta}(\gamma g k) = \tau(k)^{-1}\tilde{\beta}(g)$ for all $\gamma\in\Gamma$ and $k\in K$.

Theorem 5.7. *Let L_1 be the interval on the imaginary axis defined by* (25)*. Let $0 \leqslant j \leqslant \mu$. Then for each $\tilde{\beta}\in\mathscr{K}_\Gamma(\tau)$, on* $\operatorname{Re} s > n$ *the holomorphic function $s \mapsto \mathscr{G}_j(s,\tilde{\beta}) := \langle \boldsymbol{\delta}_{j,s}\tilde{G}_s|\tilde{\beta}\rangle$ has a meromorphic continuation to the domain $\mathbb{C}-L_1$. A point $s_0\in\mathbb{C}-L_1$ with* $\operatorname{Re} s_0 \geqslant 0$ *is a pole of the meromorphic function $\mathscr{G}_j(s,\beta)$ if and only if there exists an $m\in\mathbb{N}$ such that $(\mathsf{w}|\mathscr{J}_H(\tilde{\alpha}_m^{(\mu)}\,;e)) \neq 0$, $\langle\tilde{\alpha}_m^{(\mu)}|\tilde{\beta}\rangle \neq 0$ and $s_0^2-\lambda^2 = -\lambda_m^{(\mu)}$. In this case, the function*

$$\mathscr{G}_j(s,\beta) - \sum_{\substack{m\in\mathbb{N}\\ \lambda_m^{(\mu)}=\lambda^2-s_0^2}} \frac{4(\mathsf{w}|\mathscr{J}_H(\tilde{\alpha}_m^{(\mu)}\,;e))\langle\tilde{\alpha}_m^{(\mu)}|\tilde{\beta}\rangle}{\mu!\,(s_0^2-s^2)^{j+1}}$$

is holomorphic at $s=s_0$. We have the functional equation

$$(64)\qquad \mathscr{G}_j(-s,\tilde{\beta}) - \mathscr{G}_j(s,\tilde{\beta}) = \boldsymbol{\delta}_{j,s}\Bigg(\sum_{p=0}^{\mu}\frac{\langle\tilde{\mathscr{E}}_p^{(\mu)}(\nu_s^{(p+1)})|\tilde{\beta}\rangle}{2\,\nu_s^{(p+1)}}\Bigg).$$

Proof. We prove the assertions by downward-induction on j. Consider the case of $j=\mu$. By the same argument in [Oda and Tsuzuki 2003, §§6.2, 6.3], we see that the series $\mathscr{G}_{\mathrm{dis}}(s)$ is convergent in $\mathscr{L}^2_\Gamma(\tau)$ for any $s\in\mathbb{C}$ such that for all m, $\lambda_m^{(\mu)} \neq \lambda^2-s^2$ it gives an $\mathscr{L}^2_\Gamma(\tau)$-valued meromorphic function on $\mathbb{C}$ (namely, for each $s_0\in\mathbb{C}$ there exists $a\in\mathbb{Z}$ such that $(s-s_0)^a\mathscr{G}_{\mathrm{dis}}(s)$ is holomorphic around s_0). By Proposition 5.5, $\mathscr{G}_\mu(s)-\mathscr{G}_{\mathrm{dis}}(s)$ is the sum over p of

$$\mathscr{G}_{\mathrm{c}}^{(p,0)}(s) = \frac{1}{4\pi\sqrt{-1}}\int_{\sqrt{-1}\,\mathbb{R}} \frac{\tilde{\mathscr{E}}_p^{(\mu)}(\zeta)}{(\zeta^2-(\nu_s^{(p+1)})^2)^r}\,d\zeta,\quad 0\leqslant p\leqslant\mu.$$

Since $\operatorname{Re}\nu_s^{(p+1)} > \operatorname{Re} s$ for $\operatorname{Re} s>0$ by Lemma 3.1, the denominator of the integrand is never zero as long as $\zeta\in\sqrt{-1}\,\mathbb{R}$ and $\operatorname{Re} s>0$. Hence the same argument just cited proves the convergence of the integral $\mathscr{G}_{\rm c}^{(p,0)}(s)$ in $\mathscr{L}^2_\Gamma(\tau)$ not only on $\operatorname{Re} s > 3n-2r$ but also on the broader domain $\operatorname{Re} s > 0$, and moreover the integral defines a holomorphic function on $\operatorname{Re} s > 0$. Thus a meromorphic continuation of $\mathscr{G}_\mu(s,\beta) = \langle \mathscr{G}_{\rm dis}(s)|\tilde\beta\rangle + \sum_p \langle \mathscr{G}_{\rm c}^{(p,0)}(s)|\tilde\beta\rangle$ exists.

The next step is to obtain the analytic continuation of $\mathscr{G}_\mu(s,\beta)$ around a point $s_0 = \sqrt{-1}\,\sigma_0 \in \sqrt{-1}\,\mathbb{R} - L_1$. For that purpose, we consider the same problem for each integrals $\langle \mathscr{G}_{\rm c}^{(p,0)}(s)|\tilde\beta\rangle$. Put $\zeta_0 = \nu_{s_0}^{(p+1)}$. Note that $\zeta_0 \in \sqrt{-1}\,\mathbb{R}$. Let $a,\ b > 0$ be arbitrary numbers such that the functions $\langle \mathscr{E}_p^{(\mu)}(\zeta)|\tilde\beta\rangle$, $0 \leqslant p \leqslant \mu$, are holomorphic on the open rectangle $R_{a,b}(\zeta_0)$ having the vertices $\zeta_0 \pm a \pm \sqrt{-1}b$. Let $C_{a,b}$ be the path which, as a point set, is a union of $\sqrt{-1}\,\mathbb{R} - [\zeta_0 - b\sqrt{-1}, \zeta_0 + b\sqrt{-1}]$ and $\partial R_{a,b}(\zeta_0) \cap \{\operatorname{Re} \zeta \geqslant 0\}$, and which rounds the point ζ_0 counterclockwise. Let $U_{a,b}(\pm s_0)$ be the inverse image of $R_{a,b}(\zeta_0)$ by the map $s \mapsto \nu_s^{(p+1)}$; thus $U_{a,b}(\pm s_0)$ is an open neighborhood of $\{s_0, -s_0\}$ in $\mathbb{C} - L_1$. For $s \in U_{a,b}(\pm s_0) \cap \{\operatorname{Re} s > 0\}$, by the residue theorem,

$$
\begin{aligned}
(65)\quad & \langle \mathscr{G}_{\rm c}^{(p,0)}(s)|\tilde\beta\rangle \\
&= \frac{1}{4\pi\sqrt{-1}}\left(\int_{C_{a,b}} \frac{\langle \tilde{\mathscr{E}}_p^{(\mu)}(\zeta)|\tilde\beta\rangle}{(\zeta^2-(\nu_s^{(p+1)})^2)^r}\,d\zeta - 2\pi\sqrt{-1}\operatorname{Res}_{z=\nu_s^{(p+1)}} \frac{\langle \tilde{\mathscr{E}}_p^{(\mu)}(\zeta)|\tilde\beta\rangle}{(\zeta^2-(\nu_s^{(p+1)})^2)^r}\right) \\
&= \frac{1}{4\pi\sqrt{-1}}\int_{C_{a,b}} \frac{\langle \tilde{\mathscr{E}}_p^{(\mu)}(\zeta)|\tilde\beta\rangle}{(\zeta^2-(\nu_s^{(p+1)})^2)^r}\,d\zeta - \frac{1}{4}\boldsymbol{\delta}_{\mu,s}\left(\frac{\langle \tilde{\mathscr{E}}_p^{(\mu)}(\nu_s^{(p+1)})|\tilde\beta\rangle}{\nu_s^{(p+1)}}\right).
\end{aligned}
$$

using Lemma 5.8 to compute the residue. The integral in the first term of this expression is convergent even for $s \in U_{a,b}(\pm s_0)$ and defines a holomorphic function on $U_{a,b}(\pm s_0)$. Since the second term is also holomorphic on $U_{a,b}(\pm s_0)$, we obtain an analytic continuation of $\langle \mathscr{G}_{\rm c}^{(p,0)}(s)|\tilde\beta\rangle$ on a neighborhood of $\{s_0, -s_0\}$. The functional equation (64) for $s \in U_{a,b}(\pm s_0)$ follows if we note that the first term in the right-hand side of the second identity of (65) is invariant under the substitution $s \mapsto -s$ and also note the equation $\nu_{-s}^{(p+1)} = -\nu_s^{(p+1)}$; see Lemma 3.1. Once the functional equation (64) is established on a small open set of the form $U_{a,b}(\pm s_0)$ with $s_0 \in \sqrt{-1}\,\mathbb{R} - L_1$, we can use it to obtain a meromorphic continuation by defining the value $\mathscr{G}_\mu(s,\beta)$ for $\operatorname{Re} s < 0$ in terms of $\mathscr{G}_\mu(-s,\beta)$ which is defined above and the terms containing the derivative of $\langle \mathscr{E}^{(\mu)}(s)|\tilde\beta\rangle$ which is meromorphic on $\mathbb{C}$; see Proposition 5.4. The poles of $\langle \mathscr{G}_\mu(s)|\tilde\beta\rangle$ on $\operatorname{Re} s \geqslant 0$ stem only from the discrete part $\langle \mathscr{G}_{\rm dis}(s)|\tilde\beta\rangle$, whose series expression (61) itself proves the criterion in the theorem for s_0 to be a pole, as well as a statement on the behavior around the poles. This completes the proof of Theorem for $j = \mu$.

We prove the theorem for j assuming it holds for $j+1$. Since

$$
\mathscr{G}_{j+1}(s,\beta) = (j+1)^{-1}\frac{1}{2s}\frac{d}{ds}\mathscr{G}_j(s,\beta)
$$

for Re $s > n$ by Proposition 4.4, the function $\mathcal{G}_j(s, \beta)$ should be a primitive function of $2(j+1)s\,\mathcal{G}_{j+1}(s, \beta)$. Fix a point z_0 with Re $z_0 > n$ so that the value $\mathcal{G}_j(z_0, \beta)$ is already defined. For each $s_0 \in \sqrt{-1}\,\mathbb{R} - L_1$, let $U_{a,b}(\pm s_0)$ be its neighborhood constructed above and put $D_{a,b}(\pm s_0) = U_{a,b}(\pm s_0) \cup \{\text{Re } s > 0\}$. We take sufficiently small a and b such that $\langle \mathcal{E}_p^{(\mu)}(\nu_s^{(p+1)}|\tilde{\beta}\rangle$ are all regular on $U_{a,b}(\pm s_0)$. For a path C_s connecting z_0 and $s \in D_{a,b}(\pm s_0)$ inside $D_{a,b}(\pm s_0)$, consider the integral

$$\tilde{\mathcal{G}}(s) = 2(j+1) \int_{C_s} \zeta\, \mathcal{G}_{j+1}(\zeta, \beta)\, d\zeta + \mathcal{G}_j(z_0, \beta). \tag{66}$$

Since the residues of $\mathcal{G}_{j+1}(s, \beta)$ at any poles in Re $s \geqslant 0$ are all zero by induction-assumption and since the poles of $\mathcal{G}_{j+1}(s, \beta)$ in $D_{a,b}(\pm s_0)$ are automatically in Re $s \geqslant 0$ by the choice of a, b, this integral is independent of the choice of the path C_s and defines a meromorphic function of s on $D_{a,b}(\pm s_0)$. Since $\tilde{\mathcal{G}}(z_0) = \mathcal{G}_j(z_0, \beta)$ and $2(j+1)s\,\mathcal{G}_{j+1}(s, \beta) = d/ds\tilde{\mathcal{G}}(s)$ for Re $s > n$, we have $\tilde{\mathcal{G}}(s) = \mathcal{G}_j(s)$ at least on Re $s > n$. Thus the integral expression (66) gives us an analytic continuation of $\mathcal{G}_j(s, \beta)$ to $D_{a,b}(\pm s_0)$. The functional equation (64) for $j+1$ takes the form

$$\frac{1}{2s}\frac{d}{ds} J(s) = 0,$$

with

$$J(s) = \mathcal{G}_j(-s, \tilde{\beta}) - \mathcal{G}_j(s, \tilde{\beta}) - \boldsymbol{\delta}_{j,s}\left(\sum_{p=0}^{\mu} \frac{\langle \tilde{\mathcal{E}}_p^{(\mu)}(\nu_s^{(p+1)})|\tilde{\beta}\rangle}{2\,\nu_s^{(p+1)}} \right)$$

on $D_{a,b}(\pm s_0)$. Hence $J(s)$ is a constant on the domain $D_{a,b}(\pm s_0)$ on the one hand. On the other hand, by the functional equation of $\tilde{\mathcal{E}}_p^{(\mu)}(\nu)$ (Lemma 5.6) and that of $\nu_s^{(p+1)}$ (Lemma 3.1), we have $J(-s) = -J(s)$. Hence the constant $J(s)$ should be zero. This establishes the functional equation (64) on $D_{a,b}(\pm s_0)$. By defining the value of $\mathcal{G}_j(s, \beta)$ for Re $s < 0$ by the functional equation (64), we obtain the meromorphic continuation of $\mathcal{G}_j(s, \beta)$ to $\mathbb{C} - L_1$ keeping (64) correct. The assertion on the poles for $\mathcal{G}_j(s, \beta)$ follows from that for $\mathcal{G}_{j+1}(s, \beta)$ from (66). ☐

Lemma 5.8. *Let $1 \leqslant p \leqslant r$ and put $\nu_s = \nu_s^{(p)}$. Let U be an open domain in $\mathbb{C} - L_1$ and $F(z)$ a holomorphic function on some open neighborhood of $\{\nu_s^{(p)} \mid s \in U\}$. Then for each $j \geqslant 1$ we have*

$$\text{Res}_{z=\nu_s} \frac{F(z)}{(z^2 - \nu_s^2)^j} = \boldsymbol{\delta}_{j-1,s}\left(\frac{F(\nu_s)}{2\nu_s} \right), \quad s \in U.$$

6. Green currents

For Re $s > n$, the currents $G_s := (c_r\, \mu\, \mu!)^{-1} P(\phi_s)$ and $\Psi_s = (c_r\, \mu\, \mu!)^{-1} P(\psi_s)$ are of (μ, μ)-type and of (r, r)-type respectively. In this section we study some of the properties of the currents G_s and Ψ_s using the knowledge of the function $\tilde{G}_s$

obtained in the previous section. We put the Kähler form $\boldsymbol{\omega}$ on $\Gamma\backslash G/K$ such that $\tilde{\boldsymbol{\omega}}(g) = \omega$ for all $g \in G$. The metric on $\Gamma\backslash G/K$ corresponding to $\boldsymbol{\omega}$ defines the Laplacian $\triangle$, the Lefschetz operator and its adjoint Λ acting on the space of forms and currents on $\Gamma\backslash G/K$.

Currents defined by modular cycles. Denote by D the image of the map from $\Gamma_H\backslash H/K_H$ to $\Gamma\backslash G/K$ induced by the natural holomorphic inclusion of H/K_H into G/K. Then D, a closed complex analytic subset of $\Gamma\backslash G/K$, defines an (r, r)-current δ_D on $\Gamma\backslash G/K$ by the integration

$$\langle \delta_D, \alpha\rangle = \int_{D_{\rm ns}} j^*\alpha, \qquad \alpha \in A_{\rm c}(\Gamma\backslash G/K).$$

Here $j : D \hookrightarrow \Gamma\backslash G/K$ is the natural inclusion and $D_{\rm ns}$ is the smooth locus of D. Since δ_D is closed, it defines a cycle on $\Gamma\backslash G/K$ of real codimension $2r$ [Griffiths and Harris 1978, p. 32–33].

Proposition 6.1. *For* $\alpha \in A_{\rm c}(\Gamma\backslash G/K)$, *we have*

$$\langle \delta_D, *\bar{\alpha}\rangle = (*\mathrm{vol}_H | \mathcal{J}_H(\tilde{\alpha}; e)), \tag{67}$$

$$\langle \Lambda\delta_D, *\bar{\alpha}\rangle = (\Lambda(*\mathrm{vol}_H) | \mathcal{J}_H(\tilde{\alpha}; e)). \tag{68}$$

Proof. We give a proof assuming the natural map $p : \Gamma_H\backslash H/K_H \to \Gamma\backslash G/K$ is one-to-one. (The general case is similar.) For any $\beta \in A_c(\Gamma\backslash G/K)$, we have

$$\langle \delta_D, \beta\rangle = \int_{\Gamma_H\backslash H/K_H} p^*\beta = \int_{\Gamma_H\backslash H} (f(h)|\mathrm{vol}_H)\, d\dot{h},$$

noting that $\|\mathrm{vol}_H\| = 1$. Here $f : H \to \bigwedge(\mathfrak{h}\cap\mathfrak{p})^*_{\mathbb{C}}$ is the function on H corresponding to $p^*\beta$ which is determined by a formula similar to (48). Put $\beta = *\bar{\alpha}$. Equation (67) holds since

$$(f(h)|\mathrm{vol}_H) = (\tilde{\beta}(h)|\mathrm{vol}_H) = (\mathrm{vol}_H| *\tilde{\alpha}(h)) = (*\mathrm{vol}_H|\tilde{\alpha}(h)).$$

Thus (68) follows from (67), since $\langle \Lambda\delta_D, \beta\rangle = \langle \delta_D, \Lambda\beta\rangle$ by definition. □

Differential equations. First we show that G_s and Ψ_s satisfy some differential equations.

Theorem 6.2. *Let* Re $s > n$. *Then*

$$(\triangle + s^2 - \lambda^2)G_s = -4\Lambda\delta_D, \tag{69}$$

$$\triangle\,\Psi_s = (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1}\delta_D), \tag{70}$$

$$\partial\bar{\partial}G_s = \Psi_s - 2\sqrt{-1}\,\delta_D. \tag{71}$$

Proof. It suffices to prove these formulas for s with $\operatorname{Re} s > 3n - 2r$ because they depend on s holomorphically; see Proposition 4.4. Since $R_\Omega \tilde{\alpha} = -(\triangle\alpha)^{\sim}$ for $\alpha \in A(\Gamma\backslash G/K)$ by Kuga's formula, equation (69) follows from Theorem 4.5, the first equality in (7) and (68). To prove (71), take an arbitrary form $\alpha \in A_{\mathrm{c}}(\Gamma\backslash G/K)$. First by definition and then by an application of (50),

$$\begin{aligned}\langle \partial\bar{\partial} G_s, *\bar{\alpha}\rangle &= -\langle G_s, \bar{\partial}\partial * \bar{\alpha}\rangle = \langle G_s, *\partial^*\bar{\partial}^*\bar{\alpha}\rangle = \langle G_s, *\overline{\bar{\partial}^*\partial^*\alpha}\rangle \\ &= \frac{1}{c_r\,\mu\,\mu!}\int_0^\infty \varrho(t)\,(\phi_s(a_t)|\mathcal{J}_H(f\,;a_t))\,dt,\end{aligned} \tag{72}$$

with $f(g) = (\bar{\partial}^*\partial^*\alpha)^{\sim}(g)$. Since

$$(\bar{\partial}^*\partial^*\alpha)^{\sim}(g) = -\sum_{\alpha,\beta=0}^{n-1} e(\omega_\alpha)^* e(\bar{\omega}_\beta)^* R_{X_\alpha \bar{X}_\beta}\tilde{\alpha}(g),$$

we have

$$\mathcal{J}_H(f\,;a_t) = -\sum_{\alpha,\beta=0}^{n-1} e(\omega_\alpha)^* e(\bar{\omega}_\beta)^* R_{X_\alpha \bar{X}_\beta}\mathcal{J}_H(\tilde{\alpha}\,;a_t), \tag{73}$$

using Proposition 4.3(1). Inserting the formulas in Lemma 2.1 to the right-hand side of (73), we obtain $\mathcal{J}_H(f\,;a_t) = \mathcal{E}_t\mathcal{J}_H(f\,;a_t)$ with $\mathcal{E}_t$ the differential operator on $t > 0$ given by

$$\begin{aligned}\mathcal{E}_t = -\frac{e^*(\omega_0\wedge\bar{\omega}_0)}{4}\frac{d^2}{dt^2} + \frac{1}{2}(\tilde{\mathcal{A}}\tanh t + \tilde{\mathcal{B}}\coth t)\frac{d}{dt} \\ -\tilde{\mathcal{P}}_+\tilde{\mathcal{P}}_-\tanh^2 t - \tilde{\mathcal{R}}_+\tilde{\mathcal{R}}_-\coth^2 t - \tilde{\mathcal{P}}_+\tilde{\mathcal{R}}_- - \tilde{\mathcal{R}}_+\tilde{\mathcal{P}}_-,\end{aligned}$$

where

$$\tilde{\mathcal{P}}_+ = \sum_{i=1}^{n-r} e^*(\bar{\omega}_i)\tau(Z_i), \quad \tilde{\mathcal{P}}_- = \sum_{i=1}^{n-r} e^*(\omega_i)\tau(\bar{Z}_i),$$

$$\tilde{\mathcal{R}}_+ = \sum_{j=n-r+1}^{n-1} e^*(\bar{\omega}_j)\tau(Z_j), \quad \tilde{\mathcal{R}}_- = \sum_{j=n-r+1}^{n-1} e^*(\omega_j)\tau(\bar{Z}_j),$$

$$\tilde{\mathcal{A}} = e^*(2\sqrt{-1}\omega_H) - \tfrac{1}{2}e^*(\omega_0\wedge\bar{\omega}_0) + e^*(\omega_0)\tilde{\mathcal{P}}_+ - e^*(\bar{\omega}_0)\tilde{\mathcal{P}}_-,$$

$$\tilde{\mathcal{B}} = e^*(2\sqrt{-1}\,\eta) - \tfrac{1}{2}e^*(\omega_0\wedge\bar{\omega}_0) + e^*(\omega_0)\tilde{\mathcal{R}}_+ - e^*(\bar{\omega}_0)\tilde{\mathcal{R}}_-.$$

Let $0 < \varepsilon < R$. Integration by parts yields

$$\begin{aligned}\int_\varepsilon^R \varrho(t)\,(\phi_s(a_t)|\mathcal{J}_H(f\,;a_t))\,dt \\ = \Phi(R) - \Phi(\varepsilon) + \int_\varepsilon^R \varrho(t)\,(\mathcal{E}_t^*\phi_s(a_t)|\mathcal{J}_H(f\,;a_t))\,dt,\end{aligned} \tag{74}$$

with

$$\begin{aligned}(75)\quad \Phi(t) = &-\tfrac{1}{4}\varrho(t)\big(\omega_0\wedge\bar\omega_0\wedge\phi_s(a_t)|F'(t)\big)\\ &+\tfrac{1}{4}\varrho'(t)\big(\omega_0\wedge\bar\omega_0\wedge\phi_s(a_t)|F(t)\big)+\tfrac{1}{4}\varrho(t)\Big(\omega_0\wedge\bar\omega_0\wedge\frac{d}{dt}\phi_s(a_t)|F(t)\Big)\\ &+\tfrac{1}{2}\varrho(t)\big((\tilde{\mathscr{A}}^*\tanh t+\tilde{\mathscr{B}}^*\coth t)\phi_s(a_t)|F(t)\big).\end{aligned}$$

Here $\mathscr{E}_t^*$ is the formal adjoint operator of $\mathscr{E}_t$, uniquely determined by the relation

$$(76)\qquad \int_0^\infty (\mathscr{E}_t^*\mathsf{a}(t)|\mathsf{b}(t))\,\varrho(t)\,dt = \int_0^\infty (\mathsf{a}(t)|\mathscr{E}_t\mathsf{b}(t))\,\varrho(t)\,dt$$

for any compactly supported $\big(\bigwedge\mathfrak{p}_{\mathbb{C}}^*\big)^M$-valued functions $\mathsf{a}(t)$ and $\mathsf{b}(t)$ on $(0,\infty)$. We compute the limit of $\Phi(\varepsilon)$ as ε going zero. By (40),

$$(77)\qquad \Phi(\varepsilon) = \frac{c_r}{2}\big((e(\omega_0\wedge\bar\omega_0)+2\tilde{\mathscr{B}}^*)\mathsf{w}|F(0)\big)\big(1+O(s\,;\varepsilon,\varepsilon\log\varepsilon)\big).$$

Using the formulas in Lemma 2.1 and the relation $\tau(Z)^* = -\tau(\bar Z)$ $(Z\in\mathfrak{k}_{\mathbb{C}})$, we compute to obtain

$$\begin{aligned}\tilde{\mathscr{R}}_+^*e(\omega_0) &= -e(2\sqrt{-1}\eta+\mu\,\omega_0\wedge\bar\omega_0)+e(\omega_0)\mathscr{R}_+,\\ \tilde{\mathscr{R}}_-^*e(\bar\omega_0) &= e(2\sqrt{-1}\eta+\mu\,\omega_0\wedge\bar\omega_0)+e(\bar\omega_0)\mathscr{R}_-.\end{aligned}$$

Using these formulas, (6) and the relations $\mathscr{R}_\pm\mathsf{w}=0$, we obtain $\tilde{\mathscr{R}}_+^*e(\omega_0)\mathsf{w} = \tilde{\mathscr{R}}_-^*e(\bar\omega_0)\mathsf{w}=0$. Hence

$$\begin{aligned}\tilde{\mathscr{B}}^*\mathsf{w} &= e(2\sqrt{-1}\eta)\mathsf{w}-\tfrac{1}{2}e(\omega_0\wedge\bar\omega_0)\mathsf{w}+\tilde{\mathscr{R}}_+^*e(\omega_0)\mathsf{w}-\tilde{\mathscr{R}}_-^*e(\bar\omega_0)\mathsf{w}\\ &= 2\sqrt{-1}\eta\wedge\mathsf{w}-\tfrac{1}{2}\omega_0\wedge\bar\omega_0\wedge\mathsf{w}.\end{aligned}$$

From this formula and (77),

$$(78)\qquad \lim_{\varepsilon\to+0}\Phi(\varepsilon) = 2\sqrt{-1}c_r(\eta\wedge\mathsf{w}|F(0)) = 2\sqrt{-1}c_r\,\mu\,\mu!(*\mathrm{vol}_H|F(0))$$

using (6) to obtain the second equality. We compute the limit of $\Phi(R)$ as R tends to infinity by means of (41) noting Lemma 3.1. The result is, when $\operatorname{Re} s > n$,

$$(79)\qquad \lim_{R\to+\infty}\Phi(R)=0.$$

Putting (72), (74), (78) and (79) together and using (67), we finally obtain

$$\langle\partial\bar\partial G_s, *\bar\alpha\rangle = -2\sqrt{-1}\langle\delta_D, *\bar\alpha\rangle + \frac{1}{c_r\,\mu\,\mu!}\int_0^\infty \varrho(t)\,(\mathscr{E}_t^*\phi_s(a_t)|\mathscr{J}_H(f\,;a_t))\,dt.$$

To complete the proof of (71), we have only to prove that $\mathscr{E}_t^*\phi_s(a_t) = \psi_s(a_t)$ for $t>0$. Since $\psi_s(a_t) = \tilde{\mathscr{E}}_t\phi_s(a_t)$ with $\tilde{\mathscr{E}}_t$ the differential operator in t given by (24) in the z-coordinate, it suffices to show $\mathscr{E}_t^* = \tilde{\mathscr{E}}_t$. For that purpose, we show that

$$\int_0^\infty \big(\tilde{\mathscr{E}}_t \mathsf{a}(t)|\mathsf{b}(t)\big)\,\varrho(t)\,dt = \int_0^\infty (\mathscr{E}_t^* \mathsf{a}(t)|\mathsf{b}(t))\,\varrho(t)\,dt$$

for arbitrary compactly supported $\big(\bigwedge \mathfrak{p}_{\mathbb{C}}^*\big)^M$-valued C^∞-functions $\mathsf{a}(t)$ and $\mathsf{b}(t)$ on $(0, \infty)$. By the decomposition $G = HAK$, we can extend the functions $\mathsf{a}(t)$ and $\mathsf{b}(t)$ to smooth functions $a(g)$ and $b(g)$ belonging to C_τ^∞ by the formula

$$a(ha_t k) = \tau(k)^{-1}\mathsf{a}(t), \quad \text{and} \quad b(ha_t k) = \tau(k)^{-1}\mathsf{b}(t), \quad h \in H,\ t > 0,\ k \in K.$$

By Lemma 4.1, (76) and the definitions of $\mathscr{E}_t$ and $\tilde{\mathscr{E}}_t$, we obtain

$$\int_{H\backslash G} \bigg(\sum_{\alpha,\beta=0}^{n-1} e(\bar\omega_\beta) e(\omega_\alpha) R_{X_\alpha \bar X_\beta} a(g) \;\bigg|\; b(g) \bigg)\, d\dot g = \int_0^\infty (\tilde{\mathscr{E}}_t \mathsf{a}(t)|\mathsf{b}(t))\,\varrho(t)\,dt,$$

$$\int_{H\backslash G} \bigg(a(g) \;\bigg|\; \sum_{\alpha,\beta=0}^{n-1} e(\omega_\alpha)^* e(\bar\omega_\beta)^* R_{X_\alpha \bar X_\beta} b(g) \bigg)\, d\dot g = \int_0^\infty (\mathscr{E}_t^* \mathsf{a}(t)|\mathsf{b}(t))\,\varrho(t)\,dt.$$

The left-hand sides of these two formulas are easily seen to be identical, since

$$\int_{H\backslash G} (a_1(g)|R_X a_2(g))\,d\dot g = -\int_{H\backslash G} (R_{\bar X} a_1(g)|a_2(g))\,d\dot g, \quad a_1,\ a_2 \in C_\tau^\infty,\ X \in \mathfrak{g}_{\mathbb{C}}.$$

We now deduce (70) from (69) and (71). That the current δ_D is real and closed implies that $\partial\delta_D = 0$ and $\bar\partial\delta_D = 0$. Hence $\triangle\delta_D = \mathrm{d}\,\mathrm{d}^*\delta_D = (\partial+\bar\partial)(\partial^*+\bar\partial^*)\delta_D$. Since $\Gamma\backslash G/K$ is a Kähler manifold, we have $\partial^* = \sqrt{-1}[\Lambda, \bar\partial]$ and $\bar\partial^* = -\sqrt{-1}[\Lambda, \partial]$; see [Wells 1980, (4.5), Corollary 4.10, p. 193]. Using these equalities, we compute

$$\triangle\delta_D = \sqrt{-1}(\partial+\bar\partial)(-[\Lambda, \partial-\bar\partial])\delta_D = \sqrt{-1}(\partial+\bar\partial)(\partial-\bar\partial)\Lambda\delta_D = -2\sqrt{-1}\partial\bar\partial\Lambda\delta_D$$

to obtain

$$\triangle\delta_D + 2\sqrt{-1}\partial\bar\partial\Lambda\delta_D = 0. \tag{80}$$

Since $\Psi_s = \partial\bar\partial G_s + 2\sqrt{-1}\delta_D$ by (71), we have

$$\begin{aligned}
\triangle\Psi_s &= \triangle(\partial\bar\partial G_s + 2\sqrt{-1}\delta_D) = \partial\bar\partial \triangle G_s + 2\sqrt{-1} \triangle \delta_D \\
&= \partial\bar\partial((\lambda^2 - s^2)G_s - 4\Lambda\delta_D) + 2\sqrt{-1} \triangle \delta_D \\
&= (\lambda^2 - s^2)\partial\bar\partial G_s - 4\partial\bar\partial\Lambda\delta_D + 2\sqrt{-1} \triangle \delta_D \\
&= (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1}\delta_D) + 2\sqrt{-1}(\triangle\delta_D + 2\sqrt{-1}\partial\bar\partial\Lambda\delta_D) \\
&= (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1}\delta_D),
\end{aligned}$$

using (69) for the third equality, (71) for the fifth and (80) for the last. □

Main theorem. Let $A^{p,q}_{(2)}(\Gamma\backslash G/K)$ be the Hilbert space of the measurable (p,q)-forms on $\Gamma\backslash G/K$ with the finite L^2-norm $\|\alpha\| := (\int_{\Gamma\backslash G/K} \alpha \wedge *\bar{\alpha})^{1/2}$. For each $c \in \mathbb{C}$, let $A^{p,q}_{(2)}(\Gamma\backslash G/K; c)$ be the c-eigenspace of the Laplacian $\triangle$ acting on $A^{p,q}_{(2)}(\Gamma\backslash G/K)$. In particular,

$$\mathcal{H}^{p,q}_{(2)}(\Gamma\backslash G/K) := A^{p,q}_{(2)}(\Gamma\backslash G/K; 0)$$

is the space of the harmonic L^2-forms of (p,q)-type. For each p, let $\mathcal{E}^{(\mu)}_p(\nu)$ be the C^∞-form of (μ,μ)-type on $\Gamma\backslash G/K$ corresponding to the function $\tilde{\mathcal{E}}^{(\mu)}_p(\nu)$ on G defined by (63). Then Theorem 5.7 immediately gives us the following theorem.

Theorem 6.3. *There exists a meromorphic family of* (μ,μ)*-currents* G_s, *with* $s \in \mathbb{C} - L_1$, *on* $\Gamma\backslash G/K$ *with the following properties.*

(1) *For* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > n$, *the family is given by*

$$\langle G_s, *\bar{\alpha}\rangle = \frac{1}{(r-1)\pi^r}\int_0^\infty \varrho(t)\,(\phi_s(a_t)|\mathcal{J}_H(\tilde{\alpha}; a_t))\,dt, \quad \alpha \in A_c(\Gamma\backslash G/K).$$

(2) *A point* $s_0 \in \mathbb{C} - L_1$ *with* $\operatorname{Re} s \geqslant 0$ *is a pole of* G_s *if and only if there exists an* L^2*-form* $\alpha \in A^{r-1,r-1}_{(2)}(\Gamma\backslash G/K; (n-2r+2)^2 - s_0^2)$ *such that*

$$\int_D j^* * (\boldsymbol{\omega} \wedge \bar{\alpha}) \neq 0.$$

In this case s_0 *is a simple pole with residue*

$$\operatorname{Res}_{s=s_0} G_s = \frac{2}{s_0}\sum_m \left(\int_D j^* * (\boldsymbol{\omega} \wedge \bar{\alpha}_m)\right)\cdot \alpha_m.$$

where $\{\alpha_m\}$ *is an orthonormal basis of* $A^{r-1,r-1}_{(2)}(\Gamma\backslash G/K; (n-2r+2)^2 - s_0^2)$.

(3) *The following functional equation holds*:

$$G_{-s} - G_s = \sum_{p=0}^{r-1} \frac{\mathcal{E}^{(r-1)}_p(\nu^{(p+1)}_s)}{2\nu^{(p+1)}_s}, \quad s \in \mathbb{C} - L_1.$$

Theorem 6.4. *There exists a meromorphic family of* (r,r)*-currents* Ψ_s $s \in \mathbb{C} - L_1$, *on* $\Gamma\backslash G/K$ *such that for* $s \in \mathbb{C}$ *with* $\operatorname{Re} s > n$, *the current* Ψ_s*is given by*

$$\langle \Psi_s, *\bar{\alpha}\rangle = \frac{1}{(r-1)\pi^r}\int_0^\infty \varrho(t)\,(\psi_s(a_t)|\mathcal{J}_H(\tilde{\alpha}; a_t))\,dt, \quad \alpha \in A_c(\Gamma\backslash G/K).$$

Moreover, Ψ_s *is holomorphic at* $s = n - 2r + 2$.

Proof. The meromorphic continuation of Ψ_s follows from the differential equation (71) and the meromorphicity of G_s. Let β be the residue of G_s at $s = \lambda$. Then by Theorem 6.3(2), β is a harmonic L^2-form. Moreover $\mathrm{d}\beta$ and $\mathrm{d}^*\beta$ are also L^2-forms due to the fact that for any $D \in \mathfrak{g}_{\mathbb{C}}$, the derivative $R_D\tilde{\beta}$ is square-integrable on $\Gamma\backslash G$. Then we can conclude that $\mathrm{d}\beta = 0$ and $\mathrm{d}^*\beta = 0$ using the identity

$$0 = \| \triangle \beta\|^2 = \|\mathrm{d}\beta\|^2 + \|\mathrm{d}^*\beta\|^2.$$

Since the residue of the function $s \mapsto \partial\bar{\partial}G_s$ at its possible simple pole $s = \lambda$ is $\partial\bar{\partial}\beta = 2^{-1}\sqrt{-1}\mathrm{d_c}\mathrm{d}\beta = 0$, the function $\partial\bar{\partial}G_s$ is regular at $s = \lambda$. By (71), Ψ_s is also regular at $s = \lambda$. □

Definition. We define the $(r-1, r-1)$-current $\mathcal{G}$ on $\Gamma\backslash G/K$ to be the quarter of the constant term of the Laurent expansion of G_s at $s = \lambda$, Namely, if $\{\alpha_m\}$ is any orthonormal basis of $\mathcal{H}^{r-1,r-1}_{(2)}(\Gamma\backslash G/K)$, we put

$$\mathcal{G}(x) = \frac{1}{4}\lim_{s\to\lambda}\left(G_s(x) - \frac{2}{n-2r+2}\sum_m \int_D j^* * (\omega \wedge \bar{\alpha}_m)\,\frac{\alpha_m(x)}{s-(n-2r+2)}\right).$$

Theorem 6.5. *We have the equation*

$$\mathrm{dd_c}\,\mathcal{G} = \tfrac{\sqrt{-1}}{2}\Psi_{n-2r+2} + \delta_D, \quad \triangle\Psi_{n-2r+2} = 0.$$

The current Ψ_{n-2r+2} is represented by an element of $A^{r,r}(\Gamma\backslash G/K)$.

Proof. Since Ψ_s is regular at $s = \lambda$, the differential equation (70) gives us $\triangle\Psi_\lambda = 0$. The current Ψ_λ, which is annihilated by the elliptic differential operator $\triangle$ on $\Gamma\backslash G/K$, is then a C^∞-form by the elliptic regularity theorem. By comparing the constant terms of the Laurent expansion at $s = \lambda$ of both sides of the identity (71), we obtain the first equation in the theorem. □

7. Square-integrability of Ψ_λ

In this section we prove the square-integrability of Ψ_λ. To establish it we need the spectral expansion of the functions

$$\boldsymbol{\delta}_{j,s}((s^2-\lambda^2)^{-1}\widetilde{\Psi}_s) := (c_r\,\mu\,\mu!)^{-1}\tilde{P}(\boldsymbol{\delta}_{j,s}((s^2-\lambda^2)^{-1}\psi_s)).$$

Lemma 7.1. *Let $\mathsf{c}_\alpha(s)$ $(0 \leqslant \alpha \leqslant \mu-1)$ be the functions appearing as coefficients in the asymptotic formula* (44). *For each α there exists an even polynomial function $\tilde{\mathsf{c}}_\alpha(s)$ with degree no more than 2α such that $\mathsf{c}_\alpha(s) = (s^2-\lambda^2)\tilde{\mathsf{c}}_\alpha(s)$. We have $\tilde{\mathsf{c}}_0(s) = -(\sqrt{-1}/2)\,\mu! * \mathrm{vol}_H$, independent of s. For each $0 \leqslant j \leqslant \mu$, we have*

$$\boldsymbol{\delta}_{j,s}\left(\frac{\psi_s(a_t)}{s^2-\lambda^2}\right) = \tfrac{\tilde{\mathsf{w}}_{\mu-j}}{z^{\mu-j}}(1 + O(s\,;z, z\log z)), \quad z = \tanh^2 t, \tag{81}$$

with $\tilde{\mathsf{w}}_\alpha := (\boldsymbol{\delta}_{\alpha,s}\tilde{\mathsf{c}}_\alpha)(0)$.

Proof. For $f \in \mathcal{K}_\Gamma(\tau)$, consider the integral

$$I(f) = \int_{\Gamma\backslash G} ((s^2-\lambda^2)^{-1}\widetilde{\Psi}_s(g)|R_{\Omega+\lambda^2-\bar{s}^2} f(g))\, d\dot{g}.$$

By (70) and (67), $I(f) = -2\sqrt{-1}(*\mathrm{vol}_H|\mathcal{J}_H(f;e))$ on the one hand. On the other hand, we can compute $I(f)$ in a way similar to that in the proof of Theorem 4.5 to obtain

$$I(f) = \frac{4}{\mu!}(s^2-\lambda^2)^{-1}(\mathsf{c}_0(s)|\mathcal{J}_H(f;e)).$$

Comparing these two expressions for $I(f)$, we get

$$-2\sqrt{-1}(*\mathrm{vol}_H|\mathcal{J}_H(f;e)) = \frac{4}{\mu!}(s^2-\lambda^2)^{-1}(\mathsf{c}_0(s)|\mathcal{J}_H(f;e)). \tag{82}$$

By choosing f suitably, we can arrange for the value of $\mathcal{J}_H(f;e)$ to be any element of $(\bigwedge^{r,r}\mathfrak{p}^*_{\mathbb{C}})^M$. Hence the relation (82) implies

$$\mathsf{c}_0(s) = -\tfrac{\sqrt{-1}}{2}\mu!\, *\mathrm{vol}_H\, (s^2-\lambda^2). \tag{83}$$

Since the Schmid operators $\nabla_\pm$ commute with the Casimir operator Ω, (26) implies that ψ_s is also a C^∞-solution of (22). Hence the same argument as in the proof of Theorem 3.14 yields that the coefficients $\mathsf{c}_\alpha(s)$ must obey the same recurrence relation (43) as $\mathsf{a}_\alpha(s)$. Since the operator $\mathcal{S}' - \alpha(\mu-\alpha)$ is invertible on $\left(\bigwedge^{r,r}\mathfrak{p}^*_{\mathbb{C}}\right)^M$, the recurrence relation (43) with the initial condition (83) implies the first and the second assertions in the lemma. The last assertion follows from the expression $\mathsf{c}_\alpha(s) = (s^2-\lambda^2)\tilde{\mathsf{c}}_\alpha(s)$ just obtained. □

Theorem 7.2. *Let* $0 \leqslant j \leqslant \mu$ *be an integer. Suppose* $\mathrm{Re}\, s > 3n-2r$. *Then for any* $f \in \mathfrak{M}_\delta$ *with* $\delta \in (2rn^{-1}, 1)$, *we have*

$$\int_{\Gamma\backslash G} (\boldsymbol{\delta}_{j,s}((s^2-\lambda^2)^{-1}\widetilde{\Psi}_s)(g)|R_{(\Omega+\lambda^2-\bar{s}^2)^{j+1}} f(g))\, d\dot{g} = -2\sqrt{-1}\,(*\mathrm{vol}_H|\mathcal{J}_H(f;e)).$$

Proof. The proof is analogous to that of Theorem 4.5. We use (81). □

Theorem 7.3. *Let* $\mathrm{Re}\, s > 3n-2r$. *There exists* $\varepsilon > 0$ *such that the function* $\boldsymbol{\delta}_{\mu,s}((s^2-\lambda^2)^{-1}\widetilde{\Psi}_s)$ *belongs to the space* $\mathcal{L}^{2+\varepsilon}_\Gamma(\tau)^{(r)}$. *The spectral expansion of* $\boldsymbol{\delta}_{\mu,s}((s^2-\lambda^2)^{-1}\widetilde{\Psi}_s)$ *is*

$$\boldsymbol{\delta}_{\mu,s}\left(\frac{\widetilde{\Psi}_s}{s^2-\lambda^2}\right) = \mathcal{F}_{\mathrm{dis}}(s) + \sum_{p=0}^{r} \mathcal{F}^{(p)}_{\mathrm{c}}(s),$$

with

$$\mathcal{F}_{\mathrm{dis}}(s) = \sum_{m=0}^{\infty} \frac{-2\sqrt{-1}\,(*\mathrm{vol}_H | \mathcal{J}_H(\tilde{\alpha}_m^{(r)}; e))}{(\lambda^2 - \lambda_m^{(r)} - s^2)^r}\, \tilde{\alpha}_m^{(r)},$$

$$\mathcal{F}_{\mathrm{c}}^{(p)}(s) = \frac{1}{4\pi\sqrt{-1}} \int_{\sqrt{-1}\,\mathbb{R}} \sum_{i=1}^{h} \sum_{\mathsf{u}} \frac{-2\sqrt{-1}(*\mathrm{vol}_H | \mathcal{J}_H(\mathsf{E}^i(\zeta; \mathsf{u}); e))}{(\zeta^2 - (\nu_s^{(p+1)})^2)^r} \mathsf{E}^i(\zeta; \mathsf{u})\, d\zeta,$$

where the inner summation is over $\mathsf{u} \in \mathcal{B}_i^{(r)}(p; 0)$ *and both summations are convergent in* $\mathcal{L}_\Gamma^2(\tau)^{(r)}$.

Proof. Like that of Theorem 5.3. We use Theorem 7.2. □

Theorem 7.4. *Let* L_1 *be the interval on the imaginary axis defined by* (25). *Let* $0 \leqslant j \leqslant \mu$. *Then for each* $\tilde{\beta} \in \mathcal{K}_\Gamma(\tau)$ *the holomorphic function*

$$s \mapsto \mathcal{F}_j(s, \tilde{\beta}) := \langle \boldsymbol{\delta}_{j,s}(s^2 - \lambda^2)^{-1} \widetilde{\Psi}_s | \tilde{\beta} \rangle$$

on $\operatorname{Re} s > n$ *has a meromorphic continuation to the domain* $\mathbb{C} - L_1$. *A point* $s_0 \in \mathbb{C} - L_1$ *with* $\operatorname{Re} s_0 \geqslant 0$ *is a pole of the meromorphic function* $\mathcal{F}_j(s, \beta)$ *if and only if there exists an* $m \in \mathbb{N}$ *such that*

$$(*\mathrm{vol}_H | \mathcal{J}_H(\tilde{\alpha}_m^{(r)}; e)) \neq 0, \quad \langle \tilde{\alpha}_m^{(r)} | \tilde{\beta} \rangle \neq 0,$$

and $s_0^2 - \lambda^2 = -\lambda_m^{(r)}$. *In this case, the function*

$$\mathcal{F}_j(s, \beta) - \sum_{\substack{m \in \mathbb{N} \\ \lambda_m^{(r)} = \lambda^2 - s_0^2}} \frac{2\sqrt{-1}(*\mathrm{vol}_H | \mathcal{J}_H(\tilde{\alpha}_m^{(r)}; e)) \langle \tilde{\alpha}_m^{(r)} | \tilde{\beta} \rangle}{(s_0^2 - s^2)^{j+1}}$$

is holomorphic at $s = s_0$. *We have the functional equation*

$$\mathcal{F}_j(-s, \tilde{\beta}) - \mathcal{F}_j(s, \tilde{\beta}) = \boldsymbol{\delta}_{j,s} \left(\sum_{p=0}^{r} \frac{\langle \tilde{\mathcal{E}}_p^{(r)}(\nu_s^{(p+1)}) | \tilde{\beta} \rangle}{2\,\nu_s^{(p+1)}} \right), \tag{84}$$

where, for $g \in G$,

$$\tilde{\mathcal{E}}_p^{(r)}(\nu; g) := -2\sqrt{-1} \sum_{i=1}^{h} \sum_{\mathsf{u} \in \mathcal{B}_i^{(r)}(p;0)} (*\mathrm{vol}_H | \mathcal{J}_H(\mathsf{E}^i(-\bar{\nu}; \mathsf{u}); e))\, \mathsf{E}^i(\nu; \mathsf{u}; g).$$

Proof. The proof is the same as that of Theorem 5.7. The statement for $j = \mu$ follows from Theorem 7.3. Then we use induction to show the statement for j smaller than μ. □

Some properties of the current Ψ_s.

Theorem 7.5. (1) *A point $s_0 \in \mathbb{C} - L_1$ with* $\operatorname{Re} s_0 \geqslant 0$ *and $s_0 \neq n-2r+2$ is a pole of the current Ψ_s if and only if there exists an L^2-form*

$$\alpha \in A^{r,r}_{(2)}\big(\Gamma\backslash G/K\,;\,(n-2r+2)^2 - s_0^2\big)$$

such that

$$\int_D j^* {*}\alpha \neq 0.$$

In this case s_0 is a simple pole with residue

$$\operatorname{Res}_{s=s_0}\Psi_s = \frac{\sqrt{-1}(s_0^2-(n-2r+2)^2)}{s_0}\sum_m \left(\int_D j^* * \bar{\alpha}_m\right)\cdot \alpha_m,$$

where $\{\alpha_m\}$ is an orthonormal basis of $A^{r,r}_{(2)}(\Gamma\backslash G/K\,;\,(n-2r+2)^2 - s_0^2)$.

(2) *We have*

$$\Psi_{n-2r+2} = 2\sqrt{-1}\sum_m \left(\int_D j^* * \bar{\beta}_m\right)\cdot \beta_m,$$

with $\{\beta_m\}$ an orthonormal basis of $\mathcal{H}^{r,r}_{(2)}(\Gamma\backslash G/K)$. In particular, the current Ψ_{n-2r+2} is in $\mathcal{H}^{r,r}_{(2)}(\Gamma\backslash G/K)$.

Proof. This is a corollary of Theorem 7.4. □

By Theorem 6.5, the fundamental class $[\delta_D] \in \mathrm{H}^{r,r}(\Gamma\backslash G/K\,;\,\mathbb{C})$ of D has the harmonic L^2-representative Ψ_{n-2r+2}.

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