Pacific Journal of Mathematics

Volume 228 No. 1 November 2006

PACIFIC JOURNAL OF MATHEMATICS

http://www.pjmath.org

Founded in 1951 by

E. F. Beckenbach (1906–1982)   F. Wolf (1904–1989)

See inside back cover or www.pjmath.org for submission instructions.

Regular subscription rate for 2006: $425.00 a year (10 issues). Special rate: $212.50 a year to individual members of supporting institutions. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840 is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

A GENERALIZATION OF RANDOM MATRIX ENSEMBLE I. GENERAL THEORY

JINPENG AN, ZHENGDONG WANG AND KUIHUA YAN

We give a generalization of random matrix ensembles, which includes all classical ensembles. We derive the joint-density function of the generalized ensemble by one simple formula, giving a direct and unified way to compute the density functions for all classical ensembles and various kinds of new ensembles. An integration formula associated with the generalized ensembles is given. We propose a taxonomy of generalized ensembles encompassing all classical ensembles and some new ones not considered before.

1. Introduction

One of the most fundamental problems in the theory of random matrices is to derive the joint-density functions for the eigenvalues (or, equivalently, the measures associated with the eigenvalue distributions) of various types of matrix ensembles. Mehta [1991] summarized the classical analysis methods by which the density functions for various types of ensembles were derived case by case; but a systematic method to compute the density functions was desired.

The first achievement in this direction was made by Dyson [1970], who introduced an idea of expressing various kinds of circular ensembles in terms of symmetric spaces with invariant probability measures. From then on, guided by Dyson's idea, many authors observed new random matrix ensembles in terms of Cartan's classification of Riemannian symmetric spaces, and obtained the joint-density functions for such ensembles by using the integration formula on symmetric space (see, for example, [Altland and Zirnbauer 1997; Caselle 1994; Caselle 1996; Dueñez 2004; Ivanov 2001; Titov et al. 2001; Zirnbauer 1996]).

We briefly mention the recent work of Dueñez [2004]. He explored the random matrix ensembles that correspond to infinite families of compact irreducible Riemannian symmetric spaces of type I, including circular orthogonal and symplectic ensembles, and various kinds of Jacobi ensembles. Using an integration formula

MSC2000: primary 15A52; secondary 58C35, 57S25.
Keywords: random matrix ensemble, Lie group, integration formula.
This work is supported by the 973 Project Foundation of China (#TG1999075102).

associated with the *KAK* decomposition of compact groups, he obtained the induced measure on the space of eigenvalues associated to the underlying symmetric space, and then derived the eigenvalue distribution of the corresponding random matrix ensemble. These methods of deriving the eigenvalue distributions of random matrix ensembles by means of Riemannian symmetric spaces were summarized in the excellent article [Caselle and Magnea 2004].

In this paper we provide a generalization of random matrix ensembles, which includes all classical ensembles, and then give a unified way to derive — with one simple formula — the joint-density function for the eigenvalue distribution. The proof of this formula makes no use of an integration formula. In fact, the corresponding integration formula can be derived from this formula as a corollary. We also show how these generalized ensembles encompass all classical ensembles and some new ensembles that were not considered before.

Let $\sigma : G \times X \to X$ be a smooth action of a Lie group G on a Riemannian manifold X that preserves the induced Riemannian measure dx. Let $p(x)$ be a G-invariant smooth function on X, and consider the measure $p(x)\,dx$ on X, which is not necessarily a finite measure. We choose a closed submanifold Y of X consisting of representation points for almost all G-orbits in X. The Riemannian structure on X induces a Riemannian measure dy on Y. If K is the closed subgroup of G that fixes all points in Y, then σ reduces to a map $\varphi : G/K \times Y \to X$. Suppose there is a G-invariant measure $d\mu$ on G/K and that $\dim(G/K \times Y) = \dim X$; it can then be proved that the pull-back measure $\varphi^*(p(x)\,dx)$ of $p(x)\,dx$ is of the form $\varphi^*(p(x)\,dx) = d\mu\,d\nu$ for some measure $d\nu$ on Y, the latter being the measure associated with the eigenvalue distribution. The measure $d\nu$ can be expressed as $d\nu(y) = \mathcal{P}(y)\,dy$ for some function $\mathcal{P}(y)$ on Y, this being the joint-density function. If we write $\mathcal{P}(y)$ as $\mathcal{P}(y) = p(y)\,J(y)$, then, under some orthogonality condition (that is, $T_yY \perp T_yO_y$ for almost all $y \in Y$), we can compute the factor $J(y)$ by

$$J(y) = C\,\left|\det \Psi_y\right|, \tag{1–1}$$

where C is a constant that can be computed explicitly. This formula is the main result of this paper, and the density function $\mathcal{P}(y)$ and the eigenvalue distribution $d\nu$ are determined by it. Here, the map $\Psi_y : \mathfrak{l} \to T_yO_y$ is defined as

$$\Psi_y(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_{\exp t\xi}(y),$$

where $\mathfrak{l}$ is a linear subspace of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, with $\mathfrak{k}$ being the Lie algebra of K. We call the system $(G, \sigma, X, p(x)\,dx, Y, dy)$ a *generalized random matrix ensemble*. The measure $d\nu$ on Y is called a *generalized eigenvalue distribution* and the function $\mathcal{P}(y)$ is a *generalized joint density function*.

Using (1–1), one can derive the joint-density function for the Gaussian ensemble, chiral ensemble, new transfer matrix ensemble, circular ensemble, Jacobi ensemble, as well as some other new generalized ensembles. The precise deriving process will be the content of a sequel paper [An et al. 2005]. We should point out that the proof of formula (1–1) is not difficult, but this formula is very effective and useful; the derivation of all concrete examples in [An et al. 2005], including all classical random matrix ensembles, will be based on it.

Once the eigenvalue distribution $d\nu$ is derived from formula (1–1), under a covering condition we can get the associated integration formula. The Weyl integration formula for compact Lie groups, the Harish-Chandra integration formula for complex semisimple Lie groups and real reductive groups, the integration formulae on Riemannian symmetric spaces of noncompact and compact types that appeared in [Helgason 2000], as well as their Lie algebra versions, are all particular cases of it (see [An et al. 2005]).

We sketch the contents of this paper. In Section 2 we develop some geometrical preliminaries on the geometry of G-space, which will be required to establish the generalized ensembles. After presenting four conditions — that is, the invariance condition, the transversality condition, the dimension condition, and the orthogonality condition — on which the definition of generalized ensembles will be based, we prove in Theorem 2.5 a primary form of formula (1–1).

Section 3 is devoted to integration over G-spaces, which will be needed when we derive the integration formula associated with a generalized random matrix ensemble. Based on the four conditions from Section 2 and a covering condition, we prove an integration formula in Theorem 3.3, converting the integration over a G-space to, first, integrating over each G-orbit, and then integrating over the orbit space. Two criteria on when the covering condition holds are also given.

Prepared by Section 2 and Section 3, in Section 4 we give the precise definition of a generalized random matrix ensemble, as well as of the associated generalized eigenvalue distribution and generalized joint-density function. In Theorem 4.1 is presented formula (1–1), from which the associated eigenvalue distribution measure and density function will be derived in [An et al. 2005] for various concrete examples of the generalized ensemble, in a unified way.

In Section 5 we discuss a number of classes of generalized ensembles: the linear ensemble, the nonlinear noncompact ensemble, the compact ensemble, the group and algebra ensembles, as well as the pseudogroup and pseudoalgebra ensembles. Gaussian and chiral ensembles are included under linear ensembles; new transfer matrix ensembles are included under nonlinear noncompact ensembles; circular and Jacobi ensembles are included under compact ensembles. Some new ensembles not considered before are also mentioned.

2. Geometry of G-spaces

We develop some geometrical preliminaries needed for our theory of generalized random matrix ensembles.

We start with measures on manifolds. Let M be an n-dimensional smooth manifold. A measure dx on M is called *smooth* (or *quasi-smooth*) if on any local coordinate chart $(U; x_1, \dots, x_n)$ of M, dx has the form $dx = f(x)\, dx_1 \dots dx_n$, where f is a smooth function on U with $f > 0$ (or $f \geq 0$), and $dx_1 \dots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$. Note that the smooth measures on M are unique up to multiplication with a positive smooth function on M, so the concept of a *set of measure zero* makes sense independently of the choice of smooth measure.

Let M, N be two n-dimensional smooth manifolds, and let $\varphi : M \to N$ be a smooth map. If dy is a smooth (or quasi-smooth) measure on N, expressed locally as $dy = f(y)\, dy_1 \dots dy_n$, we can define its *pull-back* $\varphi^*(dy)$ locally as

$$\varphi^*(dy) = f(\varphi(x)) \left|\det\left(\frac{\partial y}{\partial x}\right)\right| dx_1 \dots dx_n. \tag{2–1}$$

It is easy to check that the local definitions are compatible when different coordinate charts are chosen, and that $\varphi^*(dy)$ is a quasi-smooth measure on M. Even if dy is smooth, we cannot expect $\varphi^*(dy)$ to be smooth in general, since φ may have critical points; but if φ is a local diffeomorphism and dy is smooth, then $\varphi^*(dy)$ is smooth.

If M, N are Riemannian manifolds and dx, dy are the associated Riemannian measures, then we can express the pull-back measure $\varphi^*(dy)$ globally. To do this, first we need some comments on the "determinant" of a linear map between two different inner-product vector spaces of the same dimension. Suppose V, W are two n-dimensional vector spaces with inner products. For n vectors $v_1, \dots, v_n \in V$, set $a_{ij} = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq n$, and define

$$\mathrm{Vol}(v_1, \dots, v_n) = \sqrt{\det(a_{ij})}.$$

Note that if $v_1, \dots, v_n$ is an orthogonal basis, then $\mathrm{Vol}(v_1, \dots, v_n) = |v_1| \dots |v_n|$. For vectors in W, define the same things. Supposing $A : V \to W$ is a linear map, define

$$|\det A| = \frac{\mathrm{Vol}(Av_1, \dots, Av_n)}{\mathrm{Vol}(v_1, \dots, v_n)}, \tag{2–2}$$

where $v_1, \dots, v_n$ is a basis of V. It is easy to check that the definition is independent of the choice of the basis $v_1, \dots, v_n$. In the special case when $v_1, \dots, v_n$ is an orthogonal basis of V and $Av_1, \dots, Av_n$ are mutually orthogonal, we have

$$|\det A| = \frac{|Av_1| \dots |Av_n|}{|v_1| \dots |v_n|}. \tag{2–3}$$

We can expect only the norm $|\det A|$ of the determinant to be well defined, since the sign $\pm$ depends on a choice of orientations for V and W.

Proposition 2.1. *Suppose M, N are two n-dimensional Riemannian manifolds with associated Riemannian measures dx, dy, respectively. If $\varphi : M \to N$ is a smooth map, then*

$$\varphi^*(dy) = \left|\det(d\varphi)_x\right| dx. \tag{2–4}$$

Proof. Suppose that in local-coordinate charts the Riemannian metrics on M and N are $ds^2 = \sum_{ij} g_{ij}(x)\, dx_i\, dx_j$ and $d\tilde{s}^2 = \sum_{ij} \tilde{g}_{ij}(y)\, dy_i\, dy_j$, respectively, with $g_{ij}(x) = \langle \partial/\partial x_i,\ \partial/\partial x_j\rangle$ and $\tilde{g}_{ij}(y) = \langle \partial/\partial y_i,\ \partial/\partial y_j\rangle$. By definition, the Riemannian measures dx, dy are

$$dx = \sqrt{\det(g_{ij}(x))}\, dx_1 \ldots dx_n \qquad \text{and} \qquad dy = \sqrt{\det(\tilde{g}_{ij}(y))}\, dy_1 \ldots dy_n.$$

We have:

$$\begin{aligned}
|\det(d\varphi)_x|^2 &= \frac{\mathrm{Vol}\Big((d\varphi)_x\Big(\dfrac{\partial}{\partial x_1}\Big), \ldots, (d\varphi)_x\Big(\dfrac{\partial}{\partial x_n}\Big)\Big)^2}{\mathrm{Vol}\Big(\dfrac{\partial}{\partial x_1}, \ldots, \dfrac{\partial}{\partial x_n}\Big)^2} \\
&= \frac{\det\Big\langle \sum_k \dfrac{\partial y_k}{\partial x_i}\Big(\dfrac{\partial}{\partial y_k}\Big)_{\varphi(x)},\ \sum_l \dfrac{\partial y_l}{\partial x_j}\Big(\dfrac{\partial}{\partial y_l}\Big)_{\varphi(x)}\Big\rangle}{\det\Big\langle \dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j}\Big\rangle} \\
&= \frac{\det\Big(\sum_{kl} \dfrac{\partial y_k}{\partial x_i}\dfrac{\partial y_l}{\partial x_j}\tilde{g}_{kl}\big(\varphi(x)\big)\Big)}{\det(g_{ij}(x))} = \frac{\det\Big(\Big(\dfrac{\partial y_k}{\partial x_i}\Big)^t \big(\tilde{g}_{kl}(\varphi(x))\big)\Big(\dfrac{\partial y_l}{\partial x_j}\Big)\Big)}{\det(g_{ij}(x))} \\
&= \frac{\det^2\Big(\dfrac{\partial y}{\partial x}\Big)\det\big(\tilde{g}_{ij}(\varphi(x))\big)}{\det(g_{ij}(x))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\varphi^*(dy) &= \sqrt{\det\big(\tilde{g}_{ij}(\varphi(x))\big)}\,\Big|\det\Big(\frac{\partial y}{\partial x}\Big)\Big|\, dx_1 \ldots dx_n \\
&= \left|\det(d\varphi)_x\right| \sqrt{\det(g_{ij}(x))}\, dx_1 \ldots dx_n \;=\; \left|\det(d\varphi)_x\right| dx. \qquad \square
\end{aligned}$$

We now come to the main geometric problems that will concern us in the following sections. Let G be a Lie group that acts on an n-dimensional smooth manifold X. The action is denoted by $\sigma : G \times X \to X$ and we write $\sigma_g(x) = \sigma(g, x)$. Our first goal is, roughly speaking, to choose a representation point in each G-orbit $O_x = \{\sigma_g(x) \mid g \in G\}$, depending smoothly on the orbit. In general, this aim can be only partially achieved. Hence, suppose we have a closed submanifold Y of X, consisting of chosen representation points of the orbits, such that Y intersects "almost all" orbits transversally. More precisely, supposing there are closed zero-measure subsets $X_{\mathrm{z}} \subset X$ and $Y_{\mathrm{z}} \subset Y$, set $X' = X \setminus X_{\mathrm{z}}$ and $Y' = Y \setminus Y_{\mathrm{z}}$, and assume that

(a) $X' = \bigcup_{y \in Y'} O_y$ (*invariance condition*).

(b) $T_y X = T_y O_y \oplus T_y Y$ for all $y \in Y'$ (*transversality condition*).

It is clear that (a) implies that $Y' = Y \cap X'$ and $Y_{\mathrm{z}} = Y \cap X_{\mathrm{z}}$. Notice that X' and Y' are open and dense submanifolds of X and Y, respectively. So, for all $y \in Y'$, we have $T_y X' = T_y X$ and $T_y Y' = T_y Y$.

Set $K = \{g \in G \mid \sigma_g(y) = y, \text{ for all } y \in Y\}$; it is a closed subgroup of G. For $g \in G$, write $[g] = gK$ in G/K. The map $\sigma : G \times X \to X$ reduces to a map

$$\varphi : G/K \times Y \to X \qquad \text{with} \qquad \varphi\big([g], y\big) = \sigma_g(y).$$

By restriction, the latter induces a map $G/K \times Y' \to X'$, also denoted by φ. From assumption (a), $\varphi : G/K \times Y' \to X'$ is surjective. For $x \in X$, let $G_x = \{g \in G \mid \sigma_g(x) = x\}$ be the isotropy subgroup of x. Then $K \subset G_y$ for all $y \in Y$. Let dx, dy be smooth measures on X and Y, respectively. Suppose that dx is G-invariant. In what follows we also assume that

(c) $\dim G_y = \dim K$ for all $y \in Y'$ (*dimension condition*).

This means that G_y and K have the same Lie algebra for all $y \in Y'$, and that the only difference between G_y and K is that G_y may have more components than K. Then, for some $y \in Y'$, we have

$$\begin{aligned}\dim X = \dim T_y X &= \dim T_y Y + \dim T_y O_y \\ = \dim Y + \dim G - \dim G_y &= \dim Y + \dim G - \dim K.\end{aligned}$$

So φ is a map between manifolds of the same dimension, and thus the pull-back $\varphi^*(dx)$ of dx makes sense. If there is a G-invariant smooth measure $d\mu$ on G/K, then the product measure $d\mu\, dy$ on $G/K \times Y$ is smooth, and so

$$\varphi^*(dx) = J\big([g], y\big)\, d\mu\, dy \tag{2–5}$$

for some $J \in C^\infty(G/K \times Y)$ with $J \geq 0$.

Remark. The G-invariant smooth measure $d\mu$ on G/K exists if and only if $\Delta_G|_K = \Delta_K$, where Δ_G and Δ_K are the modular functions on G and K, respectively; see, for example, [Knapp 2002, Section 8.3]. For the concrete examples in the following sections, this condition always holds.

Proposition 2.2. *The smooth function $J \in C^\infty(G/K \times Y)$ is independent of the first variable $[g] \in G/K$. So we can rewrite* (2–5) *as*

$$\varphi^*(dx) = J(y)\, d\mu\, dy, \tag{2–6}$$

where $J \in C^\infty(Y)$ with $J \geq 0$.

Proof. If we denote by l_h the natural action of $h \in G$ on G/K, then one can easily verify that $\sigma_h \circ \varphi = \varphi \circ (l_h \times \mathrm{id})$. By the G-invariance of dx and $d\mu$, we have

$$\begin{aligned} J\big([g], y\big)\, d\mu\, dy &= \varphi^*(dx) \;=\; \varphi^* \circ \sigma_h^*(dx) \;=\; (l_h \times \mathrm{id})^* \circ \varphi^*(dx) \\ &= (l_h \times \mathrm{id})^* \big(J\big([g], y\big)\, d\mu\, dy\big) \;=\; J\big(h[g], y\big)\big(l_h^*(d\mu) \times \mathrm{id}^*(dy)\big) \\ &= J\big([hg], y\big)\, d\mu\, dy. \end{aligned}$$

So $J\big([g], y\big) = J\big([hg], y\big)$ for all $g, h \in G$, which means that J is independent of the first variable. □

Corollary 2.3. *There exists a quasi-smooth measure $d\nu$ on Y such that*

$$\varphi^*(dx) = d\mu\, d\nu. \tag{2–7}$$

The measure $d\nu$ is given by

$$d\nu(y) = J(y)\, dy. \tag{2–8}$$

The factor $J(y)$ can also be given for more general smooth measures $u(x)\, dx$ and $v(y)\, dy$ on X and Y. A direct calculation yields:

Proposition 2.4. *Suppose conditions (a), (b), and (c) hold. If we replace the measures with*

$$dx' = u(x)\, dx, \qquad dy' = v(y)\, dy, \qquad \textit{and} \qquad d\mu' = \lambda\, d\mu,$$

where u, v are positive smooth functions on X, Y, respectively, and if u is G-invariant and λ is a positive constant, then $J(y)$ changes to

$$J'(y) = \frac{u(y)}{\lambda v(y)} J(y).$$

Now we suppose that there is a Riemannian structure on X such that dx and dy are the induced Riemannian measures on X and Y, respectively. We also assume that the following condition holds:

(d) $T_y Y \perp T_y O_y$ for all $y \in Y'$ (*orthogonality condition*).

In this case, the next theorem computes the factor $J(y)$ in a simple way.

Let $\mathfrak{l}$ be a linear subspace of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, where $\mathfrak{k}$ is the Lie algebra of K. If $\pi : G \to G/K$ is the natural projection, then

$$(d\pi)_e|_{\mathfrak{l}} : \mathfrak{l} \to T_{[e]}(G/K)$$

is an isomorphism. If we endow G/K with a Riemannian structure such that its associated Riemannian measure is $d\mu$, then it also induces an inner product on $T_{[e]}(G/K)$. For $y \in Y$, we define a linear map $\Psi_y : \mathfrak{l} \to T_y O_y$ by

$$\Psi_y(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_{\exp t\xi}(y) \quad \text{for all } \xi \in \mathfrak{l}. \tag{2–9}$$

If $y \in Y'$, then $\dim \mathfrak{l} = \dim T_y O_y$. We choose an inner product on $\mathfrak{l}$ and endow $T_y O_y$ with the inner product induced from the Riemannian structure on X. The "determinants" $|\det \Psi_y|$ and $\left|\det\bigl((d\pi)_e|_{\mathfrak{l}}\bigr)\right|$ now make sense.

Theorem 2.5. *Under these assumptions, we have*

$$J(y) = C\left|\det \Psi_y\right| \qquad \textit{for all } y \in Y', \tag{2–10}$$

where C is the constant $C = \left|\det\bigl((d\pi)_e|_{\mathfrak{l}}\bigr)\right|^{-1}$.

Proof. By the transversality condition (b), the tangent map of φ at the point $([e], y)$,

$$(d\varphi)_{([e],y)} : T_{([e],y)}(G/K \times Y) \to T_y X,$$

can be regarded as

$$(d\varphi)_{([e],y)} : T_{[e]}(G/K) \oplus T_y Y \to T_y O_y \oplus T_y Y.$$

Denote $(d\varphi)_{([e],y)}|_{T_{[e]}(G/K)} : T_{[e]}(G/K) \to T_y O_y$ by $\tilde{\Psi}_y$. It is then obvious that $\Psi_y = \tilde{\Psi}_y \circ (d\pi)_e|_{\mathfrak{l}}$, as one can easily prove it in its matrix form

$$(d\varphi)_{([e],y)} = \begin{pmatrix} \tilde{\Psi}_y & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Since $d\mu$ is the associated Riemannian measure on G/K, the product measure $d\mu\, dy$ is the associated Riemannian measure on the product Riemannian manifold $G/K \times Y'$. By Proposition 2.1 and the orthogonality condition (d),

$$\begin{aligned} J(y) &= \left|\det(d\varphi)_{([e],y)}\right| = \left|\begin{pmatrix} \tilde{\Psi}_y & 0 \\ 0 & \mathrm{id} \end{pmatrix}\right| \\ &= \left|\det \tilde{\Psi}_y\right| = \left|\det\bigl(\Psi_y \circ ((d\pi)_e|_{\mathfrak{l}})^{-1}\bigr)\right| = C\left|\det \Psi_y\right|, \end{aligned}$$

where $C = \left|\det\bigl((d\pi)_e|_{\mathfrak{l}}\bigr)\right|^{-1}$. □

Remark. Although formula (2–10) only holds on Y', since Y' is dense in Y and $J \in C^\infty(Y)$, we can get $J(y)$ for all $y \in Y$ by smooth continuation.

3. Integration over G-spaces

Occasionally we will be interested in some kinds of integration formulae. In this section we give some preliminaries on integration. The reader who is more interested in the eigenvalue distributions of the generalized random matrix ensembles may skip to Section 4 directly.

The next proposition generalizes the change-of-variables formula for multiple integration.

Proposition 3.1. *Let $\varphi : M \to N$ be a smooth map between two n-dimensional smooth manifolds M and N, and dy a smooth measure on N. If φ is a local diffeomorphism that is a d-sheeted covering map, then, for any $f \in C^\infty(N)$ with $f \geq 0$ or $f \in L^1(N, dy)$, we have*

$$\int_N f(y)\,dy = \frac{1}{d}\int_M f(\varphi(x))\,\varphi^*(dy). \tag{3–1}$$

Proof. It is a standard argument using a partition of unity. □

Remark. Formula (3–1) resembles a formula relating the degree of a map with the integration of volume forms on manifolds. When M, N are compact and oriented, then, under the conditions of Proposition 3.1 and up to a sign, formula (3–1) says nothing if not this. In general, however, the integration of differential forms is not suitable for us. What we need is a change-of-variables formula that ignores the negative sign.

As in the previous section, take a G-space X, where X is an n-dimensional smooth manifold and G is a Lie group. We then have a reduced map $\varphi : G/K \times Y \to X$. Suppose dx, dy, and $d\mu$ are smooth measures on X, Y, and G/K, respectively, with dx and $d\mu$ being G-invariant. Our goal is to convert the integration over X to integration over Y.

Proposition 3.2. *If conditions (a), (b), and (c) hold, then $\varphi : G/K \times Y' \to X'$ is a local diffeomorphism.*

Proof. Let e be the unit element in G. At each $([e], y) \in G/K \times Y'$ we have $d\varphi_{([e],y)}(0, v) = v$ for all $v \in T_yY'$, and so $T_yY' \subset \operatorname{Im}\big(d\varphi_{([e],y)}\big)$. Furthermore, $\varphi|_{G/K\times\{y\}} : G/K \times \{y\} \to O_y \cong G/G_y$ is a local diffeomorphism, so $T_yO_y \subset \operatorname{Im}\big(d\varphi_{([e],y)}\big)$. Thus, $d\varphi_{([e],y)}$ is surjective. But $\dim(G/K \times Y') = \dim X'$, so $d\varphi_{([e],y)}$ is in fact an isomorphism.

For general $([g], y) \in G/K \times Y'$, notice that $\varphi \circ l_g = \sigma_g \circ \varphi$, where $l_g([h], y) = ([gh], y)$, so $d\varphi_{([g],y)} \circ (dl_g)_{([e],y)} = (d\sigma_g)_{([e],y)} \circ d\varphi_{([e],y)}$, and $d\varphi_{([e],y)}$ being an isomorphism implies that $d\varphi_{([g],y)}$ is one as well. Thus, φ is everywhere regular and hence is a local diffeomorphism. □

To make Proposition 3.1 useful, we also require the following condition:

(e) The map $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map, with $d < +\infty$ (*covering condition*).

Theorem 3.3. *If conditions (a), (b), (c), and (e) hold, then*

$$\int_X f(x)\,dx = \frac{1}{d}\int_Y \left(\int_{G/K} f(\sigma_g(y))\,d\mu([g])\right) J(y)\,dy \tag{3–2}$$

for all $f \in C^\infty(X)$ with $f \geq 0$ or $f \in L^1(X, dx)$, and where $J \in C^\infty(Y)$ is determined by formula (2–6).

Proof. By Proposition 3.2, $\varphi : G/K \times Y' \to X'$ is a local diffeomorphism. By the covering condition (e), φ is a d-sheeted covering map. So, by Proposition 3.1, for $f \in C^\infty(X)$ with $f \geq 0$ or $f \in L^1(X, dx)$, we have

$$\begin{aligned}
\int_X f(x)\,dx = \int_{X'} f(x)\,dx &= \frac{1}{d}\int_{G/K\times Y'} f\big(\varphi([g], y)\big)\,\varphi^*(dx)\\
&= \frac{1}{d}\int_{G/K\times Y'} f\big(\sigma_g(y)\big)\,J(y)\,d\mu([g])\,dy\\
&= \frac{1}{d}\int_{Y'}\left(\int_{G/K} f\big(\sigma_g(y)\big)\,d\mu([g])\right) J(y)\,dy\\
&= \frac{1}{d}\int_{Y}\left(\int_{G/K} f\big(\sigma_g(y)\big)\,d\mu([g])\right) J(y)\,dy \qquad \square
\end{aligned}$$

Corollary 3.4. *Under the same conditions as in the previous theorem, if furthermore $f(\sigma_g(x)) = f(x)$ for all $g \in G$ and $x \in X$, then*

$$\int_X f(x)\,dx = \frac{\mu(G/K)}{d}\int_Y f(y)\,J(y)\,dy. \tag{3–3}$$ $\square$

To make this conclusion more useful, we give some criteria on when the map $\varphi : G/K \times Y' \to X'$ is a covering map.

Proposition 3.5. *Let M, N be smooth n-dimensional manifolds. An everywhere-regular smooth map $\varphi : M \to N$ is a d-sheeted covering map if and only if $\varphi^{-1}(y)$ has d points for each $y \in N$.*

Proof. The "only if" part is obvious; we prove the "if" part. For $y \in N$, let $\varphi^{-1}(y) = \{x_1, \dots, x_d\}$. Since φ is everywhere regular, there exist open neighborhoods U_i of x_i, $i = 1, \dots, d$, such that $U_i \cap U_j = \varnothing$ for $i \neq j$, and each $\varphi_i = \varphi|_{U_i} : U_i \to \varphi(U_i)$ is a diffeomorphism. Let $V = \bigcup_{i=1}^d \varphi(U_i)$ and $V_i = \varphi_i^{-1}(V)$. Then $\varphi|_{V_i}$ is also a diffeomorphism onto V. We conclude that $\varphi^{-1}(V) = \bigcup_{i=1}^d V_i$. In fact, if for all $z \in \varphi^{-1}(V)$ we set $z_i = \varphi_i^{-1}(\varphi(z))$, then $z_i \in \varphi^{-1}(\varphi(z))$ and $z_i \neq z_j$ for $i \neq j$. But, since $z \in \varphi^{-1}(\varphi(z))$ and $\left|\varphi^{-1}(\varphi(z))\right| = d$, this forces $z = z_{i_0}$ for some i_0. Hence $z \in \bigcup_{i=1}^d V_i$. Therefore, $\varphi^{-1}(V) = \bigcup_{i=1}^d V_i$ and the lemma is proved. $\square$

Corollary 3.6. *Suppose conditions (a), (b), and (c) hold. If furthermore there exists* $d \in \mathbb{N}$ *such that, for all* $y \in Y'$,

(1) *the isotropy subgroup* G_y *coincides with* K,

(2) $|O_y \cap Y'| = d$,

then $\varphi : G/K \times Y' \to X'$ *is a d-sheeted covering map.*

Proof. By Proposition 3.2, φ is a local diffeomorphism. So, by Proposition 3.5, we need only show that $\varphi^{-1}(x)$ has d points for each $x \in X'$.

For $x \in Y'$, suppose that $O_x \cap Y' = \{y_1, \dots, y_d\}$. Then there exists $g_i \in G$ such that $\sigma_{g_i}(y_i) = x$ for each $i \in \{1, \dots, d\}$. It follows that $([g_i], y_i) \in \varphi^{-1}(x)$. On the other hand, if $([g], y) \in \varphi^{-1}(x)$, then $y = y_{i_0}$ for some $i_0 \in \{1, \dots, d\}$. We have $\sigma_{gg_{i_0}^{-1}}(x) = \sigma_g(y_{i_0}) = x$, that is, $gg_{i_0}^{-1} \in G_x = K$, and so $[g] = [g_{i_0}]$ and $([g], y) = ([g_{i_0}], y_{i_0})$. Thus, $\varphi^{-1}(x) = \big\{([g_1], y_1), \dots, ([g_d], y_d)\big\}$.

In general, for $x \in X'$, suppose that $\sigma_h(x) \in Y'$ for some $h \in G$. Then the relation $\varphi^{-1}(\sigma_h(x)) = l_h(\varphi^{-1}(x))$ reduces the general case to the previous one. □

Both Proposition 3.5 and Corollary 3.6 will be used in a forthcoming article devoted to concrete examples [An et al. 2005].

Remark. The converse of Corollary 3.6 is not true. That is, the isotropy subgroups G_y associated to $y \in Y'$ may change "suddenly", even if Y' is connected. For example, the group $SO(n)$ acts on $\mathbb{RP}^n$ smoothly if we regard $\mathbb{RP}^n$ as a quotient space obtained by gluing opposite points on the boundary of the closed unit ball B^n. If X_z is the image of $\{0\}$ and Y is the image of the segment $\{(x, 0, \dots, 0) \mid |x| \le 1\}$, then the conditions (a), (b), (c), and (e) hold. The isotropy subgroup associated with the image of a point in Y' that is an interior point of B^n is $\mathrm{diag}(1,\ SO(n-1))$, but, for the image of the point $(1, 0, \dots, 0)$, its isotropy subgroup is $\mathrm{diag}(\pm 1,\ O^{\pm}(n-1))$; here, $O^{\pm}(n-1) = \{g \in O(n-1) \mid \det g = \pm 1\}$. Other examples exhibiting similar phenomena will appear in [An et al. 2005], where we consider the group ensembles associated with complex semisimple Lie groups. When such sudden variation of the isotropy subgroups happens, it is in general an open problem whether we can make them be the same by enlarging the set X_z.

4. Generalized random matrix ensembles

We are ready to present the generalized random matrix ensembles.

Let G be a Lie group acting on an n-dimensional smooth manifold X by $\sigma : G \times X \to X$. For convenience, suppose X is a Riemannian manifold. Assume the induced Riemannian measure dx is G-invariant (note that we do not require the Riemannian structure on X to be G-invariant). Let Y be a closed submanifold of X, endowed the induced Riemannian measure dy, and let

$$K = \big\{g \in G \;\big|\; \sigma_g(y) = y, \text{for all } y \in Y\big\}.$$

As in Section 2, we take the map $\varphi : G/K \times Y \to X$ with $\varphi\big([g], y\big) = \sigma_g(y)$. Let $X_{\mathrm{z}} \subset X$ and $Y_{\mathrm{z}} \subset Y$ be closed zero-measure subsets of X and Y, respectively. Set $X' = X \setminus X_{\mathrm{z}}$ and $Y' = Y \setminus Y_{\mathrm{z}}$. We assume that the conditions (a), (b), (c), and (d) of Section 2 hold. For the reader's convenience, we list them below.

(a) $X' = \bigcup_{y\in Y'} O_y$ (*invariance condition*).

(b) $T_yX = T_yO_y \oplus T_yY$ for all $y \in Y'$ (*transversality condition*).

(c) $\dim G_y = \dim K$ for all $y \in Y'$ (*dimension condition*).

(d) $T_yY \perp T_yO_y$ for all $y \in Y'$ (*orthogonality condition*).

Suppose $d\mu$ is a G-invariant smooth measure on G/K, and $p(x)$ is a G-invariant smooth function on X. Then, by Corollary 2.3, there is a quasi-smooth measure $d\nu$ on Y such that

$$\varphi^*(p(x)\,dx) = d\mu\,d\nu. \tag{4–1}$$

Definition. Let the conditions and notation be as above. The system

$$(G, \sigma, X, p(x)\,dx, Y, dy)$$

is called a *generalized random matrix ensemble*. The manifolds X and Y are called the *integration manifold* and the *eigenvalue manifold*, respectively. The measure $d\nu$ on Y determined by (4–1) is called the *generalized eigenvalue distribution*.

Recall that in Section 2 we have defined the map $\Psi_y : \mathfrak{l} \to T_yO_y$ by

$$\Psi_y(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_{\exp t\xi}(y) \qquad \text{for all } \xi \in \mathfrak{l},$$

where $\mathfrak{l}$ is a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$. Thanks to the preliminaries in Section 2, we can compute the generalized eigenvalue distribution directly with the next theorem, which follows directly from Proposition 2.2, Corollary 2.3, Proposition 2.4, and Theorem 2.5.

Theorem 4.1. *Let $(G, \sigma, X, p(x)\,dx, Y, dy)$ be a generalized random matrix ensemble. The generalized eigenvalue distribution $d\nu$ is given by*

$$d\nu(y) \;=\; \mathcal{P}(y)\,dy \;=\; p(y)\,J(y)\,dy, \tag{4–2}$$

where

$$J(y) = C\,\big|\det \Psi_y\big|, \tag{4–3}$$

with $C = \big|\det\big((d\pi)_e|_{\mathfrak{l}}\big)\big|^{-1}$. □

The function $\mathcal{P}(y) = p(y)\,J(y)$ in (4–2) is the *generalized joint-density function*.

One of the fundamental problems in random matrix theory is to compute the eigenvalue distribution $d\nu$. In our generalized scheme, it is given by formulae (4–2) and (4–3). Note that the power of (4–3) is reflected in the fact that it provides a direct and unified method to compute the eigenvalue distributions of various kinds of random matrix ensembles. In [An et al. 2005] we show how all classical ensembles are included in our generalized scheme, and how the corresponding eigenvalue distributions can be derived from (4–2) and (4–3). We will also present various kinds of generalized ensembles that were not considered before, and compute their eigenvalue distributions explicitly.

Now, we consider the integration formula associated with the generalized random matrix ensemble. As in Section 3, we assume the following condition holds:

(e) The map $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map, with $d < +\infty$ (*covering condition*).

Theorem 4.2. *Let* $(G, \sigma, X, p(x)\,dx, Y, dy)$ *be a generalized random matrix ensemble. If the covering condition (e) holds, then we have the integration formula*

$$\int_X f(x)\,p(x)\,dx = \frac{1}{d}\int_Y \left(\int_{G/K} f\big(\sigma_g(y)\big)\,d\mu([g])\right)d\nu(y) \tag{4–4}$$

for all $f \in C^\infty(X)$ *with* $f \geq 0$ *or* $f \in L^1(X, p(x)\,dx)$. *If moreover* $f(\sigma_g(x)) = f(x)$ *for all* $g \in G$ *and* $x \in X$, *then*

$$\int_X f(x)\,p(x)\,dx = \frac{\mu(G/K)}{d}\int_Y f(y)\,d\nu(y). \tag{4–5}$$

Proof. It is obvious from Theorem 3.3 and Corollary 3.4. □

If the measure $p(x)\,dx$ in (4–5) is a probability measure and we let $f = 1$, we get $\big(\mu(G/K)/d\big)\int_Y d\nu(y) = 1$. So, if G/K is compact, we can normalize the measure $d\mu$ such that $\mu(G/K) = d$, and then the generalized eigenvalue distribution $d\nu$ is a probability measure.

Remark. The condition $f \in C^\infty(X)$ in Theorem 4.2 is superfluous. It is sufficient to assume that f is measurable. The same is true for Proposition 3.1 and Theorem 3.3.

5. Special cases

In this section we discuss several classes of generalized random matrix ensembles: linear ensembles, nonlinear noncompact ensembles, compact ensembles, group ensembles, algebra ensembles, pseudo-group ensembles, and pseudo-algebra ensemble. These account for all kinds of classical random matrix ensembles and some new examples of generalized ensembles.

Linear ensemble and the nonlinear noncompact ensembles. Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$, in the sense of [Knapp 2002, Section 7.2]. The group G admits a global Cartan involution Θ, inducing a Cartan involution θ of $\mathfrak{g}$. Let the corresponding Cartan decomposition of $\mathfrak{g}$ be $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and set

$$K = \{g \in G \mid \Theta(g) = g\} \qquad \text{and} \qquad P = \exp(\mathfrak{p}).$$

K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, while P is a closed submanifold of G satisfying $T_e P = \mathfrak{p}$. The spaces $\mathfrak{p}$ and P are invariant under the adjoint action $\mathrm{Ad}|_K$ and the conjugate action σ of K, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let A be the connected subgroup of G with Lie algebra $\mathfrak{a}$. Set

$$M = \left\{k \in K \,\middle|\, (\mathrm{Ad}|_K)_k(\eta) = \eta, \text{ for all } \eta \in \mathfrak{a}\right\} = \left\{k \in K \,\middle|\, \sigma_k(a) = a, \text{ for all } a \in A\right\}.$$

It can be shown that there are Riemannian structures on $\mathfrak{p}$ and P inducing K-invariant Riemannian measures dX on $\mathfrak{p}$ and dx on P. They also induce Riemannian measures dY on $\mathfrak{a}$ and da on A. Further, there is a K-invariant smooth measure $d\mu$ on K/M. If $p_1(\xi)$ and $p_2(x)$ are K-invariant positive smooth functions on $\mathfrak{p}$ and P, then it can be proved that the systems

$$(K, \mathrm{Ad}|_K, \mathfrak{p}, p_1(\xi)\, dX(\xi), \mathfrak{a}, dY) \qquad \text{and} \qquad (K, \sigma, P, p_2(x)\, dx, A, da)$$

are generalized random matrix ensembles, which we call *linear ensemble* and *nonlinear noncompact ensemble*, respectively. It can be shown that the Gaussian ensemble and the chiral ensemble are particular examples of linear ensembles, while the new transfer matrix ensembles are particular examples of nonlinear noncompact ensemble.

Compact ensembles. Let G be a connected compact Lie group G with Lie algebra $\mathfrak{g}$. Suppose Θ is a global involution of G with induced involution $\theta = d\Theta$ on $\mathfrak{g}$. Let $K = \{g \in G \mid \Theta(g) = g\}$. Let $\mathfrak{p}$ be the eigenspace of θ of eigenvalue -1, and let $P = \exp(\mathfrak{p})$. Then P is invariant under the conjugate action σ of K. It was proved in [An and Wang 2006] that P is a closed submanifold of G satisfying $T_e P = \mathfrak{p}$, and that it is just the identity component of the set $\{g \in G \mid \Theta(g) = g^{-1}\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let A be the torus with Lie algebra $\mathfrak{a}$. There is a Riemannian structure on P that induces a K-invariant Riemannian measure dx on P and a Riemannian measure da on A. Let

$$M = \left\{k \in K \,\middle|\, \sigma_k(a) = a, \text{ for all } a \in A\right\}.$$

There is a K-invariant smooth measure $d\mu$ on K/M. Take $p(x)$ a K-invariant positive smooth function on P. It can be proved that the system

$$(K, \sigma, P, p(x)\, dx, A, da)$$

is a generalized random matrix ensemble, which we call *compact ensemble*. The circular ensemble and the Jacobi ensembles are particular examples of compact ensembles.

Group and algebra ensembles. Let G be a unimodular Lie group G with Lie algebra $\mathfrak{g}$. There are Riemannian structures on G and $\mathfrak{g}$ inducing a σ-invariant Riemannian measure dg on G and an Ad-invariant Riemannian measure dX on $\mathfrak{g}$, where σ denotes the conjugate action of G on itself. Let $p_1(g)$ and $p_2(\xi)$ be two functions on G and $\mathfrak{g}$, respectively, that are invariant under the corresponding actions of G. If there exists a closed submanifold Y of G such that

$$(G, \sigma, G, p(g)\,dg, Y, dy)$$

is a generalized random matrix ensemble, where dy is the induced Riemannian measure on Y, then we call it a *group ensemble*. And, if there exists a closed submanifold $\mathfrak{y}$ of $\mathfrak{g}$ such that

$$(G, \mathrm{Ad}, \mathfrak{g}, p(\xi)\,dX(\xi), \mathfrak{y}, dY)$$

is a generalized random matrix ensemble, where dY is the induced Riemannian measure on $\mathfrak{y}$, then we call it an *algebra ensemble*.

Among all unimodular Lie groups, connected compact Lie groups and connected complex semisimple Lie groups are of particular interest. For a connected compact Lie group G, we can let the submanifold Y of G be a maximal torus T of G, and let the submanifold $\mathfrak{y}$ of $\mathfrak{g}$ be the Lie algebra of T. For a connected complex semisimple Lie group G, we can let the submanifold $\mathfrak{y}$ of $\mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, and let the submanifold Y of G be the corresponding Cartan subgroup of G. For these cases, it can be proved that the systems $(G, \sigma, G, p(g)\,dg, Y, dy)$ and $(G, \mathrm{Ad}, \mathfrak{g}, p(\xi)\,dX(\xi), \mathfrak{y}, dY)$ are generalized random matrix ensembles.

Pseudogroup and pseudoalgebra ensembles. These are related to real reductive groups. Let G be a real reductive group with lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$, and $\mathfrak{h}_1, \dots, \mathfrak{h}_m$ a maximal set of mutually nonconjugate θ-stable Cartan subalgebras of $\mathfrak{g}$, with corresponding Cartan subgroups $H_1, \dots, H_m$ of G. Denote the sets of all regular elements in G and $\mathfrak{g}$ by G_r and $\mathfrak{g}_r$, respectively. Let $H_j' = H_j \cap G_r$ and $\mathfrak{h}_j' = \mathfrak{h}_j \cap \mathfrak{g}_r$. It is known that

$$G_r = \bigsqcup_{j=1}^{m} \bigcup_{g\in G} \sigma_g(H_j') \qquad \text{and} \qquad \mathfrak{g}_r = \bigsqcup_{j=1}^{m} \bigcup_{g\in G} \mathrm{Ad}_g(\mathfrak{h}_j')$$

(see [Knapp 2002, Theorem 7.108] and [Warner 1972, Proposition 1.3.4.1], respectively). Here, "$\sqcup$" means disjoint union. Each $\bigcup_{g\in G} \sigma_g(H_j')$ is an open set in

G, and each $\bigcup_{g\in G}\operatorname{Ad}_g(\mathfrak{h}_j')$ is an open set in $\mathfrak{g}$. Let

$$G_j=\overline{\bigcup_{g\in G}\sigma_g(H_j')}\qquad\text{and}\qquad \mathfrak{g}_j=\overline{\bigcup_{g\in G}\operatorname{Ad}_g(\mathfrak{h}_j')}.$$

It can be shown that some suitable Riemannian structures on G and $\mathfrak{g}$ induce, for each j, a σ-invariant measure dg_j on G_j, and an Ad-invariant measure dX_j on $\mathfrak{g}_j$, and that they also induce Riemannian measures dh_j on H_j and dY_j on $\mathfrak{h}_j$. It is known that

$$Z(H_j)=\{g\in G \mid \sigma_g(h)=h,\ \text{for all } h\in H_j\},$$
$$H_j=\{g\in G \mid \operatorname{Ad}_g(\xi)=\xi,\ \text{for all } \xi\in\mathfrak{h}_j\}.$$

Let $d\mu_j'$ and $d\mu_j$ be G-invariant measures on $G/Z(H_j)$ and G/H_j, respectively. In general, the spaces G_j and $\mathfrak{g}_j$ may have singularities, but this doesn't matter, since these spaces are closures of open submanifolds in G and $\mathfrak{g}$, whose boundaries have measure zero. If we ignore this ambiguity, it can be proved that

$$(G,\sigma,G_j,dg_j,H_j,dh_j)\qquad\text{and}\qquad(G,\operatorname{Ad},\mathfrak{g}_j,dX_j,\mathfrak{h}_j,dY_j)$$

are generalized random matrix ensembles, which we call *pseudogroup ensemble* and *pseudoalgebra ensemble*, respectively.

The classes introduced above do not exhaust all generalized ensembles. But they include all kinds of classical random matrix ensembles and some new examples of generalized ensembles, which will be studied in [An et al. 2005].

Acknowledgement

We thank the production editors for their help with English usage.

References

[Altland and Zirnbauer 1997] A. Altland and M. R. Zirnbauer, "Novel symmetry classes in mesoscopic normal conducting-superconducting hybrid structures", *Phys. Rev. B* **55** (1997), 1142–1161.

[An and Wang 2006] J. An and Z. Wang, "On the realization of Riemannian symmetric spaces in Lie groups", *Topology and its Applications* **153**:7 (2006), 1008–1015. MR 2203015 Zbl 1086.53072 math.GT/0502028

[An et al. 2005] J. An, Z. Wang, and K. Yan, "A generalization of random matrix ensemble II: concrete examples and integration formulae", preprint, 2005. math-ph/0502021

[Caselle 1994] M. Caselle, "On the distribution of transmission eigenvalues in disordered wires", preprint, 1994. cond-mat/9410097

[Caselle 1996] M. Caselle, "A new classification scheme for random matrix theories", preprint, 1996. cond-mat/9610017

[Caselle and Magnea 2004] M. Caselle and U. Magnea, "Random matrix theory and symmetric spaces", *Phys. Rep.* **394**:2-3 (2004), 41–156. MR 2005d:82049

[Dueñez 2004] E. Dueñez, "Random matrix ensembles associated to compact symmetric spaces", *Comm. Math. Phys.* **244**:1 (2004), 29–61. MR 2005g:15039 Zbl 1041.15017

[Dyson 1970] F. J. Dyson, "Correlations between eigenvalues of a random matrix", *Comm. Math. Phys.* **19** (1970), 235–250. MR 43 #4398 Zbl 0221.62019

[Helgason 2000] S. Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs **83**, American Mathematical Society, Providence, RI, 2000. MR 2001h:22001

[Ivanov 2001] D. A. Ivanov, "Random-matrix ensembles in p-wave vortices", preprint, 2001. cond-mat/0103089

[Knapp 2002] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics **140**, Birkhäuser, Boston, 2002. MR 2003c:22001 Zbl 1075.22501

[Mehta 1991] M. L. Mehta, *Random matrices*, Academic Press, Boston, 1991. MR 92f:82002 Zbl 0780.60014

[Titov et al. 2001] M. Titov, P. W. Brouwer, A. Furusaki, and C. Mudry, "Fokker–Planck equations and density of states in disordered quantum wires", *Phys. Rev. B* **63** (2001), 235–318.

[Warner 1972] G. Warner, *Harmonic analysis on semi-simple Lie groups*, vol. I, Springer, New York, 1972. MR 58 #16979 Zbl 0265.22020

[Zirnbauer 1996] M. R. Zirnbauer, "Riemannian symmetric superspaces and their origin in random-matrix theory", *J. Math. Phys.* **37**:10 (1996), 4986–5018. MR 97m:58012 Zbl 0871.58005

Received February 8, 2005.

JINPENG AN
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
THE PEOPLE'S REPUBLIC OF CHINA
anjinpeng@gmail.com

ZHENGDONG WANG
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
THE PEOPLE'S REPUBLIC OF CHINA
zdwang@pku.edu.cn

KUIHUA YAN
SCHOOL OF MATHEMATICAL SCIENCES
PEKING UNIVERSITY
BEIJING, 100871
THE PEOPLE'S REPUBLIC OF CHINA
yankh@zjnu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

QUADRATIC FORMS OVER RATIONAL FUNCTION FIELDS IN CHARACTERISTIC 2

ROBERTO ARAVIRE AND BILL JACOB

A basic result of Milnor and Scharlau determines the Witt ring of rational function fields $Wk(x)$ whenever char $k \neq 2$. An analogous result is obtained here for the Witt group of quadratic forms $W_q\mathcal{F}(x)$, where $\mathcal{F}$ is a field of characteristic 2. This generalizes earlier work by the authors where $\mathcal{F}$ was assumed to be perfect.

Milnor's determination [1970] of the Witt ring of a rational function field in terms of the Witt rings of the finite extensions of the base field is a fundamental result in the algebraic theory of quadratic forms, and was complemented by Scharlau's reciprocity law (see [Lam 1973] or [Scharlau 1972]). Here we give an analogue of these results for the Witt group of rational function fields in characteristic 2, extending earlier work where the base field was assumed to be perfect [Aravire and Jacob 2004].

All our fields will have characteristic 2. We use the notation $\mathcal{F}$ for the base field of our rational function field $F = \mathcal{F}(x)$. Whenever $p \in \mathcal{F}[x]$ is monic and irreducible, we denote by $\mathcal{F}(x)_p$ the completion at the discrete valuation $v_p : \mathcal{F}(x) \to \mathbb{Z}$ determined by p. Similarly, we denote by $\mathcal{F}(x)_{\frac{1}{x}}$ the completion at the $\frac{1}{x}$-adic (or infinite) valuation $v_{\frac{1}{x}} : \mathcal{F}(x) \to \mathbb{Z}$. We use $W_q F$ and WF to denote the Witt group and Witt ring of F, and we follow the standard notation. In particular, $[a, b]$ denotes the Witt class of quadratic form $ax^2 + xy + by^2$. These classes form an additive set of generators for $W_q F$, and $\langle a \rangle$ denotes the 1-dimensional symmetric bilinear form $(x, y) \mapsto axy$. The symbol $[\ ,\ ]$ is biadditive and $W_q F$ is a WF-module via the action $\langle a \rangle [c, d] = [ac, a^{-1}c]$. This means that $W_q F$ is also generated by the forms $\langle a \rangle [1, b]$ and when considering such an element we will refer to a as being in the *multiplicative slot* and b as being in the *additive slot*. We use the standard notation $I^n F$ for the n-th power of the fundamental ideal in WF, so that $I^n W_q F$ is generated by the forms $\langle\langle a_1, a_2, \dots, a_n \rangle\rangle [1, b]$. Arason [1979, Satz 8] gave a generator-relation description of $W_q F$ as a WF-module, and we use these relations throughout. We frequently use what we call the *fundamental relation*,

---

MSC2000: 11E04.
Keywords: Witt group, rational function fields, characteristic two.
Both authors acknowledge support from Fondecyt.

$\langle a+b\rangle [1, c] = \langle a\rangle [1, ac/(a+b)] + \langle b\rangle [1, bc/(a+b)]$, which shows how addition in the multiplicative slot can be distributed across a sum of forms.

A key component of the classical Milnor–Scharlau sequence is the second residue homomorphism $\partial_p : WF \to W\overline{F}_p$, where $\overline{F}_p$ is the residue class field of a discrete valuation $v_p : F \to \mathbb{Z}$. This map has proved to be of considerable importance in quadratic form theory. For example, if X is a variety defined over F, the kernel under all $\partial_p : WF(X) \to W\overline{F}_p$ is the *unramified* Witt group of X, which when X is a nonsingular curve coincides with the Witt group of X. This paper gives the appropriate version of ∂_p in characteristic two, and in a subsequent paper this work is applied to the study of Witt groups of curves in characteristic two.

Whenever $v_p : F \to \mathbb{Z}$ is a discrete valuation, we set

$$W_1 F_p := \operatorname{coker}(W_q \overline{F}_p \to W_q F_p),$$

where the map is induced by a Teichmüller lifting $\overline{F}_p \hookrightarrow F_p$. We show in Corollary 1.7 that the group $W_1 F_p$ is independent of the choice of a Teichmüller lifting. For such a lifting we define the second residue map $\partial_p : W_q F \to W_1 F_p$ to be the composite map induced by inclusion and projection:

$$W_q F \to W_q F_p \to W_1 F_p.$$

We are also able to identify a quotient of $W_1 F_p$ with $W_q \overline{F}_p$ where, when $p \in \mathcal{F}[x]$ is an irreducible polynomial, we have $\overline{F}_p \cong \mathcal{F}[x]/(p)$. Using this we obtain a version of Scharlau's transfer $s_p^* : W_1 F_p \to W_q \mathcal{F}$ as a composite of maps $W_1 F_p \to W_q \mathcal{F}[x]/(p) \to W_q \mathcal{F}$, where the latter map is the same transfer used by Scharlau. Both maps ∂_p and s_p^* are analogous to the classical maps, but as they depend upon choices of Teichmüller liftings and of subgroups of $W_1 F_p$, these selections must be made to meet certain compatibility requirements for our main result to hold.

With this notation, the main result of this paper is the following.

Theorem 6.2 (Analogue of the Milnor–Scharlau Sequence). *Suppose that $\mathcal{F}$ is a field of characteristic 2 and $F = \mathcal{F}(x)$ is a rational function field in one variable over $\mathcal{F}$. There exists a compatible collection of second residue and transfer maps that fit into an exact sequence*

$$0 \longrightarrow W_q \mathcal{F} \longrightarrow W_q F \xrightarrow{\bigoplus \partial_p} \bigoplus_{p, \frac{1}{x}} W_1 F_p \xrightarrow{\bigoplus s_p^*} W_q \mathcal{F} \longrightarrow 0,$$

where the direct sum is taken over discrete valuations on F.

We now provide an overview of the proof. As we do this we will recall the main features of the proof in the classical case in order to illustrate the similarities and differences. When $\operatorname{char} F \neq 2$ and F is complete with respect to a discrete valuation $v : F \to \mathbb{Z}$, a well-known result of Springer shows that $WF \cong W\overline{F} \oplus$

$\langle\pi\rangle W\overline{F}$, where π is a uniformizing parameter for v. This decomposition enables one to construct the second residue map and the transfer in the Milnor–Scharlau sequence. When $\operatorname{char} F = 2$ and F is complete it is first necessary for us to compute the Witt group $W_q F$. This is the main objective of Section 1. The main result proved there, Theorem 1.3, shows that $W_q F \cong W_q \overline{F} \oplus \mathcal{R} \oplus \langle\pi\rangle W_q \overline{F}$, where again π is a uniformizing parameter. The subgroup $\mathcal{R}$ is quite large and although its description depends upon choosing a 2-basis for F and a Teichmüller lifting $\overline{F} \hookrightarrow F$ it has adequate uniqueness properties. (When $\mathcal{F}$ is perfect, then there is a unique Teichmüller lifting, however in general, such lifts depend upon the choice of a 2-basis for $\mathcal{F}$. See [Schilling 1950, p. 236] for details.) This decomposition shows that $W_1 F \cong \mathcal{R} \oplus \langle\pi\rangle W_q \overline{F}$ and enables us to define both the second residue and Scharlau transfer maps needed for the main theorem.

After defining the second residue maps, Milnor's proof requires a filtration $L_0 \subset L_1 \subset L_2 \subset \cdots \subset WF$, where by $L_d \subset WF$ he considered the subgroup generated by all $\langle f\rangle$, where f is a polynomial of degree at most d. He then proves a key result, namely that the successive quotients L_d/L_{d-1} for $d \geq 1$ are isomorphic to the direct sum of groups $\bigoplus_{\deg p=d} W\overline{F}_p$. To do this he shows there is a well defined splitting of the sum of induced maps $\bigoplus_{\deg p=d} \partial_p : L_d/L_{d-1} \to \bigoplus_{\deg p=d} W\overline{F}_p$. In Section 2 we use the same idea and notation, except that our L_d are generated by the forms $\langle f\rangle[1, h/u^e]$, where now both f and u have degree at most d in $\mathcal{F}[x]$ and $h \in \mathcal{F}[x]$ is arbitrary. These forms are needed for two reasons. First $W_q F$ has as generators 2-dimensional forms, and second, the quotients h/u^e are needed to take into account all the extra stuff in $\mathcal{R}$. In the following section, Theorem 3.5 gives the exact analogue of Milnor's key result, namely that the map

$$\bigoplus_{\deg p=d} \partial_p : L_d/L_{d-1} \to \bigoplus_{\deg p=d} W_1 \overline{F}_p$$

is a split isomorphism.

To prove the latter result we must take several detours. First there is the complexity introduced by the existence of different ways to extend a 2-basis for $\mathcal{F}$ to a 2-basis for F and F_p. If p is separable, one can either add x or p, with x the natural choice for the rational function field F and with p the natural choice for F_p. When p is not separable, we have to specify which element we choose to omit from the 2-basis of $\mathcal{F}$ and then we must add both x and p to form a 2-basis for F_p. Since the ∂_p relate $W_q F$ to $W_1 F_p$, we need to be able to relate these choices. The bulk of Section 2 accomplishes this, by establishing the equivalence of different generating sets for the L_d in Lemma 2.5 and Proposition 2.8.

A second detour provides a generator-relation description of $W_q F$ (Theorem 3.3) needed to prove that the splitting maps are well defined (Lemma and Definition 3.4). The proof of the splitting is similar to that of the classical case, but is complicated again by the fact that 2-bases and Teichmüller liftings have to be selected carefully and in a compatible fashion. The details of these choices are set up in the discussion that follows Lemma 3.1. Finally one has to deal with the structure of L_0, which is just $W\mathcal{F}$ in the classical case. In our case it is generated by forms with polynomials $f, g \in \mathcal{F}[x]$ in the additive slots of binary forms, $[f, g]$. The result in Theorem 3.6 is that L_0 is described by an exact sequence $0 \to W_q\mathcal{F} \to L_0 \to W_1 F_{\frac{1}{x}} \to W_q\mathcal{F} \to 0$.

When Theorems 3.5 and 3.6 are combined with the definitions, we obtain a version of what Milnor did, namely that the sequence in Theorem 6.2 is exact if truncated to

$$0 \longrightarrow W_q\mathcal{F} \longrightarrow W_q F \xrightarrow{\bigoplus \partial_p} \bigoplus_{p,\text{finite}} W_1 F_p \oplus (W_1 F_{\frac{1}{x}}/\langle x\rangle W\mathcal{F}) \longrightarrow 0,$$

where the reciprocity law provided by the transfer is omitted. However, because the reciprocity law has important applications, we continue with its development in subsequent sections. Section 4 is devoted to defining the transfer maps. The subgroups of $W_1 F_p$ needed to define the maps are given in Definition 4.1 and are selected in a compatible way to ensure that the resulting s_p^* vanish on the subgroup $\mathcal{R} \subset W_1 F_p$. The definition of s_p^* when p fails to be separable, Definition 4.3(ii), is adjusted to take into account the change in the 2-basis resulting from the failure of the 2-basis of $\mathcal{F}$ to extend to F_p. In this case the exact terms necessary to make the reciprocity law work are added to the transfer of the residue form.

Having defined the s_p^*, we check the reciprocity law for elements of $L_0 + \langle p\rangle L_0$ (Theorem 5.4). This requires computing the ordinary transfer $t_p^* : W_q\mathcal{F}[x]/(p) \to W_q\mathcal{F}$ on generators $[\lambda_1 x^i, \lambda_2 x^j]$ of $W_q\mathcal{F}[x]/(p)$. There are quite a few cases to consider, but it is a straightforward computation. With this result, the main theorem, with the reciprocity law in general, is proved in Section 6, where the final stages of the proof consist of checking that the definitions involved in setting up $W_1 F_p$ and the s_p^* are arranged properly to ensure cancellation of the appropriate terms. Although the definition of s_p^* is based on the same linear functional as in the classical case, this portion of the paper differs from the approach in that case. Because of the additive nature of generators for $W_q F$ we are able to reduce to forms that vanish on all but two ∂_p's, and therefore we don't have to consider more complex transfers from algebras such as $\mathcal{F}[x]/(p_1 p_2 \cdots p_n)$, as did Scharlau.

1. Local information

If F has characteristic 2, a collection of elements $t_1, t_2, \ldots, t_n \in F$ is said to form a 2-*basis* of F if we have a strictly increasing sequence of subfields

$$F^2 \subsetneq F^2(t_1) \subsetneq F^2(t_1, t_2) \subsetneq \cdots \subsetneq F^2(t_1, t_2, \ldots, t_n) = F.$$

A field F can have many different 2-bases; if $[F : F^2] = 2^n$, every 2-basis has exactly n elements. We will assume that fields in this paper have finite 2-bases, since our main results are readily reduced to this case.

For fixed n we denote by T the set of n-tuples $I = (i_1, i_2, \dots, i_n)$, where $i_j \in \{0, 1\}$ for all j. We order T lexicographically, with minimal element $O := (0, 0, \dots, 0)$, then $(1, 0, \dots, 0)$, then $(0, 1, 0, \dots, 0)$, and so forth. It will be convenient to add elements of T as in the $\mathbb{Z}/2\mathbb{Z}$-vector space $(\mathbb{Z}/2\mathbb{Z})^n$ and let T_0 denote the nonzero elements of T. Whenever $t_1, t_2, \dots, t_n \in F$ and $I \in T$, we abbreviate $t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$ by t^I. In this notation, whenever $t_1, t_2, \dots, t_n$ form a 2-basis for F and $f \in F$, there exist *unique* elements $x_I \in F$ indexed by $I \in T$ such that

$$f = \sum_{I \in T} t^I x_I^2.$$

For the remainder of this section we assume that $v : F \to \mathbb{Z}$ is a complete discrete-valued field of characteristic 2 with residue class field $\bar{F}$ and uniformizing parameter π. We assume that $t_1, t_2, \dots, t_{n-1}$ are units in F whose residues form a 2-basis for $\bar{F}$. Since v is complete and discrete, we know that $t_1, t_2, \dots, t_{n-1}, \pi$ form a 2-basis for F. We will use the notation t^I, where $I = (I_1, I_2, \dots, I_n) \in T$, to represent elements in this 2-basis:

$$t^I = t_1^{I_1} t_2^{I_2} \cdots t_{n-1}^{I_{n-1}} \pi^{I_n}.$$

From [Schilling 1950, pp. 230–238] we also know that there is a unique monomorphism of fields $\rho : \bar{F} \hookrightarrow F$ with $\rho(\bar{t}_i) = t_i$ for $i = 1, 2, \dots, n-1$. Since we will regard this map as an inclusion, we will drop the residue bars from the t_i and view $t_1, t_2, \dots, t_{n-1}$ as lying in $\bar{F} \subset F$. With these conventions, $v(t^J) = J_n \in \{0, 1\}$ for all $J \in T$.

Since F is complete, we can view $F = \bar{F}((\pi))$ as formal Laurent series in its fixed uniformizing parameter π. We let $R := \bar{F}[\pi^{-1}] \subset F$ be the "backwards" polynomial ring, and with this, if $f \in F$ then there exists a unique $r \in \pi^{-1}R$ with $v(f - r) \geq 0$. Moreover, every element $r \in \pi^{-1}R$ can be uniquely expressed as $r = \sum_{I \in T} t^I r_I^2$, where $r_I \in R$.

Definition 1.1. We set $\mathcal{R}$ to be the subgroup of $W_q F$ of all forms

$$\sum_{I \in T} \langle t^I \rangle \Big[1, \sum_{J,\, J+I > I} t^J r_{I,J}^2\Big] \in \mathcal{R} \qquad \text{with } r_{I,J} \in \pi^{-1}R.$$

Whenever $v(a) > 0$ (that is, $a \in \bar{F}((\pi))$), we must have $a \in \wp(F)$ since F is complete with respect to v; consequently, $[1, a] = 0 \in W_q F$. We will use this fact repeatedly. When $r \in \pi^{-1}R$, however, we are in the opposite situation, since then $v(r) < 0$; in this case, if $v(r)$ is odd or the lead coefficient is not a square in $\mathcal{F}$, then r cannot lie in $\wp(F)$. This is why the module $\mathcal{R}$ is of interest.

Lemma 1.2. *Every element $\phi \in W_q F$ can be expressed as $\phi = \sum_{I \in T} \langle t^I \rangle [1, A_I]$, where $A_I = \sum_{J, J+I>I} t^J r_{I,J}^2$ and $r_{I,J} \in F$.*

Proof. Applying the W_q-relations we know that every element of $W_q F$ is a sum of elements $\langle t^I \rangle [1, t^J b^2]$ for $b \in F$. Applying the W_q-relation $\langle t \rangle [1, a] = \langle ta \rangle [1, a]$ we find for I, J with $I + J < I$ that $\langle t^I \rangle [1, t^J b^2] = \langle t^{I+J} \rangle [1, t^J b^2]$, and when $I = J$ we find that $\langle t^I \rangle [1, t^I b^2] = \langle 1 \rangle [1, t^I b^2]$. Hence every element of $\phi \in W_q F$ can be expressed as $\phi = \sum_{I \in T} \langle t^I \rangle \bigl[1, \sum_{J, J+I>I} t^J b_{I,J}^2\bigr]$, with $b_{I,J} \in F$. (In fact this much is true for any 2-basis of any field F of characteristic 2.) The statement for $\mathfrak{R}$ follows applying this same argument to that case. $\square$

Theorem 1.3. *Suppose that $v : F \to \mathbb{Z}$ is a complete discrete valued field of characteristic 2 with residue field $\bar{F} \subset F$ and uniformizing parameter π. Then every class $\phi \in W_q F$ can be expressed uniquely as*

$$\phi = \phi_1 \perp \psi \perp \langle \pi \rangle \phi_2,$$

where $\phi_1, \phi_2 \in W_q \bar{F}$ and $\psi = \sum_{I \in T} \langle t^I \rangle [1, \sum_{J, J+I>I} t^J r_{I,J}^2] \in \mathfrak{R}$ with $r_{I,J} \in \pi^{-1} R$. The classes of ϕ_1, ϕ_2 and the $r_{I,J}$ are uniquely determined by ϕ. In particular, there is a split exact sequence

$$0 \to W_q \bar{F} \to W_q F \to \bigl(\mathfrak{R} \oplus \langle \pi \rangle W_q \bar{F}\bigr) \to 0.$$

Proof. Consider $\sum_{I \in T} \langle t^I \rangle [1, \sum_{J, J+I>I} t^J b_{I,J}^2] \in W_q F$. Since F is complete and discretely valued, we can express each $b_{I,J}$ as $r_{I,J} + \bar{f}_{I,J} + b'_{I,J}$, where $r_{I,J} \in \pi^{-1} R$, $\bar{f}_{I,J} \in \bar{F} \subset F$ and $v(b'_{I,J}) > 0$. Since $v(t^J) \geq 0$ we have $t^J {b'_{I,J}}^2 \in \wp(F)$ and hence $[1, t^J {b'_{I,J}}^2] = 0$. We observe:

- If $J_n = 0$ then $t^J \in \bar{F}$ and we have $t^J r_{I,J}^2 \in \pi^{-1} R$ while $t^J \bar{f}_{I,J}^2 \in \bar{F} \subset F$. When $I_n = 0$ we have $\langle t^I \rangle [1, t^J \bar{f}_{I,J}^2] \in W_q \bar{F}$, and when $I_n = 1$ we have $\langle t^I \rangle [1, t^J \bar{f}_{I,J}^2] \in \langle \pi \rangle W_q \bar{F}$. So

$$\langle t^I \rangle [1, t^J b_{I,J}^2] = \langle t^I \rangle [1, t^J r_{I,J}^2] + \langle t^I \rangle [1, t^J \bar{f}_{I,J}^2] \in W_q \bar{F} + \mathfrak{R} + \langle \pi \rangle W_q \bar{F}$$

in this case.

- If $J_n = 1$, then we know $v(t^J) = 1$ and consequently $t^J r_{I,J}^2 \in \pi^{-1} R$ while $v(t^J \bar{f}_{I,J}^2) > 0$, so $[1, t^J \bar{f}_{I,J}^2] = 0$. Therefore, $\langle t^I \rangle [1, t^J b_{I,J}^2] = \langle t^I \rangle [1, t^J r_{I,J}^2] \in \mathfrak{R}$ in this case.

Altogether this shows that every element of $\phi \in W_q F$ can be expressed as $\phi = \phi_1 + \psi + \langle \pi \rangle \phi_2$, with $\phi_1, \phi_2 \in W_q \bar{F}$ and $\psi \in \mathfrak{R}$.

To prove the uniqueness assertions and the exactness of the sequence we need a bit more notation. We denote by n^s the set of all subsets of $\{1, 2, \ldots, n\}$ containing s elements. For any $I = (I_1, I_2, \ldots, I_n) \in T$ we use $\tilde{I}$ to denote $\{j \mid I_j \neq 0\}$. Note that $\tilde{I} \in n^s$ for some $s = 0, 1, \dots, n$, and that any subset $\tilde{I} \in n^s$ is determined by a unique $I \in T$. For any subset $S \subseteq \{1, 2, \dots, n\}$ we write

$$\langle\!\langle t_S \rangle\!\rangle = \bigotimes_{i \in S} \langle\!\langle t_i \rangle\!\rangle ,$$

where, when $S = \varnothing$, $\langle\!\langle t_\varnothing \rangle\!\rangle = \langle 1 \rangle$ by convention. Whenever $I \in T$, we define $\ell(I) = \max(\tilde{I})$. Finally, we define $[\tilde{I}]_0 := \{J \in T \mid J \neq (0, 0, \dots, 0) \text{ and } \tilde{J} \subseteq \tilde{I}\}$ and $[\tilde{I}^c] := \{J \in T \mid \tilde{J} \cap \tilde{I} = \varnothing\}$.

Lemma 1.4 [Aravire and Jacob 1996, Lemma 1.6]. *Suppose that $t_1, t_2, \dots, t_n$ are 2-independent in a field F and let $a_{\tilde{I}} \in F$. Suppose*

$$q = \sum_{\tilde{I} \in n^s} \langle\!\langle t_{\tilde{I}} \rangle\!\rangle \, [1, a_{\tilde{I}}] \in I^{s+1} W_q F.$$

Then each

$$a_{\tilde{I}} \in \Bigl(\wp(F) + \sum_{J \in [\tilde{I}]_0 + [\tilde{I}^c]} t^J F^2 \Bigr) = \Bigl(\wp(F) + \sum_{J, \tilde{J} \cap \tilde{I} \neq \varnothing} t^J F^2 \Bigr) \qquad \square$$

The next result is a modification of [Aravire and Jacob 1996, Proposition 1.7].

Proposition 1.5. *Suppose that $v : F \to \mathbb{Z}$ is a complete discrete valued field and $t_1, t_2, \dots, t_n$ are as above. Suppose*

$$q = \sum_{\tilde{I} \in n^s} \langle\!\langle t_{\tilde{I}} \rangle\!\rangle \, [1, a_{\tilde{I}}] \in I^{s+1} W_q F,$$

where $a_{\tilde{I}} \in \sum_{J+I>I} t^J (\pi^{-1} R)^2$. Then $a_{\tilde{I}} = 0$ for each I. $\square$

Proof. Assume the contrary. Let M be the maximal index among the I with $\tilde{I} \in n^s$ and $a_{\tilde{I}} \neq 0$. We express $a_{\widetilde{M}}$ as a sum $\sum_{K+M>M} t^K A^2_{K,M}$, where each $A_{K,M} \in \pi^{-1} R$. Since $a_{\widetilde{M}} \neq 0$, there is some J with $J + M > M$ and $t^J A^2_{J,M} \neq 0$. The result will be proved when we derive the contradiction that $t^J A^2_{J,M} = 0$.

Since $J + M > M$ we have $\ell(J) \notin \widetilde{M}$. We denote by $t'_1, t'_2, \dots, t'_n$ the 2-basis obtained from $t_1, t_2, \dots, t_n$ by replacing $t_{\ell(J)}$ by t^J. Then, since $\ell(J) \notin \widetilde{M}$, we have $\langle\!\langle t_{\widetilde{M}} \rangle\!\rangle = \langle\!\langle t'_{\widetilde{M}} \rangle\!\rangle$. Also, we have

$$\langle\!\langle t'_{\ell(J)} \rangle\!\rangle = \langle\!\langle t^J \rangle\!\rangle \equiv \sum_{j \in \tilde{J}} \langle\!\langle t^j \rangle\!\rangle \pmod{I^2 F}.$$

We now suppose that $K \in T$, $\tilde{K} \in n^s$, and $\ell(J) \in \tilde{K}$. We express $\tilde{K}$ as $\{\ell(J)\} \cup \tilde{Q}$ for $\tilde{Q} \in n^{s-1}$. Computing in WF modulo $I^{s+1}F$ we have

$$\begin{aligned}
\langle\langle t'_{\tilde{K}} \rangle\rangle = \langle\langle t'_{\tilde{Q}} \rangle\rangle \langle\langle t'_{\ell(J)} \rangle\rangle = \langle\langle t_{\tilde{Q}} \rangle\rangle \langle\langle t^J \rangle\rangle &\equiv \sum_{j \in \tilde{J}} \langle\langle t_{\tilde{Q}} \rangle\rangle \langle\langle t_j \rangle\rangle \\
&\equiv \langle\langle t_{\tilde{K}} \rangle\rangle + \sum_{\substack{j \in \tilde{J} \\ j \neq \ell(J)}} \langle\langle t_{\tilde{Q}} \rangle\rangle \langle\langle t_j \rangle\rangle \equiv \langle\langle t_{\tilde{K}} \rangle\rangle + \sum_{\substack{j \in \tilde{J}, j \neq \ell(J) \\ j \notin \tilde{Q}}} \langle\langle t_{\tilde{Q}} \rangle\rangle \langle\langle t_j \rangle\rangle .
\end{aligned}$$

The conditions $j \in \tilde{J}$, $j \neq \ell(J)$, and $j \notin \tilde{Q}$, are equivalent to the single condition $j \in \tilde{J} - \tilde{K}$. So, as each such $j < \ell(J)$ we find $\langle\langle t_{\tilde{Q}} \rangle\rangle \langle\langle t_j \rangle\rangle = \langle\langle t'_{\tilde{Q}} \rangle\rangle \langle\langle t'_j \rangle\rangle = \langle\langle t'_{\tilde{L}} \rangle\rangle$ for some $L \in T$ with $L < K$. Altogether this shows that whenever $K \in T$, $\tilde{K} \in n^s$, and $\ell(J) \in \tilde{K}$,

$$\langle\langle t_{\tilde{K}} \rangle\rangle \in \langle\langle t'_{\tilde{K}} \rangle\rangle + \Bigg(\sum_{\substack{L \in T \\ \tilde{L} \in n^s,\, L < K}} \langle\langle t'_{\tilde{L}} \rangle\rangle WF + I^{s+1}F \Bigg). \tag{1}$$

Expanding using (1) we can rewrite q in terms of the new 2-basis involving the t'. We find

$$\begin{aligned}
q = \sum_{\tilde{I} \in n^s} \langle\langle t_{\tilde{I}} \rangle\rangle [1, a_{\tilde{I}}] &= \Bigg(\sum_{\substack{K < M \\ \tilde{K} \in n^s}} \langle\langle t_{\tilde{K}} \rangle\rangle [1, a_{\tilde{K}}] \Bigg) + \langle\langle t_{\widetilde{M}} \rangle\rangle [1, a_{\widetilde{M}}] \\
&\equiv \Bigg(\sum_{\substack{K < M \\ \tilde{K} \in n^s}} \langle\langle t'_{\tilde{K}} \rangle\rangle [1, a'_{\tilde{K}}] \Bigg) + \langle\langle t'_{\widetilde{M}} \rangle\rangle [1, a_{\widetilde{M}}] \pmod{I^{s+1} W_q F},
\end{aligned}$$

where the $a'_{\tilde{K}}$ for $K < M$ are the elements of F that arise in the expansion using (1) repeatedly. Observe that $a_{\widetilde{M}}$ remains unchanged when passing to the 2-basis using the t'. We now apply Lemma 1.2, where the 2-basis used is the one with the t'. We find that

$$a_{\widetilde{M}} = \sum_{K+M > M} t^K A^2_{K,M} \in \wp(F) + \sum_{L, \tilde{L} \cap \widetilde{M} \neq \varnothing} t'^L F^2. \tag{2}$$

When constructing the 2-basis involving t' we replaced $t_{\ell(J)}$ by t^J, which means that $t^J = t'^{J'}$, where $\tilde{J}' = \{\ell(J)\} \in n^1$. Since $\ell(J) \notin \widetilde{M}$, this gives $\tilde{J}' \cap \widetilde{M} = \varnothing$. Therefore, moving all the other terms on the left side of (2) to the right we find

$$t^J A^2_{J,M} \in \wp(F) + \sum_{I \in T_0, I \neq J} t^I F^2.$$

We claim that this gives $A_{J,M} = 0$. As $t^J A^2_{J,M} \in \pi^{-1} R$, if $t^J A^2_{J,M} \neq 0$ we must have $v(t^J A^2_{J,M}) = s < 0$, where s is even if $J_n = 0$ and is odd if $J_n = 1$. For $K \in T$

let K' be such that $K'_n = 0$ and $K'_i = K_i$ for $1 \le i < n$. With this notation,

$$\overline{\pi^{-s} t^J A^2_{J,M}} \in t^{J'} \bar{F}^2.$$

Next, if $w = \wp(b) + \sum_{I \in T_0, I \neq J} t^I b^2_{I,J} \in \wp(F) + \sum_{I \in T_0, I \neq J} t^I F^2$ is such that $v(b) = s < 0$, then

$$\overline{pi^{-s} w} \in \begin{cases} \bar{F}^2 + \sum\limits_{\substack{I \in T_0, I \neq J \\ I_n = 0}} t^I \bar{F}^2 & \text{when } s \text{ is even,} \\ \sum\limits_{\substack{I' \in T_0, I \neq J \\ I_n = 1}} t^{I'} \bar{F}^2 & \text{when } s \text{ is odd.} \end{cases}$$

In either case, because $t_1, t_2, \ldots, t_{n-1}$ is a 2-basis for $\bar{F}$, we cannot have

$$\overline{\pi^{-s} t^J A^2_{J,M}} = \overline{\pi^{-s} w},$$

contrary to the assumption that $A_{J,M} \neq 0$. This proves the proposition. □

We may now complete the proof of Theorem 1.3. The main task is showing that if $\psi = \sum_{I \in T} \langle t^I \rangle \big[1, \sum_{J, J+I > I} t^J r^2_{I,J}\big] = 0 \in \mathfrak{R}$ with $r_{I,J} \in \pi^{-1} R$, then each $r_{I,J} = 0$. Assuming this temporarily for all such complete discrete valued fields, to prove the uniqueness statements we consider an expression $\phi = \phi_1 \perp \psi \perp \langle \pi \rangle \phi_2 = 0$. Let L be a separable finite unramified extension of F chosen so that $(\phi_1)_L = 0$ and $(\phi_2)_L = 0$. Then L is still complete and discretely valued, the 2-basis is unchanged, $R \subset R_L$, and we have that $\phi_L = \psi_L$. So our temporary assumption applies to $\psi_L = 0 \in \mathfrak{R}_L$, the $r_{I,J}$ vanish in this case, and we now have $\phi_1 \perp \langle \pi \rangle \phi_2 = 0$. Now, by valuation theory, if both ϕ_1 and ϕ_2 are anisotropic over $\bar{F}$, then $\phi_1 \perp \langle \pi \rangle \phi_2$ anisotropic as well, since π is a uniformizing parameter. So this gives $\phi_1 = \phi_2 = 0$ and the uniqueness assertion follows.

Thus we are reduced to studying $\psi = \sum_{I \in T} \langle t^I \rangle \big[1, \sum_{J, J+I > I} t^J r^2_{I,J}\big] = 0 \in \mathfrak{R}$, where we want to show that each $r_{I,J} = 0$.

Lemma 1.6 [Aravire and Jacob 1996, Lemma 1.5]. *Suppose that $t_1, t_2, \ldots, t_n$ are 2-independent in a field F and $f \in F$. Then $\langle\!\langle t_1, t_2, \ldots, t_n \rangle\!\rangle [1, f] = 0 \in W_q F$ if and only if*

$$f \in \wp(F) + \sum_{J \in T_0} t^J F^2.$$

Applying the identity in WF (symmetric bilinear forms)

$$\langle\!\langle xy \rangle\!\rangle = \langle\!\langle x \rangle\!\rangle + \langle\!\langle y \rangle\!\rangle + \langle\!\langle x, y \rangle\!\rangle,$$

we obtain

$$\langle t^I \rangle = \sum_{K, \tilde{K} \subseteq \tilde{I}} \langle\!\langle t_{\tilde{K}} \rangle\!\rangle$$

(recall that $\langle\!\langle t_\varnothing \rangle\!\rangle = \langle 1 \rangle$.) Abbreviating $a_I := \sum_{J, J+I>I} t^J r_{I,J}^2$ and rewriting ψ using this identity we obtain

$$0 = \psi = \sum_{I \in T} \langle t^I \rangle \, [1, a_I] = \sum_{I \in T} \Bigl(\sum_{K,\, \tilde{K} \subseteq \tilde{I}} \langle\!\langle t_{\tilde{K}} \rangle\!\rangle \Bigr) [1, a_I]$$
$$= \sum_{K \in T} \langle\!\langle t_{\tilde{K}} \rangle\!\rangle \Bigl[1, \sum_{I,\, \tilde{K} \subseteq \tilde{I}} a_I \Bigr] = \sum_{s=0,\dots,n} \Bigl(\sum_{\tilde{K} \in n^s} \langle\!\langle t_{\tilde{K}} \rangle\!\rangle \Bigl[1, \sum_{I,\, \tilde{K} \subseteq \tilde{I}} a_I \Bigr] \Bigr).$$

By induction on s we shall show that

$$\sum_{I,\, \tilde{K} \subseteq \tilde{I}} a_I = 0$$

whenever $K \in n^s$. When $s = 0$, $\tilde{K} = \varnothing$ and since $\sum_{I \in T} a_I$ is the Arf invariant of q we find $\sum_{I \in T} a_I \in \wp(F)$. Since nonzero elements of $\pi^{-1} R$ have negative value, by valuation theory we find $\sum_{I \in T_0} t^I (\pi^{-1} R)^2 \cap \wp(F) = \{0\}$ and we are done if $s = 0$. Assuming the result for $1, 2, \dots, s-1$, we have

$$\sum_{\tilde{K} \in n^s} \langle\!\langle t_{\tilde{K}} \rangle\!\rangle \Bigl[1, \sum_{I,\, \tilde{K} \subseteq \tilde{I}} a_I \Bigr] \in I^{s+1} W_q F.$$

We observe that if $\tilde{K} \subseteq \tilde{I}$ and $J + I > I$, then $J + K > K$. Therefore Proposition 1.5 applies and we conclude for fixed $\tilde{K} \in n^s$ that

$$\sum_{I,\, \tilde{K} \subseteq \tilde{I}} a_I = 0.$$

Using $a_I = \sum_{I+J>I} t^J r_{I,J}^2$ we obtain

$$0 = \sum_{I,\, \tilde{K} \subseteq \tilde{I}} a_I = \sum_{I,\, \tilde{K} \subseteq \tilde{I}} \Bigl(\sum_{I+J>I} t^J r_{I,J}^2 \Bigr) = \sum_{J \in T_0} t^J \Bigl(\sum_{\substack{I,\, \tilde{K} \subseteq \tilde{I} \\ I+J>I}} r_{I,J}^2 \Bigr).$$

Since $t_1, t_2, \dots, t_n$ form a 2-basis of F, for fixed K, J we find

$$\sum_{\substack{I,\, \tilde{K} \subseteq \tilde{I} \\ I+J>I}} r_{I,J}^2 = 0. \tag{3}$$

We next show that $r_{I,J}^2 = 0$ for all I, J such that $I + J > I$. We proceed by reverse induction on card$(\tilde{I})$. If card$(\tilde{I}) = n$ we have $I = (1, \dots, 1)$ and $I + J > J$ is impossible, so the conclusion is vacuous. Now suppose the desired conclusion is known for all I with card$(\tilde{I}) > r$. Fix some K with card$(\tilde{K}) = r$ and some J with $J + K > K$. If $I \neq K$, and if $\tilde{K} \subset \tilde{I}$, we have card$(\tilde{I}) > r$. Our inductive hypothesis implies that $r^2_{I,J} = 0$ for these I, J, and since these are all but one summand of (3), we find that $r^2_{K,J} = 0$ as well. This completes the induction. The definitions give $a_I = 0$ and the proof of Theorem 1.3 is complete.

Since the ring $R = \bar{F}[\pi^{-1}] \subset F$ is simply a polynomial ring over the residue field $\bar{F}$ of F, the choice of the lift to F of the 2-basis $t_1, t_2, \ldots, t_{n-1}$ of $\bar{F}$ does not affect the isomorphism type of R or $\mathcal{R}$. This, together with the uniqueness results, implies:

Corollary 1.7. *Suppose that $v : F \to \mathbb{Z}$ is a complete discrete valued field of characteristic 2 with residue field $\bar{F}$ and uniformizing parameter π. Then up to isomorphism, the submodule $\mathcal{R}$ is independent of the choice of lift of the 2-basis $t_1, t_2, \ldots, t_{n-1}$ of $\mathcal{F}$. In particular, the cokernel*

$$W_1 F := \operatorname{coker}(W_q \bar{F} \to W_q F)$$

is independent of the choice of lift of this 2-basis.

Remark 1.8. Both residue forms ϕ_1 and ϕ_2 in Theorem 1.3 depend upon the choice of the uniformizing parameter π.

When F is complete and discretely valued, the group $W_1 F$ defined in Corollary 1.7 will play the role of the "second residue forms" in characteristic 2. The projection map $\partial_v : W_q F \to W_1 F$, is the analogue of the *second residue map*. It is an immediate consequence of this definition that

$$0 \longrightarrow W_q \bar{F} \longrightarrow W_q F \xrightarrow{\partial_v} W_1 F \longrightarrow 0$$

is split exact. (This definition also coincides with the second residue map away from characteristic 2, for in that case Springer's Theorem gives a group isomorphism $WF \cong W\bar{F} \oplus \langle p \rangle W\bar{F}$, so $W_1 F \cong W\bar{F}$.)

Remark 1.9. Arason [2003] has proved a result that captures all the information in Theorem 1.3. His proof uses the generator-relation structure of the Witt group. His description of $\mathcal{R}$ is different (it uses a filtration based on negative exponents of the uniformizing parameter) and his proof does not require powers of the fundamental ideal since he directly uses the generator relation structure for the Witt group.

2. The filtration of $W_q\mathcal{F}(x)$

We now denote by $\mathcal{F}$ a fixed field of characteristic 2 with 2-basis $t_1, t_2, \ldots, t_n$. We study the Witt group of the field of rational functions $F = \mathcal{F}(x)$. The results of the previous section will be applied to the completions of F at its discrete valuations, which are trivial on $\mathcal{F}$. Following Milnor's original approach away from characteristic 2, we also filter the Witt group $W_q F$ by degree. We denote by $\mathcal{F}[x]_{\le d}$ the set of polynomials in $\mathcal{F}[x]$ of degree at most d, and by $\mathcal{F}[x]_{<d}$ those of degree less than d.

Definition 2.1. For $d \geq 1$, let L_d be the subgroup of $W_q\mathcal{F}(x)$ generated by all forms $\langle f\rangle[1, h/u^e]$, where $f, u \in \mathcal{F}[x]_{\leq d}$ and $h \in \mathcal{F}[x]$. When $d = 0$, let L_0 be the subgroup of $W_q\mathcal{F}(x)$ generated by the forms $[\lambda_1 x^i, \lambda_2 x^j]$, where $\lambda_1, \lambda_2 \in \mathcal{F}$ and $i, j \in \mathbb{N}$.

Lemma 2.2. (i) *For any polynomials $p, g, h \in \mathcal{F}[x]$ we have $\langle p\rangle[g, ph] \in L_0$.*

(ii) *For $d \geq 1$, L_d is generated by the forms $\langle ax^\epsilon\rangle[1, h/u^e]$, where $a \in \mathcal{F}$, $\epsilon \in \{0, 1\}$, $h \in \mathcal{F}[x]$ and u factors as a product of elements in $\mathcal{F}[x]_{\leq d}$.*

(iii) *If $f, u \in \mathcal{F}[x]_{<d}$ and $p \in \mathcal{F}[x]$ then $\langle pf\rangle[1, pg/u^e] \in L_{d-1}$.*

Proof. (i) The first statement follows from the identity $\langle a\rangle[b, c] = [ab, c/a]$. For then $\langle p\rangle[g, ph] = [pg, h]$, and using the biadditivity of the symbol $[\ ,\ ]$ this can be expressed as a sum of generators for L_0.

(ii) For the second statement, since the u's used as generators in this version are products of elements in $\mathcal{F}[x]_{\leq d}$, we can use apply partial fractions to h/u^e together with the additivity of $[\ ,\ ]$ to express $\langle ax^\epsilon\rangle[1, h/u^e]$ as a sum of generators of the type in specified in Definition 2.1. Conversely, given $\langle f\rangle[1, h/u^e]$ as in Definition 2.1, where $f, u \in \mathcal{F}[x]_{\leq d}$, and given $h \in \mathcal{F}[x]$, we write $f = \sum_{i=0}^{d-1} a_i x^i$ with $a_i \in \mathcal{F}$ and use the fundamental relation to express $\langle f\rangle[1, h/u^e]$ in the form $\sum_{i=0}^{d-1}\langle a_i x^i\rangle[1, ha_i x^i/fu^e]$. Since $\langle x^i\rangle = \langle x^{\epsilon_i}\rangle$, where $\epsilon_i \in \{0, 1\}$ and $i \equiv \epsilon_i \pmod 2$, and since $ha_i x^i/fu^e = ha_i x^i f^{e-1}/(fu)^e$, we have a generator of the desired type.

(iii) We apply the fundamental relation, expressing pf as $\sum_{i=1}^{n-1} a_i x^i$ with $a_i \in \mathcal{F}$, so $\langle pf\rangle[1, pg/u^e] = \sum_{i=1}^{n-1}\langle a_i x^i\rangle[1, p_i x^i pg/pfu^e] = \sum_{i=1}^{n-1}\langle a_i x^i\rangle[1, p_i x^i g/fu^e] = \sum_{i=1}^{n-1}\langle a_i x^i\rangle[1, p_i x^i gf^{e-1}/(fu)^e] \in L_{d-1}$ by part (ii). □

In particular, by the lemma, for any $\lambda \in \mathcal{F}$ and $h \in \mathcal{F}[x]$, both $\langle\lambda\rangle[1, h]$ and $\langle\lambda x\rangle[1, hx]$ lie in L_0. This will be used frequently.

Lemma 2.3. *Suppose that p is a monic irreducible polynomial of degree d. If $r \in F$ is a v_p-unit, and if s is v_p-integral, then $\partial_p(\langle r\rangle[1, s]) = 0$. Consequently, if* $\deg p = d$ *and $\phi \in L_{d-1}$, we have $\partial_p(\phi) = 0$.*

Proof. Since r is a v_p-unit we can write $r = r_0 + pr'$ with $0 \neq r_0 \in \overline{F}_p$, where r' is v_p-integral. Next, in F_p we can write $s(r_0/r) = s_0 + ps'$, where $s_0 \in \overline{F}_p$ and s' is v_p-integral in F_p. Since $v_p(s(pr'/r)) > 0$ and $v_p(ps') > 0$, we know that both $s(pr'/r)$ and ps' lie in $\wp(F_p)$. Computing in W_qF_p we find that

$$\begin{aligned}\langle r\rangle[1, s] &= \langle r_0 + pr'\rangle[1, s] = \langle r_0\rangle[1, s(r_0/r)] + \langle pr'\rangle[1, s(pr'/r)]\\ &= \langle r_0\rangle[1, s_0 + ps'] = \langle r_0\rangle[1, s_0].\end{aligned}$$

Since each of r_0 and s_0 lie in $\overline{F}_p$ we see $\langle r\rangle[1, s] \in \operatorname{im}(W_q\overline{F}_p \to W_q F_p)$ and $\partial_p(\langle r\rangle[1, s]) = 0$ follows.

Now consider a generator $\langle f\rangle[1, h/u^e]$ for L_{d-1}. As $f, u \in \mathscr{F}[x]_{<d}$, we know that h/u^e is v_p-integral and f is a v_p-unit, so $\partial_p(\langle f\rangle[1, h/u^e]) = 0$ and we are done in this case. Next consider a generator $[\lambda_1 x^i, \lambda_2 x^j] = \langle\lambda_1 x^i\rangle[1, \lambda_1\lambda_2 x^{i+j}]$ of L_0. If $p \neq x$ then $\lambda_1 x^i$ is a p-adic unit and $\lambda_1\lambda_2 x^{i+j}$ is v_p-integral. If $p = x$ and $i = 0$, then again λ_1 is an x-adic unit and $\lambda_1\lambda_2 x^j$ is v_p-integral. Otherwise if $p = x$ and $i > 0$ we know that $v_x(\lambda_1\lambda_2 x^{i+j}) > 0$ and so $\lambda_1\lambda_2 x^{i+j} \in \wp(F_{v_x})$, giving $[\lambda_1 x^i, \lambda_2 x^j] = 0 \in W_q F_{v_x}$. This proves the lemma. □

Definition 2.4. Whenever p is a monic irreducible polynomial we define S_p to be the subgroup of $W_q\mathscr{F}(x)$ generated by all forms $\langle r\rangle[1, h/p^s]$, where $r \in \mathscr{F}$, $h \in \mathscr{F}[x]$, and $s \geq 0$. When $p = \frac{1}{x}$ we denote by $S_{\frac{1}{x}}$ the subgroup generated by all forms generated by $\langle r\rangle[1, hx]$, where $r \in \mathscr{F}$ and $h \in \mathscr{F}[x]$.

We observe that $W_q\mathscr{F}(x) = \bigcup_{d=0}^{\infty} L_d$. This is because the usual additive generators for $W_q\mathscr{F}(x)$ are included in some L_d for large enough d. The next lemma describes several generating sets for the L_d.

Lemma 2.5. (i) $L_d = \sum_{p,\,\deg p \leq d}(S_p + \langle x\rangle S_p)$.

(ii) $W_q\mathscr{F}(x) = \sum_p (S_p + \langle x\rangle S_p)$.

(iii) *When* $d \geq 1$, L_d *is generated by* L_{d-1} *and* $S_p \cup \langle x\rangle S_p$, *where* $\deg p = d$.

Proof. Part (i) follows from Lemma 2.2(ii) and partial fractions. Part (ii) follows from (i) since $W_q\mathscr{F}(x) = \bigcup_{d=0}^{\infty} L_d$. Part (iii) follows using partial fractions and (i). □

We next establish a result from linear algebra needed to relate $\langle x\rangle S_p$ and $\langle p\rangle S_p$. For $p = x^d + p_1 x^{d-1} + \cdots + p_{d-1}x + p_d$ we express each p_i as $\sum_{K\in T} t^K p_{i,K}^2$, where $p_{i,J} \in \mathscr{F}$. For each $K \in T$ we define $P_K \in \overline{\mathscr{F}(x)_p}$ by

$$P_K := \begin{cases} \overline{t^K(p_{1,K}^2 x^{d-1} + p_{3,K}^2 x^{d-3} + \cdots + p_{d-1,K}^2 x)} & \text{when } d \text{ is even,} \\ \overline{t^K(x^d + p_{2,K}^2 x^{d-2} + \cdots + p_{d-1,K}^2 x)} & \text{when } d \text{ is odd.} \end{cases} \tag{4}$$

Next let M be the $2^n \times 2^n$-matrix with entries indexed by the group T and with (I, J)-th entry P_{I+J}. We show that M is invertible:

Lemma 2.6. *Assume* T *is an elementary abelian* 2*-group with* 2^n *elements and* P_K *are elements of a field of characteristic* 2 *indexed by* $K \in T$. *Suppose that* M *is the* $(2^n \times 2^n)$*-matrix with* (I, J)*-th entry* P_{I+J}. *If* $\sum_{K\in T} P_K \neq 0$, *then* M *is invertible.*

Proof. Let $\operatorname{Perm} T$ denote the set of permutations of T. We know that

$$\det M = \sum_{\sigma\in\operatorname{Perm} T}\left(\prod_{\tau\in T} P_{\tau+\sigma(\tau)}\right)$$

For each $s \in T$ we define $\sigma_s \in \operatorname{Perm} T$ by $\sigma_s(\tau) = \tau + s$. In this case $\tau + \sigma_s(\tau) = s$ for all τ and we have $\prod_{\tau \in T} P_{\tau+\sigma_s(\tau)} = P_s^{2^n}$.

Now let T act on $\operatorname{Perm} T$ via $\sigma^\epsilon(\tau) = \sigma(\tau+\epsilon)+\epsilon$ for all $\epsilon \in T$. (That this is an action is readily checked using the fact that T is abelian.) If $\sigma^\epsilon = \sigma$ for all $\epsilon \in T$, then $\sigma(\epsilon) = \sigma^\epsilon(\epsilon) = \sigma(\epsilon+\epsilon)+\epsilon = \sigma(0)+\epsilon$ for all ϵ, and we see that $\sigma = \sigma_{\sigma(0)}$ in this case. In particular, if $\sigma \neq \sigma_s$ for some $s \in T$ then the orbit of σ under T has more than one element. We next note that for any $\sigma \in \operatorname{Perm} T$ and $\epsilon \in T$ we have

$$\prod_{\tau \in T} P_{\tau+\sigma^\epsilon(\tau)} = \prod_{\tau \in T} P_{\tau+\sigma(\tau+\epsilon)+\epsilon} = \prod_{(\tau+\epsilon) \in T} P_{(\tau+\epsilon)+\sigma(\tau+\epsilon)} = \prod_{\tau \in T} P_{\tau+\sigma(\tau)},$$

and consequently for any σ different from the σ_s we have

$$\sum_{\theta \in \operatorname{Orbit}(\sigma)} \left(\prod_{\tau \in T} P_{\tau+\theta(\tau)} \right) = \operatorname{card}(\operatorname{Orbit}(\sigma)) \prod_{\tau \in T} P_{\tau+\sigma^\epsilon(\tau)} = 0,$$

since card(Orbit(σ)) is a proper power of 2. Decomposing the sum in the determinant over the orbits in $\operatorname{Perm} T$ shows that

$$\det M = \sum_{K \in T} P_K^{2^n} = \left(\sum_{K \in T} P_K \right)^{2^n}.$$

If M fails to be invertible, we have $\det M = 0$, which implies $\sum_{K \in T} P_K = 0$, contrary to our hypothesis. The lemma is proved. □

Corollary 2.7. *If p is irreducible and separable and if the P_K are defined as in* (4), *then M is invertible as a matrix over $\mathcal{F}(p) := \mathcal{F}[x]/(p)$.*

Proof. Suppose that $\det M = 0$. Then $\sum_{K \in T} P_K = 0$. Each P_K lies in $\overline{t^K x \mathcal{F}(x)_p^2}$, and since the $t_1, t_2, \ldots, t_n$ remain 2-independent in $\overline{\mathcal{F}(x)_p}$ (because p is separable), we see that each $P_K = 0$. Now, since $1, x, x^2, \ldots x^{d-1}$ are linearly independent over $\mathcal{F}$, we find for all K that each $p_{i,K}$ vanishes, where i is odd when d is even and even when d is odd. So the same follows for the p_i. The first case contradicts the separability of p and the second case contradicts the irreducibility of p. □

We are now able to apply Corollary 2.7 and relate S_p, $\langle x \rangle S_p$ and $\langle p \rangle S_p$. Whenever p is not separable we choose i so that $t_i \in \mathcal{F}(p)^2(t_1, \ldots, t_{i-1})$ and then we denote by $\tilde{S}_p$ the subgroup of $S_p + \langle x \rangle S_p$ generated by the elements $\langle t^I \rangle [1, h/p^e]$, where t^I is a product of $t_1, t_2, \ldots t_{j-1}, t_{j+1}, \ldots, t_n, x$.

Proposition 2.8. (i) *For all p, we have $S_p + \langle p \rangle S_p \subseteq S_p + \langle x \rangle S_p$.*

(ii) *If p is separable, $S_p + L_0 + \langle x \rangle S_p = S_p + L_0 + \langle p \rangle S_p$.*

(iii) *If p is not separable, $S_p + L_0 + \langle x \rangle S_p = \tilde{S}_p + L_0 + \langle p \rangle \tilde{S}_p$.*

Proof. Part (i) essentially follows from Lemma 2.2(ii), but we give a direct calculation here because it is necessary for part (ii). Consider $\langle p\rangle[1, h/p^s] \in \langle p\rangle S_p$ and apply the fundamental relation in W_q to obtain

$$\langle p\rangle\Big[1, \frac{h}{p^s}\Big] = \sum_{j=0}^{d} \langle p_j x^{d-j}\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big].$$

Whenever $d-j$ is even, we have

$$\langle p_j x^{d-j}\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big] = \langle p_j\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big] \in S_p,$$

from which we find, modulo S_p,

$$\begin{aligned}\langle p\rangle\Big[1, \frac{h}{p^s}\Big] &\equiv \sum_{d-j \text{ odd}} \langle p_j x^{d-j}\rangle\Big[1, \frac{p_j x^{d-j} h}{p^{s+1}}\Big] \equiv \sum_{d-j \text{ odd}} \Big(\sum_{K\in T} \langle t^K x\rangle\Big[1, \frac{t^K p_{j,K}^2 x^{d-j} h}{p^{s+1}}\Big]\Big)\\ &\equiv \sum_{K\in T} \langle t^K x\rangle\Big[1, \frac{t^K\big(\sum_{d-j \text{ odd}} p_{j,K}^2 x^{d-j}\big)h}{p^{s+1}}\Big] \equiv \sum_{K\in T}\langle t^K x\rangle\Big[1, \frac{P_K h}{p^{s+1}}\Big],\end{aligned}$$

where the second equivalence uses $p_j = \sum_{K\in T} t^K p_{j,K}^2$, the third changes order of summation, and the fourth uses the definition of the P_K. Part (i) follows from this.

For (ii) we consider the problem of reversing this process when p is separable. Namely, we must express an element $\langle x\rangle[1, g/p^{s+1}] \in W_q\mathscr{F}(x)$ as a sum, modulo $S_p + L_0$, of elements of the form $\sum_J \langle t^J p\rangle[1, h_J/p^s]$. For this we denote by $S_{p,s}$ the subgroup of S_p generated by the generators of S_p, where the exponent of p doesn't exceed s. We can then proceed by backwards induction on s and calculate in $\overline{F(x)_{v_p}}$. Multiplying the equivalence in the previous paragraph by $\langle t^J\rangle$ gives a system of 2^n such equivalences, one for each $J \in T$:

$$\langle t^J p\rangle\Big[1, \frac{h_J}{p^s}\Big] \equiv \sum_{K\in T} \langle t^{K+J} x\rangle\Big[1, \frac{P_K h_J}{p^{s+1}}\Big] \pmod{S_p}.$$

Taking the sum gives, again modulo S_p,

$$\sum_J \langle t^J p\rangle\Big[1, \frac{h_J}{p^s}\Big] \equiv \sum_J \sum_{K\in T} \langle t^{K+J} x\rangle\Big[1, \frac{P_K h_J}{p^{s+1}}\Big] \equiv \sum_L \langle t^L x\rangle\Big[1, \frac{\sum_J P_{L+J} h_J}{p^{s+1}}\Big],$$

where in the second sum the variable L is introduced to collect terms with like $K+J$. Since we want the latter sum to equal $\langle x\rangle[1, g/p^{s+1}]$, we obtain for the h_J the equations

$$\sum_J P_J h_J = g \quad \text{and} \quad \sum_J P_{L+J} h_J = 0 \ \text{ when } \ L \neq O$$

in $\overline{\mathcal{F}(x)_p}$. In matrix form this system is

$$M \cdot (h_J) = \begin{pmatrix} g & 0 & \cdots & 0 \end{pmatrix}^T,$$

where (h_J) means the column with 2^n entries aligned with corresponding entries of M, whose (I, J)-th entry is P_{I+J}. Since M is invertible over $\overline{\mathcal{F}(x)_p}$ by Corollary 2.7, we can find elements $h_J \in \mathcal{F}[x]_{<\deg p}$ such that

$$\langle x\rangle\Big[1, \frac{g}{p^{s+1}}\Big] \equiv \sum_J \langle t^J p\rangle\Big[1, \frac{h_J}{p^s}\Big] \quad (\text{mod } \; S_p + \langle x\rangle S_{p,s}).$$

By backwards induction on s we can reduce to $s = 0$. When $s = 0$, the error terms are sums $\sum_J \langle t^J p\rangle[1, h_J p]$ lying in L_0 by Lemma 2.2(i). The result in (ii) follows.

For part (iii) we write $p = \sum_j p_j x^{d-j}$ and note that since p is not separable, each $d-j$ is even, and we have a 2-dependence between $t_1, t_2, \ldots t_{n-1}, t_n, p$. Reordering $t_1, t_2, \ldots, t_n$ we can assume that $t_n \in \mathcal{F}(x)^2(t_1, t_2, \ldots, t_{n-1}, p)$. This relabeling guarantees that $t_1, t_2, \ldots t_{n-1}, x, p$ is a basis for $\mathcal{F}(x)$ as well as $\mathcal{F}(x)_p$. We express each p_j as $p_{0,j} + p_{1,j}t_n$, where $p_{i,j} \in \mathcal{F}(x)^2(t_1, t_2, \ldots, t_{n-1})$, and further express each $p_{1,j}$ as $\sum_J t^J p_{1,j,J}^2$; here each J_n vanishes. Then we can form

$$\tilde{P}_K = t^K (p_{1,0,K}^2 x^d + p_{1,2,K}^2 x^{d-2} + \cdots + p_{1,d,K}^2)$$

and note that $\sum_K \tilde{P}_K = \partial p / \partial t_n \neq 0 \in \mathcal{F}^2(t_1, t_2, \ldots, t_{n-1})[x^2]$.

We can write

$$\langle p\rangle\Big[1, \frac{h}{p^e}\Big] = \sum_i \langle p_i\rangle\Big[1, \frac{p_i x^{d-i} h}{p^{e+1}}\Big] = \sum_i \langle p_{0,i} + p_{1,i}t_n\rangle\Big[1, \frac{p_i x^{d-i} h}{p^{e+1}}\Big],$$

and so, modulo $\tilde{S}_p$,

$$\begin{aligned} \langle p\rangle\Big[1, \frac{h}{p^e}\Big] &\equiv \sum_i \langle p_{1,i}t_n\rangle\Big[1, \frac{p_{1,i}t_n x^{d-i} h}{p^{e+1}}\Big] \\ &\equiv \sum_{i,K} \langle t^K t_n\rangle\Big[1, \frac{t^K p_{1,i,K}^2 t_n x^{d-i} h}{p^{e+1}}\Big] \equiv \sum_K \langle t^K t_n\rangle\Big[1, \frac{\tilde{P}_K t_n h}{p^{e+1}}\Big]. \end{aligned}$$

What we must do is reverse this process and solve, modulo $\tilde{S}_p + L_0$, the congruence

$$\begin{aligned} \langle t_n\rangle\Big[1, \frac{g}{p^{e+1}}\Big] &\equiv \sum_L \langle t^L p\rangle\Big[1, \frac{h_L}{p^e}\Big] \equiv \sum_{K,L} \langle t^{L+K} t_n\rangle\Big[1, \frac{\tilde{P}_K t_n h_L}{p^{e+1}}\Big] \\ &\equiv \sum_J \langle t^J t_n\rangle\Big[1, \frac{\sum_L \tilde{P}_{J+L} t_n h_L}{p^{e+1}}\Big], \end{aligned}$$

for appropriate polynomials h_L. As in part (ii) it suffices to reduce the exponent e by 1. This system is equivalent to solving the system in 2^{n-1} variables in $\mathscr{F}(p)$, $g = \sum_L \tilde{P}_L t_n h_L$ and $0 = \sum_L \tilde{P}_{K+L} t_n h_L$, where $K \neq O$. This can be written in matrix form as

$$\tilde{M} \cdot (h_L) = \begin{pmatrix} g & 0 & \cdots & 0 \end{pmatrix}^T,$$

for the $h_L \in \mathscr{F}(p)$. Here $\tilde{M}$ is the matrix with (K, L)-th entry $\tilde{P}_{K+L} t_n \in \mathscr{F}(p)$. However, we have noted that $\sum_K \tilde{P}_K \neq 0$, so the invertibility of $\tilde{M}$ follows from Lemma 2.6. This gives what is needed. □

3. The maps ∂_p and their splitting

From now on, unless stated otherwise, p denotes either a monic irreducible polynomial in $\mathscr{F}[x]$ or $\frac{1}{x}$. Then v_p denotes the associated valuation and we continue to use F_p to denote the completion of $F = \mathscr{F}(x)$ at v_p. We use F_p to denote the completion of $F = \mathscr{F}(x)$ at v_p. We continue to assume that $\mathscr{F}$ has a finite 2-basis $t_1, t_2, \ldots, t_n$. To apply the results from Section 1 we will need to specify a 2-basis for $\mathscr{F}(p) := \mathscr{F}[x]/(p) = \overline{F}_p$. So in this section we will have to be careful and keep track of separability conditions. We recall a well-known result, whose proof is embedded in the subsequent discussion, where we set up notation.

Lemma 3.1. *A 2-basis for $\mathscr{F}$ is a 2-basis for $\mathscr{F}(p)$ if and only if p is separable.*

Since $t_1, t_2, \ldots, t_n$ is the fixed 2-basis for $\mathscr{F}$, $t_1, t_2, \ldots, t_n, x$ is a 2-basis for $F = \mathscr{F}(x)$. We express the monic irreducible $p \in \mathscr{F}[x]$ as

$$p = \sum_{I \in T} t^I (p_I(x))^2,$$

where $p_I(x) \in \mathscr{F}[x]$ and the multiindices t^I refer to the 2-basis for F (which includes x). Since $\overline{F}_p \cong \mathscr{F}[x]/(p)$, we find that $t_1, t_2, \ldots, t_n$ remain 2-independent in $\overline{F}_p$ if and only if for some I with $I_{n+1} \neq 0$ we also have $p_I(x) \neq 0$. But this happens if and only if $p(x)$ has a nonzero summand of odd degree, i.e., if and only if p is separable. Now, when p is separable, if $I_{n+1} \neq 0$ for some $p_I(x) \neq 0$ we take $t_1, t_2, \ldots, t_n$ as our 2-basis for $\overline{F}_p \subset F_p$ and then we can use $t_1, t_2, \ldots, t_n, p$ as our 2-basis for F_p.

Otherwise, when p is not separable, we choose j maximal with $I_j \neq 0$ for some $p_I(x) \neq 0$ and we note that in this case $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n, \bar{x}$ is a 2-basis for $\overline{F}_p$. We then take $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n, \bar{x}, p$ as our 2-basis for F_p. In this case we use the lifting to $t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n-1}, x \in F$ to define the embedding $\overline{F}_p \hookrightarrow F_p$ needed to define $\partial_p : W_q F \to W_1 F_p$.

We will need to keep track of products of elements of these various 2-bases. This will be accomplished by using three different notations for multiindex sets:

• We use $\{t^I \mid I \in T\}$ to denote products of elements of the original 2-basis for $\mathscr{F}$.

• We use $\{t^I \mid I \in T_p\}$ to denote products of elements of the residue 2-basis of $\mathscr{F}(p)$ as described above (which vary depending upon whether p is separable).

• We use $\{t^I \mid I \in \tilde{T}_p\}$ to denote products of elements of the local 2-basis of F_p as described above.

In our study of $W_q F$ we will need to understand how $W_q F$ elements map to elements of $W_1 F_p$, so we need to represent $W_1 F_p$ classes in a special way. The next result is a consequence of Theorem 1.3. Whenever $s \in \mathscr{F}[x]$ we denote by $\bar{s} \in \mathscr{F}[x]_{<\deg p}$ the unique polynomial with $s \equiv \bar{s} \pmod{p}$. When applying Theorem 1.3, the Teichmüller lifting used is the one associated with the 2-basis $\tilde{T}_p$.

Theorem 3.2. (i) *If p is monic and irreducible, every class ϕ in $W_1 F_p$ can be expressed uniquely as*

$$\phi = \psi \perp \langle p \rangle \phi_2,$$

where

$$\psi = \sum_{I \in \tilde{T}_p} \langle t^I \rangle \Big[1, \sum_{J, J+I>I} t^J u_{I,J} \Big],$$

$u_{I,J} = \sum_{r \geq 1} \overline{s^2_{I,J,r}}/p^{2r}$ *with* $s_{I,J,r} \in \mathscr{F}[x]_{<\deg p}$ *and* $\phi_2 \in W_q \overline{v} F_p$, *and where* $s_{I,J,r}$ *and the Witt classes of ψ in $W_q F_p$ and ϕ_2 in $W_q \overline{v} F_p$ are uniquely determined by the class of ϕ.*

(ii) *Every class ϕ in $W_1 F_{\frac{1}{x}}$ can be expressed as*

$$\phi = \psi \perp \Big\langle \frac{1}{x} \Big\rangle \phi_2,$$

where $\psi = \sum_{I \in \tilde{T}_{\frac{1}{x}}} \langle t^I \rangle \big[1, \sum_{J,J+I>I} t^J u^2_{I,J} \big]$ *with* $u_{I,J} \in x \cdot \mathscr{F}[x]$ *and* $\phi_2 \in W_q \mathscr{F}$, *and where the Witt classes of ψ in WF_p and ϕ_2 in $W_q \mathscr{F}$ are uniquely determined by the class of ϕ. In this expression we note that* $\sum_{J,J+I>I} t^J u^2_{I,J} \in x \cdot \mathscr{F}[x]$.

Proof. In (i), since $\overline{F_p} = \mathscr{F}(p)$, applying valuation theory we conclude that every element $f \in \overline{F_p} \subset F_p$ can be expressed as $f_1 + f_2$, where $f_1 \in \mathscr{F}[x]_{<\deg p}$, $f_2 \in F_p$, and $v_p(f_2) > 0$. Consequently, any element $r \in R = \overline{\mathscr{F}(x)_p}[p^{-1}]$ can be expressed in the form $r = s_1 + s_2$, where $s_1 \in \mathscr{F}[x]^2_{<\deg p}[p^{-1}]$ and $v_p(s_2) > 0$. Part (i) now follows, interpreting Theorem 1.3 in this setting and making the appropriate substitutions. Part (ii) is a direct consequence of Theorem 1.3, since there is no ambiguity about viewing the residue field as a subfield of $F_{\frac{1}{x}}$. □

We next digress slightly and give a generator-relation structure for the Witt group. A very similar characterization was found by Arason [2003].

Theorem 3.3. *For any field F of characteristic 2 the Witt group $W_q F$ is isomorphic to $(F^+ \otimes F^+)/\mathcal{W}$, where $\mathcal{W}$ is the subgroup generated by*

(i) *$a \otimes b$ for $ab \in \wp(F)$,*

(ii) *$a \otimes b + b \otimes a$ for all $a, b \in F$, and*

(iii) *$a \otimes b + c \otimes ab/c$ whenever $c \in D_F[a, b]$.*

Proof. We show that the epimorphism $F^+ \otimes F^+ \to W_q F$ defined by $a \otimes b \mapsto [a, b]$ has kernel $\mathcal{W}$. First we note that the generators of $\mathcal{W}$ map to trivial elements of $W_q F$. For type (i) generators, the Arf invariant of $ax^2 + xy + by^2$ is ab, and therefore $[a, b] = 0 \in W_q F$ if and only if $ab \in \wp(F)$. Type (ii) generators vanish by the symmetry of $[a, b]$. For type (iii) generators, since $c \in D_F[a, b]$, we know the form $[a, b] \perp \langle c\rangle[1, ab] = \langle a, c\rangle[1, ab]$ is isotropic, hence hyperbolic. So $[a, b] = \langle c\rangle[1, ab] = [c, ab/c]$, which is what we want.

We next note that whenever $[a, b] = [c, d] \in W_q F$, and since $c \in D_F[a, b]$, we find $[a, b] = [c, ab/c] = [c, d]$, the later equality being equivalent to $[c, ab/c+d] = 0$. By our observations about (i) this is equivalent to $ab + cd \in \wp(F)$. This shows that all equalities $[a, b] = [c, d] \in W_q F$ are a consequence of multilinearity and the relations (i), (ii), and (iii).

Next, given an isomorphism $q = [a_1, b_1] \perp \psi \cong [c, d] \perp \chi$, we must show that it follows from the relations defining $\mathcal{W}$. For this we view each representation q as having the same underlying vector space $V = F^{2n}$ and we let $v, w \in V$ denote the first two symplectic basis elements in the second expression. Hence $q(v) = c$, $q(w) = d$ and the inner product $(v, w)_q$ equals 1. We suppose $\psi = [a_2, b_2] \perp \cdots \perp [a_n, b_n]$. If we view $v \in V$ according to the decomposition given by the first form, we can express c as a sum $c_1 + c_2 + \cdots + c_n$, where $c_i \in D_F[a_i, b_i]$. Applying (iii) at each summand, we can write $[a_1, b_1] \perp [a_2, b_2] \perp \cdots \perp [a_n, b_n] \cong [c_1, e_1] \perp [c_2, e_2] \perp \cdots \perp [c_n, e_n]$ for $e_i = a_i b_i / c_i \in F$. Using the bilinearity of the symbol $[\ ,\ ]$, and since $c = c_1 + c_2 + \cdots + c_n$, we have $[c_1, e_1] \perp [c_2, e_2] \perp \cdots \perp [c_n, e_n] \cong [c, e_1] \perp [c_2, e_1 + e_2] \perp \cdots \perp [c_n, e_1 + e_n]$. If $v_1 = v, w_1, v_2, w_2, \ldots, v_n, w_n$ is the symplectic basis corresponding to this new decomposition $q = [c, e_1] \perp [c_2, e_1 + e_2] \perp \cdots \perp [c_n, e_1 + e_n]$, we can express w as $z_1 + z_2 + \cdots + z_n$ with each z_i a linear combination of v_i and w_i. This means that if $d_i = q(z_i)$ for $1 \le i \le n$, then $d = q(w) = d_1 + d_2 + \cdots + d_n$. Since $(v, w)_q = 1$ while $(v, z_i)_q = 0$ for $2 \le i \le n$, we see that $(v, z_1)_q = 1$. Since $\operatorname{span}(v, w_1) = \operatorname{span}(v, z_1)$, restricting our attention to this subspace we see that in fact $[c, e_1] \cong [c, d_1]$. We now apply relation (iii) to the other summands to obtain $[c_i, e_1 + e_i] \cong [c_i', d_i]$ for $c_i' = c_i(e_1 + e_i)/d_i \in F$. Using bilinearity again we find $[c, e_1] \perp [c_2, e_1 + e_2] \perp \cdots \perp [c_n, e_1 + e_n] \cong [c, d_1] \perp [c_2', d_2] \perp \cdots \perp [c_n', d_n] \cong [c, d] \perp [c + c_2', d_2] \perp \cdots \perp [c + c_n', d_n]$. Altogether, using only bilinearity and the rules (i), (ii), (iii), we have shown that our original $[a_1, b_1] \perp \psi$ is Witt equivalent to $[c, d] \perp \psi'$ for some ψ'. By Witt cancellation we now have $\psi' \cong \chi$. By induction on n we can reduce to the case $[a,b] \cong [c,d]$ already considered. This proves the theorem. $\square$

We now define the Milnor splittings.

Lemma and Definition 3.4. *Suppose p is a monic irreducible polynomial of degree $d \geq 1$ and that $\phi \in W_1 F_p$ is of the form*

$$\phi = \psi \perp \langle p \rangle \phi_2,$$

where

$$\psi = \sum_{I \in T_p} \langle t^I \rangle \Big[1, \sum_{J, J+I > I} t^J r_{I,J}^2 \Big]$$

with $r_{I,J} \in p^{-1}\mathscr{F}[x]_{< \deg p}[p^{-1}]$ and $\phi_2 \in W_q \bar{v} F_p$. Here the t^I depend upon the 2-basis for $\mathscr{F}(x)_p$, which has last element p and will include x in the case where p is not separable. We further write $\phi_2 = \sum_i [r_i(\bar{x}), s_i(\bar{x})]$, where $\bar{x}$ denotes x modulo $p(x)$ and the $r_i(x), s_i(x)$ lie in $\mathscr{F}[x]_{<\deg p}$. Then the map τ_p defined by

$$\tau_p(\phi) = \psi \perp \langle p \rangle \Big(\sum_i [r_i(x), s_i(x)] \Big) \pmod{L_{d-1}} \in L_d / L_{d-1}$$

is a well defined homomorphism $\tau_p : W_1 F_p \to L_d / L_{d-1}$.

Proof. According to Theorem 3.2(i) every class ϕ in $W_1 F_p$ can be expressed as stated, and the Witt classes of $\psi \in W_q F_p$ and $\phi_2 \in W_q \overline{F_p}$ are uniquely determined. Further, the Witt class of ψ uniquely determines the $r_{I,J}$ as elements of $p^{-1}\mathscr{F}[x]_{< \deg p}[p^{-1}]$.

The expression of ϕ_2 as $\sum_i [r_i(\bar{x}), s_i(\bar{x})]$ need not be unique so we suppose also that $\phi_2 = \sum_j [u_j(\bar{x}), v_j(\bar{x})] \in W_q \mathscr{F}(p)$, where each $u_j(x)$ and $v_j(x)$ lies in $\mathscr{F}[x]_{<\deg p}$. By Theorem 3.3, using the biadditivity of the symbol [,] we have the expansion

$$\sum_i [r_i(\bar{x}), s_i(\bar{x})] + \sum_j [u_j(\bar{x}), v_j(\bar{x})] = \sum_k [a_k(\bar{x}), b_k(\bar{x})],$$

where the latter sum is a sum of relations of the form given in Theorem 3.3(i), (ii) or (iii). Since we only used the biadditivity of [,] in the expansion, we know that each $a_k(x), b_k(x) \in \mathscr{F}[x]_{<\deg p}$ and we also have $\sum_i [r_i(x), s_i(x)] + \sum_j [u_j(x), v_j(x)] = \sum_k [a_k(x), b_k(x)] \in W_q \mathscr{F}(x)$. Checking for each of the types of relations given in Theorem 3.3 we will show that $\langle p \rangle$ times this sum lies in L_{d-1}.

Suppose first we have a summand $[a,b]$ with $a, b \in \mathscr{F}[x]_{<\deg p}$, where $\bar{a}\bar{b} \in \wp(\mathscr{F}(p))$. Then we can write $ab = \wp(z) + pg$ in $\mathscr{F}[x]$ and we find that $[a,b] = \langle a \rangle [1, ab] = \langle a \rangle [1, \wp(z) + pg] = \langle a \rangle [1, pg]$ in $W_q F$. By Lemma 2.2(iii) the form $\langle p \rangle \langle a \rangle [1, pg]$ lies in L_{d-1} since $a \in \mathscr{F}[x]_{<\deg p}$. Next, any pair in the sum of the form $[a,b] + [b,a]$ is zero in $W_q F$ as well. Finally, suppose we have a pair in the sum $[a, b] + [c, d]$, where $\bar{c} \in D_{\mathscr{F}(p)}[\bar{a}, \bar{b}]$ and $\bar{d} = \overline{ab/c}$ where $a, b, c, d \in \mathscr{F}[x]_{<\deg p}$. Then we can write $c = ar^2 + rs + bs^2 + pg$ and $ab = cd + ph$ in $\mathscr{F}[x]$ where $r, s \in \mathscr{F}[x]_{<\deg p}$ also. Since $(s/r + bs^2/r^2)b \in \wp(F)$, we obtain $[s/r + bs^2/r^2, b] = 0$. Rewriting the expression for c, we find $a = c/r^2 + s/r + bs^2/r^2 + pg/r^2$, so in $W_q F$ we have

$$\begin{aligned}[a, b] + [c, d] &= [c/r^2, b] + [s/r + bs^2/r^2, b] + [pg/r^2, b] + [c, d]\\ &= \langle c\rangle[1/r^2, bc] + [pg/r^2, b] + \langle c\rangle[1, cd]\\ &= \langle c\rangle[1, bc/r^2] + \langle c\rangle[1, cd] + [pg/r^2, b].\end{aligned}$$

Next, substituting $bc/r^2 = ab + bs/r + b^2s^2/r^2 + bpg/r^2$ and $cd = ab + ph$ we find

$$\begin{aligned}[a,b]+[c,d] &= \langle c\rangle[1, ab+bs/r+b^2s^2/r^2+pgb/r^2]+\langle c\rangle[1, ab+ph]+[pg/r^2, b]\\ &= \langle c\rangle[1, pgb/r^2+ph]+[pg/r^2, b],\end{aligned}$$

since $bs/r+b^2s^2/r^2 \in \wp(F)$. Applying Lemma 2.2(iii) to each of these latter forms we find that $\langle p\rangle([a, b] + [c, d])$ lies in L_{d-1}. It follows that τ_p is well defined. It is clear from the defining formula that τ_p is additive in ψ, and the proof that the lift of ϕ_2 is well defined shows that τ_p is additive in that term as well. Hence τ_p is a homomorphism. □

Our goal is to prove the surjectivity of the Milnor splitting. This requires the information provided in Proposition 2.8 and is given next.

Theorem 3.5. *The map $\bigoplus \tau_p : \bigoplus_{p,\, \deg p=d} W_1 F(x)_{v_p} \to L_d/L_{d-1}$ is an isomorphism for $d \geq 1$.*

Proof. It suffices to show the map is surjective, since for any p the composite $W_1 F(x)_{v_p} \to L_d/L_{d-1} \to W_1 F(x)_{v_p}$ is the identity. (Here, the first map is τ_p and the second map is ∂_p, which vanishes on L_{d-1} by Lemma 2.3.) By the definition of τ_p, every element $\psi = \sum_{I\in T_p} \langle t^I\rangle\big[1, \sum_{J, J+I>I} t^J r_{I,J}^2\big]$ with $r_{I,J} \in p^{-1}\mathscr{F}[x]_{<\deg p}[p^{-1}]$ lies in the image. When p is separable, these elements generate $S_p + \langle p\rangle S_p$ so we have $S_p + \langle p\rangle S_p \subseteq \operatorname{im}(\tau_p)$, and when p is not separable, these elements generate $\tilde{S}_p + \langle p\rangle \tilde{S}_p$ and we have $\tilde{S}_p + \langle p\rangle \tilde{S}_p \subseteq im(\tau_p)$. Further, if p is separable, then $S_p + \langle p\rangle S_p + L_0 = S_p + \langle x\rangle S_p + L_0$ by Proposition 2.8(ii) and in case p is not separable we have $\tilde{S}_p + \langle p\rangle \tilde{S}_p + L_0 = S_p + \langle x\rangle S_p + L_0$ by Proposition 2.8(iii). However, L_d is generated by $S_p \cup \langle x\rangle S_p$ for p with $\deg p = d$ together with L_{d-1} by Lemma 2.5. From this the theorem is proved. □

The definition of L_0 combined with Theorem 3.2(ii) gives:

Theorem 3.6. *There is an exact sequence*

$$0 \to W_q\mathscr{F} \to L_0 \to W_1 F_{\frac{1}{x}} \to W_q\mathscr{F} \to 0,$$

where the first two maps are induced by inclusion and the last map is $\phi \mapsto \phi_2$, where ϕ_2 is as given in Theorem 3.2(ii).

Proof. Since L_0 is generated by the forms $[\lambda_1 x^i, \lambda_2 x^j] = \langle \lambda_1 x^i \rangle [1, \lambda_1\lambda_2 x^{i+j}]$, applying the relations in $W_q F$ we see that every element $\phi \in L_0$ can be expressed as $\phi = \phi_1 \perp \psi$, where $\phi_1 \in W_q\mathcal{F}$ and $\psi = \sum_{I+J>I} \langle t^I \rangle [1, t^J r_{I,J}^2]$ with $r_{I,J} \in x \cdot \mathcal{F}[x]$. Moreover, the expression of ψ as such a sum is unique, according to the local theory at the $v_{\frac{1}{x}}$-adic valuation as given in Theorem 3.2(ii). This means that the natural map from L_0 to $W_1 F_{\frac{1}{x}}$ has kernel $W_q\mathcal{F}$ and cokernel the elements in $\langle \frac{1}{x} \rangle W_q\mathcal{F}$. The result follows. $\square$

4. The transfer maps s_p^*

We continue to use the 2-bases for $\mathcal{F}(p)$ and $\mathcal{F}(x)_p$ defined in the discussion following Lemma 3.1, as well as the notation T, T_p and $\tilde{T}_p$. When p fails to be separable and $I \in T_p$, we denote by I_x the entry corresponding to the exponent of x (so x occurs in t^I if and only if $I_x = 1$.) In the next definition, we define subgroups $S_{p,r}$ of $S_p + \langle p \rangle S_p$, for each $r \geq 1$. We include $p = \frac{1}{x}$ in our list. We set $d = \deg p$ and $d = 1$ when $p = \frac{1}{x}$. As in the last section, when considering elements of S_p of the form $\langle t^I \rangle [1, h/p^r]$ for $h \in \mathcal{F}[x]$, we write $\bar{h} \in \mathcal{F}[x]_{<\deg p}$ for the unique element with $h \equiv \bar{h} \pmod{p}$.

Definition 4.1. (i) Suppose p is separable or is $\frac{1}{x}$. We define $S_{p,r} \subset S_p + \langle p \rangle S_p$ as the subgroup generated by elements of two types: those of the form

$$\langle t^I \rangle [1, \overline{t^J s_{I,J}^2}/p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, and where $I + J > I$ whenever r is even, and those of the form

$$\langle t^I p \rangle [1, \overline{t^J s_{I,J}^2}/p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, r is even, and where $I + J > I$.

(ii) Suppose p is not separable. We define $S_{p,r} \subset \tilde{S}_p + \langle p \rangle \tilde{S}_p$ as the subgroup generated by elements of two types: those of the form

$$\langle t^I \rangle [1, \overline{t^J s_{I,J}^2}/p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, and where $I + J > I$ whenever r is even, and those of the form

$$\langle t^I p \rangle [1, \overline{t^J s_{I,J}^2}/p^r],$$

where $I, J \in T = T_p$, $s_{I,J} \in \mathcal{F}[x]_{<d}$, r is even, and where $I + J > I$.

(iii) Suppose p is not separable. We define $S^0_{p,r} \subset S_{p,r}$ as the subgroup generated by elements of two types: those of the form

$$\langle t^I \rangle [1, t^J \overline{s^2_{I,J}/p^r}],$$

where $I_x = 0$, $I, J \in T_p$, $s_{I,J} \in \mathscr{F}[x]_{<d}$, and where $I + J > I$ whenever r is even, and those of the form

$$\langle t^I p \rangle [1, t^J \overline{s^2_{I,J}/p^r}],$$

where $I_x = 0$, $I, J \in T_p$, r is even, $s_{I,J} \in \mathscr{F}[x]_{<d}$, and where $I + J > I$.

(iv) We define $U_p := \sum_{r \geq 1} S_{p,r}$ for all p and $U^0_p := \sum_{r \geq 1} S^0_{p,r}$ when p is not separable.

The definitions in (i) and (ii) are formally the same, except that the T_p's differ according to whether p is separable or not, which also accounts for using S_p or $\tilde{S}_p$. In part (iii) the listed elements are a subcollection of those listed in (ii), and are precisely those without an x in the t^I. The reason for the restrictions on I, J, r in the definition will become clear in the proof of the next lemma, where we apply Theorem 3.2.

Lemma 4.2. (i) *For each* p, $S_p + \langle p \rangle S_p \subseteq U_p + L_0 + \langle p \rangle L_0$.

(ii) *The group*

$$U_p = \bigoplus_{r \geq 1} S_{p,r} \subset S_p + \langle p \rangle S_p$$

is a direct sum.

(iii) *Let the image of this group in* $W_1 F_p$ *also be denoted by* U_p. *Then*

$$W_1 F_p / U_p \cong \langle p \rangle \cdot W_q \mathscr{F}(p).$$

Thus every element in $W_1 F_p / U_p$ *can be represented by an element of* $\langle p \rangle L_0$.

Proof. (i) This follows from the additive property of the symbol $[1, a]$, expanding elements of S_p into sums of $S_{p,r}$ elements one power of p at a time, leaving an element of the form $[1, g]$ where $g \in \mathscr{F}[x]$. The last summands lie in L_0.

(ii) The summands from ψ in Theorem 3.2 can be uniquely expressed as a sum of elements of the form

$$\langle t^I \rangle [1, t^J \overline{s^2_{I,J,r}/p^{2r}}],$$

where $I, J \in \tilde{T}_p$ satisfy $I + J > I$ and $s_{I,J,r} \in \mathscr{F}[x]_{<\deg p}$. Given this, we claim that the generators identified in Definition 4.1 are equivalent to those required to apply Theorem 3.2. There are four cases, depending upon whether $I, J \in T_p$ or not. In case both $I, J \in T_p$, then the condition that $I + J > I$ is the same for T_p elements as for $\tilde{T}_p$ elements, and this is recorded in Definition 4.1 in the first type where r is even. In case $I \in T_p$ but $J \notin T_p$, then $I + J > I$ is automatic and this corresponds to the first type in Definition 4.1 where r is odd. In case, $I \notin T_p$ but $J \in T_p$ then $I + J > I$ in $\tilde{T}_p$ is equivalent to $I' + J > I'$ in T_p where I' is the same as I except the p component is deleted, and this is the case of the second type of generator in Definition 4.1. Finally, if both I, $J \notin T_p$, then $I + J > I$ is impossible as elements of $\tilde{T}_p$, and this case is ignored by Definition 4.1. So we are ready to apply Theorem 3.2.

First, every sum of elements of $S_{p,r}$ can be represented as

$$\sum_{r\geq 1} \sum_{I+J>I} \langle t^I \rangle \left[1, \frac{t^J \overline{s^2_{I,J,r}}}{p^r}\right] + \sum_{r\geq 2,\ \text{even}} \sum_{I+J>I} \langle t^I p \rangle \left[1, \frac{t^J \overline{s'^2_{I,J,r}}}{p^r}\right],$$

where $s_{I,J,r}$, $s'_{I,J,r}$ lie in $\mathcal{F}[x]_{<d}$ and as elements of T_p we have $I + J > I$ in each summand. These expressions can be rewritten as

$$\sum_{I+J>I} \langle t^I \rangle \left[1, \sum_{r\geq 1} \frac{t^J \overline{s^2_{I,J,r}}}{p^r}\right] + \sum_{I+J>I} \langle t^I p \rangle \left[1, \sum_{r\geq 2,\ \text{even}} \frac{t^J \overline{s'^2_{I,J,r}}}{p^r}\right].$$

Now, in Theorem 3.2 the 2-basis $\hat{T}_p$ is used, which is in this case consists of the elements of T_p and $p \cdot T_p$. So each of the sums in the additive slots of the last expression,

$$\sum_{r\geq 1} \frac{t^J \overline{s^2_{I,J,r}}}{p^r} \quad \text{and} \quad \sum_{r\geq 2,\ \text{even}} \frac{t^J \overline{s'^2_{I,J,r}}}{p^r},$$

correspond uniquely to elements listed as $t^J s^2_{I,J,r}$ in the statement of Theorem 3.2(i). The directness of the sum is follows. The statement in (iii) is also a consequence of Theorem 3.2, proving the lemma. $\square$

We are now able to define the transfer maps s_p^*.

Definition 4.3. For $\psi \in W_1 F_p$ we define $\theta_p(\psi) \in W_q \mathcal{F}(p)$ to be the unique element of $W_q \bar{v} F_p$ for which $\psi \equiv \langle p \rangle \theta_p(\psi) \pmod{U_p}$. We denote by $t_p^* : \mathcal{F}(p) \to \mathcal{F}$ the Scharlau transfer associated to the linear functional t_p for which $t_p(x^{d-1}) = 1$ but $t_p(x^i) = 0$ when $0 \leq i < d - 1$. Finally, whenever $f \in \mathcal{F}[x]$ is a polynomial we denote by f_c its constant term.

(i) Suppose p is separable or $\frac{1}{x}$. Then we define $s_p^*(\psi) := t_p^*(\theta_p(\psi)) \in W_q \mathcal{F}$.

(ii) If p is not separable we express $\psi - \theta_p(\psi)$ modulo U_p^0 as

$$\sum_{I_x=1,r} \langle t^I \rangle \left[1, \frac{t^J \overline{s^2_{I,J,r}}}{p^r}\right] + \sum_{I_x=1,r} \langle (t^I) p \rangle \left[1, \frac{t^J \overline{s'^2_{I,J,r}}}{p^r}\right].$$

Then we define $s_p^*(\psi) \in\in W_q\mathcal{F}$ by

$$s_p^*(\psi) := t_p^*(\theta_p(\psi)) + \sum_{I_x=1,r} \langle t^I/x \rangle \left[1, \frac{(t^J s_{I,J,r}^2)_c}{p_c^r}\right] + \sum_{I_x=1,r} \langle t^I p_c/x \rangle \left[1, \frac{(t^J s'^2_{I,J,r})_c}{p_c^r}\right].$$

(Note that $t^I/x \in \mathcal{F}$ since $I_x = 1$.)

To prove reciprocity, we must check it for each $\langle p \rangle L_0$ and each U_p. This will be done in the next two sections.

5. The reciprocity law for $L_0 + \langle p \rangle L_0$

In this section we prove the reciprocity law in a critical special case. We assume $p = x^d + p_1 x^{d-1} + \cdots + p_d \in \mathcal{F}[x]$ is a monic irreducible polynomial of degree $d = 2e$ when d is even, and of degree $d = 2e+1$ when d is odd. Since we will be calculating in both $\mathcal{F}[x]$ and $\mathcal{F}[x]/(p)$ we will use x to represent the variable in $\mathcal{F}[x]$ as well as its residue in $\mathcal{F}[x]/(p)$, since no confusion will arise. For $h \in \{0, 1\}$, $\lambda_1, \lambda_2 \in \mathcal{F}$, and $k \geq 0$ we will compute the transfer $t_p^*([\lambda_1 x^h, \lambda_2 x^k]) = \phi \in W_q\mathcal{F}$ described in Definition 4.3. We note that in case $h' > 1$ we can write $h' = h + 2h_0$ for $h \in \{0, 1\}$ and as $[\lambda_1 x^{h'}, \lambda_2 x^k] = \langle \lambda_1 x^{h'} \rangle [1, \lambda_1\lambda_2 x^{k+h'}] = \langle \lambda_1 x^h \rangle [1, \lambda_1\lambda_2 x^{k+h'}] = [\lambda_1 x^h, \lambda_2 x^{k+h'-h}]$ we see there is no loss of generality in our restriction on h.

For $0 \leq j \leq e$ we define the polynomials $f_j = x^{e+j} + p_1 x^{e+j-1} + \cdots + p_{2j} x^{e-j}$ and $g_j = x^{e+j} + p_1 x^{e+j-1} + \cdots + p_{2j+1} x^{e-j-1}$. Thus $g_j = f_j + p_{2j+1} x^{e-j-1}$. We next define $\gamma_i \in \mathcal{F}$ by expressing $x^{d+i-1} = \gamma_i x^{d-1} + G_i \in \mathcal{F}[x]/(p)$ where $G_i \in \mathcal{F}[x]$ is a polynomial of degree at most $d-2$. Note that this means that $t_p(x^{d+i-1}) = \gamma_i$. Clearly, $\gamma_0 = 1$, and using the equation $x^d = p_1 x^{d-1} + \cdots + p_d \in \mathcal{F}(p) := \mathcal{F}[x]/(p)$ we see that $\gamma_1 = p_1$ and by induction for $i \geq 1$,

$$\begin{aligned} x^{d+i-1} &= p_1 x^{d+i-2} + p_2 x^{d+i-3} + \cdots + p_d x^{i-1} \\ &= (p_1\gamma_{i-1} p_2 \gamma_{i-2} + \cdots + p_i \gamma_0) x^{d-1} + p_1 G_{i-1} p_2 G_{i-2} + \cdots + p_i G_0. \end{aligned}$$

This shows that in general the γ_i satisfy the recurrence relation

$$\gamma_i = \gamma_{i-1} p_1 + \gamma_{i-2} p_2 + \cdots \gamma_0 p_i \quad \text{for } i \geq 1.$$

In fact, this recurrence relation is the same as the relation that guarantees that, as power series, $(1 + p_1 X + p_2 X^2 + \cdots)(1 + \gamma_1 X + \gamma_2 X^2 + \cdots) = 1 \in \mathcal{F}((X))$, and which we use below in proving Lemma 5.3.

Lemma 5.1. (i) *Suppose $d = 2e$. We have $t_p(f_0^2) = \gamma_1 = p_1$ and $t_p(x f_0^2) = \gamma_2 = p_1^2 + p_2$. For all j with $1 \leq j \leq e$ we have $t_p(f_j^2) = p_{2j+1}$, $t_p(x f_j^2) = p_{2j+2}$. For all $k \geq 0$ we have $t_p(x^k g_0^2) = \gamma_{k+1} + p_1^2 \gamma_{k-1}$.*

(ii) *Suppose* $d = 2e + 1$. *We have* $t_p(f_0^2) = \gamma_0 = 1$ *and* $t_p(xf_0^2) = \gamma_1 = p_1$. *For all* j *with* $1 \le j \le e$ *we have* $t_p(f_j^2) = p_{2j}$, $t_p(xf_j^2) = p_{2j+1}$. *For all* $k \ge 0$ *we have and* $t_p(x^k g_0^2) = \gamma_k + p_1^2\gamma_{k-2}$.

Proof. Since $f_0 = x^e$ we have $t(x^h f_0^2) = t(x^{h+2e})$ so when $d = 2e$ we find $t(x^h f_0^2) = \gamma_{h+1}$ and when $d = 2e + 1$ we find $t(x^h f_0^2) = \gamma_h$ as required. When $j > 0$ and $d = 2e$ we have in $\mathscr{F}(p)$

$$
\begin{aligned}
f_j^2 &= (x^{e+j} + p_1x^{e+j-1} + \cdots + p_{2j}x^{e-j})^2 \\
&= (x^{2e} + p_1x^{2e-1} + \cdots + p_{2j}x^{2e-2j})(x^{2j} + p_1x^{2j-1} + \cdots + p_{2j}) \\
&= (p_{2j+1}x^{d-2j-1} + \cdots + p_d)(x^{2j} + p_1x^{2j-1} + \cdots + p_{2j}) \\
&= p_{2j+1}x^{d-1} + (p_{2j+1}p_1 + p_{2j+2})x^{d-2} + \cdots + p_d p_{2j}.
\end{aligned}
$$

We find that $t_p(f_j^2) = t_p(p_{2j+1}x^{d-1}) = p_{2j+1}\gamma_0 = p_{2j+1}$ and that $t_p(xf_j^2) = t_p(p_{2j+1}x^d + (p_{2j+1}p_1 + p_{2j+2})x^{d-1}) = p_{2j+1}\gamma_1 + (p_{2j+1}p_1 + p_{2j+2})\gamma_0 = p_{2j+2}$ as $\gamma_0 = 1$ and $\gamma_1 = p_1$.

When $j > 0$ and $d = 2e + 1$ we have

$$
\begin{aligned}
f_j^2 &= (x^{e+j} + p_1x^{e+j-1} + \cdots + p_{2j}x^{e-j})^2 \\
&= (x^{2e+1} + p_1x^{2e} + \cdots + p_{2j}x^{2e-2j+1})(x^{2j-1} + p_1x^{2j-2} + \cdots + p_{2j-1}) \\
&\qquad + (x^{2e} + p_1x^{2e-1} + \cdots + p_{2j}x^{2e-2j})p_{2j} \\
&= (p_{2j+1}x^{d-2j-1} + \cdots + p_d)(x^{2j-1} + p_1x^{2j-2} + \cdots + p_{2j-1}) \\
&\qquad + (x^{2e} + p_1x^{2e-1} + \cdots + p_{2j}x^{2e-2j})p_{2j} \\
&= p_{2j}x^{d-1} + (p_{2j+1} + p_1p_{2j})x^{d-2} + \cdots + p_d p_{2j-1}.
\end{aligned}
$$

So we find $t_p(f_j^2) = p_{2j}$ and $t_p(xf_j^2) = p_{2j}\gamma_1 + (p_{2j}p_1 + p_{2j+1})\gamma_0 = p_{2j+1}$, as $\gamma_0 = 1$ and $\gamma_1 = p_1$.

Finally, as $g_0 = x^e + p_1x^{e-1}$, we find that $x^k g_0^2 = x^{2e+k} + p_1^2x^{2e+k-2}$ and therefore $t_p(x^k g_0^2) = \gamma_{k+1} + p_1^2\gamma_{k-1}$ when $d = 2e$ and $t_p(x^k g_0^2) = \gamma_k + p_1^2\gamma_{k-2}$ when $d = 2e+1$. This proves the lemma. □

The next lemma calculates the transfers.

Lemma 5.2. *Let* $\lambda_1, \lambda_2 \in \mathscr{F}$, $h \in \{0, 1\}$, *and* $k \ge 0$. *When* $d = 2e$, *the transfer is given by*

$$
t_p^*([\lambda_1x^h, \lambda_2x^k]) = [\lambda_1h, \lambda_2(\gamma_{k+1} + p_1^2\gamma_{k-1})] + \sum_{i=1}^{e-1}[\lambda_1p_{2(e-i)+h+1}, \lambda_2\gamma_{k+2i-d-1}] \\
+ [\lambda_1\gamma_{h+1}, \lambda_2\gamma_{k-1}]
$$

and when $d = 2e+1$ by

$$t_p^*([\lambda_1 x^h, \lambda_2 x^k]) = \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)+h}, \lambda_2 \gamma_{k+2i-d+1}].$$

Proof. We first give a symplectic basis for $t_p^*([\lambda_1 x^h, \lambda_2 x^k])$. For $(r, s) \in \mathcal{F}[x]/(p) \times \mathcal{F}[x]/(p)$, applying the quadratic form we have

$$t_p^*([\lambda_1 x^h, \lambda_2 x^j])(r, s) = t_p(\lambda_1 x^h r^2 + rs + \lambda_2 x^j s^2).$$

If $d = 2e$ is even we consider the basis

$$\{(1, 0), (0, g_{e-1});\ (x, 0), (0, g_{e-2});\ \ldots;\ (x^{e-1}, 0), (0, g_0);\\ (f_{e-1}, 0), (0, 1);\ (f_{e-2}, 0), (0, x);\ \ldots;\ (f_0, 0), (0, x^{e-1})\}.$$

If $d = 2e+1$ is odd we consider the basis

$$\{(1, 0), (0, g_e);\ (x, 0), (0, g_{e-1});\ \ldots;\ (x^{e-1}, 0), (0, g_1);\\ (f_e, 0), (0, 1);\ (f_{e-1}, 0), (0, x);\ \ldots;\ (0, x^e), (f_0, 0)\}.$$

In each case we claim the basis is symplectic. When $d = 2e$, since each polynomial $x^{i-1} f_{e-i}$ and $x^{i-1} g_{e-i}$ is monic of degree $d-1$ when $1 \le i \le e$ we see that the inner products $((x^{i-1}, 0), (0, g_{e-i}))$ and $((0, x^{i-1}), (f_{e-i}, 0))$ equal 1. Likewise, when $d = 2e+1$ each of $((x^i, 0), (0, g_{e-i}))$ and $((0, x^i), (f_{e-i}, 0))$ equal 1. Next, whenever $i + j \le d-2$ we have $((x^i, 0), (0, x^j)) = 0$, showing that such pairs are always orthogonal. Since the product $x^i f_{e-j}$ has degree $i + 2e - j$, when $d = 2e$ and $i < j$ we have $t_p(x^{i-1} f_{e-j}) = 0$, and when $d = 2e+1$ with $i < j$ we have $t_p(x^i f_{e-j}) = 0$. For $i > j$, when $d = 2e$, we have

$$\begin{aligned} x^{i-1} f_{e-j} &= x^{i-1}(x^{2e-j} + \cdots + p_{2(e-j)} x^j) \\ &= (x^d + \cdots + p_{2(e-j)} x^{2j}) x^{i-j-1} = (p_{2(e-j)+1} x^{2j-1} + \cdots + p_d) x^{i-j-1}, \end{aligned}$$

this last having degree $i + j - 2 < d - 1$. For $i > j$, when $d = 2e+1$, we have $x^i f_{e-j} = x^{i-1}(x^{2e-j} + \cdots + p_{2(e-j)} x^j) = (x^d + \cdots + p_{2(e-j)} x^{2j}) x^{i-j-1} = (p_{2(e-j)+1} x^{2j-1} + \cdots + p_d) x^{i-j-1}$, this last having degree $i + j - 2 < d - 1$. So we see that the inner products $((f_{e-j}, 0), (0, x^{i-1}))$ vanish whenever $i \ne j$ and $d = 2e$, while $((f_{e-j}, 0), (0, x^i))$ vanish whenever $i \ne j$ and $d = 2e+1$. This also shows that when $i \ne j$ the inner product $((0, g_{e-j}), (x^{i-1}, 0))$ vanishes as well when $d = 2e$:

$$\begin{aligned} ((0, g_{e-j}), (x^{i-1}, 0)) &= ((0, f_{e-j} + p_{2(e-j)+1} x^{j-1}), (x^{i-1}, 0)) \\ &= t_p(p_{2(e-j)+1} x^{i+j-2}) = 0. \end{aligned}$$

Similarly $((0, g_{e-j}), (x^i, 0)) = 0$ when $d = 2e+1$.

So to check the required orthogonality we must calculate $t_p(g_i f_j)$ where $0 \le i, j \le e-1$ and $j = e$ as well as when $d = 2e+1$. When $d = 2e$ and $j > i$ or $d = 2e+1$ and $j > i+1$ we calculate

$$\begin{aligned} t_p(g_i f_j) &= t_p((x^{e+i} + \cdots + p_{2i+1}x^{e-i-1})(x^{e+j} + \cdots + p_{2j}x^{e-j})) \\ &= t_p((x^{2i+1} + \cdots + p_{2i+1})(x^{2e+j-i-1} + \cdots + p_{2j}x^{2e-j-i-1})) \\ &= t_p((x^{2i+1} + \cdots + p_{2i+1})(p_{2j+1}x^{2e-j-i-2} + \cdots + p_d x^{j-i-1})) = 0, \end{aligned}$$

because the latter polynomial has degree $2e+i-j-1 < d-1$. When $d = 2e$ and $j \le i$ or $d = 2e+1$ and $i \le j+1$ we calculate

$$\begin{aligned} t_p(g_i f_j) &= t_p((x^{e+i} + \cdots + p_{2i+1}x^{e-i-1})x^{e-j}(x^{2j} + \cdots + p_{2j})) \\ &= t_p((x^{2e+i-j} + \cdots + p_{2i+1}x^{2e-i-j-1})(x^{2j} + \cdots + p_{2j})) \\ &= t_p((p_{2i+2}x^{2e-i-j-2} + \cdots + p_d x^{i-j})(x^{2j} + \cdots + p_{2j})) = 0, \end{aligned}$$

because the latter polynomial has degree $2e-i+j-2 < d-1$. This shows that the inner products $((f_i, 0), (0, g_j)) = 0$ and therefore both bases listed are symplectic.

We are now able to compute the transfer $t_p^*([\lambda_1 x^h, \lambda_2 x^k])$, where $h \in \{0, 1\}$. When $d = 2e$, since $h \le 1$ the vectors $(x^i, 0)$ are isotropic as long as $0 \le i < e-1$. So we can apply the previous lemma and we only need to use the portion of the symplectic basis that involves g_0 and the f_j. We find

$$\begin{aligned} t_p^*([\lambda_1 x^h, \lambda_2 x^k]) &= [t_p(\lambda_1 x^h x^{2(e-1)}), t_p(\lambda_2 x^k g_0^2)] + \sum_{i=1}^{e} [t_p(\lambda_1 x^h f_{e-i}^2), t_p(\lambda_2 x^k x^{2(i-1)})] \\ &= [\lambda_1 h, \lambda_2(\gamma_{k+1} + p_1^2 \gamma_{k-1})] \\ &\qquad + \sum_{i=1}^{e-1} [\lambda_1 p_{2(e-i)+h+1}, \lambda_2 \gamma_{k+2i-d-1}] + [\lambda_1 \gamma_{h+1}, \lambda_2 \gamma_{k-1}], \end{aligned}$$

where in this last summand we have used $t(x^h f_0^2) = \gamma_{h+1}$. When $d = 2e+1$ we note that, since $h \le 1$, each of the vectors $(x^i, 0)$, where $0 \le i \le e-1$, is isotropic. Hence we need only consider the part of the symplectic basis involving the f_j and we find

$$\begin{aligned} t_p^*([\lambda_1 x^h, \lambda_2 x^k]) &= \sum_{i=0}^{e} [t_p(\lambda_1 x^h f_{e-i}^2), t_p(\lambda_2 x^k x^{2i})] \\ &= \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)+h}, \lambda_2 \gamma_{k+2i-d+1}], \end{aligned}$$

where in the latter sum we have used $\gamma_h = p_h$ for $h = 0, 1$ when $i = e$. This proves the lemma. $\square$

We next have to compute $\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1 x^h, \lambda_2 x^k]) = \partial_{\frac{1}{x}}(\langle px^h\rangle[\lambda_1, \lambda_2 x^{h+k}])$. There are four cases, depending upon the parity of h and d:

Lemma 5.3.

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k])) = \begin{cases} \sum_{j=0}^{e-1}[\lambda_1 p_{2j+1}, \lambda_2\gamma_{k-2j-1}] & \text{if } d \text{ is even},\\ \sum_{j=0}^{e}[\lambda_1 p_{2j}, \lambda_2\gamma_{k-2j}] & \text{if } d \text{ is odd};\end{cases}$$

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle px\rangle[\lambda_1, \lambda_2 x^{k+1}])) = \begin{cases} \sum_{j=0}^{e}[\lambda_1 p_{2j}, \lambda_2\gamma_{k-2j+1}] & \text{if } d \text{ is even},\\ \sum_{j=0}^{e}[\lambda_1 p_{2j+1}, \lambda_2\gamma_{k-2j}] & \text{if } d \text{ is odd}.\end{cases}$$

Proof. As $p = x^d + p_1x^{d-1} + \cdots + p_d$ we are able to express $\langle px^h\rangle[1, x^{h+k}]$ as

$$\langle x^{d+h}\rangle[1, x^{h+k+d}p^{-1}] + \langle p_1x^{d+h-1}\rangle[1, p_1x^{h+k+d-1}p^{-1}] + \cdots + \langle p_dx^h\rangle[1, p_dx^{h+k}p^{-1}].$$

Since $p = x^d(1 + p_1x^{-1} + p_2x^{-2} + \cdots p_dx^{-d})$, inside the completion $\mathcal{F}(x)_{\frac{1}{x}}$ we can write $p^{-1} = x^{-d}(1+\gamma_1x^{-1}+\gamma_2x^{-2}+\cdots)$. When $d+h-i$ is even we have $s^*_{\frac{1}{x}}(\langle p_ix^{d+h-i}\rangle[\lambda_1, \lambda_2p_ix^{h+k+d-i}p^{-1}]) = 0$ and when $d+h-i$ is odd we have

$$\begin{aligned} s^*_{\frac{1}{x}}(\langle p_ix^{d+h-i}\rangle&[\lambda_1, \lambda_2p_ix^{h+k+d-i}p^{-1}])\\ &= s^*_{\frac{1}{x}}(\langle p_ix^{-1}\rangle[\lambda_1, \lambda_2p_ix^{h+k-i}(1+\gamma_1x^{-1}+\cdots)])\\ &= \langle p_i\rangle[\lambda_1, \lambda_2p_i\gamma_{h+k-i}]. \end{aligned}$$

So when $h = 0$ and d is even we have

$$\begin{aligned} s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2x^k])) &= \langle p_1\rangle[\lambda_1, \lambda_2p_1\gamma_{k-1}] + \langle p_3\rangle[\lambda_1, \lambda_2p_3\gamma_{k-3}]\\ &\qquad + \cdots + \langle p_{d-1}\rangle[\lambda_1, \lambda_2p_{d-1}\gamma_{k-d+1}]\\ &= \sum_{j=0}^{e-1}[\lambda_1p_{2j+1}, \lambda_2\gamma_{k-2j-1}]. \end{aligned}$$

Similarly, if $h = 1$ and d is even we find

$$\begin{aligned} s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle px\rangle[\lambda_1, \lambda_2x^{k+1}])) &= \langle p_0\rangle[\lambda_1, \lambda_2p_0\gamma_{k+1}] + \langle p_2\rangle[\lambda_1, \lambda_2p_2\gamma_{k-1}]\\ &\qquad + \cdots + \langle p_d\rangle[\lambda_1, \lambda_2p_d\gamma_{k-d+1}]\\ &= \sum_{j=0}^{e}[\lambda_1p_{2j}, \lambda_2\gamma_{k-2j+1}]. \end{aligned}$$

Next, if $h = 0$ and d is odd,

$$\begin{aligned} s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2x^{k+1}])) &= \langle p_0\rangle[\lambda_1, \lambda_2p_0\gamma_k] + \langle p_2\rangle[\lambda_1, \lambda_2p_2\gamma_{k-2}]\\ &\qquad + \cdots + \langle p_{2e}\rangle[\lambda_1, \lambda_2p_{2e}\gamma_{k-2e}]\\ &= \sum_{j=0}^{e}[\lambda_1p_{2j}, \lambda_2\gamma_{k-2j}]. \end{aligned}$$

Finally, if $h=1$ and d is odd,

$$\begin{aligned} s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle px\rangle[\lambda_1,\lambda_2x^k])) &= \langle p_1\rangle[\lambda_1,\lambda_2p_1\gamma_k]+\langle p_3\rangle[\lambda_1,\lambda_2p_3\gamma_{k-2}] \\ &\qquad +\cdots+\langle p_{d-1}\rangle[\lambda_1,\lambda_2p_{d-1}\gamma_{k-d+1}] \\ &= \sum_{j=0}^{e-1}[\lambda_1p_{2j+1},\lambda_2\gamma_{k-2j}]. \qquad \square \end{aligned}$$

Theorem 5.4. *The reciprocity law* $\sum_q s^*_q(\partial_q(\phi))=0$ *holds for all* $\phi\in L_0+\langle p\rangle L_0$.

Proof. We first consider a generator $\phi=[\lambda_1x^i,\lambda_2x^j]$ of L_0. By Lemma 2.3, $\partial_p(\phi)$ vanishes for all $p\neq\frac{1}{x}$. When $p=\frac{1}{x}$, Theorem 3.6 shows that $s^*_{\frac{1}{x}}(\phi)=0$. So the reciprocity law holds for elements of L_0. We next note that for any generator $\phi=\langle p\rangle[\lambda_1x^h,\lambda_2x^k]$ of $\langle p\rangle L_0$ we have $s^*_q(\partial_q(\phi))=0$ as long as $q\neq p,\frac{1}{x}$. So we must check that $s^*_p(\partial_p(\phi))=s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\phi))$ for all such generators.

When $d=2e$ and $h=0$, we have by Lemma 5.2

$$t^*_p([\lambda_1,\lambda_2x^k])=\sum_{i=1}^{e-1}[\lambda_1p_{2(e-i)+1},\lambda_2\gamma_{k+2i-d-1}]+[\lambda_1\gamma_1,\lambda_2\gamma_{k-1}].$$

Also in this case by Lemma 5.3 we have

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1,\lambda_2x^k]))=\sum_{j=0}^{e-1}[\lambda_1p_{2j+1},\lambda_2\gamma_{k-2j-1}],$$

But $p_1=\gamma_1$; therefore the terms in these sums match exactly, which shows that $s^*_p(\partial_p(\langle p\rangle[\lambda_1,\lambda_2x^k]))=t^*_p([\lambda_1,\lambda_2x^k])=s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1,\lambda_2x^k]))$ in this case.

When $d=2e$ and $h=1$, we have by Lemma 5.2

$$\begin{aligned} t^*_p([\lambda_1x,\lambda_2x^k]) = [\lambda_1,\lambda_2(\gamma_{k+1}+p_1^2\gamma_{k-1})]+\sum_{i=1}^{e-1}[\lambda_1p_{2(e-i)+2},\lambda_2\gamma_{k+2i-d-1}] \\ +[\lambda_1\gamma_2,\lambda_2\gamma_{k-1}]. \end{aligned}$$

Also in this case by Lemma 5.3 we have

$$s^*_{\frac{1}{x}}(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1x,\lambda_2x^k]))=\sum_{j=0}^{e}[\lambda_1p_{2j},\lambda_2\gamma_{k-2j+1}].$$

These two expressions will be equal provided we can show

$$[\lambda_1,\lambda_2(\gamma_{k+1}+p_1^2\gamma_{k-1})]+[\lambda_1\gamma_2,\lambda_2\gamma_{k-1}]=[\lambda_1,\lambda_2\gamma_{k+1}]+[\lambda_1p_2,\lambda_2\gamma_{k-1}]$$

since the summands $\sum_{i=1}^{e-1}[\lambda_1p_{2(e-i)+2},\lambda_2\gamma_{k+2i-d-1}]$ correspond exactly to the summands $\sum_{j=2}^{e}[\lambda_1p_{2j},\lambda_2\gamma_{k-2j+1}]$. So we need that

$$[\lambda_1,\lambda_2p_1^2\gamma_{k-1}]+[\lambda_1\gamma_2,\lambda_2\gamma_{k-1}]=[\lambda_1p_2,\lambda_2\gamma_{k-1}]$$

which follows because $[\lambda_1, \lambda_2 p_1^2\gamma_{k-1}] = \langle p_1^2\rangle \cdot [\lambda_1 p_1^2, \lambda_2\gamma_{k-1}]$ and $\gamma_2 = p_1^2 + p_2$.

When $d = 2e+1$ and $h = 0$, we have by Lemma 5.2

$$t_p^*([\lambda_1, \lambda_2 x^k]) = \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)}, \lambda_2\gamma_{k+2i-d+1}].$$

Also in this case by Lemma 5.3 we have

$$s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k])) = \sum_{j=0}^{e} [\lambda_1 p_{2j}, \lambda_2\gamma_{k-2j}],$$

which shows that $t_p^*([\lambda_1, \lambda_2 x^k]) = s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k]))$, since the summations are the same apart from indexing.

When $d = 2e+1$ and $h = 1$, we have by Lemma 5.2

$$t_p^*([\lambda_1 x, \lambda_2 x^k]) = \sum_{i=0}^{e} [\lambda_1 p_{2(e-i)+1}, \lambda_2\gamma_{k+2i-d+1}].$$

Also in this case by Lemma 5.3 we have

$$s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1 x, \lambda_2 x^k])) = \sum_{j=0}^{e} [\lambda_1 p_{2j+1}, \lambda_2\gamma_{k-2j}],$$

which shows that $t_p^*([\lambda_1, \lambda_2 x^k]) = s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle p\rangle[\lambda_1, \lambda_2 x^k]))$ again since the summations are the same. This gives the reciprocity law for $\langle p\rangle L_0$. $\square$

6. The reciprocity law and the analogue of Milnor's theorem

We next turn to the reciprocity law for $W_q\mathcal{F}(x)$.

Theorem 6.1. *The composite* $W_q F \xrightarrow{\bigoplus \partial_p} \bigoplus_{p,\frac{1}{x}} W_1 F_p \xrightarrow{\bigoplus s_p^*} W_q\mathcal{F}$ *is zero.*

Proof. According to Lemma 2.5(ii), $W_q\mathcal{F}(x) = \sum_p (S_p + \langle x\rangle S_p)$. So it suffices to check the composite vanishes on $S_p + \langle x\rangle S_p$ for each p. In case $p = \frac{1}{x}$ then $S_{\frac{1}{x}} + \langle x\rangle S_{\frac{1}{x}} \subset L_0 + \langle \frac{1}{x}\rangle L_0 = L_0 + \langle x\rangle L_0$. Since the reciprocity law holds for $L_0 + \langle x\rangle L_0$ by Theorem 5.4, we can assume that p is monic and irreducible. By Lemma 4.2, we know that $S_p + \langle x\rangle S_p \subset U_p + L_0 + \langle p\rangle L_0$. By Theorem 5.4 we know the composite vanishes on $L_0 + \langle p\rangle L_0$. Since $U_p = \sum_r S_{p,r}$, therefore, it suffices to verify the composite vanishes on each generator of $S_{p,r}$. If q is not one of $p, x, \frac{1}{x}$, we know by Lemma 2.3 that ∂_p vanishes on $S_{p,r}$, so we only need to worry about those three primes.

We consider first the generators of $S_{p,r}$ of the form $\phi = \langle t^I \rangle [1, h/p^r] \in S_p$ or $\phi = \langle t^I p \rangle [1, h/p^r] \in \langle p \rangle S_p$ where $\deg h < \deg p$ and $r \geq 1$. Since $t^I \in \mathcal{F}$ we can assume $I = 0$ by Frobenius reciprocity. If $p \neq x$ then p is both an x-unit and a $\frac{1}{x}$-adic unit. This means $v_x(h/p^r) \geq 0$ and since $\deg h < \deg p$ we must have $v_{\frac{1}{x}}(h/p^r) > 0$. So by Lemma 2.3 we find that $\partial_x(\phi) = \partial_{\frac{1}{x}}(\phi) = 0$ in these cases. However by definition we know that $s_p^*(\partial_p(\phi)) = 0$ for these particular generators of U_p so reciprocity is established in these cases. In case $p = x$ then $v_{\frac{1}{x}}(h/x) > 0$ so $\partial_x(\phi) = \partial_{\frac{1}{x}}(\phi) = 0$ in this case as well. In case p is separable we know that all generators for U_p are of the form just considered so we are done when p is separable. When p is not separable, the generators just considered are the generators in U_p^0, so we are done in that case as well.

Finally, when p is not separable, we must consider generators of U_p that don't lie in U_p^0. These have the form $\langle t^I x \rangle [1, h/p^r] \in \tilde{S}_p$ or $\langle t^I px \rangle [1, h/p^r] \in \langle p \rangle \tilde{S}_p$ where $\deg h < \deg p$ and $r \geq 1$. Again, since $t^I \in \mathcal{F}$ we can by Frobenius reciprocity assume $I = 0$. In these cases we have by Definition 4.3(ii) that $s_x^*(\partial_x(\langle x \rangle [1, h/p^r])) = [1, h_c/p_c^r]$, and $s_x^*(\partial_x(\langle px \rangle [1, h/p^r])) = \langle p_c \rangle [1, h_c/p_c^r]$. Since $v_{\frac{1}{x}}(h/p^r) > 0$ we have $s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle x \rangle [1, h/p^r])) = 0$, and $s_{\frac{1}{x}}^*(\partial_{\frac{1}{x}}(\langle px \rangle [1, h/p^r])) = 0$. By the definition of s_p^*, we know that $s_p^*(\partial_p(\langle x \rangle [1, h/p^r]) = [1, h_c/p_c^r]$ and $s_p^*(\partial_p(\langle px \rangle [1, h/p^r]) = \langle p_0 \rangle [1, h_c/p_c^r]$, giving the reciprocity law in this case. □

Putting everything together gives the main result of the paper.

Theorem 6.2 (Analogue of the Milnor–Scharlau Sequence). *Suppose that $\mathcal{F}$ is a field of characteristic 2 and $F = \mathcal{F}(x)$ is a rational function field in one variable over $\mathcal{F}$. There exists a compatible collection of second residue and transfer maps that fit into an exact sequence*

$$0 \longrightarrow W_q\mathcal{F} \longrightarrow W_qF \xrightarrow{\bigoplus \partial_p} \bigoplus_{p,\frac{1}{x}} W_1F_p \xrightarrow{\bigoplus s_p^*} W_q\mathcal{F} \longrightarrow 0,$$

where the direct sum is taken over discrete valuations on F.

Proof. Everything completed previously applies when $\mathcal{F}$ has a finite 2-basis. In that case Theorem 3.5 shows that

$$W_qF/L_0 \cong \bigoplus_{d \geq 1} L_d/L_{d-1} \to \bigoplus_p W_1F_p$$

is an isomorphism. Theorem 3.6 shows that

$$0 \to W_q\mathcal{F} \to L_0 \to W_1F_{\frac{1}{x}} \to W_q\mathcal{F} \to 0$$

is an exact sequence. Theorem 6.1 shows we can patch the two sequences together and obtain the result. When $\mathcal{F}$ does not have a finite 2-basis, the result follows from the finite 2-basis case because any element in any group in the sequence lies in the same sequence defined for a finitely generated subfield of $\mathcal{F}$. This proves the theorem. $\square$

References

[Arason 1979] J. K. Arason, "Wittring und Galoiscohomologie bei Charakteristik 2", *J. Reine Angew. Math.* **307/308** (1979), 247–256. MR 80f:10020 Zbl 0396.12019

[Arason 2003] J. K. Arason, "Generators and Relations for $W_q(K((S)))$", manuscript, 2003.

[Aravire and Jacob 1996] R. Aravire and B. Jacob, "Versions of Springer's theorem for quadratic forms in characteristic 2", *Amer. J. Math.* **118**:2 (1996), 235–261. MR 97d:11064 Zbl 0853.11028

[Aravire and Jacob 2004] R. Aravire and B. Jacob, "The Milnor sequence for $W_q\mathcal{F}(x)$ in characteristic 2 when $\mathcal{F}$ is perfect.", pp. 1–17 in *Algebraic and arithmetic theory of quadratic forms* (Talca and Pucón, Chile), edited by R. Baeza et al., Contemporary Mathematics **344**, American Math. Soc., Providence, RI, 2004. MR 2005f:11064 Zbl 1092.11022

[Lam 1973] T. Y. Lam, *The algebraic theory of quadratic forms*, Benjamin, Reading, MA, 1973. Revised reprint, 1980. MR 53 #277 Zbl 0437.10006

[Milnor 1970] J. Milnor, "Algebraic K-theory and quadratic forms", *Invent. Math.* **9** (1970), 318–344. MR 41 #5465 Zbl 0199.55501

[Scharlau 1972] W. Scharlau, "Quadratic reciprocity laws", *J. Number Theory* **4** (1972), 78–97. MR 45 #1835 Zbl 0241.12005

[Schilling 1950] O. F. G. Schilling, *The theory of valuations*, Mathematical Surveys **4**, American Math. Soc., New York, 1950. MR 13,315b Zbl 0037.30702

Received February 25, 2005. Revised September 15, 2005.

ROBERTO ARAVIRE
DEPARTAMENTO DE CIENCIAS FÍSICAS Y MATEMÁTICAS
UNIVERSIDAD ARTURO PRAT
CASILLA 121
IQUIQUE
CHILE
raravire@unap.cl

BILL JACOB
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA SANTA BARBARA
SANTA BARBARA, CA 93106
UNITED STATES
jacob@math.ucsb.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

DISTORTION OF WREATH PRODUCTS IN SOME FINITELY PRESENTED GROUPS

SEAN CLEARY

Wreath products such as $\mathbb{Z} \wr \mathbb{Z}$ are not finitely presentable yet can occur as subgroups of finitely presented groups. Here we compute the distortion of $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of Thompson's group F and as a subgroup of Baumslag's metabelian group G. We find that $\mathbb{Z} \wr \mathbb{Z}$ is undistorted in F but is at least exponentially distorted in G.

1. Introduction

We consider aspects of the question of the distortion of infinitely related groups as subgroups of finitely presented groups. Higman [1961] showed that every recursively presentable group occurs as a subgroup of a finitely presented group, but it is not clear in general what happens to the geometry of the group since this embedding uses complicated algebraic methods and methods from recursive function theory which may affect the geometry of the group severely. Ol'shanskiĭ [1997] constructed isometric embeddings of recursively presentable groups into finitely presented groups using difficult methods that do not lead to easily constructed examples. In the particular concrete cases here, we consider concrete embeddings of one of the simplest finitely generated but not finitely presentable groups, $\mathbb{Z} \wr \mathbb{Z}$. We consider two embeddings of $\mathbb{Z} \wr \mathbb{Z}$ into finitely presented groups. The first is as a subgroup of Thompson's group F and the second is as subgroup of Baumslag's remarkable finitely presented metabelian group which contains $\mathbb{Z} \wr \mathbb{Z}$ and thus a free abelian subgroup of infinite rank. The distortion of the metric of $\mathbb{Z} \wr \mathbb{Z}$ is linear in Thompson's group F but is at least exponential in Baumslag's group.

2. Background

Metrics of wreath products. Two of the simplest infinite wreath products are the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}$. Cleary and Taback [2005] analyzed aspects of

MSC2000: 20F65.
Keywords: subgroup distortion, Thompson's group.
The author gratefully acknowledges support from PSC-CUNY grant #64459-0033 and the hospitality of the Centre de Recerca Matemàtica.

the metric geometry of those groups and other wreath products. There are natural normal forms for elements in these groups which lead to geodesic words for elements in these groups with respect to their standard generating sets.

For $\mathbb{Z} \wr \mathbb{Z}$, we consider the standard presentation

$$\langle a, t \mid [a^{t^i}, a^{t^j}],\ \text{for } i, j \in \mathbb{Z} \rangle,$$

where a^b denotes the conjugate $b^{-1}ab$ and $[a, b]$ the commutator $aba^{-1}b^{-1}$.

Geometrically, we can think of this wreath product by imagining a string of counters arranged from left to right and infinite in both directions, with one counter distinguished as the origin. As in the lamplighter group, we imagine a cursor that moves along the string of counters and will point to a particular one of these counters as being of current interest. The generator a acts as a generator of $\mathbb{Z}$ in the factor to which the cursor currently points and increases the counter in that factor, and the generator t moves the cursor to the right to the next counter. A typical such word is illustrated in Figure 1.

The starting configuration of these counters, corresponding to the identity element in $\mathbb{Z}\wr\mathbb{Z}$, is with all of the counters at zero and the cursor resting at the counter designated at the origin. We consider a word in these generators as a sequence of instructions to move the cursor and change the counter in the current factor. After application of a long string of the generators, we will be in a state where a finite number of counters are nonzero and the cursor points at a particular counter, called the final position of the cursor for that word.

We define $a_n = a^{t^n}$ and note that a_n is a generator of the conjugate copy of $\mathbb{Z}$ indexed by n. These a_n commute and, as described in [Cleary and Taback 2005], we can put any word in the generators into one of two normal forms:

$$rf(w) = a_{i_1}^{e_1} a_{i_2}^{e_2} \ldots a_{i_k}^{e_k} a_{-j_1}^{f_1} a_{-j_2}^{f_2} \ldots a_{-j_l}^{f_l} t^m \quad \text{(right-first)},$$
$$lf(w) = a_{-j_1}^{f_1} a_{-j_2}^{f_2} \ldots a_{-j_l}^{f_l} a_{i_1}^{e_i} a_{i_2}^{e_i} \ldots a_{i_k}^{e_k} t^m \quad \text{(left-first)},$$

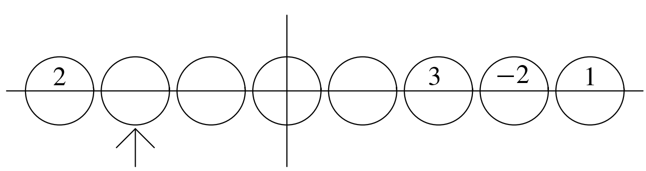

Figure 1. Diagram for $w = a_2^3 a_3^{-2} a_4 a_{-3}^2 t^{-2}$. The origin in the wreath product direction is indicated by a vertical line, empty circles denote counters which are zero, and the final cursor position is indicated by the arrow.

with $i_k > \dots i_2 > i_1 \geq 0$ and $j_l > \dots j_2 > j_1 > 0$ and $e_i, f_j \neq 0$.

The final resting position of the cursor is easily seen to be m for either of these normal forms, and we can see that the leftmost nonzero counter is in position $-j_l$ and the rightmost nonzero counter is in position i_k.

In the right-first form, $rf(w)$, the cursor moves first to the right from the origin, changing the counters in the appropriate factors as the cursor moves to the right. Then the cursor moves back to the origin not affecting any of the counters until passing the origin. Past the origin, the cursor continues to work leftwards, again changing the counters in the appropriate factors. Finally, the cursor moves to its ending location from the leftmost nonzero counter to the left of the origin.

The left-first form is similar, but instead of initially moving to the right, the cursor begins by moving toward the left.

At least one of these normal forms will lead to minimal-length representation for w, depending upon the final location of the cursor. If m is nonnegative, then the left-first normal form will lead to a geodesic representative, and if m is nonpositive, the right-first normal form will lead to a geodesic representative, giving the following measurement of length:

Proposition 2.1 [Cleary and Taback 2005, Proposition 3.8]. *If a word $w \in \mathbb{Z} \wr \mathbb{Z}$ is in either right-first or left-first normal form, the word length of w with respect to $\{a, t\}$ satisfies*

$$|w| = \sum_{n=1}^{k} |e_{i_n}| + \sum_{n=1}^{l} |f_{j_n}| + \min\left\{2j_l + i_k + |m - i_k|,\ 2i_k + j_l + |m + j_l|\right\}.$$

The first two terms are the minimum number of applications of $a^{\pm 1}$ needed to put all of the counters into their desired states and the last term is the minimum amount of movement required to visit the leftmost and rightmost nonzero counters and then the final position of the cursor, counting the required applications of $t^{\pm 1}$.

The word $w = a_2^3 a_3^{-2} a_4 a_{-3}^2 t^{-2}$ pictured in Figure 1 has geodesic representatives in right-first normal form since the final position of the cursor is to the left of the origin. One such minimal length representative is $t^2 a^3 t a^{-2} t a t^{-7} a^2 t$, of length 20.

3. $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of Thompson's group F

Thompson's group F is a remarkable finitely generated, finitely presented group that can be understood via a wide range of perspectives. For an excellent overview of its properties, see [Cannon et al. 1996]. The standard infinite presentation of F is

$$\langle x_0, x_1, \dots \mid x_n^{x_i} = x_{n+1} \text{ for } i < n \rangle .$$

Since $x_2 = x_1^{x_0}$ and so on, F is generated by the first two generators and we can define $x_{n+1} = x_n^{x_0}$ to express all generators and thus all group elements in terms of

x_0 and x_1. Furthermore, all of these infinitely many relations are consequences of the first two nontrivial relations, so we have the standard finite presentation

$$\langle x_0, x_1 \mid x_2^{x_1} = x_3, x_3^{x_1} = x_4 \rangle .$$

Thompson's group F can be described in terms of rooted tree pair diagrams, and there is a straightforward method of converting between words in a normal form with respect to the infinite generating set and tree pair diagrams, via the method of leaf exponents, as described in [Cannon et al. 1996]. There is also an easy method of converting from tree pair diagrams to piecewise-linear homeomorphisms of the unit interval where F can be regarded as the subgroup of elements with dyadic breakpoints and slopes which are powers of 2. There is a natural notion of a reduced tree pair diagram described there and there are efficient means to convert between the unique normal form for an element of F and the unique reduced tree pair diagram for that word.

We consider a rooted binary tree with n leaves as being constructed of $n-1$ *carets*, which are interior nodes of the tree together with the two downward directed edges from that node. The *left side* of a tree consists of nodes and edges connected to the root by a path consisting only of left edges, and similarly for the *right side* A tree pair diagram (S, T) is made up of a "positive" tree T and a "negative" tree S with the same number of leaves.

To understand the metric properties of F, we consider expressing words with respect to the finite generating set. Burillo, Cleary and Stein [2001] estimated the word length in terms of the number of carets and showed that the number of carets is quasiisometric to the word length. Fordham [2003] developed a remarkable method using tree pair diagrams to efficiently compute exact word length and find minimal length representatives of words.

We can understand word length of elements represented as tree pair diagrams by understanding how the generators change the tree pair diagram for w to that for wg for the generators, as described in [Fordham 2003; Cleary and Taback 2004]. The right actions of the generators can be described as 'rotations' which change the negative tree in a possibly unreduced representative of the element.

The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is a subgroup of F and can be realized in many different ways. Perhaps the simplest is as the subgroup H generated by x_0 and $h = x_1 x_2 x_1^{-2}$, pictured in Figure 2. The element h, regarded as a piecewise-linear homeomorphism of the unit interval, has support $\left[\frac{1}{2}, \frac{3}{4}\right]$. The conjugates $h^{x_0^n}$ have support

$$\left[\frac{2^{n+1}-1}{2^{n+1}}, \frac{2^{n+2}-1}{2^{n+2}}\right] \quad \text{for } n \geq 0,$$

$$\left[\frac{1}{2^{1-n}}, \frac{1}{2^{-n}}\right] \quad \text{for } n < 0.$$

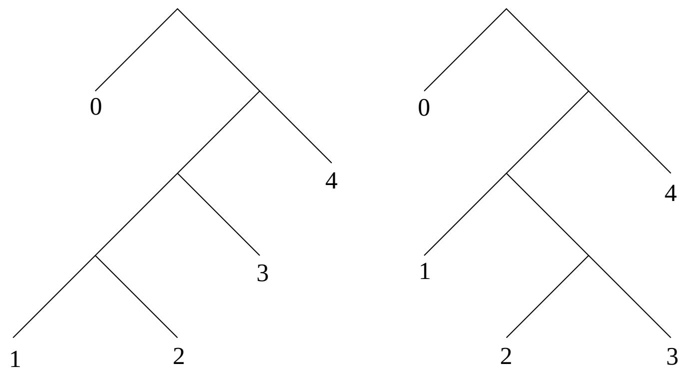

Figure 2. Tree pair diagram for $x_1 x_2 x_1^{-2}$, the image of a under ϕ.

The isomorphism between this subgroup and $\mathbb{Z} \wr \mathbb{Z}$ is given by the homomorphism $\phi(w) : \mathbb{Z} \wr \mathbb{Z} \to F$ where $\phi(a) = h = x_1 x_2 x_1^{-2}$ and $\phi(t) = x_0$. The conjugates of $\phi(a)$ by x_0 generate infinite cyclic groups, and these conjugates each have interiors of their supports which are disjoint from the interiors of the supports of the other conjugates. Thus they freely generate a free abelian group of countable rank. Since x_0 conjugates each of these abelian factors to the next, the isomorphism is readily established with $\mathbb{Z} \wr \mathbb{Z}$.

To understand the distortion of the subgroup H in F, we compare the word length of an element $w = a_{i_1}^{e_1} a_{i_2}^{e_2} \dots a_{i_k}^{e_k} a_{-j_1}^{f_1} a_{-j_2}^{f_2} \dots a_{-j_l}^{f_l} t^m$ with its image in F.

Theorem 3.1. *The subgroup H isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ in F generated by $x_0 = \phi(t)$ and $h = x_1^2 x_2^{-1} x_1^{-1} = \phi(a)$ is undistorted.*

Proof. We count the number of carets of the image of a word w. First, we consider the case when $m = 0$ and then the cases where m is nonzero.

Case $m = 0$. Here, the image of the word as a tree pair diagram has a characteristic form where the root of the positive tree is paired with the root of the negative tree, such as that shown in Figure 3. In the general case, where both k and l are positive, we have

- a single root caret,
- $i_k + 1$ right carets,
- $\sum_{n=1}^{k} (|e_n| + 1)$ interior carets below the right arm of the tree,
- j_l left carets, and
- $\sum_{n=1}^{l} (|f_n| + 1)$ interior carets below the left arm of the tree.

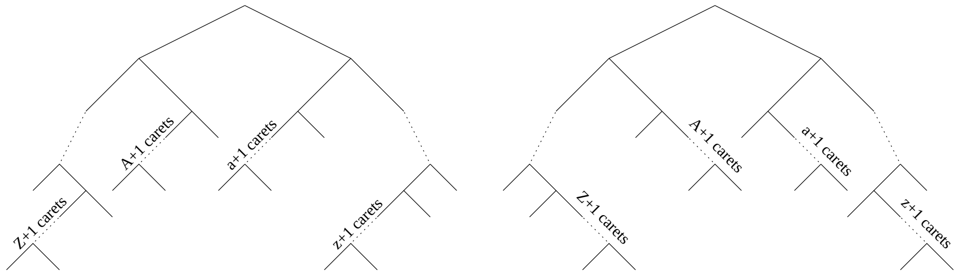

Figure 3. The tree pair diagram for the image $\phi(w)$ of a word $w = a_0^a a_1^b \ldots a_n^z a_{-1}^A a_{-2}^B \ldots a_{-m}^Z$ with t exponent sum 0 and all positive exponents for a_i.

This gives a total of

$$N(\phi(w)) = i_k + j_l + 2 + \sum_{n=1}^{k} (|e_n| + 1) + \sum_{n=1}^{l} (|f_n| + 1)$$

carets in the image of w. By [Burillo et al. 2001], the number of carets is quasi-isometric to the word length in F with respect to $\{x_0, x_1\}$ and since the length of w in $\mathbb{Z} \wr \mathbb{Z}$ is

$$2j_l + 2i_k + \sum_{n=1}^{k} |e_n| + \sum_{n=1}^{l} |f_n|,$$

we see that these lengths are quasiisometric.

The image of a typical word with all e_n and f_n positive is shown in Figure 3, corresponding to a series of rightward rotations at nodes distance one from the sides of the tree.

Case $m > 0$. Here we start with the same tree pair diagram for the $m = 0$ case and apply x_0 on the right m times. Each application of x_0 will change the negative tree by moving the root caret to a right caret and the topmost left caret to the root, if there is a left caret. If there is no left caret, a new caret will need to be added for each such application. For each application of x_0 which requires a new caret, in the negative tree, that new caret will become the root caret and in the positive tree, the new caret will be added as the left child of the leftmost caret. Since there are j_l left carets, if $m \le j_l$, we do not need to add any carets and the number of carets is

$$i_k + j_l + 2 + \sum_{n=1}^{k} (|e_n| + 1) + \sum_{n=1}^{l} (|f_n| + 1)$$

as before. If $m > j_l$, we will need to add $m - j_l$ new carets and will have

$$i_k + j_l + 2 + \sum_{n=1}^{k} (|e_n| + 1) + \sum_{n=1}^{l} (|f_n| + 1) + m - j_l$$

carets. Again, these quantities give lengths which are comparable to word length in $\mathbb{Z} \wr \mathbb{Z}$.

Case $m < 0$. This works in the same way as the case $m > 0$.

Thus in all cases ϕ does not distort distances more than linearly, so the subgroup H isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ is undistorted in F. □

We can obtain more precise estimates of the quasiisometry constants using Fordham's method [2003] for computing exact lengths in F. We can keep track of the particular caret pairings and their weights and we find that the caret pairings that occur are easily computed. Caret pairing types are described in [Fordham 2003; Cleary and Taback 2004]. For example, in the case where $m = 0$ and both l and k are positive, we have the following caret pairs:

- One caret pair of type (L_0, L_0) from the leftmost carets, contributing no weight.
- j_l caret pairs of type (L_L, L_L) from the left side and root, contributing weight $2j_l$.
- $i_k - 1$ caret pairs of types (R_*, R_*) not of type (R_0, R_0), contributing weight $2(i_k - 1)$.
- One caret pair of type (R_0, R_0) from the rightmost carets, contributing no weight.
- For each $e_n > 0$, there will be a single pairing of type (I_0, I_0) contributing weight 2 and $e_n - 1$ pairings of type (I_0, I_R), contributing weight $4(e_n - 1)$.
- For each $e_n < 0$, there will be a single pairing of type (I_0, I_0) contributing weight 2 and $|e_n| - 1$ pairings of type (I_R, I_0), contributing weight $4(|e_n| - 1)$.
- Similarly, for the interior carets from the left side of the tree, we have for each f_n, there will be a single pairing of type (I_0, I_0) contributing weight 2 and $|f_n| - 1$ pairings of type (I_0, I_R) or (I_R, I_0), contributing weight $4(|f_n| - 1)$.

These will give a total weight of

$$\begin{aligned}2j_l + 2(i_k - 1) + 2k + 4\sum |e_n| + 2l + 4\sum |f_n| \\ = 2j_l + 2i_k + 2k + 2l + 4\sum |e_n| + 4\sum |f_n| - 2\end{aligned}$$

in the case when $m = 0$, which compares to the corresponding length in $\mathbb{Z} \wr \mathbb{Z}$ of $2j_l + 2i_k + \sum |e_n| + \sum |f_n|$.

Again, these give lengths comparable to word length in $\mathbb{Z} \wr \mathbb{Z}$. After a similar analysis for other cases, we see that for a word w in $\mathbb{Z} \wr \mathbb{Z}$, we have

$$|w|_{\mathbb{Z}\wr\mathbb{Z}} - 2 \le |\phi(w)|_F \le 4|w|_{\mathbb{Z}\wr\mathbb{Z}}.$$

4. $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of Baumslag's metabelian group

Baumslag [1972] introduced the group $G = \langle a, s, t \mid [s, t], [a^t, a], a^s = aa^t \rangle$ to show that a finitely presented metabelian group can contain free abelian subgroups of infinite rank. This group in fact contains $\mathbb{Z} \wr \mathbb{Z}$: all relators of the form $[a^{t^i}, a^{t^j}]$ are consequences of these three, so the subgroup H generated by a and t is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

Here we examine the distortion of this subgroup in G.

Theorem 4.1. *The subgroup H has at least exponential distortion in G.*

Proof. First, s conjugates elements in H to other elements in H in a manner illustrated here:

$$a^{(s^2)} = (a^s)^s = (aa^t)^s = a^s(a^s)^t = aa^t(aa^t)^t = aa^ta^ta^{t^2} = a_0a_1^2a_2.$$

In terms of the notation described above, we have $a_n^s = a_na_{n+1}$. Further conjugation by s leads to increasingly long words, such as

$$a^{(s^3)} = (a_0a_1^2a_2)^s = a_0a_1a_1^2a_2^2a_2a_3 = a_0a_1^3a_2^3a_3,$$

and we notice the occurrence of the binomial coefficients with repeated iteration, formalized below:

Lemma 4.2. *Higher conjugates of a by s in G give elements of the following form*:

$$a^{s^n} = a_0^{\binom{n}{0}}a_1^{\binom{n}{1}} \dots a_n^{\binom{n}{n}}.$$

Proof. We work by induction. The cases with $n = 1, 2$ and 3 are described above, and by assuming it is true for n we derive

$$a^{s^{n+1}} = (a_0^{\binom{n}{0}}a_1^{\binom{n}{1}} \dots a_n^{\binom{n}{n}})^s = a_0^{\binom{n}{0}}a_1^{\binom{n}{0}}a_1^{\binom{n}{1}}a_2^{\binom{n}{1}} \dots a_n^{\binom{n}{n}}a_{n+1}^{\binom{n}{n}} = a_0^{\binom{n+1}{0}} \dots a_{n+1}^{\binom{n+1}{n+1}},$$

using the commutativity of the a_i and the fact that $a_i^s = a_ia_{i+1}$. □

Returning to the proof of the theorem, we see that a^{s^n} has length $2n + 1$ as an element of G, and that it lies in the subgroup H, as there is a representative with no occurrences of s.

To compute the length of this element in the subgroup H with respect to its generators a and t, we use the method described in Section 2 on the expression with the binomial coefficients and find that

$$|a^{s^n}|_H = 2n + \sum_{i=0}^{n} \binom{n}{i} = 2n + 2^n.$$

Thus we have $|a^{s^n}|_H = 2n + 2^n$ while $|a^{s^n}|_G = 2n + 1$, so the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is at least exponentially distorted in G. □

References

[Baumslag 1972] G. Baumslag, “A finitely presented metabelian group with a free abelian derived group of infinite rank”, *Proc. Amer. Math. Soc.* **35** (1972), 61–62. MR 45 #8710 Zbl 0269.20029

[Burillo et al. 2001] J. Burillo, S. Cleary, and M. I. Stein, “Metrics and embeddings of generalizations of Thompson’s group *F*”, *Trans. Amer. Math. Soc.* **353**:4 (2001), 1677–89. MR 2001k:20087 Zbl 0989.20030

[Cannon et al. 1996] J. W. Cannon, W. J. Floyd, and W. R. Parry, “Introductory notes on Richard Thompson’s groups”, *Enseign. Math.* (2) **42**:3-4 (1996), 215–256. MR 98g:20058 Zbl 0880.20027

[Cleary and Taback 2004] S. Cleary and J. Taback, “Combinatorial properties of Thompson’s group *F*”, *Trans. Amer. Math. Soc.* **356**:7 (2004), 2825–2849. MR 2005b:20074 Zbl 1065.20052

[Cleary and Taback 2005] S. Cleary and J. Taback, “Dead end words in lamplighter groups and other wreath products”, *Q. J. Math.* **56**:2 (2005), 165–178. MR 2006h:20055 Zbl 02230593

[Fordham 2003] S. B. Fordham, “Minimal length elements of Thompson’s group *F*”, *Geom. Dedicata* **99** (2003), 179–220. MR 2004g:20045 Zbl 1039.20014

[Higman 1961] G. Higman, “Subgroups of finitely presented groups”, *Proc. Roy. Soc. Ser. A* **262** (1961), 455–475. MR 24 #A152 Zbl 0104.02101

[Ol’shanskiĭ 1997] A. Y. Ol’shanskiĭ, “On subgroup distortion in finitely presented groups”, *Mat. Sb.* **188**:11 (1997), 51–98. In Russian; translated in *Sb. Math.* **188** (1997), 1617-1664. MR 99a:20038 Zbl 0905.20020

Received January 28, 2005.

SEAN CLEARY
DEPARTMENT OF MATHEMATICS R8133
THE CITY COLLEGE OF NEW YORK
NEW YORK, NY 10031

cleary@sci.ccny.cuny.edu
http://www.sci.ccny.cuny.edu/∼cleary

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

COMMUTATION RELATIONS FOR ARBITRARY QUANTUM MINORS

KENNETH R. GOODEARL

Complete sets of commutation relations for arbitrary pairs of quantum minors are computed, with explicit coefficients in closed form.

1. Introduction

The title of this paper begins with what may seem to be a misnomer; the term *commutation relation*, in current usage, does not refer to a commutativity condition $xy = yx$, but has evolved to encompass various "skew commutativity" conditions that have proved to be useful replacements for commutativity. Older types of commutation relations include conditions of the form $xy - yx = z$, used in defining Weyl algebras and enveloping algebras. In quantized versions of classical algebras appear relations such as $xy = qyx$ (known as *q-commutation*), along with mixtures of both types. Thus, it has become common to refer to any equation of the form $xy = \lambda yx + z$, where λ is a nonzero scalar, as a *commutation relation for x and y*. One important use of such a relation, especially in enveloping algebras, is that, if the algebra supports a filtration such that $\deg(z) < \deg(x)+\deg(y)$, then the images of x and y in the associated graded algebra — call them $\tilde{x}$ and $\tilde{y}$ — commute up to a scalar: $\tilde{x}\tilde{y} = \lambda\tilde{y}\tilde{x}$. Similarly, the cosets of x and y modulo the ideal generated by z commute up to λ. Such coset relations are key ingredients in the work on quantized coordinate rings of Soibelman [1990], Hodges and Levasseur [1993; 1994], Joseph [1995], and others.

In many quantized algebras, the available commutation relations are homogeneous and quadratic, of the form $xy = \lambda yx + \sum_i \mu_i x_i y_i$ (where λ and the μ_i are nonzero scalars). Relations of this type are particularly important in establishing a (noncommutative) standard basis of monomials in generators that include the elements x, y, x_i, y_i. Namely, if the generators are ordered in such a way that each $x_i \le y_i$ but $x > y$, then the given relation allows one to rewrite monomials involving xy as linear combinations of monomials closer to standard form. For example, noncommutative standard bases have been constructed by Lakshmibai

MSC2000: 16W35, 20G42.
Keywords: quantum matrices, quantum minors, commutation, quasicommutation, q-commutation.
This research was partially supported by a grant from the National Science Foundation.

and Reshetikhin [1991; 1992] (for quantized coordinate rings of flag varieties and Schubert schemes), by Goodearl and Lenagan [2000] (for quantum matrix algebras), and by Lenagan and Rigal [2006] (for quantum Grassmannians and quantum determinantal rings).

In order to work effectively with quantized coordinate rings of matrices, Grassmannians, special or general linear groups, and related algebras, one needs explicit commutation relations for quantum minors and related elements. Such relations have often been derived for special cases as needed, either by induction on the size of the minors, using quantum Laplace relations as in Parshall and Wang [1991] and Taft and Towber [1991], or by applying the quasitriangular structure of $U_q(\mathfrak{sl}_n(k))$ (that is, its universal R-matrix) to coordinate functions in $\mathbb{O}_q(SL_n(k))$, as in the work of Lakshmibai and Reshetikhin [1991; 1992], Soibelman [1990], and Hodges and Levasseur [1993; 1994]. Along the former line, the most complete results to date were obtained by Fioresi [1999; 2004], who developed an algorithm that yields a commutation relation for any pair of quantum minors. This algorithm is an iterative procedure, in which certain products of quantum minors may appear multiple times; explicit coefficients are produced, but are not expressed as closed formulas. Via the quasitriangular approach, general commutation relations for pairs of coordinate functions in quantized coordinate rings $\mathbb{O}_q(G)$, where G is a semisimple Lie group, have been derived in special cases (for example, see [Lakshmibai and Reshetikhin 1991; 1992; Soibelman 1990; Hodges and Levasseur 1993; 1994]), not all with explicit coefficients. (Quantum minors in $\mathbb{O}_q(SL_n(k))$ are special coordinate functions.) Perhaps the largest group of explicit commutation relations obtained in this way appeared in Hodges et al. [1997] (see also [Brown and Goodearl 2002]). However, to make these fully explicit, canonical elements for the Rosso–Tanisaki Killing form on $U_q(\mathfrak{sl}_n(k))$ had to be computed.

Here we introduce a new method — new only in the sense that it has apparently not been used for this purpose before — with which we derive complete commutation relations for arbitrary pairs of quantum minors, with explicit coefficients in closed form. Our method is dual to the quasitriangular approach, as it relies on the coquasitriangular (or braided) bialgebra structure on the quantized coordinate ring of $n \times n$ matrices. Representation-theoretically, the two approaches are based on equivalent information, in that a quasitriangular (respectively, coquasitriangular) structure on a bialgebra encodes braiding isomorphisms

$$V \otimes W \xrightarrow{\cong} W \otimes V$$

for finite dimensional modules (respectively, comodules) V and W. To record such isomorphisms, one typically requires formulas for matrix entries. However, in the case of a coquasitriangular bialgebra A, the above isomorphism information is stored more compactly, in a bilinear form $\mathbf{r}$ on A. The braiding isomorphism for left A-comodules V and W is then given by the formula

$$v \otimes w \longmapsto \sum_{(v),(w)} \mathbf{r}(v_0, w_0)\, w_1 \otimes v_1,$$

where we have used Sweedler's notation: $v \mapsto \sum_{(v)} v_0 \otimes v_1$ for the comodule structure map $V \to A \otimes V$, and similarly for W. The resulting commutation relations are equations with values of $\mathbf{r}$ as coefficients, namely,

$$\sum_{(a),(b)} \mathbf{r}(a_1, b_1)\, a_2 b_2 = \sum_{(a),(b)} \mathbf{r}(a_2, b_2)\, b_1 a_1 \tag{1-1}$$

for $a, b \in A$, using now Sweedler's notation for the comultiplication map $A \to A \otimes A$.

When A is the bialgebra $\mathcal{O}_q(M_n(k))$, and $a = [I \mid J]$ and $b = [M \mid N]$ are quantum minors (see below for the notation), equation (1-1) becomes

$$\sum_{\substack{|S|=|I| \\ |T|=|M|}} \mathbf{r}\big([I \mid S], [M \mid T]\big)\, [S \mid J]\, [T \mid N] = \sum_{\substack{|S|=|J| \\ |T|=|N|}} \mathbf{r}\big([S \mid J], [T \mid N]\big)\, [M \mid T]\, [I \mid S]. \tag{1-2}$$

Observe that $[I \mid J]\, [M \mid N]$ occurs on the left hand side of (1-2) when $S = I$ and $T = M$, while $[M \mid N]\, [I \mid J]$ occurs on the right when $S = J$ and $T = N$. As we shall see, the coefficients for these terms — namely, $\mathbf{r}\big([I \mid I], [M \mid M]\big)$ and $\mathbf{r}\big([J \mid J], [N \mid N]\big)$ — are nonzero (in fact, they are powers of q). Thus, to obtain explicit commutation relations for $[I \mid J]$ and $[M \mid N]$, we only need to compute the values $\mathbf{r}\big([I \mid S], [M \mid T]\big)$ and $\mathbf{r}\big([S \mid J], [T \mid N]\big)$. This is precisely what we do in this paper, see especially Theorems 5.6 and 6.2. Additional relations follow from these by various symmetries, or by using quantum Laplace relations. (Quantum Plücker relations in quantum Grassmannians can also be used for this purpose.) See Theorems 6.7 and 7.3, and Corollaries 6.3, 6.8 and 7.5.

Our notation and conventions are collected in Section 2. In particular, the relations we use for $\mathcal{O}_q(M_n(k))$ are displayed in (2-6), so that the reader may compare them with other papers in which q is replaced by q^{-1} or q^2. Our computations of the values of the form $\mathbf{r}$ on pairs of quantum minors occupy Sections 3 and 5; the intermediate Section 4 provides a first set of commutation relations to illustrate our methods. The general commutation relations are derived in Sections 6 and 7, and we conclude, in Section 8, by using these relations to evaluate the standard Poisson bracket on pairs of classical minors.

2. Notation and conventions

Fix a positive integer n, a base field k, and a nonzero scalar $q \in k^\times$. We work within the standard single-parameter quantized coordinate ring of $n \times n$ matrices over k, which we denote $\mathcal{O}_q(M_n(k))$, as defined in Section 2.2 below. We use the abbreviation

$$\hat{q} = q - q^{-1}, \tag{2-1}$$

since this scalar appears in numerous formulas.

2.1. ***R-matrix.*** The standard R-matrix of type A_{n-1} can be presented in the form

$$R = q\sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{n} e_{ii} \otimes e_{jj} + \hat{q} \sum_{\substack{i,j=1\\ i>j}}^{n} e_{ij} \otimes e_{ji}; \tag{2-2}$$

see [Reshetikhin et al. 1989, Equation 1.5, p. 200]. We view R as a linear automorphism of $k^n \otimes k^n$, which acts on the standard basis vectors $x_i \otimes x_j$ according to the formula

$$R(x_l \otimes x_m) = \sum_{i,j=1}^{n} R^{ij}_{lm} x_i \otimes x_j, \tag{2-3}$$

using the conventions of [Klimyk and Schmüdgen 1997]. The entries of the $n^2 \times n^2$ matrix R^{ij}_{lm} are

$$\begin{aligned} R^{ii}_{ii} &= q \quad \text{for all } i, & R^{ij}_{ij} &= 1 \quad \text{for } i \neq j,\\ R^{ij}_{ji} &= \hat{q} \quad \text{for } i > j, & R^{ij}_{lm} &= 0 \quad \text{otherwise}; \end{aligned} \tag{2-4}$$

see [Klimyk and Schmüdgen 1997, Equation 9.13, p. 309].

2.2. ***Generators, relations, and grading.*** The algebra $A = \mathcal{O}_q(M_n(k))$ is obtained from (2-4) by the Faddeev–Reshetikhin–Takhtadzhyan construction, that is, as the k-algebra $A(R)$ presented by generators X_{ij} (for $i, j = 1, \dots, n$) and relations

$$\sum_{s,t=1}^{n} R^{ij}_{st} X_{sl} X_{tm} = \sum_{s,t=1}^{n} X_{jt} X_{is} R^{st}_{lm} \tag{2-5}$$

for all $i, j, l, m = 1, \dots, n$ (see [Reshetikhin et al. 1989, Definition 1, p. 197] and [Klimyk and Schmüdgen 1997, Section 9.1.1]; we have written X_{ij} for the generators labelled t_{ij} in [Reshetikhin et al. 1989] and u^i_j in [Klimyk and Schmüdgen 1997]). As is well known, the relations (2-5) are equivalent to

$$\begin{aligned} X_{ij}X_{lj} &= qX_{lj}X_{ij} && \text{for } i < l,\\ X_{ij}X_{im} &= qX_{im}X_{ij} && \text{for } j < m,\\ X_{ij}X_{lm} &= X_{lm}X_{ij} && \text{for } i < l \text{ and } j > m,\\ X_{ij}X_{lm} - X_{lm}X_{ij} &= \hat{q}X_{im}X_{lj} && \text{for } i < l \text{ and } j < m \end{aligned} \tag{2-6}$$

(see [Klimyk and Schmüdgen 1997, Equations 9.17, p. 310]). Some authors define quantum matrices using relations as in (2-6) but with q replaced by q^{-1}; thus, the algebras they define match what we would label $\mathbb{O}_{q^{-1}}(M_n(k))$. See, for example, [Larson and Towber 1991, p. 3317] or [Parshall and Wang 1991, Equation 3.5a, p. 37]. In comparing our work with those papers, we must be careful to interchange q and q^{-1}; however, $\hat{q}$ is defined to be $q^{-1}-q$ in [Parshall and Wang 1991, p. 38], and so we do not change $\hat{q}$ when carrying over results from that paper.

Because of the homogeneity of the relations (2-6), A carries a natural $(\mathbb{Z}^n \times \mathbb{Z}^n)$-grading, such that each X_{ij} is homogeneous of degree (ϵ_i, ϵ_j), where $\epsilon_1, \ldots, \epsilon_n$ are the standard basis elements for $\mathbb{Z}^n$.

2.3. ***Coquasitriangular structure.*** We follow [Hayashi 1992, Section 1] in defining a *coquasitriangular bialgebra* (also called a *bialgebra with braiding structure* [Larson and Towber 1991, Theorem 2.7] or a *cobraided bialgebra* [Kassel 1995, Definition VIII.5.1]) to be a bialgebra B equipped with a convolution-invertible bilinear form $\mathbf{r} : B \otimes B \to k$ such that

$$\sum_{(a),(b)} \mathbf{r}(a_1, b_1)\, a_2 b_2 = \sum_{(a),(b)} \mathbf{r}(a_2, b_2)\, b_1 a_1, \tag{2-7}$$

$$\mathbf{r}(ab, c) = \sum_{(c)} \mathbf{r}(a, c_1)\, \mathbf{r}(b, c_2), \tag{2-8}$$

$$\mathbf{r}(a, bc) = \sum_{(a)} \mathbf{r}(a_1, c)\, \mathbf{r}(a_2, b), \tag{2-9}$$

$$\mathbf{r}(a, 1) = \mathbf{r}(1, a) = \varepsilon(a), \tag{2-10}$$

for all $a, b, c \in B$, where $\mathbf{r}(x, y)$ stands for $\mathbf{r}(x \otimes y)$ for convenience and we use Sweedler's notation for comultiplication, in the form $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$. Condition (2-10) is redundant by [Klimyk and Schmüdgen 1997, Proposition 10.2(ii), p. 333]. Thus, the above definition agrees with [Kassel 1995, Definition VIII.5.1], [Klimyk and Schmüdgen 1997, Definition 10.1, pp. 331–2], and [Lambe and Radford 1997, Definition 7.3.1], but not with the conditions in [Larson and Towber 1991, Theorem 2.7]. However, the latter conditions match those of (2-10) if one uses the form $\langle \cdot \mid \cdot \rangle$ given by $\langle a \mid b \rangle = \mathbf{r}(b, a)$.

By [Klimyk and Schmüdgen 1997, Theorem 10.7, p. 337], whenever R is an invertible R-matrix satisfying the original form of the quantum Yang–Baxter equation, the FRT-algebra $A(R)$ is coquasitriangular with respect to the form $\mathbf{r}$ determined by

$$\mathbf{r}(X_{ij}, X_{lm}) = R^{il}_{jm} \tag{2-11}$$

for all i, j, l, m. (By "original quantum Yang–Baxter equation" we mean the equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ [Reshetikhin et al. 1989, Equation 0.7, p. 195], as opposed to the form exhibiting the braid relation, namely $R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}$.) Note that, in view of (2-11), if we put $a = X_{il}$ and $b = X_{jm}$ into (2-7), we recover relations (2-5).

It is well known that the R-matrix given in (2-2) satisfies the original quantum Yang–Baxter equation (see, for example, [Klimyk and Schmüdgen 1997, Section 8.1.2, pp. 246–7, and Equation 8.60, p. 270]). Consequently:

Theorem 2.4. *The algebra $A = \mathcal{O}_q(M_n(k))$ is a coquasitriangular bialgebra with respect to the bilinear form $\mathbf{r} : A \otimes A \to k$ determined by the following conditions*:

$$\text{(2-12)} \qquad \begin{aligned} &\mathbf{r}(X_{ii}, X_{ii}) = q \quad \text{for all } i, && \mathbf{r}(X_{ii}, X_{jj}) = 1 \quad \text{for } i \neq j, \\ &\mathbf{r}(X_{ij}, X_{ji}) = \hat{q} \quad \text{for } i > j, && \mathbf{r}(X_{ij}, X_{lm}) = 0 \quad \text{otherwise.} \end{aligned}$$

2.5. ***Quantum minors.*** We write $[I \mid J]$ for the quantum minor in A with row-index set I and column-index set J; this minor is just the quantum determinant in the subalgebra $k\langle X_{ij} \mid i \in I,\ j \in J\rangle$, which is naturally isomorphic to $\mathcal{O}_q(M_{|I|}(k))$. Specifically, if we write the elements of I and J in ascending order, say,

$$I = \{i_1 < \cdots < i_t\} \qquad \text{and} \qquad J = \{j_1 < \cdots < j_t\},$$

then

$$\text{(2-13)} \qquad \begin{aligned} [I \mid J] &= \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_{\sigma(1)}, j_1} X_{i_{\sigma(2)}, j_2} \cdots X_{i_{\sigma(t)}, j_t} \\ &= \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} X_{i_2, j_{\sigma(2)}} \cdots X_{i_t, j_{\sigma(t)}}, \end{aligned}$$

where $\ell(\sigma)$ denotes the length of the permutation $\sigma \in S_t$ as a product of simple transpositions $(l, l+1)$ (see [Klimyk and Schmüdgen 1997, Equations 9.18 and 9.20, pp. 311–312], [Parshall and Wang 1991, p. 43]). Note that $[I \mid J]$ is homogeneous of degree

$$(\epsilon_{i_1} + \cdots + \epsilon_{i_t},\ \epsilon_{j_1} + \cdots + \epsilon_{j_t})$$

with respect to the grading of Section 2.2.

Comultiplication of quantum minors is given by the rule

$$\text{(2-14)} \qquad \Delta\big([I \mid J]\big) = \sum_{\substack{K \subseteq \{1,\dots,n\} \\ |K| = |I|}} [I \mid K]\,[K \mid J]$$

(see, for example, [Klimyk and Schmüdgen 1997, Proposition 9.7(ii), p. 312]).

2.6. ***Transpose and antitranspose.*** As observed in [Parshall and Wang 1991, Proposition 3.7.1(1)], there is a k-algebra automorphism τ on A such that $\tau(X_{ij}) = X_{ji}$ for all i, j. We refer to τ as the *transpose automorphism*. There is also a k-algebra anti-automorphism τ_2 on A sending $X_{ij} \mapsto X_{n+1-i,\, n+1-j}$ for all i, j [Parshall and Wang 1991, Proposition 3.7.1(2)]. This proposition also shows that τ is a coalgebra anti-automorphism, while τ_2 is a coalgebra automorphism; that is,

$$\Delta \circ \tau = \phi \circ (\tau \otimes \tau) \circ \Delta \qquad \text{and} \qquad \Delta \circ \tau_2 = (\tau_2 \otimes \tau_2) \circ \Delta,$$

where ϕ is the *flip* automorphism on $A \otimes A$, sending $a \otimes b \mapsto b \otimes a$ for all $a, b \in A$. Hence,

$$\Delta\tau(a) = \sum_{(a)} \tau(a_2) \otimes \tau(a_1) \qquad \text{and} \qquad \Delta\tau_2(a) = \sum_{(a)} \tau_2(a_1) \otimes \tau_2(a_2)$$

for $a \in A$. Consequently, when writing out $\Delta\tau(a)$ and $\Delta\tau_2(a)$ in Sweedler's notation, we may take

$$\begin{aligned} \tau(a)_1 &= \tau(a_2), & \tau(a)_2 &= \tau(a_1), \\ \tau_2(a)_1 &= \tau_2(a_1), & \tau_2(a)_2 &= \tau_2(a_2). \end{aligned} \tag{2-15}$$

We recall from [Parshall and Wang 1991, Lemma 4.3.1] that

$$\tau\big([I \mid J]\big) = [J \mid I] \qquad \text{and} \qquad \tau_2\big([I \mid J]\big) = [\omega_0 I \mid \omega_0 J] \tag{2-16}$$

for all quantum minors $[I \mid J]$ in A, where ω_0 is the longest element of S_n, that is, the permutation $i \mapsto n+1-i$.

As discussed in [Parshall and Wang 1991, Remark 3.7.2], there is an isomorphism (of bialgebras) $\mathcal{O}_q(M_n(k)) \to \mathcal{O}_{q^{-1}}(M_n(k))$ that sends $X_{ij} \mapsto X'_{n+1-i,\, n+1-j}$ for all i, j, where the $X'_{\bullet,\bullet}$ are the standard generators for $\mathcal{O}_{q^{-1}}(M_n(k))$. We call this isomorphism β and use the notation $[I \mid J]'$ for quantum minors in $\mathcal{O}_{q^{-1}}(M_n(k))$. It was shown in [Goodearl and Lenagan 2003, proof of Corollary 5.9] that

$$\beta\big([I \mid J]\big) = [\omega_0 I \mid \omega_0 J]' \tag{2-17}$$

for all quantum minors $[I \mid J]$ in A.

Lemma 2.7. *The form* $\mathbf{r}$ *satisfies* $\mathbf{r}(a, b) = \mathbf{r}(\tau(b), \tau(a)) = \mathbf{r}(\tau_2(b), \tau_2(a))$ *for all* $a, b \in A$. *In particular*,

$$\mathbf{r}\big([I \mid J], [M \mid N]\big) = \mathbf{r}\big([N \mid M], [J \mid I]\big) = \mathbf{r}\big([\omega_0 M \mid \omega_0 N], [\omega_0 I \mid \omega_0 J]\big) \tag{2-18}$$

for all quantum minors $[I \mid J]$ *and* $[M \mid N]$ *in* A.

Proof. Set $\mathbf{r}'(a, b) = \mathbf{r}(\tau(b), \tau(a))$ and $\mathbf{r}''(a, b) = \mathbf{r}(\tau_2(b), \tau_2(a))$ for all $a, b \in A$. From (2-12), note that $\mathbf{r}'(X_{ij}, X_{lm}) = \mathbf{r}''(X_{ij}, X_{lm}) = \mathbf{r}(X_{ij}, X_{lm})$ for all i, j, l, m. To prove that $\mathbf{r}'$ and $\mathbf{r}''$ coincide with $\mathbf{r}$, it suffices to show that these forms agree on all monomials in the X_{ij}. This will be clear by induction on the lengths of the monomials, once we show that $\mathbf{r}'$ and $\mathbf{r}''$ satisfy (2-8) and (2-9). With the aid of (2-15), these identities are routine; we give one sample:

$$\begin{aligned}\mathbf{r}'(ab,c) &= \mathbf{r}\big(\tau(c),\tau(a)\tau(b)\big)\\ &= \sum_{(\tau(c))}\mathbf{r}\big(\tau(c)_1,\tau(b)\big)\,\mathbf{r}\big(\tau(c)_2,\tau(a)\big) = \sum_{(c)}\mathbf{r}\big(\tau(c_2),\tau(b)\big)\,\mathbf{r}\big(\tau(c_1),\tau(a)\big)\\ &= \sum_{(c)}\mathbf{r}'(b,c_2)\,\mathbf{r}'(a,c_1) = \sum_{(c)}\mathbf{r}'(a,c_1)\,\mathbf{r}'(b,c_2),\end{aligned}$$

for all $a,b,c \in A$. □

2.8. ***Definition of the quantities $\ell(S;T)$.*** Many formulas concerning quantum minors require powers of q or $-q$ whose exponents are quantities that might be called the number of inversions between two sets. We follow [Noumi et al. 1993] in defining

$$\ell(S;T) = \big|\{(s,t)\in S\times T \mid s>t\}\big| \tag{2-19}$$

for any subsets $S,T\subseteq\{1,\dots,n\}$.

2.9. ***Quantum Laplace relations.*** We shall need the following *q-Laplace relations* from [Noumi et al. 1993, Proposition 1.1], for index sets $I,J\subseteq\{1,\dots,n\}$ of the same cardinality. If I_1, I_2 are nonempty subsets of I with $|I_1|+|I_2|=|I|$, then

$$\sum_{\substack{J=J_1\sqcup J_2\\ |J_l|=|I_l|}}(-q)^{\ell(J_1;J_2)}[I_1\,|\,J_1]\,[I_2\,|\,J_2] = \begin{cases}(-q)^{\ell(I_1;I_2)}[I\,|\,J] & \text{if } I_1\cap I_2=\varnothing,\\ 0 & \text{if } I_1\cap I_2\neq\varnothing;\end{cases} \tag{2-20}$$

while if J_1, J_2 are nonempty subsets of J with $|J_1|+|J_2|=|J|$, then

$$\sum_{\substack{I=I_1\sqcup I_2\\ |I_l|=|J_l|}}(-q)^{\ell(I_1;I_2)}[I_1\,|\,J_1]\,[I_2\,|\,J_2] = \begin{cases}(-q)^{\ell(J_1;J_2)}[I\,|\,J] & \text{if } J_1\cap J_2=\varnothing,\\ 0 & \text{if } J_1\cap J_2\neq\varnothing.\end{cases} \tag{2-21}$$

Observe that (2-20) holds trivially in case I_1 or I_2 is empty, and that (2-21) holds trivially in case J_1 or J_2 is empty

Reduction formulas for the values of the form $\mathbf{r}$ can be obtained by combining (2-8) and (2-9) with (2-20) and (2-21). For example, if $J=J_1\sqcup J_2$, then (2-21) together with (2-8) yields

$$(2\text{-}22)\quad (-q)^{\ell(J_1;J_2)}\,\mathbf{r}\big([I\,|\,J],[M\,|\,N]\big) = \sum_{I=I_1\sqcup I_2}\sum_{L}(-q)^{\ell(I_1;I_2)}\,\mathbf{r}\big([I_1\,|\,J_1],[M\,|\,L]\big)\,\mathbf{r}\big([I_2\,|\,J_2],[L\,|\,N]\big)$$

for all $[M\,|\,N]$.

2.10. ***Some further notation.*** To simplify notation for operations on index sets, we often omit braces from singletons; in particular, we write

$$(2\text{-}23)\qquad I\setminus i = I\setminus\{i\},\qquad I\sqcup l = I\sqcup\{l\},\qquad I\setminus i\sqcup l = \big(I\setminus\{i\}\big)\sqcup\{l\},$$

for $i\in I$ and $l\notin I$. The Kronecker delta symbol will be applied to index sets as well as to individual indices; thus, $\delta(I,J)=1$ when $I=J$, while $\delta(I,J)=0$ when $I\neq J$. In the case of an index versus an index set, the Kronecker symbol will be used to indicate membership, that is, $\delta(i,I)=1$ means $i\in I$, while $\delta(i,I)=0$ means $i\notin I$.

Finally, we shall need the following partial order on index sets of the same cardinality. If I and J are t-element subsets of $\{1,\dots,n\}$, write their elements in ascending order, say,

$$I=\{i_1<i_2<\cdots<i_t\}\qquad\text{and}\qquad J=\{j_1<j_2<\cdots<j_t\},$$

and then define

$$(2\text{-}24)\qquad I\le J\qquad\text{if and only if}\qquad i_l\le j_l\quad\text{for } l=1,\dots,t.$$

3. Initial computations

Throughout this section, let i and j denote indices in $\{1,\dots,n\}$, and let I, J, M, N denote index sets contained in $\{1,\dots,n\}$, with $|I|=|J|$ and $|M|=|N|$.

Lemma 3.1. $\mathbf{r}\big(X_{ii},[I\,|\,J]\big)=\mathbf{r}\big([I\,|\,J],X_{ii}\big)=q^{\delta(i,I)}\delta(I,J)$.

Proof. Write $I=\{i_1<\cdots<i_t\}$ and $J=\{j_1<\cdots<j_t\}$, and, using (2-13) and (2-8), note that

$$(3\text{-}1)\quad \mathbf{r}\big([I\,|\,J],X_{ii}\big) = \sum_{\sigma\in S_t}(-q)^{\ell(\sigma)}\sum_{l_1,\dots,l_{t-1}}^{n}\mathbf{r}(X_{i_1j_{\sigma(1)}},X_{il_1})\,\mathbf{r}(X_{i_2j_{\sigma(2)}},X_{l_1l_2})\cdots\mathbf{r}(X_{i_tj_{\sigma(t)}},X_{l_{t-1}i}).$$

In view of (2-12), a nonzero term can occur in the second summation of (3-1) only when $i\le l_1\le l_2\le\cdots\le l_{t-1}\le i$, that is, when $l_1=\cdots=l_{t-1}=i$. Hence, (3-1) reduces to

$$\text{(3-2)}\quad \mathbf{r}\big([I\,|\,J], X_{ii}\big) = \sum_{\sigma\in S_t} (-q)^{\ell(\sigma)} \mathbf{r}(X_{i_1 j_{\sigma(1)}}, X_{ii})\, \mathbf{r}(X_{i_2 j_{\sigma(2)}}, X_{ii}) \cdots \mathbf{r}(X_{i_t j_{\sigma(t)}}, X_{ii}).$$

In (3-2), a nonzero term can occur in the sum only when $i_s = j_{\sigma(s)}$ for $s = 1, \dots, t$. Since the i_s and j_s are arranged in ascending order, this situation only happens when $I = J$ and $\sigma = \mathrm{id}$. Thus, $\mathbf{r}\big([I\,|\,J], X_{ii}\big) = 0$ when $I \neq J$, and

$$\mathbf{r}\big([I\,|\,I], X_{ii}\big) = \mathbf{r}(X_{i_1 i_1}, X_{ii})\, \mathbf{r}(X_{i_2 i_2}, X_{ii}) \cdots \mathbf{r}(X_{i_t i_t}, X_{ii}) = q^{\delta(i,I)}.$$

The formula for $\mathbf{r}\big(X_{ii}, [I\,|\,J]\big)$ follows via Lemma 2.7. □

Lemma 3.2. $\mathbf{r}(X_{ij}, \cdot\,) \equiv 0$ *when* $i < j$, *and* $\mathbf{r}(\,\cdot\,, X_{ij}) \equiv 0$ *when* $i > j$.

Proof. Consider any monomial $a = X_{i(1),j(1)}\, X_{i(2),j(2)} \cdots X_{i(t),j(t)} \in A$. By (2-8),

$$\mathbf{r}(a, X_{ij}) = \sum_{l_1,\dots,l_{t-1}}^{n} \mathbf{r}(X_{i(1),j(1)}, X_{il_1}) \mathbf{r}(X_{i(2),j(2)}, X_{l_1 l_2}) \cdots \mathbf{r}(X_{i(t),j(t)}, X_{l_{t-1} j}).$$

If some term $\mathbf{r}(X_{i(1),j(1)}, X_{il_1})\, \mathbf{r}(X_{i(2),j(2)}, X_{l_1 l_2}) \cdots \mathbf{r}(X_{i(t),j(t)}, X_{l_{t-1} j})$ does not vanish, we must have $i \le l_1 \le \cdots \le l_{t-1} \le j$. This shows that $\mathbf{r}(\,\cdot\,, X_{ij})$ can fail to vanish only when $i \le j$. The first statement of the lemma follows via Lemma 2.7. □

Corollary 3.3. $\mathbf{r}\big([I\,|\,J], \cdot\,\big) \equiv 0$ *when* $I \not\geq J$, *and* $\mathbf{r}\big(\,\cdot\,, [I\,|\,J]\big) \equiv 0$ *when* $I \not\leq J$.

Proof. Write $I = \{i_1 < \cdots < i_t\}$ and $J = \{j_1 < \cdots < j_t\}$, and suppose that $\mathbf{r}\big([I\,|\,J], c\big) \neq 0$ for some $c \in A$. Then, by (2-13) and (2-8),

$$\sum_{(c)} \mathbf{r}(X_{i_1 j_{\sigma(1)}}, c_1)\, \mathbf{r}(X_{i_2 j_{\sigma(2)}}, c_2) \cdots \mathbf{r}(X_{i_t j_{\sigma(t)}}, c_t) \neq 0$$

for some $\sigma \in S_t$. Lemma 3.2 then implies that $i_s \ge j_{\sigma(s)}$ for $s = 1, \dots, t$.

First, $i_1 \ge j_{\sigma(1)} \ge j_1$. Now let $1 < s \le t$. If $\sigma(s) \ge s$, then $i_s \ge j_{\sigma(s)} \ge j_s$. If $\sigma(s) < s$, then $\sigma(u) \ge s$ for some $u < s$, whence $i_s > i_u \ge j_{\sigma(u)} \ge j_s$. Thus, $i_s \ge j_s$ for all s, and therefore $I \ge J$. Similarly, if $\mathbf{r}\big(\,\cdot\,, [I\,|\,J]\big)$ does not vanish, then $I \le J$. □

Proposition 3.4. *If* $i < j$, *then*

$$\text{(3-3)}\quad \mathbf{r}\big([I\,|\,J], X_{ij}\big) = \hat{q}(-q)^{|[1,i)\cap J| - |[1,j)\cap I|}\, \delta(i,J)\, \delta(j,I)\, \delta(I \setminus j, J \setminus i)$$

$$\text{(3-4)}\quad = \hat{q}(-q)^{-|(i,j)\cap I\cap J|}\, \delta(i,J)\, \delta(j,I)\, \delta(I \setminus j, J \setminus i).$$

Proof. Note first that (3-4) follows from (3-3). For, if the right-hand side of (3-3) is nonzero, then $I = (I \cap J) \sqcup j$ and $J = (I \cap J) \sqcup i$, whence $[1, i) \cap J = [1, i) \cap I \cap J = [1, i] \cap I \cap J$ and $[1, j) \cap I = [1, j) \cap I \cap J$.

We induct on $|I|$, the case $|I| = 1$ being clear from (2-12). Assume that $|I| > 1$, and suppose that $\mathbf{r}\big([I \mid J], X_{ij}\big) \neq 0$.

Choose $s \in I$, and write $I = I_1 \sqcup I_2$ with $I_1 = \{s\}$ and $I_2 = I \setminus \{s\}$. The q-Laplace relation (2-20) yields

$$(-q)^{|[1,s)\cap I|}[I \mid J] = \sum_{t\in J}(-q)^{|[1,t)\cap J|} X_{st}[I \setminus s \mid J \setminus t]. \tag{3-5}$$

For each $t \in J$, we have

$$\mathbf{r}\big(X_{st}[I \setminus s \mid J \setminus t], X_{ij}\big) = \sum_{l=1}^{n} \mathbf{r}(X_{st}, X_{il})\, \mathbf{r}\big([I \setminus s \mid J \setminus t], X_{lj}\big). \tag{3-6}$$

Since $\mathbf{r}\big([I \mid J], X_{ij}\big) \neq 0$, we must have $\mathbf{r}(X_{st}, X_{il})\, \mathbf{r}\big([I \setminus s \mid J \setminus t], X_{lj}\big) \neq 0$ for some $l \in \{1, \dots, n\}$ and $t \in J$.

Suppose that $i \notin J$. Then $t \neq i$, and so, because $\mathbf{r}(X_{st}, X_{il}) \neq 0$, we must have $t = s$ and $l = i$. Then $\mathbf{r}\big([I \setminus s \mid J \setminus s], X_{ij}\big) \neq 0$, which contradicts the induction hypothesis because $i \notin J \setminus s$. Therefore $i \in J$.

Next, suppose that $j \notin I \setminus s$. If $l < j$, we would have $\mathbf{r}\big([I \setminus s \mid J \setminus t], X_{lj}\big) = 0$ by the induction hypothesis. Since $\mathbf{r}(\,\cdot\,, X_{lj})$ would vanish if $l > j$, we must have $l = j$. Now $\mathbf{r}(X_{st}, X_{ij}) \neq 0$, and so $s = j$ and $t = i$. Thus, either $j \in I \setminus s$ or $j = s$, and so in any case we conclude that $j \in I$.

We may now assume that $s = j$. Since $j \notin I \setminus j$, we have $\mathbf{r}\big([I \setminus j \mid J \setminus t], X_{ij}\big) = 0$ for all $t \in J$ by the induction hypothesis. On the other hand, $\mathbf{r}(X_{jt}, X_{il}) = 0$ for $l \neq i, j$, and $\mathbf{r}(X_{jt}, X_{ij}) = 0$ for $t \neq i$. Hence, the right-hand side of (3-6) vanishes when $t \neq i$, and it equals $\hat{q}\, \mathbf{r}\big([I \setminus j \mid J \setminus i], X_{jj}\big)$ when $t = i$. Combining (3-5) and (3-6) thus yields

$$(-q)^{|[1,j)\cap I|}\, \mathbf{r}\big([I \mid J], X_{ij}\big) = (-q)^{|[1,i)\cap J|}\, \hat{q}\, \mathbf{r}\big([I \setminus j \mid J \setminus i], X_{jj}\big). \tag{3-7}$$

Since the left-hand side of (3-7) is nonzero by assumption, Lemma 3.1 implies that $I \setminus j = J \setminus i$ and $\mathbf{r}\big([I \setminus j \mid J \setminus i], X_{jj}\big) = 1$. The formula (3-3) follows, and the induction step is established. □

Corollary 3.5. *If $i > j$, then*

$$\mathbf{r}\big(X_{ij}, [I \mid J]\big) = \hat{q}(-q)^{|[1,j)\cap I| - |[1,i)\cap J|}\, \delta(i, J)\, \delta(j, I)\, \delta(I \setminus j, J \setminus i) \tag{3-8}$$

$$= \hat{q}(-q)^{-|(j,i)\cap I\cap J|}\, \delta(i, J)\, \delta(j, I)\, \delta(I \setminus j, J \setminus i). \tag{3-9}$$

Proof. Apply Lemma 2.7 to Proposition 3.4. □

Proposition 3.6. $\mathbf{r}\big([I \mid I], [M \mid N]\big) = \mathbf{r}\big([M \mid N], [I \mid I]\big) = q^{|I\cap M|}\, \delta(M, N)$.

Proof. This is parallel to the proof of Lemma 3.1. Write $M=\{m_1<\cdots<m_t\}$ and $N=\{n_1<\cdots<n_t\}$, and note that

$$\mathbf{r}\big([M\,|\,N],[I\,|\,I]\big)=\sum_{\sigma\in S_t}(-q)^{\ell(\sigma)}\mathbf{r}\big(X_{m_1n_{\sigma(1)}}X_{m_2n_{\sigma(2)}}\cdots X_{m_tn_{\sigma(t)}},[I\,|\,I]\big), \tag{3-10}$$

while for each $\sigma\in S_t$ we have

$$\begin{aligned}&\mathbf{r}\big(X_{m_1n_{\sigma(1)}}X_{m_2n_{\sigma(2)}}\cdots X_{m_tn_{\sigma(t)}},[I\,|\,I]\big)\\ &\quad=\sum_{L_1,\dots,L_{t-1}}\mathbf{r}\big(X_{m_1n_{\sigma(1)}},[I\,|\,L_1]\big)\,\mathbf{r}\big(X_{m_2n_{\sigma(2)}},[L_1\,|\,L_2]\big)\cdots\mathbf{r}\big(X_{m_tn_{\sigma(t)}},[L_{t-1}\,|\,I]\big).\end{aligned} \tag{3-11}$$

Consider the right-hand side of (3-11). By Corollary 3.3, a nonzero term can occur in that sum only when $I\le L_1\le\cdots\le L_{t-1}\le I$, and so only when all $L_s=I$. Thus,

$$\begin{aligned}&\mathbf{r}\big([M\,|\,N],[I\,|\,I]\big)\\ &\quad=\sum_{\sigma\in S_t}(-q)^{\ell(\sigma)}\mathbf{r}\big(X_{m_1n_{\sigma(1)}},[I\,|\,I]\big)\,\mathbf{r}\big(X_{m_2n_{\sigma(2)}},[I\,|\,I]\big)\cdots\mathbf{r}\big(X_{m_tn_{\sigma(t)}},[I\,|\,I]\big).\end{aligned} \tag{3-12}$$

In view of Lemma 3.2 and Corollary 3.5, $\mathbf{r}\big(X_{ij},[I\,|\,I]\big)=0$ for all $i\neq j$. Hence, a nonzero term can occur in the right-hand side of (3-12) only when $m_s=n_{\sigma(s)}$ for all s, that is, only when $M=N$ and $\sigma=\mathrm{id}$. Therefore, $\mathbf{r}\big([M\,|\,N],[I\,|\,I]\big)=0$ when $M\neq N$, while, in view of Lemma 3.1,

$$\mathbf{r}\big([M\,|\,M],[I\,|\,I]\big)=\mathbf{r}\big(X_{m_1m_1},[I\,|\,I]\big)\,\mathbf{r}\big(X_{m_2m_2},[I\,|\,I]\big)\cdots\mathbf{r}\big(X_{m_tm_t},[I\,|\,I]\big)=q^{|I\cap M|}.$$

The formula for $\mathbf{r}\big([I\,|\,I],[M\,|\,N]\big)$ follows via Lemma 2.7. □

4. Initial commutation relations

We now use the computations of $\mathbf{r}(\,\cdot\,,\,\cdot\,)$ obtained so far to derive some commutation relations, both to illustrate the method and to double-check the results against known relations in the literature. As in the previous section, let i and j denote indices in $\{1,\dots,n\}$, and let I, J, M, N denote index sets contained in $\{1,\dots,n\}$, with $|I|=|J|$ and $|M|=|N|$.

4.1. ***Direct application of*** **(2-7).** If we set $a=X_{ij}$ and $b=[I\,|\,J]$ in (2-7), we obtain

$$\sum_{l,L}\mathbf{r}\big(X_{il},[I\,|\,L]\big)\,X_{lj}\,[L\,|\,J]=\sum_{l,L}\mathbf{r}\big(X_{lj},[L\,|\,J]\big)\,[I\,|\,L]\,X_{il}. \tag{4-1}$$

We claim that (4-1) reduces to

$$\begin{aligned} q^{\delta(i,I)} X_{ij}[I\,|\,J] + \bigl(1-\delta(i,I)\bigr)\,\hat{q} \sum_{\substack{l\in I\\ l<i}} (-q)^{-|(l,i)\cap I|} X_{lj}[I\setminus l\sqcup i\,|\,J] \\ = q^{\delta(j,J)}[I\,|\,J]X_{ij} + \bigl(1-\delta(j,J)\bigr)\,\hat{q} \sum_{\substack{l\in J\\ l>j}} (-q)^{-|(j,l)\cap J|}[I\,|\,J\setminus l\sqcup j]X_{il}. \end{aligned} \tag{4-2}$$

According to Lemma 3.2 and Corollary 3.3, $\mathbf{r}\bigl(X_{il}, [I\,|\,L]\bigr) = 0$ unless $i \geq l$ and $I \leq L$. By Lemma 3.1, $\mathbf{r}\bigl(X_{ii}, [I\,|\,L]\bigr) = 0$ unless $L = I$, and $\mathbf{r}\bigl(X_{ii}, [I\,|\,I]\bigr) = q^{\delta(i,I)}$. When $i > l$, Corollary 3.5 shows that $\mathbf{r}\bigl(X_{il}, [I\,|\,L]\bigr)$ is nonzero only when $i \in L$, $l \in I$, and $I \setminus l = L \setminus i$. In such cases, $i \notin I$ and $L = I \setminus l \sqcup i$, and the exponent of $-q$ that appears in (3-9) is $-|(l,i)\cap I\cap L| = -|(l,i)\cap I|$. Thus, the left-hand sides of (4-1) and (4-2) agree.

Similarly, $\mathbf{r}\bigl(X_{lj}, [L\,|\,J]\bigr) = 0$ unless $l \geq j$ and $L \leq J$, while $\mathbf{r}\bigl(X_{jj}, [L\,|\,J]\bigr) = 0$ unless $L = J$, and $\mathbf{r}\bigl(X_{jj}, [J\,|\,J]\bigr) = q^{\delta(j,J)}$. When $l > j$, Corollary 3.5 shows that $\mathbf{r}\bigl(X_{lj}, [L\,|\,J]\bigr)$ is nonzero only when $l \in J$, $j \in L \setminus J$, and $L = J \setminus l \sqcup j$. In such cases, the exponent of $-q$ that appears in (3-9) is $-|(j,l)\cap L\cap J| = -|(j,l)\cap J|$. Therefore, the right-hand sides of (4-1) and (4-2) agree. This establishes (4-2).

4.2. ***Application of the transpose automorphism.*** There are several ways to obtain a second commutation relation of a kind similar to (4-2). First, we could set $a = [I\,|\,J]$ and $b = X_{ij}$ in (2-7) and proceed as above. Alternatively, we could apply the automorphism τ, the anti-automorphism τ_2, or the isomorphism β of Section 2.6 to (4-2) itself. As we shall see in Remark 4.4, the first three ways are equivalent, up to some relabelling. The use of β is discussed in Section 4.5.

Among the first three alternatives above, the most convenient choice is to apply the transpose automorphism τ to (4-2). If we do this, and then relabel the terms by interchanging $i \leftrightarrow j$ and $I \leftrightarrow J$, we obtain

$$\begin{aligned} q^{\delta(j,J)} X_{ij}[I\,|\,J] + \bigl(1-\delta(j,J)\bigr)\,\hat{q} \sum_{\substack{l\in J\\ l<j}} (-q)^{-|(l,j)\cap J|} X_{il}[I\,|\,J\setminus l\sqcup j] \\ = q^{\delta(i,I)}[I\,|\,J]X_{ij} + \bigl(1-\delta(i,I)\bigr)\,\hat{q} \sum_{\substack{l\in I\\ l>i}} (-q)^{-|(i,l)\cap I|}[I\setminus l\sqcup i\,|\,J]X_{lj}. \end{aligned} \tag{4-3}$$

4.3. ***Some known cases.*** We now compare some cases of (4-2) and (4-3) with the literature.

When $i \in I$ and $j \in J$, (4-2) and (4-3) both yield $qX_{ij}[I\,|\,J] = q[I\,|\,J]X_{ij}$ — the well-known fact that X_{ij} and $[I\,|\,J]$ commute in that case (this is just the centrality of the quantum determinant in the subalgebra $k\langle X_{st} \mid s \in I,\ t \in J\rangle$). If $i \in I$ and

$j \notin J$, then (4-2) yields

$$qX_{ij}[I \mid J] = [I \mid J]X_{ij} + \hat{q} \sum_{\substack{l \in J \\ l > j}} (-q)^{-|(j,l) \cap J|} [I \mid J \setminus l \sqcup j] X_{il}. \tag{4-4}$$

Multiply (4-4) by q^{-1}, and note that

$$q^{-1}(-q)^{-|(j,l) \cap J|} = -(-q)^{-|[j,l] \cap J|}.$$

Thus modified, (4-4) recovers [Goodearl and Lenagan 2000, Lemma A.1(b)] (this is the second equation of [Parshall and Wang 1991, Lemma 4.5.1(2)], rewritten using the present notation). Similarly, consider the case that $i \notin I$ and $j \in J$. Then (4-3) yields

$$qX_{ij}[I \mid J] = [I \mid J]X_{ij} + \hat{q} \sum_{\substack{l \in I \\ l > i}} (-q)^{-|(i,l) \cap I|} [I \setminus l \sqcup i \mid J] X_{lj}. \tag{4-5}$$

We again multiply by q^{-1}, and note that

$$q^{-1}(-q)^{-|(i,l) \cap I|} = -(-q)^{-|[i,l] \cap I|}.$$

Thus, (4-5) recovers [Goodearl and Lenagan 2002, Lemma A.2(c), Equation (A.3)] (this is the second equation of [Parshall and Wang 1991, Lemma 4.5.1(4)], in present notation).

Finally, consider the case when $i \notin I$ and $j \notin J$. We may assume that $I \sqcup i = J \sqcup j = \{1, \dots, n\}$. If we write $\hat{s} = \{1, \dots, n\} \setminus \{s\}$ for $s = 1, \dots, n$, then (4-2) yields

$$X_{ij}[\hat{i} \mid \hat{j}] + \hat{q} \sum_{\substack{l \in I \\ l < i}} (-q)^{l+1-i} X_{lj}[\hat{l} \mid \hat{j}] = [\hat{i} \mid \hat{j}]X_{ij} + \hat{q} \sum_{\substack{l \in J \\ l > j}} (-q)^{j+1-l} [\hat{i} \mid \hat{l}] X_{il}. \tag{4-6}$$

Multiplying (4-6) by q^{-1} and then interchanging $q \leftrightarrow q^{-1}$ recovers the fourth equation of [Parshall and Wang 1991, Lemma 5.1.2].

Remark 4.4. As mentioned above, (4-3) could also have been obtained by setting $a = [I \mid J]$ and $b = X_{ij}$ in (2-7) and proceeding as with (4-2). In fact, interchanging any choice of a and b in (2-7) has the same effect as applying τ, as we now explain.

First, apply τ to (2-7), and use (2-15) for both a and b. This yields

$$\sum_{(a),(b)} \mathbf{r}(a_1, b_1)\, \tau(a)_1\, \tau(b)_1 = \sum_{(a),(b)} \mathbf{r}(a_2, b_2)\, \tau(b)_2\, \tau(a)_2. \tag{4-7}$$

Invoking Lemma 2.7 and setting $a' = \tau(a)$ and $b' = \tau(b)$, (4-7) becomes

$$\sum_{(a'),(b')} \mathbf{r}(b'_2, a'_2)\, a'_1\, b'_1 = \sum_{(a'),(b')} \mathbf{r}(b'_1, a'_1)\, b'_2\, a'_2. \tag{4-8}$$

Equation (4-8) is nothing but (2-7) with a and b replaced by b' and a', respectively.

Similarly, applying the anti-automorphism τ_2 to (2-7) and relabelling again recovers (2-7) with a and b interchanged.

4.5. *Two further commutation relations.* Each case of commutation relations for X_{ij} and $[I \mid J]$ derived in [Parshall and Wang 1991] has four subcases — two pairs in which one equation of each pair is obtained from the other by inserting a q-Laplace relation. Two commutation relations from each group of four correspond to our equations (4-2) and (4-3). It is more efficient to derive the remaining two by applying the isomorphism β of Section 2.6, as follows. Set

$$A' = \mathbb{O}_{q^{-1}}(M_n(k))$$

and recall the notation X'_{ij} and $[I \mid J]'$ for generators and quantum minors in A'.

First, consider the relation (4-2) in A', but replace i, j, I, J by $\tilde{i}$, $\tilde{j}$, $\tilde{I}$, $\tilde{J}$, respectively. The result is

$$q^{-\delta(\tilde{i},\tilde{I})} X'_{\tilde{i}\tilde{j}}[\tilde{I} \mid \tilde{J}]' + \big(1 - \delta(\tilde{i}, \tilde{I})\big)(-\hat{q}) \sum_{\substack{\tilde{l}\in\tilde{I} \\ \tilde{l}<\tilde{i}}} (-q)^{|(\tilde{l},\tilde{i})\cap\tilde{I}|} X'_{\tilde{l}\tilde{j}}[\tilde{I} \setminus \tilde{l} \sqcup \tilde{i} \mid \tilde{J}]'$$
$$= q^{-\delta(\tilde{j},\tilde{J})}[\tilde{I} \mid \tilde{J}]' X'_{\tilde{i}\tilde{j}} + \big(1 - \delta(\tilde{j}, \tilde{J})\big)(-\hat{q}) \sum_{\substack{\tilde{l}\in\tilde{J} \\ \tilde{l}>\tilde{j}}} (-q)^{|(\tilde{j},\tilde{l})\cap\tilde{J}|} [\tilde{I} \mid \tilde{J} \setminus \tilde{l} \sqcup \tilde{j}]' X'_{\tilde{i}\tilde{l}}. \tag{4-9}$$

Now set

$$\tilde{i} = \omega_0(i), \qquad \tilde{j} = \omega_0(j), \qquad \tilde{l} = \omega_0(l),$$
$$\tilde{I} = \omega_0(I), \qquad \tilde{J} = \omega_0(J),$$

and apply β^{-1} to (4-9). This yields

$$q^{-\delta(i,I)} X_{ij}[I \mid J] + \big(\delta(i, I) - 1\big)\, \hat{q} \sum_{\substack{l\in I \\ l>i}} (-q)^{|(i,l)\cap I|} X_{lj}[I \setminus l \sqcup i \mid J]$$
$$= q^{-\delta(j,J)}[I \mid J] X_{ij} + \big(\delta(j, J) - 1\big)\, \hat{q} \sum_{\substack{l\in J \\ l<j}} (-q)^{|(l,j)\cap J|} [I \mid J \setminus l \sqcup j] X_{il}. \tag{4-10}$$

Similarly, the relation (4-3) in A' can be written

$$q^{-\delta(\tilde{j},\tilde{J})} X'_{\tilde{i}\tilde{j}}[\tilde{I} \mid \tilde{J}]' + \big(1 - \delta(\tilde{j}, \tilde{J})\big)(-\hat{q}) \sum_{\substack{\tilde{l}\in\tilde{J} \\ \tilde{l}<\tilde{j}}} (-q)^{|(\tilde{l},\tilde{j})\cap\tilde{J}|} X'_{\tilde{i}\tilde{l}}[\tilde{I} \mid \tilde{J} \setminus \tilde{l} \sqcup \tilde{j}]'$$
$$= q^{-\delta(\tilde{i},\tilde{I})}[\tilde{I} \mid \tilde{J}]' X'_{\tilde{i}\tilde{j}} + \big(1 - \delta(\tilde{i}, \tilde{I})\big)(-\hat{q}) \sum_{\substack{\tilde{l}\in\tilde{I} \\ \tilde{l}>\tilde{i}}} (-q)^{|(\tilde{i},\tilde{l})\cap\tilde{I}|} [\tilde{I} \setminus \tilde{l} \sqcup \tilde{i} \mid \tilde{J}]' X'_{\tilde{l}\tilde{j}}. \tag{4-11}$$

Applying β^{-1} to (4-11) as above, we conclude that

$$(4\text{-}12)\quad q^{-\delta(j,J)}X_{ij}[I\,|\,J]+\big(\delta(j,J)-1\big)\,\hat{q}\sum_{\substack{l\in J\\ l>j}}(-q)^{|(j,l)\cap J|}X_{il}[I\,|\,J\setminus l\sqcup j]$$
$$=q^{-\delta(i,I)}[I\,|\,J]X_{ij}+\big(\delta(i,I)-1\big)\,\hat{q}\sum_{\substack{l\in I\\ l<i}}(-q)^{|(l,i)\cap I|}[I\setminus l\sqcup i\,|\,J]X_{lj}.$$

4.6. *Quasicommutation.* Elements $a, b \in A$ are said to *quasicommute* or *q-commute* provided they commute up to a power of q, that is, $ab = q^m ba$ for some integer m. The relations (2-6) say that two of the standard generators for A which have the same row (or column) indices must quasicommute, and it is natural to expect other instances of this in A. From the results above, we can recover the quasicommutation relations for quantum minors given by Krob and Leclerc [1995]. These apply to certain quantum minors whose row (or column) index sets are disjoint. Cases allowing nondisjoint index sets were obtained by Leclerc and Zelevinsky [1998, Lemmas 2.1–2.3] by applying quantum Plücker relations. Building on the results of [Leclerc and Zelevinsky 1998], Scott [2005, Theorems 1 and 2] determined exactly which pairs of quantum minors quasicommute, and calculated the corresponding relations. We will recover some other cases of his results in Corollary 6.5.

First, consider X_{ij} and $[M\,|\,N]$, with $i \in M$. If $j < \min(N)$, then either (4-3) or (4-10) implies that $X_{ij}[M\,|\,N] = q[M\,|\,N]X_{ij}$, while if $j > \max(N)$, then, by either (4-2) or (4-12), $X_{ij}[M\,|\,N] = q^{-1}[M\,|\,N]X_{ij}$. Of course, if $j \in N$, then $X_{ij}[M\,|\,N] = [M\,|\,N]X_{ij}$.

Now suppose that $I \subseteq M$ and that J and N are *separated* in the following sense: there is a partition $J = J' \sqcup J''$ such that

$$\max(J') < \min(N) \le \max(N) < \min(J'').$$

Each of the generators $X_{i_{\sigma(l)},j_l}$ occurring in (2-13) quasicommutes with $[M\,|\,N]$ as in the previous paragraph, whence

$$X_{i_{\sigma(1)},j_1}X_{i_{\sigma(2)},j_2}\cdots X_{i_{\sigma(t)},j_t}[M\,|\,N] = q^{|J'|-|J''|}[M\,|\,N]X_{i_{\sigma(1)},j_1}X_{i_{\sigma(2)},j_2}\cdots X_{i_{\sigma(t)},j_t}$$

for all $\sigma \in S_t$. Consequently, under the present hypotheses,

$$(4\text{-}13)\qquad [I\,|\,J]\,[M\,|\,N] = q^{|J'|-|J''|}[M\,|\,N]\,[I\,|\,J].$$

This recovers [Krob and Leclerc 1995, Lemma 3.7] (after interchanging q and q^{-1}). In fact, (4-13) holds when $I \subseteq M$, and J and N are *weakly separated* in the sense of [Leclerc and Zelevinsky 1998], meaning that there is a partition $J \setminus N = J' \sqcup J''$ such that $\max(J') < \min(N \setminus J) \le \max(N \setminus J) < \min(J'')$ [Leclerc and Zelevinsky 1998, Lemma 2.1].

Applying τ to (4-13) and relabelling, we find that

$$[I \mid J]\,[M \mid N] = q^{|I'|-|I''|}\,[M \mid N]\,[I \mid J] \tag{4-14}$$

when $J \subseteq N$ and $I = I' \sqcup I''$ with $\max(I') < \min(M) \le \max(M) < \min(I'')$.

5. Computation of $\mathbf{r}\big([I \mid J], [M \mid N]\big)$

Throughout this section, let I, J, M, N denote index sets contained in the interval $\{1, \dots, n\}$, with $|I| = |J|$ and $|M| = |N|$. Our goal is to develop a formula for $\mathbf{r}\big([I \mid J], [M \mid N]\big)$.

Lemma 5.1. *If $\mathbf{r}\big([I \mid J], [M \mid N]\big) \neq 0$, then $I \cap M = J \cap N$ and $I \cup M = J \cup N$.*

Proof. We induct on $|I|$, starting with the case $[I \mid J] = X_{ij}$. If $i = j$, Lemma 3.1 implies that $M = N$, and the conclusion is clear. If $i \neq j$, then $i > j$ by Lemma 3.2, whence Corollary 3.5 implies that $i \in N$, $j \in M$, and $M \setminus j = N \setminus i$. Consequently, $I \cap M = J \cap N = \varnothing$ and $I \cup M = J \cup N$.

Now suppose that $|I| \ge 2$. If $I = J$, then Proposition 3.6 implies that $M = N$, and we are done. Hence, we may assume that $I \neq J$. Since $|I| = |J|$, there must exist an element $j \in J \setminus I$. Set $J = J_1 \sqcup J_2$ with $J_1 = \{j\}$ and $J_2 = J \setminus j$, and write (2-22) in the form

$$\pm q^{\bullet}\, \mathbf{r}\big([I \mid J], [M \mid N]\big) = \sum_{i \in I} \sum_{L} \pm q^{\bullet}\, \mathbf{r}\big(X_{ij}, [M \mid L]\big)\, \mathbf{r}\big([I \setminus i \mid J \setminus j], [L \mid N]\big). \tag{5-1}$$

Since $\mathbf{r}\big([I \mid J], [M \mid N]\big) \neq 0$, (5-1) implies that

$$\mathbf{r}\big(X_{ij}, [M \mid L]\big)\, \mathbf{r}\big([I \setminus i \mid J \setminus j], [L \mid N]\big) \neq 0 \tag{5-2}$$

for some $i \in I$ and some L.

Note that $i \neq j$, because $j \notin I$. Equation (5-2) and Lemma 3.2 now show that $i > j$, and then Corollary 3.5 implies that $i \in L$, $j \in M$, and $L \setminus i = M \setminus j$. Consequently, $i \notin M$ and $j \notin L$, while $L = (L \cap M) \sqcup i$ and $M = (L \cap M) \sqcup j$. Since the second factor of (5-2) is nonzero, our induction implies that $(I \setminus i) \cap L = (J \setminus j) \cap N$ and $(I \setminus i) \cup L = (J \setminus j) \cup N$. Now,

$$I \cup (L \cap M) = (I \setminus i) \cup i \cup (L \cap M) = (I \setminus i) \cup L = (J \setminus j) \cup N,$$

and so $I \cup M = I \cup (L \cap M) \cup j = J \cup N$. Since $j \notin I \cup L$, we see from the equation $(I \setminus i) \cup L = (J \setminus j) \cup N$ that $j \notin N$. Consequently,

$$I \cap M = I \cap (M \setminus j) = I \cap (L \setminus i) = (I \setminus i) \cap L = (J \setminus j) \cap N = J \cap N.$$

This establishes the induction step. □

Lemma 5.2. *If $I \cap M = J \cap N$ and $I \cup M = J \cup N$, then:*

(a) $I \setminus J = N \setminus M$ *and* $J \setminus I = M \setminus N$;

(b) $\mathbf{r}\big([I \mid J], [M \mid N]\big) = q^{|I\cap M|}(-q)^{\ell(I;\,J\cap N)-\ell(J;\,I\cap M)}\mathbf{r}\big([I \setminus M \mid J \setminus N], [M \mid N]\big)$.

Proof. (a) This follows easily from the hypotheses.

(b) Write $J = J_1 \sqcup J_2$ with $J_1 = J \setminus N$ and $J_2 = J \cap N = I \cap M$, and recall equation (2-22). We focus first on the term on the right-hand side of (2-22) with $I_2 = J_2$ and $L = N$, in which case $I_1 = I \setminus M$. For this term, we have

$$(5\text{-}3)\qquad (-q)^{\ell(I_1;\,I_2)}\mathbf{r}\big([I_1 \mid J_1], [M \mid L]\big)\,\mathbf{r}\big([I_2 \mid J_2], [L \mid N]\big) \\ = (-q)^{\ell(I\setminus M;\,J\cap N)}q^{|I\cap M|}\,\mathbf{r}\big([I \setminus M \mid J \setminus N], [M \mid N]\big),$$

in view of Proposition 3.6. We claim that all other terms on the right-hand side of (2-22) vanish.

Suppose that $\mathbf{r}\big([I_1 \mid J_1], [M \mid L]\big)\,\mathbf{r}\big([I_2 \mid J_2], [L \mid N]\big)$ is nonzero for some I_1, I_2, L. Lemma 5.1 implies that $I_2 \cap L = J_2 \cap N = J_2$, and then, because $|I_2| = |J_2|$, we must have $I_2 = J_2$. Consequently, Proposition 3.6 implies that $L = N$, verifying the claim. Equations (2-22) and (5-3) thus yield

$$(5\text{-}4)\qquad (-q)^{\ell(J\setminus N;\,I\cap M)}\mathbf{r}\big([I \mid J], [M \mid N]\big) \\ = (-q)^{\ell(I\setminus M;\,J\cap N)}q^{|I\cap M|}\,\mathbf{r}\big([I \setminus M \mid J \setminus N], [M \mid N]\big).$$

Finally, we have

$$\ell(I;\, J\cap N) = \ell(I \setminus M;\, J \cap N) + \ell(I \cap M;\, J \cap N),$$
$$\ell(J;\, I\cap M) = \ell(J \setminus N;\, I \cap M) + \ell(J \cap N;\, I \cap M),$$

and, since $I \cap M = J \cap N$, we obtain

$$(5\text{-}5)\qquad \ell(I \setminus M;\, J\cap N) - \ell(J \setminus N;\, I \cap M) = \ell(I;\, J \cap N) - \ell(J;\, I \cap M).$$

Part (b) follows from (5-4) and (5-5). □

Lemma 5.3. *If* $I \cap M = J \cap N = \varnothing$ *and* $I \cup M = J \cup N$, *then*

$$\mathbf{r}\big([I \mid J], [M \mid N]\big) = (-q)^{\ell(I\cup N;\,I\setminus J)-\ell(J\cup M;\,J\setminus I)}\mathbf{r}\big([I \setminus J \mid J \setminus I], [M \setminus N \mid N \setminus M]\big).$$

Proof. Write $J = J_1 \sqcup J_2$ with $J_1 = I \cap J$ and $J_2 = J \setminus I$, and recall (2-22). Consider the term with $I_1 = J_1$ and $L = M$, in which case $I_2 = I \setminus J$. Since $I_1 \cap M = \varnothing$, Proposition 3.6 implies that $\mathbf{r}\big([I_1 \mid J_1], [M \mid L]\big) = 1$. Thus, for this term of (2-22), we have

$$(5\text{-}6)\qquad (-q)^{\ell(I_1;\,I_2)}\mathbf{r}\big([I_1 \mid J_1], [M \mid L]\big)\,\mathbf{r}\big([I_2 \mid J_2], [L \mid N]\big) \\ = (-q)^{\ell(I\cap J;\,I\setminus J)}\mathbf{r}\big([I \setminus J \mid J \setminus I], [M \mid N]\big).$$

We next claim that all other terms on the right-hand side of (2-22) vanish. Hence, suppose that $\mathbf{r}\big([I_1 \,|\, J_1], [M \,|\, L]\big)\, \mathbf{r}\big([I_2 \,|\, J_2], [L \,|\, N]\big) \neq 0$ for some I_1, I_2, L. Lemma 5.1 implies that $I_2 \cap L = J_2 \cap N = \varnothing$ and $I_2 \cup L = J_2 \cup N = (J \setminus I) \cup N$, from which it follows that $I_2 = N \setminus L$. Now, $I_2 \cap J \subseteq N \cap J = \varnothing$, and so $I_2 \subseteq I \setminus J$. Since also

$$|I_2| = |J_2| = |J \setminus I| = |I \setminus J|,$$

we must have $I_2 = I \setminus J$. Consequently, $I_1 = J_1$, and then Proposition 3.6 implies that $L = M$. This verifies the claim. As a result, (2-22) and (5-6) combine to yield

$$\mathbf{r}\big([I \,|\, J], [M \,|\, N]\big) = (-q)^{\ell(I \cap J; I \setminus J) - \ell(I \cap J; J \setminus I)}\, \mathbf{r}\big([I \setminus J \,|\, J \setminus I], [M \,|\, N]\big). \tag{5-7}$$

Note that $(I \setminus J) \cap M = (J \setminus I) \cap N = \varnothing$ and $(I \setminus J) \cup M = M \cup N = (J \setminus I) \cup N$. Hence, (5-7) also holds with I, J, M, N replaced by N, M, $J \setminus I$, $I \setminus J$, respectively. That is,

$$\begin{aligned} &\mathbf{r}\big([N \,|\, M], [J \setminus I \,|\, I \setminus J]\big) \\ &\qquad = (-q)^{\ell(N \cap M; N \setminus M) - \ell(N \cap M; M \setminus N)}\, \mathbf{r}\big([N \setminus M \,|\, M \setminus N], [J \setminus I \,|\, I \setminus J]\big). \end{aligned} \tag{5-8}$$

In view of Lemma 2.7, (5-8) can be rewritten as

$$\begin{aligned} &\mathbf{r}\big([I \setminus J \,|\, J \setminus I], [M \,|\, N]\big) \\ &\qquad = (-q)^{\ell(N \cap M; N \setminus M) - \ell(N \cap M; M \setminus N)}\, \mathbf{r}\big([I \setminus J \,|\, J \setminus I], [M \setminus N \,|\, N \setminus M]\big). \end{aligned} \tag{5-9}$$

Combining (5-7) and (5-9), we obtain

$$\mathbf{r}\big([I \,|\, J], [M \,|\, N]\big) = (-q)^{\lambda}\, \mathbf{r}\big([I \setminus J \,|\, J \setminus I], [M \setminus N \,|\, N \setminus M]\big), \tag{5-10}$$

where (recalling Lemma 5.2 (a))

$$\begin{aligned} \lambda &= \ell(I \cap J;\, I \setminus J) - \ell(I \cap J;\, J \setminus I) \\ &\qquad\qquad + \ell(N \cap M;\, N \setminus M) - \ell(N \cap M;\, M \setminus N) \\ &= \ell\big((I \cap J) \sqcup (M \cap N);\, I \setminus J\big) - \ell\big((I \cap J) \sqcup (M \cap N);\, J \setminus I\big). \end{aligned} \tag{5-11}$$

Next, observe that

$$I \cup N = (I \setminus J) \sqcup (I \cap J) \sqcup (M \cap N), \qquad J \cup M = (J \setminus I) \sqcup (I \cap J) \sqcup (M \cap N).$$

Because $|I \setminus J| = |J \setminus I|$, we have $\ell(I \setminus J;\, I \setminus J) = \ell(J \setminus I;\, J \setminus I)$, and therefore

$$\lambda = \ell(I \cup N;\, I \setminus J) - \ell(J \cup M;\, J \setminus I). \tag{5-12}$$

Equations (5-10) and (5-12) establish the lemma. $\square$

In view of Lemmas 5.1–5.3, it remains only to calculate $\mathbf{r}\big([I\,|\,J],\,[M\,|\,N]\big)$ in the case when

$$(I\cup N)\cap(J\cup M)=\varnothing \qquad\text{and}\qquad I\cup M=J\cup N,$$

whence $I=N$ and $J=M$. Further, because of Corollary 3.3, we may assume that $I>J$. In these cases, certain sums of powers of $-q$ appear in $\mathbf{r}\big([I\,|\,J],\,[M\,|\,N]\big)$, and we introduce the following notation to deal with them:

5.4. ***Definition of $\xi_q(I;\,J)$.*** Recall that, for $d\in\mathbb{N}$, the $(-q)$-*integer* $[d]_{-q}$ is given by

$$\begin{aligned}[d]_{-q}=\frac{(-q)^d-(-q)^{-d}}{(-q)-(-q)^{-1}}&=(-q)^{d-1}+(-q)^{d-3}+\cdots+(-q)^{-(d-1)}\\&=(-q)^{1-d}(1+q^2+q^4+\cdots+q^{2d-2}).\end{aligned}$$

Hence, $1+q^2+q^4+\cdots+q^{2d-2}=(-q)^{d-1}[d]_{-q}$.

For index sets $I\ge J$, we define a scalar $\xi_q(I;\,J)$ as follows: First, set $m=|I|$ and write $I=\{r_1<\cdots<r_m\}$. Then, set $d_l=|[1,r_l]\cap J|-l+1$ for $l=1,\dots,m$, noting that $d_l\ge 1$ because $J\le I$. Finally, define

$$\xi_q(I;\,J)=[d_1]_{-q}\,[d_2]_{-q}\cdots[d_m]_{-q},$$

with the convention that $\xi_q(\varnothing;\,\varnothing)=1$. When $I\cap J=\varnothing$, as in the next lemma, each $d_l=\ell(r_l;\,J)-l+1$. Note that $[d]_{-q^{-1}}=[d]_{-q}$ for all $d\in\mathbb{N}$, whence $\xi_{q^{-1}}(I;\,J)=\xi_q(I;\,J)$.

Lemma 5.5. *If $I>J$ and $I\cap J=\varnothing$, then*

$$\mathbf{r}\big([I\,|\,J],\,[J\,|\,I]\big)=\hat{q}^{|I|}(-q)^{\ell(J;I)-\ell(I;I)}\xi_q(I;\,J). \tag{5-13}$$

Proof. Set $m=|I|=|J|$, write $I=\{r_1<\cdots<r_m\}$, and set $d_l=\ell(r_l;\,J)-l+1$ for $l=1,\dots,m$ as in Section 5.4.

We proceed by induction on m. If $m=1$, then $J=\{j\}$ for some $j<r_1$, whence $\ell(J;\,I)=\ell(I;\,I)=0$. Moreover, $d_1=1$ and so $\xi_q(I;\,J)=1$. By (2-12), $\mathbf{r}\big([I\,|\,J],\,[J\,|\,I]\big)=\mathbf{r}(X_{r_1j},\,X_{jr_1})=\hat{q}$, which verifies (5-13) in this case.

Now suppose that $m>1$. Write $I=I_1\sqcup I_2$ with $I_1=\{r_1\}$ and $I_2=\{r_2,\dots,r_m\}$. Since $\ell(I_1;\,I_2)=0$, equation (2-20) implies that

$$[I\,|\,J]=\sum_{j\in J}(-q)^{\ell(j;\,J\setminus j)}X_{r_1j}[I_2\,|\,J\setminus j].$$

Applying (2-8), we obtain

$$\begin{aligned}\mathbf{r}\big([I\,|\,J],\,[J\,|\,I]\big)\\=\sum_{j\in J}\sum_{L}(-q)^{|[1,j)\cap J|}\mathbf{r}\big(X_{r_1j},\,[J\,|\,L]\big)\,\mathbf{r}\big([I_2\,|\,J\setminus j],\,[L\,|\,I]\big).\end{aligned} \tag{5-14}$$

According to Lemma 3.2 and Corollary 3.5, a nonzero term can occur on the right-hand side of (5-14) only if $r_1 > j$ and $r_1 \in L$, as well as $J \setminus j = L \setminus r_1$, in which case

$$\mathbf{r}\big(X_{r_1 j}, [J \mid L]\big) = \hat{q}\,(-q)^{|[1,j)\cap J| - |[1,r_1)\cap L|}.$$

Now, $\big|[1,r_1)\cap L\big| = \big|[1,r_1)\cap(L\setminus r_1)\big| = \big|[1,r_1)\cap(J\setminus j)\big| = d_1 - 1$, and so

$$\mathbf{r}\big(X_{r_1 j}, [J \mid L]\big) = \hat{q}\,(-q)^{1+|[1,j)\cap J| - d_1}. \tag{5-15}$$

Next, note that $L = J \setminus j \sqcup r_1$, whence $L \cap I = \{r_1\}$. Consequently, $I_2 \cap L = (J\setminus j)\cap I = \varnothing$ and $I_2 \cup L = I \cup L = (J \setminus j)\cup I$. Lemma 5.3 now implies that

$$\mathbf{r}\big([I_2 \mid J\setminus j], [L \mid I]\big) = (-q)^{\lambda}\,\mathbf{r}\big([I_2 \mid J\setminus j], [J\setminus j \mid I_2]\big) \tag{5-16}$$

where

$$\begin{aligned} \lambda &= \ell(I; I_2) - \ell(L; J\setminus j) \\ &= \ell(I_2; I_2) - \ell(J\setminus j; J\setminus j) + \ell(r_1; I_2) - \ell(r_1; J\setminus j) \;=\; -d_1 + 1. \end{aligned} \tag{5-17}$$

Combining equations (5-14)–(5-17), we obtain

$$\mathbf{r}\big([I \mid J], [J \mid I]\big) = \hat{q} \sum_{\substack{j\in J\\ j<r_1}} (-q)^{2+2|[1,j)\cap J| - 2d_1}\,\mathbf{r}\big([I_2 \mid J\setminus j], [J\setminus j \mid I_2]\big). \tag{5-18}$$

It remains to compute $\mathbf{r}\big([I_2 \mid J\setminus j], [J\setminus j \mid I_2]\big)$ for $j \in J$ with $j < r_1$. Observe that $I_2 > J \setminus j$ for any such j, so that our induction hypothesis will apply. Now,

$$\begin{aligned} \ell(J\setminus j; I_2) &= \ell(J; I_2) = \ell(J; I) - \ell(J; r_1) = \ell(J; I) - m + d_1, \\ \ell(I_2; I_2) &= \ell(I; I_2) = \ell(I; I) - m + 1, \end{aligned}$$

whence $\ell(J\setminus j; I_2) - \ell(I_2; I_2) = \ell(J; I) - \ell(I; I) + d_1 - 1$. For $l = 1, \dots, m-1$, observe that

$$\ell(r_{l+1}; J\setminus j) - l + 1 = \ell(r_{l+1}; J) - l = d_{l+1},$$

and consequently $\xi_q(I_2; J\setminus j) = [d_2]_{-q}\,[d_3]_{-q} \cdots [d_m]_{-q}$. Thus, our induction hypothesis implies that

$$\begin{aligned} &\mathbf{r}\big([I_2 \mid J\setminus j], [J\setminus j \mid I_2]\big) \\ &\qquad = \hat{q}^{m-1}(-q)^{\ell(J;I)-\ell(I;I)+d_1-1}[d_2]_{-q}\,[d_3]_{-q}\cdots[d_m]_{-q}. \end{aligned} \tag{5-19}$$

Inserting (5-19) in (5-18), we obtain

$$\begin{aligned} &\mathbf{r}\big([I \mid J], [J \mid I]\big) \\ &\qquad = \hat{q}^{m}(-q)^{\ell(J;I)-\ell(I;I)+1-d_1}[d_2]_{-q}\,[d_3]_{-q}\cdots[d_m]_{-q} \sum_{\substack{j\in J\\ j<r_1}} q^{2|[1,j)\cap J|}. \end{aligned} \tag{5-20}$$

The summation appearing in (5-20) is just $\sum_{t=1}^{d_1} q^{2(t-1)} = (-q)^{d_1-1}[d_1]_{-q}$, whence

$$[d_2]_{-q}\,[d_3]_{-q}\cdots[d_m]_{-q}\sum_{\substack{j\in J\\ j<r_1}} q^{2|[1,j)\cap J|} = (-q)^{d_1-1}\xi_q(I;\,J). \tag{5-21}$$

Equations (5-20) and (5-21) establish (5-13), completing the induction step. $\square$

Theorem 5.6. *Let $I, J, M, N \subseteq \{1,\ldots,n\}$ with $|I| = |J|$ and $|M| = |N|$.*

(a) *If* $\mathbf{r}\big([I\,|\,J],[M\,|\,N]\big) \neq 0$, *then*

$$I \geq J, \qquad I\cap M = J\cap N, \qquad I\cup M = J\cup N. \tag{5-22}$$

(b) *If conditions* (5-22) *hold*, *then*

$$\mathbf{r}\big([I\,|\,J],[M\,|\,N]\big) = q^{|I\cap M|}\hat{q}^{|I\setminus J|}(-q)^{\lambda}\xi_q(I\setminus J;\,J\setminus I), \tag{5-23}$$

where $\lambda = \ell\big((J\setminus N)\cup(M\setminus I);\,I\setminus J\big) - \ell\big((J\setminus N)\cup(M\setminus I);\,J\setminus I\big)$.

Proof. (a) Follows from Corollary 3.3 and Lemma 5.1.

(b) Recall from Lemma 5.2 that $I\setminus J = N\setminus M$ and $J\setminus I = M\setminus N$. If $I = J$, then we must have $M = N$. In this case, $\mathbf{r}\big([I\,|\,J],[M\,|\,N]\big) = q^{|I\cap M|}$ by Proposition 3.6, and we are done. Now assume that $I \neq J$, and note that $I\setminus J > J\setminus I$. We shall need the observations that

$$\begin{aligned} (I\setminus M)\cup N &= I\cup N, &\qquad (I\setminus M)\setminus(J\setminus N) &= I\setminus J,\\ (J\setminus N)\cup M &= J\cup M, &\qquad (J\setminus N)\setminus(I\setminus M) &= J\setminus I. \end{aligned}$$

Applying successively Lemmas 5.2, 5.3 and 5.5, we obtain

$$\mathbf{r}\big([I\,|\,J],[M\,|\,N]\big) = q^{|I\cap M|}\hat{q}^{|I\setminus J|}(-q)^{\lambda}\xi_q(I\setminus J;\,J\setminus I), \tag{5-24}$$

where

$$\begin{aligned}\lambda = \ell(I;\,J\cap N) - \ell(J;\,I\cap M) + \ell(I\cup N;\,I\setminus J)&\\ - \ell(J\cup M;\,J\setminus I) + \ell(J\setminus I;\,I\setminus J) - \ell(I\setminus J;\,I\setminus J)&.\end{aligned}$$

Observe that $(I\cup N)\sqcup(J\setminus I) = J\cup N = I\cup M = (J\cup M)\sqcup(I\setminus J)$, whence

$$\begin{aligned}\ell(I\cup N;\,I\setminus J) - \ell(J\cup M;\,J\setminus I) + \ell(J\setminus I;\,I\setminus J) - \ell(I\setminus J;\,I\setminus J)&\\ = \ell(J\cup M;\,I\setminus J) - \ell(J\cup M;\,J\setminus I)&.\end{aligned} \tag{5-25}$$

Next, observe that $I\setminus N = J\setminus M$ and $N\setminus I = M\setminus J$. Moreover,

$$\begin{aligned} I\cup M &= I\cup M\cup N = I\sqcup(N\setminus I)\sqcup(M\setminus N),\\ J\cup N &= J\cup M\cup N = J\sqcup(M\setminus J)\sqcup(N\setminus M), \end{aligned}$$

and consequently

$$\ell(I;\, J\cap N) + \ell(N\setminus I;\, J\cap N) + \ell(M\setminus N;\, J\cap N) = \ell(I\cup M;\, J\cap N),$$
$$\ell(J;\, I\cap M) + \ell(M\setminus J;\, I\cap M) + \ell(N\setminus M;\, I\cap M) = \ell(J\cup N;\, I\cap M).$$

It follows that

$$\begin{aligned}\ell(I;\, J\cap N) - \ell(J;\, I\cap M) &= \ell(N\setminus M;\, I\cap M) - \ell(M\setminus N;\, J\cap N)\\ &= |N\setminus M|\cdot|I\cap M| - \ell(I\cap M;\, N\setminus M)\\ &\qquad - |M\setminus N|\cdot|J\cap N| + \ell(J\cap N;\, M\setminus N)\\ &= \ell(I\cap M;\, J\setminus I) - \ell(I\cap M;\, I\setminus J).\end{aligned} \tag{5-26}$$

Finally, since

$$(J\cup M)\setminus(I\cap M) = \big(J\setminus(J\cap N)\big)\cup\big(M\setminus(I\cap M)\big) = (J\setminus N)\cup(M\setminus I),$$

we conclude from (5-25) and (5-26) that

$$\lambda = \ell\big((J\setminus N)\cup(M\setminus I);\, I\setminus J\big) - \ell\big((J\setminus N)\cup(M\setminus I);\, J\setminus I\big). \tag{5-27}$$

In view of (5-24) and (5-27), the theorem is proved. □

Example 5.7. Let $[I\,|\,J] = [45678\,|\,12345]$ and $[M\,|\,N] = [123459\,|\,456789]$, where we have omitted commas between elements of the index sets. It is clear that $I \geq J$; moreover, $I\cap M = \{4, 5\} = J\cap N$ and $I\cup M = \{1, \dots, 9\} = J\cup N$. Hence, conditions (5-22) hold. Now $I\setminus J = \{6, 7, 8\}$ and $J\setminus I = \{1, 2, 3\}$, while $(J\setminus N)\cup(M\setminus I) = \{1, 2, 3, 9\}$, whence

$$\ell\big((J\setminus N)\cup(M\setminus I);\, I\setminus J\big) - \ell\big((J\setminus N)\cup(M\setminus I);\, J\setminus I\big) = 3 - 6 = -3.$$

Since all the elements of $I\setminus J$ are greater than all the elements of $J\setminus I$, we have

$$\xi_q(I\setminus J;\, J\setminus I) = [3]_{-q}\,[2]_{-q}\,[1]_{-q} = (q^2+1+q^{-2})(-q-q^{-1}).$$

Thus, we conclude from (5-23) that

$$\mathbf{r}\big([I\,|\,J], [M\,|\,N]\big) = q^2\hat{q}^3(-q)^{-3}(q^2+1+q^{-2})(-q-q^{-1}).$$

6. General commutation relations

Now that we have formulas for the value of the braiding form $\mathbf{r}$ on pairs of quantum minors, commutation relations follow readily from property (2-7). The following notation for certain index sets and exponents will be helpful in displaying the results. Recall the quantities $\ell(\,\cdot\,,\,\cdot\,)$ and $\xi_q(\,\cdot\,,\,\cdot\,)$ from Section 2.8 and Section 5.4.

6.1. ***Definitions of index sets*** $\{<\!X\|Y\}$ ***and*** $\{>\!X\|Y\}$ ***and numerical quantities*** $\mathscr{L}(S,X,Y)$ ***and*** $\mathscr{L}^\natural(T,X,Y)$**.** For any subsets X and Y of $\{1,\dots,n\}$, define

$$\{<\!X\|Y\} = \bigl\{S \subseteq X\cup Y \bigm| X\cap Y \subseteq S;\ |S| = |X|;\ S < X\bigr\},$$
$$\{>\!X\|Y\} = \bigl\{T \subseteq X\cup Y \bigm| X\cap Y \subseteq T;\ |T| = |X|;\ T > X\bigr\}. \tag{6-1}$$

In Section 7, we shall need index sets $\{\le\!X\|Y\}$ and $\{\ge\!X\|Y\}$, defined in a similar manner. For any set $S \subseteq X\cup Y$ such that $X\cap Y\subseteq S$, define

$$S^\natural = S^\natural_{X,Y} = (X\cap Y)\sqcup\bigl((X\cup Y)\setminus S\bigr). \tag{6-2}$$

Note that, if $S\in\{<\!X\|Y\}$ or $S\in\{>\!X\|Y\}$, then $|S^\natural| = |Y|$. Finally, for $S\in\{<\!X\|Y\}$ and $T\in\{>\!X\|Y\}$, define

$$\mathscr{L}(S,X,Y) = \ell\bigl((S\setminus S^\natural)\cup(Y\setminus X);\, X\setminus S\bigr) - \ell\bigl((S\setminus S^\natural)\cup(Y\setminus X);\, S\setminus X\bigr),$$
$$\mathscr{L}^\natural(T,X,Y)$$
$$= \ell\bigl((T^\natural\setminus T)\cup(X\setminus Y);\, T\setminus X\bigr) - \ell\bigl((T^\natural\setminus T)\cup(X\setminus Y);\, X\setminus T\bigr). \tag{6-3}$$

For example, suppose that $X=\{2,3,4,6\}$ and $Y=\{1,3,5\}$. Then $\{<\!X\|Y\}$ consists of those 4-element subsets S of $\{1,\dots,6\}$ such that $3\in S$ and $S<X$. There are six such sets:

$$\{1,2,3,4\},\quad \{1,2,3,5\},\quad \{1,2,3,6\},$$
$$\{1,3,4,5\},\quad \{1,3,4,6\},\quad \{2,3,4,5\}.$$

Similarly, $\{>\!X\|Y\}$ consists of those 4-element subsets T of $\{1,\dots,6\}$ such that $3\in T$ and $T>X$. There are two: $\{3,4,5,6\}$ and $\{2,3,5,6\}$. Finally, consider the set $S=\{1,2,3,4\}\in\{<\!X\|Y\}$. Then $S^\natural=\{3,5,6\}$, and so

$$\mathscr{L}(S,X,Y) = \ell\bigl(\{1,2,4,5\};\{6\}\bigr) - \ell\bigl((\{1,2,4,5\};\{1\}\bigr) = 0-3.$$

Theorem 6.2. *If* $I,J,M,N\subseteq\{1,\dots,n\}$ *with* $|I|=|J|$ *and* $|M|=|N|$, *then*

$$q^{|I\cap M|}[I\,|\,J]\,[M\,|\,N] + q^{|I\cap M|}\sum_{S\in\{<I\|M\}} \lambda_S\,[S\,|\,J]\,[S^\natural\,|\,N]$$
$$= q^{|J\cap N|}[M\,|\,N]\,[I\,|\,J] + q^{|J\cap N|}\sum_{T\in\{>J\|N\}} \mu_T[M\,|\,T^\natural]\,[I\,|\,T], \tag{6-4}$$

where

$$\lambda_S = \hat q^{|I\setminus S|}(-q)^{\mathscr{L}(S,I,M)}\,\xi_q(I\setminus S;\, S\setminus I)$$
$$\mu_T = \hat q^{|T\setminus J|}(-q)^{\mathscr{L}^\natural(T,J,N)}\,\xi_q(T\setminus J;\, J\setminus T) \tag{6-5}$$

for $S\in\{<\!I\|M\}$ *and* $T\in\{>\!J\|N\}$.

Proof. Taking $a = [I \mid J]$ and $b = [M \mid N]$ in (2-7), we obtain

$$\sum_{\substack{|S|=|I| \\ |S'|=|M|}} \mathbf{r}\big([I \mid S], [M \mid S']\big)\, [S \mid J]\, [S' \mid N] = \sum_{\substack{|T|=|J| \\ |T'|=|N|}} \mathbf{r}\big([T \mid J], [T' \mid N]\big)\, [M \mid T']\, [I \mid T]. \tag{6-6}$$

In view of Corollary 3.3 and Lemma 5.1, the left-hand summation in (6-6) can be restricted to index sets S and S' such that

$$\begin{aligned} |S| &= |I|, & I &\geq S, \\ I \cap M &= S \cap S', & I \cup M &= S \cup S'. \end{aligned} \tag{6-7}$$

Proposition 3.6 shows that the coefficient of the term with $S = I$ and $S' = M$ is $q^{|I \cap M|}$, and that the terms with $S = I$ and $S' \neq M$ vanish.

The index sets S and S' such that $S \neq I$ and (6-7) hold are precisely those for which $S \in \{<I \,\|\, M\}$ and $S' = S^\natural$. For these index sets, Theorem 5.6 shows that

$$\mathbf{r}\big([I \mid S], [M \mid S']\big) = q^{|I \cap M|} \lambda_S.$$

Thus, the left-hand side of (6-6) reduces to the left-hand side of (6-4).

Similarly, the right-hand side of (6-6) reduces to the right-hand side of (6-4), and the theorem is proved. □

Corollary 6.3. *If $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$, then*

$$q^{|J \cap N|} [I \mid J]\, [M \mid N] + q^{|J \cap N|} \sum_{S \in \{<J \,\|\, N\}} \lambda_S\, [I \mid S]\, [M \mid S^\natural] = q^{|I \cap M|} [M \mid N]\, [I \mid J] + q^{|I \cap M|} \sum_{T \in \{>I \,\|\, M\}} \mu_T\, [T^\natural \mid N]\, [T \mid J], \tag{6-8}$$

where

$$\begin{aligned} \lambda_S &= \hat{q}^{|J \setminus S|} (-q)^{\mathscr{L}(S,J,N)}\, \xi_q(J \setminus S;\, S \setminus J) \\ \mu_T &= \hat{q}^{|T \setminus I|} (-q)^{\mathscr{L}^\natural(T,I,M)}\, \xi_q(T \setminus I;\, I \setminus T) \end{aligned} \tag{6-9}$$

for $S \in \{<J \,\|\, N\}$ and $T \in \{>I \,\|\, M\}$.

Proof. Interchange the index sets in the statement of Theorem 6.2 as follows: $I \leftrightarrow J$ and $M \leftrightarrow N$. Then apply the automorphism τ to the resulting version of (6-4) to obtain (6-8) (recall (2-16)).

This corollary can also be obtained from Theorem 6.2 by interchanging $I \leftrightarrow M$ and $J \leftrightarrow N$, in which case one should also interchange $S \leftrightarrow T^\natural$ and $T \leftrightarrow S^\natural$. □

6.4. ***Further quasicommutation.*** In particular, Theorem 6.2 yields quasicommutation relations of the form $q^{|I\cap M|}[I\,|\,J]\,[M\,|\,N] = q^{|J\cap N|}[M\,|\,N]\,[I\,|\,J]$ in cases where the index sets $\{<\!I\,\|\,M\}$ and $\{>\!J\,\|\,N\}$ are empty. This occurs, for instance, if either

$$[I\,|\,J] = [1,\ldots,r\,|\,n{+}1{-}r,\ldots,n] \quad \text{or} \quad [M\,|\,N] = [n{+}1{-}r,\ldots,n\,|\,1,\ldots,r],$$

recovering the well-known fact that the northeastern-most and southwestern-most quantum minors are normal elements of A. Moreover,

$$[1,\ldots,r\,|\,J]\,[M\,|\,1,\ldots,s] = q^{|J\cap[1,s]|-|[1,r]\cap M|}[M\,|\,1,\ldots,s]\,[1,\ldots,r\,|\,J], \tag{6-10}$$

which is part of [Hodges and Levasseur 1994, Proposition 1.1] (with q^2 replaced by q). Also, (6-10) immediately implies the type A case of [Berenstein and Zelevinsky 2005, Equation 10.3].

We record the general quasicommutation relations of the above type in the next corollary. Part (a) recovers one case of [Scott 2005, Theorem 2]. It does not seem, however, that the relations (4-13) and (4-14) follow directly from equations such as (6-4) or (6-8).

Corollary 6.5. *Let* $I, J, M, N \subseteq \{1,\ldots,n\}$ *with* $|I| = |J|$ *and* $|M| = |N|$.

(a) *If* $\max(M \setminus I) < \min(I \setminus M)$ *and* $\max(J \setminus N) < \min(N \setminus J)$, *then*

$$[I\,|\,J]\,[M\,|\,N] = q^{|I\cap M|-|J\cap N|}[M\,|\,N]\,[I\,|\,J]. \tag{6-11}$$

(b) *If* $\max(I \setminus M) < \min(M \setminus I)$ *and* $\max(N \setminus J) < \min(J \setminus N)$, *then*

$$[I\,|\,J]\,[M\,|\,N] = q^{|J\cap N|-|I\cap M|}[M\,|\,N]\,[I\,|\,J]. \tag{6-12}$$

Proof. (a) If $S \in \{<\!J\,\|\,N\}$, then $S \setminus (J \cap N) < J \setminus N$, whence

$$\max\bigl(S \setminus (J \cap N)\bigr) \le \max(J \setminus N) < \min(N \setminus J).$$

But then S is disjoint from $N \setminus J$. Since $J \cap N \subseteq S \subseteq J \cup N$ and $|S| = |J|$, this forces $S = J$, which is ruled out by the assumption that $S < J$. Thus, $\{<\!J\,\|\,N\} = \varnothing$. Similarly, $\{>\!I\,\|\,M\} = \varnothing$, and thus (6-11) follows from (6-8).

(b) Interchange $I \leftrightarrow M$ and $J \leftrightarrow N$, and apply part (a). □

Example 6.6 ($n = 6$). Let $J = N = \{1, 2, 3\}$, and take $I = \{1, 4, 5\}$ and $M = \{2, 3, 6\}$. We first apply Theorem 6.2. Note that $\{>\!J\,\|\,N\}$ is empty because $J = N$. For $S \in \{<\!I\,\|\,M\}$, we make the calculations in Table 1, where commas have been deleted for the sake of abbreviation (for instance, $\{123\}$ stands for the index set $\{1, 2, 3\}$).

S	$\{123\}$	$\{124\}$	$\{125\}$	$\{134\}$	$\{135\}$
$S^\natural$	$\{456\}$	$\{356\}$	$\{346\}$	$\{256\}$	$\{246\}$
$I\setminus S$	$\{45\}$	$\{5\}$	$\{4\}$	$\{5\}$	$\{4\}$
$S\setminus I$	$\{23\}$	$\{2\}$	$\{2\}$	$\{3\}$	$\{3\}$
$(S\setminus S^\natural)\cup(M\setminus I)$	$\{1236\}$	$\{12346\}$	$\{12356\}$	$\{12346\}$	$\{12356\}$
$\ell\big((S\setminus S^\natural)\cup(M\setminus I);\, I\setminus S\big)$	2	1	2	1	2
$\ell\big((S\setminus S^\natural)\cup(M\setminus I);\, S\setminus I\big)$	3	3	3	2	2
$\mathscr{L}(S,I,M)$	-1	-2	-1	-1	0
$\xi_q(I\setminus S;\, S\setminus I)$	$-q-q^{-1}$	1	1	1	1

Table 1

Consequently, Theorem 6.2 implies that

$$\begin{aligned}(6\text{-}13)\quad & q^3[236\,|\,J]\,[145\,|\,J]\\ &= [145\,|\,J]\,[236\,|\,J]+\hat{q}^2(-q)^{-1}(-q-q^{-1})[123\,|\,J]\,[456\,|\,J]\\ &\quad +\hat{q}(-q)^{-2}[124\,|\,J]\,[356\,|\,J]+\hat{q}(-q)^{-1}[125\,|\,J]\,[346\,|\,J]\\ &\quad +\hat{q}(-q)^{-1}[134\,|\,J]\,[256\,|\,J]+\hat{q}[135\,|\,J]\,[246\,|\,J].\end{aligned}$$

The relation (6-13) matches the one calculated by Fioresi [1999, Example 2.22] (see the first display on page 435, where one must replace q by q^{-1} to account for the difference between (2-6) and the relations used in that paper).

For contrast, we record the relation obtained from Corollary 6.3 for the current choices of I, J, M, N:

$$\begin{aligned}(6\text{-}14)\quad & q^3[145\,|\,J]\,[236\,|\,J]\\ &= [236\,|\,J]\,[145\,|\,J]+\hat{q}[235\,|\,J]\,[146\,|\,J]\\ &\quad +\hat{q}(-q)^{-1}[234\,|\,J]\,[156\,|\,J]+\hat{q}[136\,|\,J]\,[245\,|\,J]\\ &\quad +\hat{q}^2[135\,|\,J]\,[246\,|\,J]+\hat{q}^2(-q)^{-1}[134\,|\,J]\,[256\,|\,J]\\ &\quad +\hat{q}(-q)^{-1}[126\,|\,J]\,[345\,|\,J]+\hat{q}^2(-q)^{-1}[125\,|\,J]\,[346\,|\,J]\\ &\quad +\hat{q}^2(-q)^{-2}[124\,|\,J]\,[356\,|\,J]+\hat{q}(-q)^{-4}[123\,|\,J]\,[456\,|\,J].\end{aligned}$$

We derive two further relations from Theorem 6.2 and Corollary 6.3 with the help of the isomorphism β of Section 2.6, as in Section 4.5. For use in the upcoming proof, note that, since ω_0 reverses inequalities of integers, it also reverses the ordering on index sets: if U and V are subsets of $\{1,\dots,n\}$ with $|U|=|V|$, then $U\le V$ if and only if $\omega_0 U\ge \omega_0 V$.

Theorem 6.7. *If $I, J, M, N\subseteq\{1,\dots,n\}$ with $|I|=|J|$ and $|M|=|N|$, then*

$$\text{(6-15)}\quad q^{|J\cap N|}[I\,|\,J]\,[M\,|\,N]+q^{|J\cap N|}\sum_{S\in\{>I\|M\}}\tilde\mu_S\,[S\,|\,J]\,[S^\natural\,|\,N] = q^{|I\cap M|}[M\,|\,N]\,[I\,|\,J]+q^{|I\cap M|}\sum_{T\in\{<J\|N\}}\tilde\lambda_T\,[M\,|\,T^\natural]\,[I\,|\,T],$$

where

$$\text{(6-16)}\quad \begin{aligned}\tilde\mu_S&=(-\hat q)^{|S\setminus I|}(-q)^{-\mathscr{L}^\natural(S,I,M)}\,\xi_q(S\setminus I;\,I\setminus S)\\ \tilde\lambda_T&=(-\hat q)^{|J\setminus T|}(-q)^{-\mathscr{L}(T,J,N)}\,\xi_q(J\setminus T;\,T\setminus J)\end{aligned}$$

for $S\in\{>I\|M\}$ and $T\in\{<J\|N\}$.

Proof. Just for this proof, write $\tilde U=\omega_0 U$ for index sets U, and observe that

$$\omega_0\big(\{>I\|M\}\big)=\{<\tilde I\|\tilde M\}\qquad\text{and}\qquad \omega_0\big(\{<J\|N\}\big)=\{>\tilde J\|\tilde N\}.$$

Note also that $\tilde S^\natural=\widetilde{S^\natural}$ for $S\in\{>I\|M\}$, and similarly $\tilde T^\natural=\widetilde{T^\natural}$ for $T\in\{<J\|N\}$.

Set $A'=\mathbb{O}_{q^{-1}}(M_n(k))$, with generators X'_{ij} and braiding form $\mathbf{r}'$, and label the quantum minors in A' by $[I\,|\,J]'$. Recall the isomorphism $\beta: A\to A'$ from Section 2.6, and equation (2-17). Note that, when specializing general results to A', the scalars q and $\hat q$ change to q^{-1} and $-\hat q$, respectively.

Now apply Theorem 6.2 to the quantum minors $[\tilde I\,|\,\tilde J]'$ and $[\tilde M\,|\,\tilde N]'$ in A':

$$\text{(6-17)}\quad q^{-|\tilde I\cap\tilde M|}[\tilde I\,|\,\tilde J]'\,[\tilde M\,|\,\tilde N]'+q^{-|\tilde I\cap\tilde M|}\sum_{S\in\{>I\|M\}}\lambda'_{\tilde S}\,[\tilde S\,|\,\tilde J]'\,[\tilde S^\natural\,|\,\tilde N]' = q^{-|\tilde J\cap\tilde N|}[\tilde M\,|\,\tilde N]'\,[\tilde I\,|\,\tilde J]'+q^{-|\tilde J\cap\tilde N|}\sum_{T\in\{<J\|N\}}\mu'_{\tilde T}\,[\tilde M\,|\,\tilde T^\natural]'\,[\tilde I\,|\,\tilde T]',$$

where

$$\begin{aligned}\lambda'_{\tilde S}&=(-\hat q)^{|I\setminus S|}(-q)^{-\mathscr{L}(\tilde S,\tilde I,\tilde M)}\,\xi_q(\tilde I\setminus\tilde S;\,\tilde S\setminus\tilde I),\\ \mu'_{\tilde T}&=(-\hat q)^{|T\setminus J|}(-q)^{-\mathscr{L}^\natural(\tilde T,\tilde J,\tilde N)}\,\xi_q(\tilde T\setminus\tilde J;\,\tilde J\setminus\tilde T),\end{aligned}$$

for $S\in\{>I\|M\}$ and $T\in\{<J\|N\}$. (Here we have simplified the exponents of the $-\hat q$ terms and used the observation that $\xi_{q^{-1}}(U;V)=\xi_q(U;V)$ for any U, V.) Applying the isomorphism β^{-1} to (6-17) yields, in A,

$$\text{(6-18)}\quad q^{-|I\cap M|}[I\,|\,J]\,[M\,|\,N]+q^{-|I\cap M|}\sum_{S\in\{>I\|M\}}\lambda'_{\tilde S}\,[S\,|\,J]\,[S^\natural\,|\,N] = q^{-|J\cap N|}[M\,|\,N]\,[I\,|\,J]+q^{-|J\cap N|}\sum_{T\in\{<J\|N\}}\mu'_{\tilde T}\,[M\,|\,T^\natural]\,[I\,|\,T].$$

Equation (6-15) will follow from (6-18) once we see that $\lambda'_{\tilde{S}} = \tilde{\mu}_S$ and $\mu'_{\tilde{T}} = \tilde{\lambda}_T$ for all S and T.

Take $S \in \{>I \| M\}$ and observe that

$$S \cap S^\natural = I \cap M, \qquad S \cup S^\natural = I \cup M,$$
$$S^\natural \setminus M = I \setminus S, \qquad M \setminus S^\natural = S \setminus I. \tag{6-19}$$

It follows from Theorem 5.6 and Lemma 2.7 that

$$q^{-|I \cap M|} \lambda'_{\tilde{S}} = q^{-|\tilde{I} \cap \tilde{M}|} \lambda'_{\tilde{S}} = \mathbf{r}'\big([\tilde{I} \,|\, \tilde{S}]', [\tilde{M} \,|\, \tilde{S}^\natural]'\big) = \mathbf{r}'\big([M \,|\, S^\natural]', [I \,|\, S]'\big).$$

With the help of (6-19), a second application of Theorem 5.6 shows that

$$\mathbf{r}'\big([M \,|\, S^\natural]', [I \,|\, S]'\big) = q^{-|I \cap M|} \tilde{\mu}_S,$$

and therefore $\lambda'_{\tilde{S}} = \tilde{\mu}_S$. Similarly, $\mu'_{\tilde{T}} = \tilde{\lambda}_T$ for all $T \in \{<J \| N\}$, and the theorem is proved. □

The next corollary is obtained from Theorem 6.7 in the same way as was Corollary 6.3 from Theorem 6.2.

Corollary 6.8. *If $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$, then*

$$q^{|I \cap M|} [I \,|\, J] [M \,|\, N] + q^{|I \cap M|} \sum_{S \in \{>J \| N\}} \tilde{\mu}_S [I \,|\, S] [M \,|\, S^\natural]$$
$$= q^{|J \cap N|} [M \,|\, N] [I \,|\, J] + q^{|J \cap N|} \sum_{T \in \{<I \| M\}} \tilde{\lambda}_T [T^\natural \,|\, N] [T \,|\, J], \tag{6-20}$$

where

$$\tilde{\mu}_S = (-\hat{q})^{|S \setminus J|} (-q)^{-\mathscr{L}^\natural(S,J,N)} \xi_q(S \setminus J;\, J \setminus S)$$
$$\tilde{\lambda}_T = (-\hat{q})^{|I \setminus T|} (-q)^{-\mathscr{L}(T,I,M)} \xi_q(I \setminus T;\, T \setminus I) \tag{6-21}$$

for $S \in \{>J \| N\}$ and $T \in \{<I \| M\}$. □

7. Some variants

Consider the general form of a commutation relation for quantum minors $[I \,|\, J]$ and $[M \,|\, N]$, namely, an equation that allows a product $[I \,|\, J] [M \,|\, N]$ to be replaced by a scalar multiple of the reverse product $[M \,|\, N] [I \,|\, J]$, at the cost of some additional terms. In an equation such as (6-4), the additional terms are of two types: scalar multiples of $[S \,|\, J] [S^\natural \,|\, N]$ and of $[M \,|\, T^\natural] [I \,|\, T]$. In some applications, one type may be more useful than the other. For instance, the *prefered bases* constructed in [Goodearl and Lenagan 2000] consist of certain products of quantum minors in which quantum minors with larger index sets must occur to the left of those with smaller index sets. Thus, if $|I| < |M|$, then $[M \,|\, N] [I \,|\, J]$ and the terms $[M \,|\, T^\natural] [I \,|\, T]$ are in preferred order, but $[I \,|\, J] [M \,|\, N]$ and the terms $[S \,|\, J] [S^\natural \,|\, N]$

are not. A commutation relation in which all the extra terms are in preferred order can be achieved by iteration: after a first application of (6-4), apply (6-4) to any products $[S\,|\,J]\,[S^\natural\,|\,N]$ that appear, and continue until all terms have the desired form. This produces a relation in which $q^{|I\cap M|}[I\,|\,J]\,[M\,|\,N]$ is expressed as $q^{|J\cap N|}[M\,|\,N]\,[I\,|\,J]$ plus a linear combination of products $[S^\natural\,|\,T^\natural]\,[S\,|\,T]$, where $S \in \{\leq I \,\|\, M\}$ and $T \in \{\geq J \,\|\, N\}$. We begin by illustrating the iteration process in the next example.

The aim of this section is to derive closed formulas (that is, without iterations) for commutation relations of the type just discussed.

Example 7.1 ($n = 4$). Consider $[I\,|\,J] = [23|12]$ and $[M\,|\,N] = [14|23]$. First, (6-4) leads to the relation

$$\begin{aligned}(7\text{-}1)\quad & [23\,|\,12]\,[14\,|\,23] - q\,[14\,|\,23]\,[23\,|\,12] \\ &= q\,\hat{q}\,[14\,|\,12]\,[23\,|\,23] - \hat{q}(-q)^{-1}[12\,|\,12]\,[34\,|\,23] - \hat{q}\,[13\,|\,12]\,[24\,|\,23].\end{aligned}$$

The last two terms on the right-hand side of (7-1) must now be treated. Applying (6-4) in each case, we obtain

$$\begin{aligned}(7\text{-}2)\qquad & [12\,|\,12]\,[34\,|\,23] = q[34\,|\,23]\,[12\,|\,12] + q\hat{q}[34\,|\,12]\,[12\,|\,23] \\ (7\text{-}3)\qquad & [13\,|\,12]\,[24\,|\,23] = q[24\,|\,23]\,[13\,|\,12] + q\hat{q}[24\,|\,12]\,[13\,|\,23] \\ &\qquad\qquad -\hat{q}[12\,|\,12]\,[34\,|\,23].\end{aligned}$$

Note that (7-3) contains a term involving $[12\,|\,12]\,[34\,|\,23]$. Hence, we first substitute that equation into (7-1), and then combine the two $[12\,|\,12]\,[34\,|\,23]$-terms, before substituting (7-2) into the result. The final relation is

$$\begin{aligned}(7\text{-}4)\quad & [23\,|\,12]\,[14\,|\,23] - q[14\,|\,23]\,[23\,|\,12] \\ &= q\,\hat{q}\,[14\,|\,12]\,[23\,|\,23] - \hat{q}\,q\,[24\,|\,23]\,[13\,|\,12] - \hat{q}^2 q\,[24\,|\,12]\,[13\,|\,23] \\ &\qquad + \hat{q}\,q^2[34\,|\,23]\,[12\,|\,12] + \hat{q}^2 q^2[34\,|\,12]\,[12\,|\,23].\end{aligned}$$

In each of the terms on the right-hand side of (7-4), the second factor is of the form $[S\,|\,T]$, where $S \in \{23, 13, 12\} = \{\leq I \,\|\, M\}$ and $T \in \{23, 12\} = \{\geq J \,\|\, N\}$.

Lemma 7.2. *Let $s \in \{1, \ldots, n-1\}$, and let B and C be the subalgebras of $A = \mathcal{O}_q(M_n(k))$ given by*

$$B = k\langle X_{ij} \mid 1 \leq i \leq n,\ 1 \leq j \leq s\rangle \quad \textit{and} \quad C = k\langle X_{ij} \mid 1 \leq i \leq n,\ s+1 \leq j \leq n\rangle.$$

The multiplication map $\mu : B \otimes_k C \to A$ is a vector space isomorphism.

Proof. Let X, Y, and Z be the standard PBW bases of the respective algebras B, C, and A. Thus,

$$\begin{aligned}
X &= \bigl\{(X_{11}^{b_{11}}\cdots X_{1s}^{b_{1s}})(X_{21}^{b_{21}}\cdots X_{2s}^{b_{2s}})\cdots(X_{n1}^{b_{n1}}\cdots X_{ns}^{b_{ns}}) \bigm| b_{ij}\in\mathbb{Z}^+\bigr\},\\
Y &= \bigl\{(X_{1,s+1}^{c_{1,s+1}}\cdots X_{1n}^{c_{1n}})(X_{2,s+1}^{c_{2,s+1}}\cdots X_{2n}^{c_{2n}})\cdots(X_{n,s+1}^{c_{n,s+1}}\cdots X_{nn}^{c_{nn}}) \bigm| c_{ij}\in\mathbb{Z}^+\bigr\},\\
Z &= \bigl\{(X_{11}^{a_{11}}\cdots X_{1n}^{a_{1n}})(X_{21}^{a_{21}}\cdots X_{2n}^{a_{2n}})\cdots(X_{n1}^{a_{n1}}\cdots X_{nn}^{a_{nn}}) \bigm| a_{ij}\in\mathbb{Z}^+\bigr\},
\end{aligned}$$

where the variables occur in each monomial in lexicographic order. Observe that the monomials $X_{i1}^{b_{i1}}\cdots X_{is}^{b_{is}}$ and $X_{l,s+1}^{c_{l,s+1}}\cdots X_{ln}^{c_{ln}}$ commute whenever $i>l$. Hence, any product of a monomial from X with a monomial from Y can be rewritten as

$$\begin{aligned}
&\bigl((X_{11}^{b_{11}}\cdots X_{1s}^{b_{1s}})(X_{21}^{b_{21}}\cdots X_{2s}^{b_{2s}})\cdots(X_{n1}^{b_{n1}}\cdots X_{ns}^{b_{ns}})\bigr)\\
&\qquad\times\bigl((X_{1,s+1}^{c_{1,s+1}}\cdots X_{1n}^{c_{1n}})\cdot(X_{2,s+1}^{c_{2,s+1}}\cdots X_{2n}^{c_{2n}})\cdots(X_{n,s+1}^{c_{n,s+1}}\cdots X_{nn}^{c_{nn}})\bigr)\\
&= (X_{11}^{b_{11}}\cdots X_{1s}^{b_{1s}})(X_{1,s+1}^{c_{1,s+1}}\cdots X_{1n}^{c_{1n}})(X_{21}^{b_{21}}\cdots X_{2s}^{b_{2s}})\\
&\qquad\times(X_{2,s+1}^{c_{2,s+1}}\cdots X_{2n}^{c_{2n}})\cdots(X_{n1}^{b_{n1}}\cdots X_{ns}^{b_{ns}})(X_{n,s+1}^{c_{n,s+1}}\cdots X_{nn}^{c_{nn}}).
\end{aligned}$$

Consequently, μ maps the set $\{x\otimes y \mid x\in X,\ y\in Y\}$ bijectively onto Z, and the lemma follows. ☐

Theorem 7.3. *If $I, J, M, N\subseteq\{1,\dots,n\}$ with $|I|=|J|$ and $|M|=|N|$, then*

$$\begin{aligned}
q^{|I\cap M|}[I\,|\,J]\,[M\,|\,N] \\
= q^{|J\cap N|}[M\,|\,N]\,[I\,|\,J] + q^{|J\cap N|}\sum_{\substack{S\in\{\le I\|M\}\\ T\in\{\ge J\|N\}\\ (S,T)\ne(I,J)}}\tilde{\lambda}_S\mu_T\,[S^\natural\,|\,T^\natural]\,[S\,|\,T],
\end{aligned}\tag{7-5}$$

where

$$\begin{aligned}
\tilde{\lambda}_S &= (-\hat{q})^{|I\setminus S|}(-q)^{-\mathscr{L}(S,I,M)}\xi_q(I\setminus S;\,S\setminus I)\\
\mu_T &= \hat{q}^{|T\setminus J|}(-q)^{\mathscr{L}^\natural(T,J,N)}\xi_q(T\setminus J;\,J\setminus T)
\end{aligned}\tag{7-6}$$

for $S\in\{\le I\|M\}$ and $T\in\{\ge J\|N\}$.

Remark 7.4. We have isolated the term $q^{|J\cap N|}[M\,|\,N]\,[I\,|\,J]$ on the right-hand side of (7-5) to emphasize that this equation is a commutation relation. It may, of course, be incorporated in the given summation as a term where $(S,T)=(I,J)$, since $\tilde{\lambda}_I\mu_J=1$.

Proof of Theorem 7.3. Note that the coefficients λ_S and μ_T defined in (6-5) also depend on I, J, M, N. For purposes of the present proof, we record that dependence by writing

$$\begin{aligned}
\lambda_S^{X,Y} &= \hat{q}^{|X\setminus S|}(-q)^{\mathscr{L}(S,X,Y)}\xi_q(X\setminus S;\,S\setminus X),\\
\mu_T^{J,N} &= \hat{q}^{|T\setminus J|}(-q)^{\mathscr{L}^\natural(T,J,N)}\xi_q(T\setminus J;\,J\setminus T),
\end{aligned}$$

for $S \in \{\leq X \| Y\}$ and $T \in \{\geq J \| N\}$. Note that $\lambda_X^{X,Y} = 1$ and $\mu_J^{J,N} = 1$. For $S \in \{<I \| M\}$, set

$$\alpha_S^{I,M} = \sum_{\substack{S_1 \in \{<I \| M\} \\ S_2 \in \{<S_1 \| S_1^\natural\} \\ \cdots \\ S \in \{<S_{i-1} \| S_{i-1}^\natural\}}} (-1)^i \lambda_{S_1}^{I,M} \lambda_{S_2}^{S_1, S_1^\natural} \cdots \lambda_S^{S_{i-1}, S_{i-1}^\natural},$$

where, in terms with $i = 1$, we interpret $S_0 = I$ and $S_0^\natural = M$. Finally, set $\alpha_I^{I,M} = 1$. We claim that

$$q^{|I \cap M|} [I \mid J] [M \mid N] = q^{|J \cap N|} \sum_{\substack{S \in \{\leq I \| M\} \\ T \in \{\geq J \| N\}}} \alpha_S^{I,M} \mu_T^{J,N} [S^\natural \mid T^\natural] [S \mid T]. \tag{7-7}$$

Let $t = |I|$, and let $\mathcal{N}_t$ denote the collection of t-element subsets of $\{1, \ldots, n\}$, partially ordered as in Section 2.10. For proving (7-7), we proceed by induction on I relative to the ordering in $\mathcal{N}_t$. To start, suppose that I is minimal in $\mathcal{N}_t$ (that is, $I = \{1, \ldots, t\}$). In this case, $\{<I \| M\}$ is empty, and so Theorem 6.2 implies that

$$q^{|I \cap M|} [I \mid J] [M \mid N] = q^{|J \cap N|} [M \mid N] [I \mid J] + q^{|J \cap N|} \sum_{T \in \{>J \| N\}} \mu_T^{J,N} [M \mid T^\natural] [I \mid T],$$

which verifies (7-7).

Now suppose that I is not minimal in $\mathcal{N}_t$, but that (7-7) holds whenever I is replaced by an index set $I' < I$. Theorem 6.2 implies that

$$\begin{aligned} q^{|I \cap M|} &[I \mid J] [M \mid N] \\ &= q^{|J \cap N|} \sum_{T \in \{\geq J \| N\}} \mu_T^{J,N} [M \mid T^\natural] [I \mid T] - q^{|I \cap M|} \sum_{S_1 \in \{<I \| M\}} \lambda_{S_1}^{I,M} [S_1 \mid J] [S_1^\natural \mid N]. \end{aligned} \tag{7-8}$$

Recall that $S_1 \cap S_1^\natural = I \cap M$ for $S_1 \in \{<I \| M\}$, by definition of $S_1^\natural$. Hence, our induction hypothesis yields

$$q^{|I \cap M|} [S_1 \mid J] [S_1^\natural \mid N] = q^{|J \cap N|} \sum_{\substack{S \in \{\leq S_1 \| S_1^\natural\} \\ T \in \{\geq J \| N\}}} \alpha_S^{S_1, S_1^\natural} \mu_T^{J,N} [S^\natural \mid T^\natural] [S \mid T] \tag{7-9}$$

for all $S_1 \in \{<I \| M\}$. Substitute (7-9) in (7-8), which yields

$$\begin{aligned} q^{|I \cap M|} [I \mid J] [M \mid N] = q^{|J \cap N|} &\sum_{T \in \{\geq J \| N\}} \mu_T^{J,N} [M \mid T^\natural] [I \mid T] \\ &- q^{|J \cap N|} \sum_{\substack{S_1 \in \{<I \| M\} \\ S \in \{\leq S_1 \| S_1^\natural\} \\ T \in \{\geq J \| N\}}} \lambda_{S_1}^{I,M} \alpha_S^{S_1, S_1^\natural} \mu_T^{J,N} [S^\natural \mid T^\natural] [S \mid T]. \end{aligned} \tag{7-10}$$

Since $\alpha_I^{I,M} = 1$, the coefficients in the first summation of (7-10) match the corresponding coefficients in (7-7). The second summation of (7-10) may be rewritten in the form

$$q^{|J\cap N|} \sum_{\substack{S\in\{<I\|M\}\\ T\in\{\geq J\|N\}}} \beta_S \mu_T^{J,N} [S^\natural \,|\, T^\natural]\,[S\,|\,T],$$

where each

$$\beta_S = -\sum_{\substack{S_1\in\{<I\|M\}\\ S\in\{\leq S_1\|S_1^\natural\}}} \lambda_{S_1}^{I,M} \alpha_S^{S_1,S_1^\natural} = \alpha_S^{I,M}.$$

Thus, (7-10) yields (7-7), establishing the induction step. This proves (7-7).

It remains to show that $\alpha_S^{I,M} = \tilde{\lambda}_S$ for $S \in \{\leq I\|M\}$.

Observe that all quantities appearing in (7-7) involve index sets contained in the union $I \cup J \cup M \cup N$, and so they remain the same if we work in $\mathbb{O}_q(M_\nu(k))$ for some $\nu > n$. Hence, there is no loss of generality in assuming that $n \geq |I| + |M|$. If we set

$$J^* = \{n-|I|+1, \dots, n\} \qquad \text{and} \qquad N^* = \{1, \dots, |M|\},$$

we have $\max(N^*) < \min(J^*)$. Note also that J^* is maximal among $|I|$-element subsets of $\{1, \dots, n\}$. The quantum minors $[U\,|\,N^*]$, for $U \subseteq \{1, \dots, n\}$ with $|U| = |M|$, are homogeneous elements of distinct degrees with respect to the grading on A discussed in Section 2.2. Hence, the $[U\,|\,N^*]$ are linearly independent over k. Similarly, the $[V\,|\,J^*]$, for $V \subseteq \{1, \dots, n\}$ with $|V| = |I|$, are linearly independent, and thus it follows from Lemma 7.2 that the products $[U\,|\,N^*]\,[V\,|\,J^*]$ are linearly independent over k.

Now apply (7-7) to the quantum minors $[I\,|\,J^*]$ and $[M\,|\,N^*]$. Since $\{>J^*\|N^*\}$ is empty, we obtain

$$q^{|I\cap M|}[I\,|\,J^*]\,[M\,|\,N^*] = \sum_{S\in\{\leq I\|M\}} \alpha_S^{I,M} [S^\natural\,|\,N^*]\,[S\,|\,J^*]. \tag{7-11}$$

However, we also have a relation of this type from Corollary 6.8, which may be written in the form

$$q^{|I\cap M|}[I\,|\,J^*]\,[M\,|\,N^*] = \sum_{T\in\{\leq I\|M\}} \tilde{\lambda}_T [T^\natural\,|\,N^*]\,[T\,|\,J^*]. \tag{7-12}$$

Since the products $[S^\natural\,|\,N^*]\,[S\,|\,J^*]$ are linearly independent, it follows from (7-11) and (7-12) that $\alpha_S^{I,M} = \tilde{\lambda}_S$ for all $S \in \{\leq I\|M\}$. Therefore (7-7) implies (7-5), as desired. □

As is easily checked, Theorem 7.3 directly yields equation (7-4).

We next consider the derivation of new relations from Theorem 7.3. Unlike the situation in Section 5, however, the methods used there to prove Corollary 6.3 and

Theorem 6.7 yield the same result when applied to Theorem 7.3. Hence, we use the method of Corollary 6.3.

Corollary 7.5. *If* $I, J, M, N \subseteq \{1, \dots, n\}$ *with* $|I| = |J|$ *and* $|M| = |N|$, *then*

$$q^{|J\cap N|}[I\,|\,J]\,[M\,|\,N] = q^{|I\cap M|}[M\,|\,N]\,[I\,|\,J] + q^{|I\cap M|} \sum_{\substack{S\in\{\ge I\|M\}\\ T\in\{\le J\|N\}\\ (S,T)\ne(I,J)}} \mu_S \tilde{\lambda}_T\,[S^\natural\,|\,T^\natural]\,[S\,|\,T], \tag{7-13}$$

where

$$\begin{aligned} \mu_S &= \hat{q}^{|S\setminus I|}(-q)^{\mathscr{L}^\natural(S,I,M)}\,\xi_q(S\setminus I;\, I\setminus S)\\ \tilde{\lambda}_T &= (-\hat{q})^{|J\setminus T|}(-q)^{-\mathscr{L}(T,J,N)}\,\xi_q(J\setminus T;\, T\setminus J) \end{aligned} \tag{7-14}$$

for $S \in \{\ge I\|M\}$ *and* $T \in \{\le J\|N\}$.

Proof. Interchange $I \leftrightarrow J$ and $M \leftrightarrow N$ in the statement of Theorem 7.3, and also interchange the roles of S and T in the summation. This yields

$$q^{|J\cap N|}[J\,|\,I]\,[N\,|\,M] = q^{|I\cap M|}[N\,|\,M]\,[J\,|\,I] + q^{|I\cap M|} \sum_{\substack{T\in\{\le J\|N\}\\ S\in\{\ge I\|M\}\\ (T,S)\ne(J,I)}} \tilde{\lambda}_T^{J,N} \mu_S^{I,M}[T^\natural\,|\,S^\natural]\,[T\,|\,S], \tag{7-15}$$

where we have placed the superscripts on $\tilde{\lambda}_T^{J,N}$ and $\mu_S^{I,M}$ as reminders of the changes required when carrying over (7-6) to the present situation. Thus, observe that $\tilde{\lambda}_T^{J,N}$ and $\mu_S^{I,M}$ are equal to the scalars denoted $\tilde{\lambda}_T$ and μ_S in (7-14). Consequently, an application of the automorphism τ to (7-15) yields (7-13) (recall (2-16)). □

Remark 7.6. In addition to (7-5) and (7-13), one can derive two commutation relations for quantum minors $[I\,|\,J]$ and $[M\,|\,N]$ in which the additional terms involve products in the same order as $[I\,|\,J]\,[M\,|\,N]$, rather than in reverse order. To obtain such results, simply interchange the roles of $[I\,|\,J]$ and $[M\,|\,N]$ in Theorem 7.3 and Corollary 7.5. One may wish to simplify the coefficients; for instance, with the help of observations such as (6-19), one sees that

$$\mathscr{L}(S^\natural, M, I) = \mathscr{L}^\natural(S, I, M).$$

We leave this to the interested reader.

Example 7.7 ($n = 4$). We close the section by applying Corollary 7.5 to the quantum minors $[I\,|\,J] = [23|13]$ and $[M\,|\,N] = [14|24]$. In this case, equation (7-13) becomes

$$\begin{aligned}
\text{(7-16)}\quad [23|13]\,[14|24] &= [14|24]\,[23|13] + \hat{q}[13|24]\,[24|13] + \hat{q}(-q)^{-1}[12|24]\,[34|13] \\
&\quad + (-\hat{q})[14|34]\,[23|12] + \hat{q}(-\hat{q})[13|34]\,[24|12] \\
&\quad + \hat{q}(-q)^{-1}(-\hat{q})[12|34]\,[34|12].
\end{aligned}$$

Equation (7-16) matches the relation calculated by Fioresi [2004, Example 6.2] (after replacing q by q^{-1}).

8. Poisson brackets

In this final section, we use the commutation relations for quantum minors obtained above to derive expressions for the standard Poisson bracket on pairs of classical minors in $\mathcal{O}(M_n(k))$. In particular, we recover, for the case of the standard bracket, a formula calculated by Kupershmidt [1994]. Although the study of Poisson brackets is often restricted to characteristic zero, that restriction is not needed for the results below.

8.1. ***Standard Poisson bracket on $\mathcal{O}(M_n(k))$.*** Recall that a *Poisson bracket* on a commutative k-algebra B is a k-bilinear map $\{\cdot,\cdot\} : B \times B \to B$ such that

- B is a Lie algebra with respect to $\{\cdot,\cdot\}$; and
- $\{b,\cdot\}$ is a derivation for each $b \in B$.

Note that a Poisson bracket is uniquely determined by its values on pairs of elements from a k-algebra generating set for B.

Write $\mathcal{O}(M_n(k))$ as a commutative polynomial ring over k in indeterminates x_{ij} for $i, j = 1, \dots, n$. The *standard Poisson bracket* on this algebra is the unique Poisson bracket such that

$$\text{(8-1)}\qquad \begin{aligned}
\{x_{ij}, x_{lj}\} &= x_{ij}x_{lj} && \text{if } i < l, \\
\{x_{ij}, x_{im}\} &= x_{ij}x_{im} && \text{if } j < m, \\
\{x_{ij}, x_{lm}\} &= 0 && \text{if } i < l,\ j > m, \\
\{x_{ij}, x_{lm}\} &= 2x_{im}x_{lj} && \text{if } i < l,\ j < m.
\end{aligned}$$

8.2. ***$\mathcal{O}_q(M_n)$ as a quantization of $\mathcal{O}(M_n)$.*** It is well known that $\mathcal{O}_q(M_n(K))$, for a rational function field $K = k(q)$, is a quantization of the Poisson algebra $\mathcal{O}(M_n(k))$, in the sense that the Poisson bracket on $\mathcal{O}(M_n(k))$ is the "semiclassical limit" (as $q \to 1$) of the scaled commutator bracket $\frac{1}{q-1}[\cdot,\cdot]$ on $\mathcal{O}_q(M_n(K))$; we indicate the details below.

For the remainder of this section, replace the scalar q by an indeterminate, and consider the quantum matrix algebra

$$\mathcal{O}_q\big(M_n(k(q))\big),$$

defined over the rational function field $k(q)$.

The $k[q^{\pm 1}]$-subalgebra A_0 of $\mathcal{O}_q\big(M_n(k(q))\big)$ generated by the X_{ij} can be presented, as a $k[q^{\pm 1}]$-algebra, by the generators X_{ij} and relations (2-6), from which it follows that there is an isomorphism

$$A_0/(q-1)A_0 \xrightarrow{\cong} \mathcal{O}(M_n(k)) \tag{8-2}$$

sending the cosets

$$X_{ij} + (q-1)A_0 \longmapsto x_{ij} \qquad \text{for all } i, j.$$

We identify $A_0/(q-1)A_0$ with $\mathcal{O}(M_n(k))$ via (8-2). Since $\mathcal{O}(M_n(k))$ is commutative, the additive commutator $[\,\cdot\,,\,\cdot\,]$ on A_0 takes all its values in $(q-1)A_0$, and so $\frac{1}{q-1}[\,\cdot\,,\,\cdot\,]$ is well-defined on A_0. It follows that the latter bracket induces a well-defined Poisson bracket on $\mathcal{O}(M_n(k))$, such that

$$\{\overline{a}, \overline{b}\} = \overline{(ab-ba)/(q-1)} \tag{8-3}$$

for $a, b \in A_0$, where overbars denote cosets modulo $(q-1)A_0$. This induced bracket is nothing but the standard Poisson bracket on $\mathcal{O}(M_n(k))$, as one easily sees by computing its values on pairs of generators x_{ij}, x_{lm}.

We shall apply (8-3) when $\overline{a}$ and $\overline{b}$ are minors. In order to reserve the notation $[I \,|\, J]$ for classical minors, we denote the quantum minors in $\mathcal{O}_q\big(M_n(k(q))\big)$ by

$$[I \,|\, J]_q.$$

Note that $[I \,|\, J]_q$ is an element of A_0, and that the isomorphism (8-2) maps the coset of $[I \,|\, J]_q$ to $[I \,|\, J]$. Hence, for pairs of minors, (8-3) can be written as

$$\big\{[I \,|\, J], [M \,|\, N]\big\} = \overline{\big([I \,|\, J]_q[M \,|\, N]_q - [M \,|\, N]_q[I \,|\, J]_q\big)/(q-1)}. \tag{8-4}$$

Combining (8-4) with formulas for additive commutators of quantum minors thus yields formulas for Poisson brackets of classical minors. For instance, from (6-10) we obtain

$$\begin{aligned}\big\{[1,\dots,r \,|\, J], [M \,|\, 1,\dots,s]\big\} &= \\ &\big(|[1,r]\cap J| - |M\cap[1,s]|\big)\,[1,\dots,r \,|\, J]\,[M \,|\, 1,\dots,s],\end{aligned} \tag{8-5}$$

which recovers some cases of [Kogan and Zelevinsky 2002, Theorem 2.6].

Theorem 8.3. *If $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$, then*

$$\begin{aligned}(8\text{-}6)\quad & \big\{[I\,|\,J], [M\,|\,N]\big\} \\ &= \big(|J\cap N| - |I\cap M|\big)\,[I\,|\,J]\,[M\,|\,N] \\ &\qquad + 2\sum_{\substack{j\in J\setminus N\\ n\in N\setminus J\\ j<n}} (-1)^{|(J\Delta N)\cap(j,n)|}\,[I\,|\,J\sqcup n\setminus j]\,[M\,|\,N\sqcup j\setminus n] \\ &\qquad - 2\sum_{\substack{i\in I\setminus M\\ m\in M\setminus I\\ i>m}} (-1)^{|(I\Delta M)\cap(m,i)|}\,[I\sqcup m\setminus i\,|\,J]\,[M\sqcup i\setminus m\,|\,N].\end{aligned}$$

Proof. Write (6-4) in the form

$$\begin{aligned}(8\text{-}7)\quad & [I\,|\,J]_q\,[M\,|\,N]_q - [M\,|\,N]_q\,[I\,|\,J]_q \\ &= \big(q^{|J\cap N|-|I\cap M|} - 1\big)\,[M\,|\,N]_q\,[I\,|\,J]_q \\ &\quad + q^{|J\cap N|-|I\cap M|}\sum_{T\in\{>J\|N\}} \mu_T\,[M\,|\,T^\natural]_q\,[I\,|\,T]_q - \sum_{S\in\{<I\|M\}} \lambda_S\,[S\,|\,J]_q\,[S^\natural\,|\,N]_q.\end{aligned}$$

Since $\hat{q}^2/(q-1)$ vanishes modulo $q-1$, we only need to consider the terms in the sums for $T \in \{>J\|N\}$ with $|T\setminus J| = 1$, and $S \in \{<I\|M\}$ with $|I\setminus S| = 1$. Any such T has the form

$$T = J\sqcup n\setminus j$$

with $j \in J\setminus N$ and $n \in N\setminus J$ such that $j < n$, whence

$$T^\natural = N\sqcup j\setminus n \qquad\text{and}\qquad (T^\natural\setminus T)\cup(J\setminus N) = (J\Delta N)\setminus n,$$

and so

$$\begin{aligned}\mathscr{L}^\natural(T, J, N) &= \ell\big((J\Delta N)\setminus n; n\big) - \ell\big((J\Delta N)\setminus n; j\big) \\ &= \ell(J\Delta N; n) - \ell(J\Delta N; j) + 1 = -|(J\Delta N)\cap(j,n)|.\end{aligned}$$

Similarly, the indices S that appear have the form

$$S = I\sqcup m\setminus i$$

with $i \in I\setminus M$ and $m \in M\setminus I$ such that $i > m$, whence

$$S^\natural = M\sqcup i\setminus m \qquad\text{and}\qquad \mathscr{L}(S, I, M) = -|(I\Delta M)\cap(m,i)|.$$

Consequently, dividing (8-7) by $q-1$, and then reducing the resulting equation modulo $q-1$ yields (8-6). $\square$

Similarly, Corollary 6.3 yields:

Theorem 8.4. *If $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$, then*

$$\begin{aligned} \text{(8-8)} \quad & \big\{[I \,|\, J], [M \,|\, N]\big\} \\ &= \big(|I \cap M| - |J \cap N|\big)\,[I \,|\, J]\,[M \,|\, N] \\ &\qquad + 2 \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i < m}} (-1)^{|(I \,\Delta\, M) \cap (i,m)|}\,[I \sqcup m \setminus i \,|\, J]\,[M \sqcup i \setminus m \,|\, N] \\ &\qquad - 2 \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j > n}} (-1)^{|(J \,\Delta\, N) \cap (n,j)|}\,[I \,|\, J \sqcup n \setminus j]\,[M \,|\, N \sqcup j \setminus n]. \end{aligned}$$

Finally, provided k does not have characteristic 2, we can average equations (8-6) and (8-8) to obtain:

Corollary 8.5. *Let $I, J, M, N \subseteq \{1, \dots, n\}$ with $|I| = |J|$ and $|M| = |N|$. If $\operatorname{char} k \neq 2$, then*

$$\begin{aligned} \text{(8-9)} \quad \big\{[I \,|\, J], [M \,|\, N]\big\} &= \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i < m}} (-1)^{|(I \,\Delta\, M) \cap (i,m)|}\,[I \sqcup m \setminus i \,|\, J]\,[M \sqcup i \setminus m \,|\, N] \\ &\quad - \sum_{\substack{i \in I \setminus M \\ m \in M \setminus I \\ i > m}} (-1)^{|(I \,\Delta\, M) \cap (m,i)|}\,[I \sqcup m \setminus i \,|\, J]\,[M \sqcup i \setminus m \,|\, N] \\ &\quad + \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j < n}} (-1)^{|(J \,\Delta\, N) \cap (j,n)|}\,[I \,|\, J \sqcup n \setminus j]\,[M \,|\, N \sqcup j \setminus n] \\ &\quad - \sum_{\substack{j \in J \setminus N \\ n \in N \setminus J \\ j > n}} (-1)^{|(J \,\Delta\, N) \cap (n,j)|}\,[I \,|\, J \sqcup n \setminus j]\,[M \,|\, N \sqcup j \setminus n]. \end{aligned}$$

Equation (8-9) is the standard case of the formula of Kupershmidt [1994, Equation (9)]. To obtain the standard Poisson bracket in his setting, make the following choices for the structure constants:

$$r^{ij}_{lm} = \begin{cases} 1 & \text{if } i > j,\ l = j,\ m = i, \\ -1 & \text{if } i < j,\ l = j,\ m = i, \\ 0 & \text{otherwise.} \end{cases}$$

Acknowledgments

We thank T. H. Lenagan, L. Rigal, A. Zelevinsky, and the referee for their comments and suggestions concerning this project.

References

[Berenstein and Zelevinsky 2005] A. Berenstein and A. Zelevinsky, "Quantum cluster algebras", *Adv. Math.* **195**:2 (2005), 405–455. MR 2006a:20092 Zbl 02189412

[Brown and Goodearl 2002] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2002. MR 2003f:16067 Zbl 1027.17010

[Fioresi 1999] R. Fioresi, "Quantum deformation of the Grassmannian manifold", *J. Algebra* **214**:2 (1999), 418–447. MR 2000j:20092 Zbl 0955.17007

[Fioresi 2004] R. Fioresi, "Commutation relations among quantum minors in $O_q(M_n(k))$", *J. Algebra* **280**:2 (2004), 655–682. MR 2005f:20086 Zbl 1072.16035

[Goodearl and Lenagan 2000] K. R. Goodearl and T. H. Lenagan, "Quantum determinantal ideals", *Duke Math. J.* **103**:1 (2000), 165–190. MR 2001k:16080 Zbl 0958.16025

[Goodearl and Lenagan 2002] K. R. Goodearl and T. H. Lenagan, "Prime ideals invariant under winding automorphisms in quantum matrices", *International J. Math.* **13**:5 (2002), 497–532. MR 2003h:20091 Zbl 1054.16030

[Goodearl and Lenagan 2003] K. R. Goodearl and T. H. Lenagan, "Winding-invariant prime ideals in quantum 3×3 matrices", *J. Algebra* **260**:2 (2003), 657–687. MR 2004g:20069 Zbl 1059.16035

[Hayashi 1992] T. Hayashi, "Quantum groups and quantum determinants", *J. Algebra* **152**:1 (1992), 146–165. MR 93j:17031 Zbl 0906.17011

[Hodges and Levasseur 1993] T. J. Hodges and T. Levasseur, "Primitive ideals of $\mathbf{C}_q[\mathrm{SL}(3)]$", *Comm. Math. Phys.* **156**:3 (1993), 581–605. MR 94k:17023 Zbl 0801.17012

[Hodges and Levasseur 1994] T. J. Hodges and T. Levasseur, "Primitive ideals of $\mathbf{C}_q[\mathrm{SL}(n)]$", *J. Algebra* **168**:2 (1994), 455–468. MR 95i:16038 Zbl 0814.17012

[Hodges et al. 1997] T. J. Hodges, T. Levasseur, and M. Toro, "Algebraic structure of multiparameter quantum groups", *Adv. Math.* **126**:1 (1997), 52–92. MR 98e:17022 Zbl 0878.17009

[Joseph 1995] A. Joseph, *Quantum groups and their primitive ideals*, vol. 29, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Springer, Berlin, 1995. MR 96d:17015 Zbl 0808.17004

[Kassel 1995] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics **155**, Springer, New York, 1995. MR 96e:17041 Zbl 0808.17003

[Klimyk and Schmüdgen 1997] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, Berlin, 1997. MR 99f:17017 Zbl 0891.17010

[Kogan and Zelevinsky 2002] M. Kogan and A. Zelevinsky, "On symplectic leaves and integrable systems in standard complex semisimple Poisson-Lie groups", *Int. Math. Res. Not.* 32 (2002), 1685–1702. MR 2003k:53107 Zbl 1006.22015

[Krob and Leclerc 1995] D. Krob and B. Leclerc, "Minor identities for quasi-determinants and quantum determinants", *Comm. Math. Phys.* **169**:1 (1995), 1–23. MR 96g:15015 Zbl 0829.15024

[Kupershmidt 1994] B. A. Kupershmidt, "Poisson relations between minors and their consequences", *J. Phys. A* **27**:13 (1994), L507–L513. MR 95g:22033 Zbl 0842.17018

[Lakshmibai and Reshetikhin 1991] V. Lakshmibai and N. Reshetikhin, "Quantum deformations of flag and Schubert schemes", *Comptes Rendus de l'Académie des Sciences. Sér. I Math.* **313**:3 (1991), 121–126. MR 92f:14053 Zbl 0753.17021

[Lakshmibai and Reshetikhin 1992] V. Lakshmibai and N. Reshetikhin, "Quantum flag and Schubert schemes", pp. 145–181 in *Deformation theory and quantum groups with applications to mathematical physics* (Amherst, MA, 1990), edited by M. Gerstenhaber and J. Stasheff, Contemp. Math. **134**, Amer. Math. Soc., Providence, RI, 1992. MR 94a:14055 Zbl 0792.17012

[Lambe and Radford 1997] L. A. Lambe and D. E. Radford, *Introduction to the quantum Yang–Baxter equation and quantum groups: an algebraic approach*, Mathematics and its Applications **423**, Kluwer Academic Publishers, Dordrecht, 1997. MR 1483071 (98k:16054) Zbl 0888.16021

[Larson and Towber 1991] R. G. Larson and J. Towber, "Two dual classes of bialgebras related to the concepts of 'quantum group' and 'quantum Lie algebra' ", *Comm. Algebra* **19**:12 (1991), 3295–3345. MR 93b:16070 Zbl 0751.16014

[Leclerc and Zelevinsky 1998] B. Leclerc and A. Zelevinsky, "Quasicommuting families of quantum Plücker coordinates", pp. 85–108 in *Kirillov's seminar on representation theory*, edited by G. I. Olshanski, Amer. Math. Soc. Transl. Ser. 2 **181**, Amer. Math. Soc., Providence, RI, 1998. MR 99g:14066 Zbl 0894.14021

[Lenagan and Rigal 2006] T. H. Lenagan and L. Rigal, "Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum Grassmannians", *J. Algebra* **301**:2 (2006), 670–702. MR MR2236763

[Noumi et al. 1993] M. Noumi, H. Yamada, and K. Mimachi, "Finite-dimensional representations of the quantum group $\mathrm{GL}_q(n;\mathbf{C})$ and the zonal spherical functions on $\mathrm{U}_q(n-1)\backslash\mathrm{U}_q(n)$", *Japan. J. Math.* (*N.S.*) **19**:1 (1993), 31–80. MR 94i:17023 Zbl 0806.17016

[Parshall and Wang 1991] B. Parshall and J. P. Wang, "Quantum linear groups", *Mem. Amer. Math. Soc.* **89**:439 (1991), vi+157. MR 91g:16028 Zbl 0724.17011

[Reshetikhin et al. 1989] N. Y. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, "Quantization of Lie groups and Lie algebras", *Algebra i Analiz* **1**:1 (1989), 178–206. MR 90j:17039 Zbl 0715.17015

[Scott 2005] J. Scott, "Quasi-commuting families of quantum minors", *J. Algebra* **290**:1 (2005), 204–220. MR 2006d:16064 Zbl 1079.14058

[Soibelman 1990] Y. S. Soĭbel'man, "Algebra of functions on a compact quantum group and its representations", *Algebra i Analiz* **2**:1 (1990), 190–212. In Russian; correction in *Algebra i Analiz* **2**:3 (1990), 256; translation in *Leningrad Math. J.* **2** (1991), 161–178. MR 91i:58053a

[Taft and Towber 1991] E. Taft and J. Towber, "Quantum deformation of flag schemes and Grassmann schemes, I: A q-deformation of the shape-algebra for $\mathrm{GL}(n)$", *J. Algebra* **142**:1 (1991), 1–36. MR 93e:17028 Zbl 0739.17007

Received February 15, 2005. Revised September 18, 2005.

KENNETH R. GOODEARL
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA SANTA BARBARA
SANTA BARBARA, CA 93106
UNITED STATES

goodearl@math.ucsb.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

APPROXIMATING THE MODULUS OF AN INNER FUNCTION

GEIR ARNE HJELLE AND ARTUR NICOLAU

We show that the modulus of an inner function can be uniformly approximated in the unit disk by the modulus of an interpolating Blaschke product.

1. Introduction

Let H^∞ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$. A function in H^∞ is called inner if it has radial limit of modulus one at almost every point of the unit circle. A Blaschke product is an inner function of the form

$$B(z) = z^m \prod_{n=1}^{\infty} \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z},$$

where m is a nonnegative integer and $\{z_n\}$ is a sequence of points in $\mathbb{D} \setminus \{0\}$ satisfying the Blaschke condition $\sum_n (1 - |z_n|) < \infty$. A classical result of O. Frostman tells that for any inner function f, there exists an exceptional set $E = E(f) \subset \mathbb{D}$ of logarithmic capacity zero such that the Möbius shift

$$\frac{f - \alpha}{1 - \bar{\alpha} f}$$

is a Blaschke product for any $\alpha \in \mathbb{D} \setminus E$. See [Frostman 1935] or [Garnett 1981, p. 79]. Hence any inner function can be uniformly approximated by a Blaschke product.

A Blaschke product B is called an interpolating Blaschke product if its zero set $\{z_n\}$ forms an interpolating sequence, that is, if for any bounded sequence of complex numbers $\{w_n\}$, there exists a function $f \in H^\infty$ such that $f(z_n) = w_n$, $n = 1, 2, \ldots$. A celebrated result by L. Carleson [1958] (or see [Garnett 1981, p. 287]) tells us that this holds precisely when two conditions are satisfied:

$$\inf_{n \neq m} \left| \frac{z_n - z_m}{1 - \bar{z}_m z_n} \right| > 0, \tag{1}$$

MSC2000: 30D50, 30E10.
Keywords: interpolating Blaschke product, Carleson contour, discretization, inner function, modulus.
Hjelle is partially supported by grants from the Research Council of Norway, projects #155060 and #166395. Nicolau is partially supported by MTM2005-00544 and 2005SGR00774.

and there exists a constant C such that $\sum_{z_n \in Q}(1-|z_n|) < C\ell(Q)$ for any Carleson square Q of the form

$$Q = \left\{ re^{i\theta} : 0 < 1-r < \ell(Q),\ |\theta - \theta_0| < \pi\ell(Q) \right\} \tag{2}$$

where $\theta_0 \in [0, 2\pi)$ and $0 < \ell(Q) < 1$. Although interpolating Blaschke products comprise a small subset of all Blaschke products, they play a central role in the theory of the algebra H^∞. See the last three chapters of [Garnett 1981].

D. Marshall [1976] proved that any function $f \in H^\infty$ can be uniformly approximated by finite linear combinations of Blaschke products. That is, for any $\varepsilon > 0$ there are constants $c_1, \dots, c_N$ and Blaschke products $B_1, \dots, B_N$ such that

$$\left\| f - \sum_{i=1}^{N} c_i B_i \right\|_\infty < \varepsilon.$$

Here the ∞-norm is given by $\|g\|_\infty = \sup\{|g(z)| : z \in \mathbb{D}\}$. This result was improved in [Garnett and Nicolau 1996] by showing that one can take each of $B_1, \dots, B_N$ to be an interpolating Blaschke product. However the following problem remains open.

For any inner function B and $\varepsilon > 0$, is there an interpolating Blaschke product I such that $\|B - I\|_\infty < \varepsilon$?

This question was posed in [Garnett 1981, p. 430; Havin and Nikol'skiĭ 1994, pp. 268–269; Jones 1981; Nikol'skiĭ 1986, p. 202]. Here we provide a positive answer if one restricts attention to the modulus.

Theorem 1. *Let B be an inner function and $\varepsilon > 0$. There exists an interpolating Blaschke product I such that*

$$\big| |B(z)| - |I(z)| \big| < \varepsilon \quad \textit{for all } z \in \mathbb{D}.$$

The proof may be described as follows. The first step consists of constructing a system $\Gamma = \bigcup_i \Gamma_i$ of disjoint closed curves $\Gamma_i \subset \mathbb{D}$ such that arclength of Γ is a Carleson measure, and verifying that

(a) $|B(z)|$ is uniformly small on hyperbolic disks of fixed radius centered at points of Γ, and

(b) in any hyperbolic disk of fixed radius centered at a point outside the union of the interiors of Γ_i, $\bigcup_i \operatorname{int}\Gamma_i$, there is a point z where $|B(z)|$ is not small.

Write $B = B_1 \cdot B_2$, where B_1 is the Blaschke product formed with the zeros of B which are in $\bigcup_i \operatorname{int}\Gamma_i$. Statement (b) implies that B_2 is a finite product of interpolating Blaschke products. Since D. Marshall and A. Stray [1996] proved that any finite product of interpolating Blaschke products may be approximated by a single interpolating Blaschke product, the relevant zeros of B lie in $\bigcup_i \operatorname{int}\Gamma_i$: they are those of B_1. The construction of Γ is a variation of the original corona construction introduced by L. Carleson [1962] (or see [Garnett 1981, pp. 342–347]).

Next, for each $i = 1, 2, \dots$, let μ_i be the sum of harmonic measures in $\operatorname{int}\Gamma_i$ from the zeros of B_1 contained in $\operatorname{int}\Gamma_i$. Then the mass $\mu_i(\Gamma_i)$ is the total number of zeros of B_1 contained in $\operatorname{int}\Gamma_i$. The second step consists of splitting Γ_i as $\bigcup_k \Gamma_{i,k}$, where the pieces $\Gamma_{i,k}$ satisfy $\mu_i(\Gamma_{i,k}) = 1$, $k = 1, 2, \dots$, and choosing points $\xi_{i,k} \in \Gamma_{i,k}$ matching a certain moment of the measure μ_i on $\Gamma_{i,k}$. This choice may be compared with that of [Lyubarskii and Malinnikova 2001], where a related discretization argument is performed in a different context. Let I_1 be the Blaschke product with zeros $\xi_{i,k}$, $i, k = 1, 2, \dots$. The last step of the proof is to use (b) above to show that I_1 is an interpolating Blaschke product and to use the location of $\{\xi_{i,k}\}$, as well as (a) above, to show that $|I_1(z) \cdot B_2(z)|$ approximates $|B(z)|$.

Besides the individual problem mentioned above, some questions concerning approximation by arguments of interpolating Blaschke products remain open. Let B be an inner function.

A. *Given* $\varepsilon > 0$, *is there an interpolating Blaschke product I such that*

$$\|\operatorname{Arg} B - \operatorname{Arg} I\|_{\mathrm{BMO}(\partial\mathbb{D})} < \varepsilon?$$

B. *Is there an interpolating Blaschke product I such that* $\operatorname{Arg} B - \operatorname{Arg} I = \tilde{v}$, *where* $v \in L^\infty(\partial\mathbb{D})$?

C. *Is there an interpolating Blaschke product I such that* $\operatorname{Arg} B - \operatorname{Arg} I = u + \tilde{v}$, *where* $u, v \in L^\infty(\partial\mathbb{D})$ *and* $\|u\|_\infty < \pi/2$?

A positive answer to Problem A would imply the main result of this note. Problem C was posed by in [Havin and Nikol'skiĭ 1994; Nikol'skiĭ 1986] in connection with Toeplitz operators and complete interpolating sequences in model spaces. Problems B and C are discussed in the nice monograph by K. Seip [2004, p. 92].

2. Construction of the contour

The hyperbolic distance between two points $z, w \in \mathbb{D}$ is

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

where $\rho(z, w)$ is the pseudohyperbolic distance,

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w} z} \right|.$$

Recall that a positive measure μ in the unit disk is called a Carleson measure if there exists a constant $M = M(\mu) > 0$ such that $\mu(Q) \leq M\ell(Q)$ for any Carleson square of the form (2). The infimum of the constants M satisfying the inequality above is called the Carleson norm of the measure μ and it is denoted by $\|\mu\|_C$.

The main result of this section is a variant of the classical construction of the Carleson contour introduced by L. Carleson in his original proof of the corona theorem [1962] (or see [Garnett 1981, pp. 342–347]).

Lemma 2. *Let $B \in H^\infty$ with $\|B\|_\infty = 1$. Let $0 < \varepsilon < 1$ and $K > 0$ be fixed constants. Then, there exist a constant $\delta = \delta(\varepsilon, K) > 0$ and a system $\Gamma = \bigcup \Gamma_i$ of disjoint closed curves Γ_i contained in $\mathbb{D}$ such that*

(a) $|B(z)| \leq \varepsilon$ *if* $\inf_i \beta(z, \operatorname{int} \Gamma_i) \leq K$;

(b) $\sup\{|B(w)| : \beta(w, z) \leq K + 14\} > \delta$ *if* $z \notin \bigcup \operatorname{int} \Gamma_i$; *and*

(c) *arclength* $\mathrm{d}s_{|\Gamma}$ *on* Γ *is a Carleson measure with* $\|\mathrm{d}s_{|\Gamma}\|_C \leq 68$.

Proof. The proof is essentially contained in [Nicolau and Suárez $\geq$ 2006], but we sketch it for the convenience of the reader. Given a set $E \subset \mathbb{D}$, let $\Omega_K(E)$ denote the set of points that are at most at hyperbolic distance K from the set E, that is,

$$\Omega_K(E) = \Big\{z : \inf_{w \in E} \beta(z, w) \leq K\Big\}.$$

Consider dyadic Carleson squares of the form

$$Q_{n,j} = \big\{re^{i\theta} : 1 - 2^{-n} < r < 1,\ 2\pi j 2^{-n} < \theta < 2\pi(j+1)2^{-n}\big\},$$

for $j = 0, 1, \ldots, 2^n - 1$ and $n = 1, 2, \ldots$, and their top halves

$$T(Q_{n,j}) = \{re^{i\theta} \in Q_{n,j} : r < 1 - 2^{-n-1}\}.$$

Let $0 < \delta < \varepsilon$ be a constant to be fixed later. A dyadic Carleson square Q will be called good if

$$\sup\big\{|B(z)| : z \in \Omega_K\big(T(Q)\big)\big\} > \varepsilon.$$

The collection of good dyadic Carleson squares will be denoted by

$$\{Q_j^G : j = 1, 2, \ldots\}.$$

A dyadic Carleson square Q will be called bad if

$$\sup\big\{|B(z)| : z \in \Omega_K\big(T(Q)\big)\big\} < \delta.$$

We denote the collection of bad dyadic Carleson squares by $\{Q_j^B : j = 1, 2, \ldots\}$. The construction goes as follows.

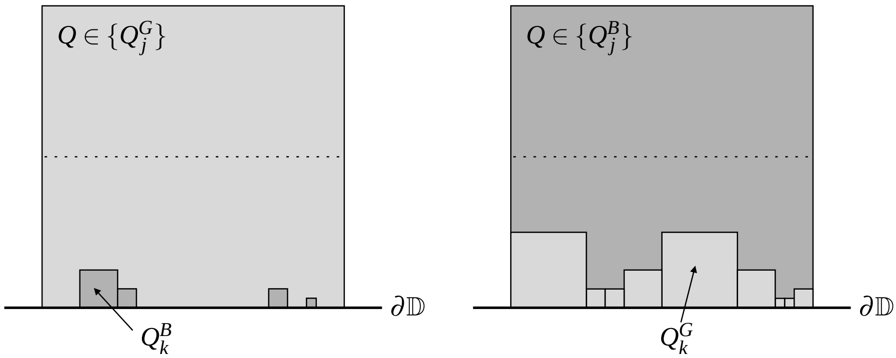

Figure 1. Choosing good and bad squares for constructing the contour.

Step 1. For each good dyadic Carleson square $Q = Q_j^G$, we choose the maximal bad dyadic Carleson squares Q_k^B contained in Q. The main estimate needed is

$$\sum_{Q_k^B \subset Q} \ell(Q_k^B) \leq \tfrac{1}{2}\ell(Q). \tag{3}$$

Since $|B(z)| < \delta$ if $z \in T(Q_k^B)$, while $|B(z)| > \varepsilon$ for some $z \in \Omega_K(T(Q))$, taking $\delta = \delta(\varepsilon, K)$ sufficiently small, standard arguments lead to (3). See [Nicolau and Suárez $\geq$ 2006, Lemma 2.1] for details.

Step 2. For each bad dyadic Carleson square $Q = Q_j^B$, we choose the maximal good dyadic Carleson squares Q_k^G contained in Q. This family is denoted by $G(Q) = \{Q_k^G : k = 1, 2, \ldots\}$.

So, from each good dyadic Carleson square we move to bad ones fulfilling the estimate (3) and from each bad one we again move to good ones. See Figure 1. Now for each bad square $Q = Q_j^B$, let

$$R(Q) = Q \setminus \overline{\bigcup_{G(Q)} Q_k^G} \quad \text{and} \quad R = \bigcup_j R(Q_j^B).$$

Finally, decompose R into its connected components R_i and denote $\Gamma_i = \partial R_i$, $i = 1, 2, \ldots$. Observe that each Γ_i consists of pieces of boundaries of dyadic Carleson squares. See Figure 2. By construction if $z \in R$ we have

$$\sup\big\{|B(w)| : \beta(w, z) \leq K\big\} \leq \varepsilon$$

and hence part (a) in the statement follows. Similarly, if $z \notin R$, the point z is not in the top part of a bad dyadic Carleson square. As the hyperbolic diameter of a top part of a Carleson square is uniformly bounded, say by 14, we deduce that there exists $w \in \mathbb{D}$ with $\beta(z, w) \leq K + 14$ such that $|B(w)| > \delta$. Hence statement (b)

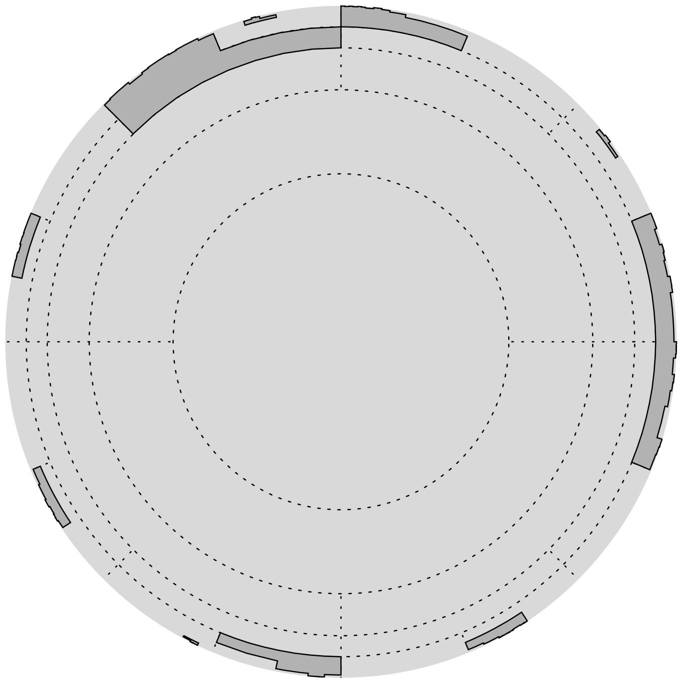

Figure 2. The unit disk, some dyadic Carleson contours and an example of a contour.

follows. Since the length of $\partial R(Q)$ is bounded by $17\ell(Q)$, the scaling (3) shows that for any bad dyadic square Q, one has

$$\sum_{Q_j^B \subsetneq Q} |\partial R(Q_j^B)| \leq 17\ell(Q).$$

Then easy geometric considerations show that arclength on $\bigcup \Gamma_i$ is a Carleson measure and its Carleson norm is smaller than 68. □

3. Construction of the interpolating Blaschke product

We now use Lemma 2 to construct a contour Γ. Note that by Frostman's Theorem we can assume that B is a Blaschke product. Given $\varepsilon > 0$, let N be a big constant dependent on ε to be fixed later. Apply Lemma 2 with $\varepsilon/2$ and $2N$ instead of ε and K to obtain Γ and $\delta > 0$ such that

(a) $|B(z)| < \varepsilon/2$ if $\beta(z, \operatorname{int}\Gamma) \leq 2N$,

(b) $\sup\{|B(w)| : \beta(w, z) \leq 2N + 14\} > \delta$ if $z \notin \operatorname{int}\Gamma$, and

(c) arclength on Γ is a Carleson measure with Carleson norm $\|\mathrm{d}s_{|\Gamma}\|_C \leq 68$.

With the contour Γ in place, we want to construct the interpolating Blaschke product I. Split B into two Blaschke products B_1 and B_2. That is $B = B_1 \cdot B_2$, where B_1 is formed with the zeros $\{z_n\}$ of B that lie inside $\operatorname{int}\Gamma$ and at hyperbolic distance more than 1 from the contour Γ. For each zero z of B_2, part (b) provides a point $w \in \mathbb{D}$, $\beta(w, z) \le 2N + 15$ such that $|B_2(w)| \ge |B(w)| > \delta$. This implies that B_2 is a finite product of interpolating Blaschke products; [Mortini and Nicolau 2004, Theorem 2.2].

Hence the dangerous part of B will be B_1, which has all its zeros contained deeply inside the contour Γ. We want to mimic the behavior of $|B_1|$ by constructing a Blaschke product I_1 with zeros on Γ. To this end, for each component Γ_i of the contour we consider the measure

$$\mathrm{d}\mu_i(\xi) = \sum_{\substack{z_n \in \operatorname{int}\Gamma_i \\ \beta(z_n,\Gamma_i)>1}} \omega(z_n, \xi; \operatorname{int}\Gamma_i)$$

defined for $\xi \in \Gamma_i$. Here $\omega(z, \xi; \Omega)$ denotes the harmonic measure from the point $z \in \Omega$ in the domain $\Omega \subseteq \mathbb{D}$. Clearly $\mu_i(\Gamma_i)$ will be equal to the number of zeros z_n of B_1 inside Γ_i. Next we split Γ_i into disjoint arcs $\Gamma_{i,k}$ such that $\mu_i(\Gamma_{i,k}) = 1$ for each k. This is illustrated in Figure 3. On each such arc we locate one zero $\xi_{i,k}$ of I_1 such that

$$1 - |\xi_{i,k}|^2 = \int_{\Gamma_{i,k}} \left(1 - |\xi|^2\right) \mathrm{d}\mu_i(\xi). \tag{4}$$

This will in general not determine the points $\xi_{i,k}$ uniquely. However, there seems to be a lot of freedom for placing the zeros of I_1 in this construction, and the condition (4) will be sufficient for our purposes.

Let I_1 be the Blaschke product with the zeros $\xi_{i,k}$, and factor $I_1 = I_1^o \cdot I_1^e$ where I_1^o is the Blaschke product with zeros $\xi_{i,k}$ with k odd, while I_1^e is the Blaschke product with zeros $\xi_{i,k}$ with k even. In Figure 3, I_1^o has its zeros placed in the dark arcs, while the zeros of I_1^e are placed in the light arcs. We claim that both I_1^o and I_1^e are interpolating Blaschke products, and hence I_1 can be approximated by an interpolating Blaschke product [Marshall and Stray 1996]. To show this claim we will observe that their zero sets satisfy the two conditions of Carleson's theorem [1958] stated in the introduction.

In this case, the existence of a constant C as in Carleson's criterion (see top of page 104) follows from the fact that arclength is a Carleson measure on Γ, while inequality (1) follows from the following lemma and the geometry of the contour.

Lemma 3. *The hyperbolic length*, $\ell_\beta(\Gamma_{i,k})$, *of* $\Gamma_{i,k}$ *is bounded from below*:

$$\ell_\beta(\Gamma_{i,k}) \ge \delta^{2\exp(2(2N+14))}.$$

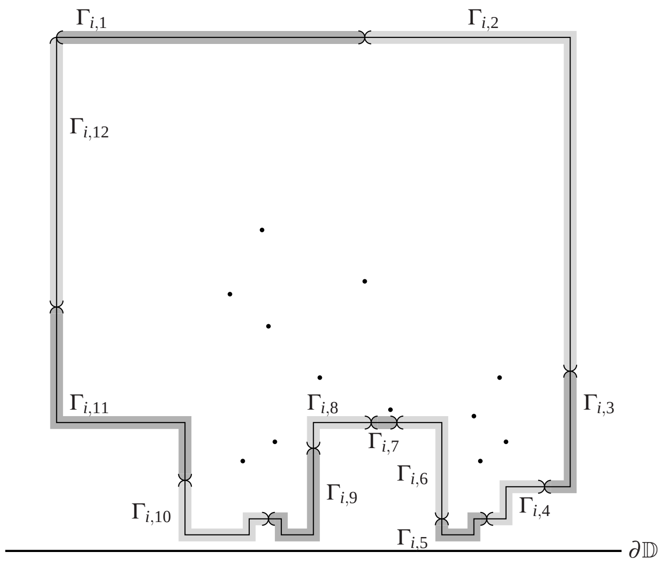

Figure 3. Each component Γ_i of the contour is split into arcs $\Gamma_{i,k}$ such that the μ-measure of each arc is 1.

Proof. We first show that for any point $w \in \Gamma$, $|B_1(w)|$ is bounded from below by some constant depending only on δ and N. To see this, recall that there is a point ζ such that $\beta(\zeta, w) \le 2N + 14$ and $|B_1(\zeta)| \ge |B(\zeta)| > \delta$. Consider

$$\log\bigl|B_1(w)\bigr|^{-1} = \sum \log \rho(w, z_n)^{-1},$$

where the sum is taken over all zeros z_n of B_1. As w is separated from the zeros of B_1,

$$\log \rho(w, z_n)^{-1} \le 1 - \rho(w, z_n)^2.$$

Furthermore,

$$\rho(w, z_n) \ge \frac{\rho(z_n, \zeta) - \rho(\zeta, w)}{1 - \rho(z_n, \zeta)\rho(\zeta, w)} \ge \frac{\rho(z_n, \zeta) - C}{1 - C\rho(z_n, \zeta)},$$

where $C = \dfrac{e^{2(2N+14)} - 1}{e^{2(2N+14)} + 1} < 1$. Hence

$$\begin{aligned}\log \rho(w, z_n)^{-1} &\le \frac{\bigl(1 - \rho(z_n, \zeta)^2\bigr)\bigl(1 - C^2\bigr)}{\bigl(1 - C\rho(z_n, \zeta)\bigr)^2} \le \frac{1 + C}{1 - C}\bigl(1 - \rho(z_n, \zeta)^2\bigr) \\ &\le 2\frac{1 + C}{1 - C} \log \rho(z_n, \zeta)^{-1} = 2e^{2(2N+14)} \log \rho(z_n, \zeta)^{-1},\end{aligned}$$

and we see that $|B_1(w)| \ge \delta^{2\exp(2(2N+14))}$.

Intuitively, this lower bound for the values of $|B_1|$ should imply that the arcs $\Gamma_{i,k}$ cannot be too short hyperbolically. To make this observation rigorous we argue as follows. Using that the harmonic measure ω is positive and harmonic, we have that for any $z \in \operatorname{int}\Gamma_i$,

$$\omega(z, \Gamma_{i,k}; \operatorname{int}\Gamma_i) \leq \omega(z, \Gamma_{i,k}; \mathbb{D} \setminus \Gamma_{i,k}) \leq \frac{\displaystyle\int_{\Gamma_{i,k}} \log\left|\frac{z-w}{1-\overline{w}z}\right|^{-1} \frac{|dw|}{1-|w|^2}}{\displaystyle\min_{z\in\Gamma_{i,k}} \int_{\Gamma_{i,k}} \log\left|\frac{z-w}{1-\overline{w}z}\right|^{-1} \frac{|dw|}{1-|w|^2}}$$

and

$$\begin{aligned} 1 = \mu_i(\Gamma_{i,k}) &= \sum_{z_n \in \operatorname{int}\Gamma_i} \omega(z_n, \Gamma_{i,k}; \operatorname{int}\Gamma_i) \\ &\leq \frac{1}{C_{i,k}} \int_{\Gamma_{i,k}} \log\Bigg(\prod_{z_n\in\operatorname{int}\Gamma_i} \Big|\frac{z_n - w}{1-\overline{w}z_n}\Big|^{-1} \Bigg) \frac{|dw|}{1-|w|^2}, \end{aligned}$$

where

$$C_{i,k} = \min_{z\in\Gamma_{i,k}} \int_{\Gamma_{i,k}} \log\Big|\frac{z-w}{1-\overline{w}z}\Big|^{-1} \frac{|dw|}{1-|w|^2}$$

is a constant dependent on $\Gamma_{i,k}$. Let $B_{1,i}$ denote the Blaschke product with the zeros of B_1 that fall inside the component Γ_i. Then, for $w \in \Gamma_i$,

$$\log\Bigg(\prod_{z_n\in\operatorname{int}\Gamma_i} \Big|\frac{z_n - w}{1-\overline{w}z_n}\Big|^{-1} \Bigg) = \log|B_{1,i}(w)|^{-1} \leq \log|B_1(w)|^{-1} \leq 2e^{2(2N+14)} \log\delta^{-1}.$$

Thus

$$1 \leq \frac{1}{C_{i,k}} 2e^{2(2N+14)} \log\delta^{-1} \int_{\Gamma_{i,k}} \frac{|dw|}{1-|w|^2} = \frac{1}{C_{i,k}} 2e^{2(2N+14)} \log\delta^{-1} \ell_\beta(\Gamma_{i,k})$$

such that

$$\ell_\beta(\Gamma_{i,k}) \geq \frac{C_{i,k}}{2e^{2(2N+14)}\log\delta^{-1}}.$$

To estimate $C_{i,k}$ we use the substitution $\xi = \varphi_z(w) = (z-w)/(1-\overline{w}z)$ and the conformal invariance of the hyperbolic metric. A calculation then gives that

$$C_{i,k} \geq \log\big(\tanh \ell_\beta(\Gamma_{i,k})\big)\ell_\beta(\Gamma_{i,k}),$$

which implies the desired bound, $\ell_\beta(\Gamma_{i,k}) \geq \delta^{2\exp(2(2N+14))}$. □

4. Proof of the approximation

In this section we will show that the constructed function, $I = I_1 \cdot B_2$, approximates the given Blaschke product uniformly in modulus. We first claim that it suffices to prove Theorem 1 for points $z \in \mathbb{D}$ far away from the contour. Indeed, assume that we can prove that

$$\big||B_1(z)| - |I_1(z)|\big| < \varepsilon/2 \tag{5}$$

for all z such that $\beta(z, \operatorname{int}\Gamma) \geq 2N$, where N is as in the construction of the contour. Then, for points z with $\beta(z, \operatorname{int}\Gamma) = 2N$,

$$\begin{aligned}|I(z)| &= \big(|I_1(z)| - |B_1(z)| + |B_1(z)|\big)|B_2(z)| \\ &\leq \big||B_1(z)| - |I_1(z)|\big| + |B(z)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

By the maximum principle $|I(z)| < \varepsilon$ for all $z \in \Omega_{2N}(\operatorname{int}\Gamma)$ as well. Hence

$$\big||B(z)| - |I(z)|\big| = \big||B_1(z)| - |I_1(z)|\big||B_2(z)| < \begin{cases}\varepsilon/2 & \text{if } \beta(z, \operatorname{int}\Gamma) \geq 2N, \\ \varepsilon & \text{if } \beta(z, \operatorname{int}\Gamma) < 2N.\end{cases}$$

So Theorem 1 follows from (5).

The rest of the paper will be dedicated to prove that (5) holds. Fix a point z such that $\beta(z, \operatorname{int}\Gamma) \geq 2N$. We will consider the logarithm of $|B_1|$. Since all the zeros of B_1 lie inside the contour Γ, $\log\big|(z - z_n)/(1 - \bar{z}_n z)\big|$ is harmonic inside Γ as a function of z_n. Hence

$$\log|B_1(z)| = \sum_j \log\left|\frac{z - z_n}{1 - \bar{z}_n z}\right| = \int_\Gamma \log\left|\frac{z - \xi}{1 - \bar{\xi} z}\right| \mathrm{d}\mu(\xi),$$

where $\mathrm{d}\mu = \sum_i \mathrm{d}\mu_i$. As the μ-measure of each arc $\Gamma_{i,k}$ is 1, we have

$$\begin{aligned}(6)\quad \log|B_1(z)| - \log|I_1(z)| &= \int_\Gamma \log\left|\frac{z-\xi}{1-\bar{\xi}z}\right| \mathrm{d}\mu(\xi) - \sum_{i,k} \log\left|\frac{z-\xi_{i,k}}{1-\bar{\xi}_{i,k}z}\right| \\ &= \sum_{i,k} \int_{\Gamma_{i,k}} \left(\log\left|\frac{z-\xi}{1-\bar{\xi}z}\right| - \log\left|\frac{z - \xi_{i,k}}{1 - \bar{\xi}_{i,k}z}\right|\right) \mathrm{d}\mu(\xi) \\ &= \sum_{i,k} \int_{\Gamma_{i,k}} \log\frac{\rho(z, \xi)}{\rho(z, \xi_{i,k})} \mathrm{d}\mu(\xi) \stackrel{\text{def}}{=} \sum_{i,k} H_{i,k}(z).\end{aligned}$$

To estimate this sum we consider different types of arcs. By Q_z we denote the Carleson square with z as the midpoint on the top-side. We say that an arc $\Gamma_{i,k}$ is in the class $\mathcal{B}$ if $\Gamma_{i,k} \subset 2^N Q_z$. Note that since $\beta(z, \operatorname{int}\Gamma) \geq 2N$, this implies that such an arc lies very close to the boundary. The rest of the arcs we split into short and long arcs. For $n \geq N + 1$ define

$$\begin{aligned}\mathcal{S}_n &= \big\{\Gamma_{i,k} : \ell_\beta(\Gamma_{i,k}) < 1,\ \Gamma_{i,k} \subset 2^n Q_z\big\} \setminus \big(\mathcal{B} \cup \bigcup_{i<n} \mathcal{S}_i\big), \\ \mathcal{L}_n &= \big\{\Gamma_{i,k} : \ell_\beta(\Gamma_{i,k}) \geq 1,\ \Gamma_{i,k} \subset 2^n Q_z\big\} \setminus \big(\mathcal{B} \cup \bigcup_{i<n} \mathcal{L}_i\big).\end{aligned}$$

Consult Figure 4 for some examples of this classification. This partition is such that each arc $\Gamma_{i,k}$ belongs to one and only one of the classes $\mathcal{B}$, $\mathcal{S}_n$ and $\mathcal{L}_n$, with

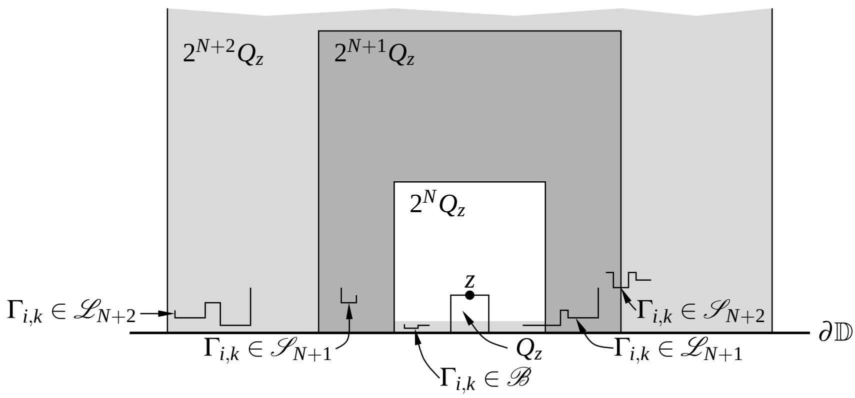

Figure 4. We divide the arcs $\Gamma_{i,k}$ into classes denoted $\mathscr{B}$, $\mathscr{S}_n$ and $\mathscr{L}_n$.

$n \geq N+1$. Hence we may decompose the sum (6) as

$$\sum_{i,k} H_{i,k}(z) = \sum_{\Gamma_{i,k}\in\mathscr{B}} H_{i,k}(z) + \sum_{n=N+1}^{\infty}\Bigg(\sum_{\Gamma_{i,k}\in\mathscr{S}_n} H_{i,k}(z) + \sum_{\Gamma_{i,k}\in\mathscr{L}_n} H_{i,k}(z)\Bigg).$$

Our goal is to show that the absolute value of the left hand side is small. To accomplish this we will show that each of the terms

$$\Bigg|\sum_{\Gamma_{i,k}\in\mathscr{B}} H_{i,k}(z)\Bigg|, \quad \Bigg|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{S}_n} H_{i,k}(z)\Bigg| \quad \text{and} \quad \Bigg|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n} H_{i,k}(z)\Bigg|$$

are small.

We begin with the boundary arcs $\Gamma_{i,k} \in \mathscr{B}$. Using that $\log(1-t) = -t + \mathcal{O}(t^2)$ we get

$$\begin{aligned}\sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \log \frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\, \mathrm{d}\mu(\xi)&\\ = -\frac{1}{2}\sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} &\Bigg(1 - \frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2} + \mathcal{O}\Bigg(\Big(1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\Big)^2\Bigg)\Bigg)\, \mathrm{d}\mu(\xi).\end{aligned}$$

Taking absolute values,

$$\begin{aligned}(7)\quad \Bigg|\sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \log \frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\, \mathrm{d}\mu(\xi)\Bigg| &\leq \frac{1}{2}\Bigg|\sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} 1 - \frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\, \mathrm{d}\mu(\xi)\Bigg|\\ + \frac{1}{2}\Bigg|\sum_{\Gamma_{i,k}\in\mathscr{B}} \int_{\Gamma_{i,k}} \mathcal{O}\Bigg(\Big(1-&\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\Big)^2\Bigg)\, \mathrm{d}\mu(\xi)\Bigg| \stackrel{\mathrm{def}}{=} E_{\mathscr{B},1} + E_{\mathscr{B},2},\end{aligned}$$

where we define $E_{\mathcal{B},1}$ and $E_{\mathcal{B},2}$ for convenience. At first we focus on the term $E_{\mathcal{B},1}$. Note that since z is far away from $\xi_{i,k} \in \Gamma_i$, the expression $\rho(z, \xi_{i,k})^{-2}$ is bounded above, say by 2. By expanding $1 - \rho(z, \xi)^2$ and $1 - \rho(z, \xi_{i,k})^2$, we can write

$$\begin{aligned} (8) \quad E_{\mathcal{B},1} &\le \sum_{\Gamma_{i,k} \in \mathcal{B}} \left| \int_{\Gamma_{i,k}} (1-|z|^2)\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2} - \frac{1-|\xi_{i,k}|^2}{|1-\bar{\xi}_{i,k}z|^2} \right) \mathrm{d}\mu(\xi) \right| \\ &= \sum_{\Gamma_{i,k} \in \mathcal{B}} \left| \int_{\Gamma_{i,k}} (1-|z|^2)\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2} - \frac{1-|\xi|^2}{|1-\bar{\xi}_{i,k}z|^2} + \frac{|\xi_{i,k}|^2-|\xi|^2}{|1-\bar{\xi}_{i,k}z|^2} \right) \mathrm{d}\mu(\xi) \right|. \end{aligned}$$

By the placement, (4), of the zeros $\xi_{i,k}$, the integral of the last term is zero. We now move the modulus under the integral to get

$$(9) \qquad E_{\mathcal{B},1} \le \left(1-|z|^2\right) \sum_{\Gamma_{i,k} \in \mathcal{B}} \int_{\Gamma_{i,k}} \left(1-|\xi|^2\right) \left| \frac{1}{|1-\bar{\xi}z|^2} - \frac{1}{|1-\bar{\xi}_{i,k}z|^2} \right| \mathrm{d}\mu(\xi).$$

Because ξ and $\xi_{i,k}$ should be close to each other in some sense, compared to z, we suspect some cancellation. Therefore we use the estimate

$$(10) \qquad \left| \frac{1}{|1-\bar{\xi}z|^2} - \frac{1}{|1-\bar{\xi}_{i,k}z|^2} \right| \le \frac{2|\xi - \xi_{i,k}|}{(1-|z|)^3}$$

and the more trivial inequalities $|\xi - \xi_{i,k}| \le \ell(\Gamma_{i,k})$ and $1 - |z|^2 \le 2(1 - |z|)$ to obtain

$$E_{\mathcal{B},1} \le 2^3 \left(1-|z|\right)^{-2} \sum_{\Gamma_{i,k} \in \mathcal{B}} \ell(\Gamma_{i,k}) \int_{\Gamma_{i,k}} \left(1-|\xi|\right) \mathrm{d}\mu(\xi).$$

All the arcs $\Gamma_{i,k} \in \mathcal{B}$ are contained in a rectangle at the boundary with height $2^{-2N}(1 - |z|)$ and width $2^N(1 - |z|)$. Using that $1 - |\xi| \le 2^{-2N}(1 - |z|)$ and that the arclength $\mathrm{d}s_{|\Gamma}$ is a Carleson measure, we then get

$$E_{\mathcal{B},1} \le 2^3 \, \|\mathrm{d}s_{|\Gamma}\|_C \cdot 2^{-N}$$

where $\|\mathrm{d}s_{|\Gamma}\|_C$ is the Carleson norm of arclength on Γ.

Next we focus our attention on the higher-order terms, and give the estimate for $E_{\mathcal{B},2}$. From (7) and (8) and the inequality $(a+b)^2 \le 2(a^2+b^2)$ we see that $E_{\mathcal{B},2}$ is bounded by a fixed multiple of

$$\begin{aligned} &\left(1-|z|^2\right)^2 \sum_{\Gamma_{i,k} \in \mathcal{B}} \int_{\Gamma_{i,k}} \left(1-|\xi|^2\right)^2 \left| \frac{1}{|1-\bar{\xi}z|^2} - \frac{1}{|1-\bar{\xi}_{i,k}z|^2} \right|^2 \mathrm{d}\mu(\xi) \\ &\qquad + \left(1-|z|^2\right)^2 \sum_{\Gamma_{i,k} \in \mathcal{B}} \int_{\Gamma_{i,k}} \frac{(|\xi_{i,k}|^2 - |\xi|^2)^2}{|1-\bar{\xi}_{i,k}z|^4} \mathrm{d}\mu(\xi). \end{aligned}$$

For the first term, we use as above the estimate (10) as well as $1-|\xi|\le 2^{-2N}(1-|z|)$ and $|\xi-\xi_{i,k}|\le 2\cdot 2^N(1-|z|)$. Then we find

$$\begin{aligned}&\big(1-|z|^2\big)^2\sum_{\Gamma_{i,k}\in\mathcal{B}}\int_{\Gamma_{i,k}}\big(1-|\xi|^2\big)^2\left|\frac{1}{|1-\bar{\xi}z|^2}-\frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|^2\mathrm{d}\mu(\xi)\\&\qquad\le 2^4\cdot 2^{-N}\cdot\big(1-|z|^2\big)\sum_{\Gamma_{i,k}\in\mathcal{B}}\int_{\Gamma_{i,k}}\big(1-|\xi|^2\big)\left|\frac{1}{|1-\bar{\xi}z|^2}-\frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|\mathrm{d}\mu(\xi).\end{aligned}$$

The last sum is just (9), and by the earlier argument the last expression is bounded by $2^7\,\|\mathrm{d}s_{|\Gamma}\|_C\cdot 2^{-2N}$.

For the second term we use that $|1-\bar{\xi}_{i,k}z|\ge 1-|z|$, $\big||\xi_{i,k}|-|\xi|\big|\le 2^{-2N}(1-|z|)$ and $\big||\xi_{i,k}|-|\xi|\big|\le \ell(\Gamma_{i,k})$ to arrive at

$$\begin{aligned}&\big(1-|z|^2\big)^2\sum_{\Gamma_{i,k}\in\mathcal{B}}\int_{\Gamma_{i,k}}\frac{(|\xi_{i,k}|^2-|\xi|^2)^2}{|1-\bar{\xi}_{i,k}z|^4}\,\mathrm{d}\mu(\xi)\\&\qquad\le 2^4\big(1-|z|\big)^{-2}\sum_{\Gamma_{i,k}\in\mathcal{B}}\int_{\Gamma_{i,k}}\big||\xi_{i,k}|-|\xi|\big|^2\,\mathrm{d}\mu(\xi)\\&\qquad\le 2^4\cdot 2^{-2N}\big(1-|z|\big)^{-1}\sum_{\Gamma_{i,k}\in\mathcal{B}}\ell(\Gamma_{i,k})\le 2^4\,\|\mathrm{d}s_{|\Gamma}\|_C\cdot 2^{-N}.\end{aligned}$$

Thus we get $E_{\mathcal{B},2}\le C\,(2^4+1)\,\|\mathrm{d}s_{|\Gamma}\|_C\cdot 2^{-N}$ for big N.

For the short arcs $\Gamma_{i,k}\in\mathcal{S}_n$, $n\ge N+1$, we will use similar estimates as above, but we do not need to be as delicate. For these arcs we can use the inequality $|\log x|\le|1-x^2|$, which holds for x far away from zero, to obtain

$$\begin{aligned}E_{\mathcal{S}}&\overset{\text{def}}{=}\left|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathcal{S}_n}\int_{\Gamma_{i,k}}\log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\,\mathrm{d}\mu(\xi)\right|\\&\qquad\le\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathcal{S}_n}\int_{\Gamma_{i,k}}\Big|1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\Big|\,\mathrm{d}\mu(\xi).\end{aligned}$$

The same calculations that gave (8) show that

$$\Big|1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\Big|\le 2\big(1-|z|^2\big)\left(\left|\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}-\frac{1-|\xi|^2}{|1-\bar{\xi}_{i,k}z|^2}\right|+\frac{\big||\xi_{i,k}|^2-|\xi|^2\big|}{|1-\bar{\xi}_{i,k}z|^2}\right).$$

For $\xi\in\Gamma_{i,k}\in\mathcal{S}_n$, using $|1-\bar{\xi}z|\ge 2^{n-3}(1-|z|)$ we get

$$\big(1-|\xi|^2\big)\left|\frac{1}{|1-\bar{\xi}z|^2}-\frac{1}{|1-\bar{\xi}_{i,k}z|^2}\right|\le 2^{11}\frac{\big(1-|\xi|\big)|\xi-\xi_{i,k}|}{2^{3n}\big(1-|z|\big)^3}\le 2^{11}\frac{|\xi-\xi_{i,k}|}{2^{2n}\big(1-|z|\big)^2}.$$

Similarly,

$$\frac{\big||\xi_{i,k}|^2-|\xi|^2\big|}{|1-\bar{\xi}_{i,k}z|^2}\le 2^7\frac{|\xi-\xi_{i,k}|}{2^{2n}\big(1-|z|\big)^2}.$$

Adding up, we obtain

$$\left|1-\frac{\rho(z,\xi)^2}{\rho(z,\xi_{i,k})^2}\right|\le 2^{14}\frac{|\xi-\xi_{i,k}|}{2^{2n}\big(1-|z|\big)}.$$

Hence

$$E_{\mathscr{S}}\le 2^{14}\sum_{n=N+1}^{\infty}\frac{1}{2^{2n}\big(1-|z|\big)}\sum_{\Gamma_{i,k}\in\mathscr{S}_n}\ell(\Gamma_{i,k})\le 2^{14}\,\|\mathrm{d}s_{|\Gamma}\|_C\cdot 2^{-N}.$$

Finally, we estimate the long arcs $\Gamma_{i,k}\in\mathscr{L}_n$, for $n\ge N+1$. As the zeros on these arcs are well separated, one can expect only a small contribution from these arcs. We will use an auxiliary interpolating Blaschke product to find a bound for the $\mathscr{L}_n$-terms of (6). By the same reasoning that led to (8) and the triangle inequality,

$$\begin{aligned}
E_{\mathscr{L}}&\overset{\text{def}}{=}\left|\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n}\int_{\Gamma_{i,k}}\log\frac{\rho(z,\xi)}{\rho(z,\xi_{i,k})}\,\mathrm{d}\mu(\xi)\right|\\
&\le 2\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n}\int_{\Gamma_{i,k}}\big(1-|z|^2\big)\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2}+\frac{1-|\xi_{i,k}|^2}{|1-\bar{\xi}_{i,k}z|^2}\right)\mathrm{d}\mu(\xi)\\
&\le 2^2\sum_{n=N+1}^{\infty}\sum_{\Gamma_{i,k}\in\mathscr{L}_n}\max_{\xi\in\overline{\Gamma}_{i,k}}\frac{\big(1-|z|^2\big)\big(1-|\xi|^2\big)}{|1-\bar{\xi}z|^2}.
\end{aligned}$$

For each $\Gamma_{i,k}\in\mathscr{L}_n$, let $\zeta_{i,k}\in\Gamma_{i,k}$ be such that

$$\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2}=\max_{\xi\in\overline{\Gamma}_{i,k}}\frac{1-|\xi|^2}{|1-\bar{\xi}z|^2},$$

and define B_ζ to be the Blaschke product with $\{\zeta_{i,k}\}$ as zeros. Now we reorder the summation, and sum with respect to the placement of the $\zeta_{i,k}$ instead. Then

$$E_{\mathscr{L}}\le 2^3\big(1-|z|\big)\sum_{n=0}^{\infty}\sum_{\zeta_{i,k}\in U_n}\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2}$$

where $U_0=Q_z$ and $U_n=2^nQ_z\setminus 2^{n-1}Q_z$ for $n\ge 1$. The scaling property (3) implies that at most four of the points $\zeta_{i,k}$ are contained in $2^{N-1}Q_z$. These must be close to the boundary, so

$$2^3\big(1-|z|\big)\sum_{n=0}^{N-1}\sum_{\zeta_{i,k}\in U_n}\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2}\le 4\cdot 2^4\cdot 2^{-2N}.$$

For the rest of the terms, we then get

$$2^3\big(1-|z|\big)\sum_{n=N}^{\infty}\sum_{\zeta_{i,k}\in U_n}\frac{1-|\zeta_{i,k}|^2}{|1-\bar{\zeta}_{i,k}z|^2}\le 2^8\sum_{n=N}^{\infty}\frac{1}{2^n}\sum_{\zeta_{i,k}\in U_n}\frac{1-|\zeta_{i,k}|}{2^n\big(1-|z|\big)}\le 2^9\,C_\zeta\cdot 2^{-N},$$

where C_ζ is the Carleson norm of the measure $\sum(1-|\zeta_{i,k}|)\delta_{\zeta_{i,k}}$, which is bounded by a fixed multiple of $\|\mathrm{d}s_{|\Gamma}\|_C$. Thus $E_{\mathscr{L}}\le 2^9\,(C_\zeta+1)\cdot 2^{-N}$.

We have now estimated the contribution from all the arcs $\Gamma_{i,k}$, and we have found that for some constant C,

$$\big|\log|B_1(z)|-\log|I_1(z)|\big|\le C\cdot 2^{-N}.$$

This means that given $\varepsilon>0$, taking N so that $C\cdot 2^{-N}<\varepsilon/2$, we obtain

$$\big||B_1(z)|-|I_1(z)|\big|<\varepsilon/2,$$

which is what we needed.

Acknowledgments

We are indebted to Arne Stray for his valuable comments on an earlier version of this paper. Part of this work was done while Hjelle was visiting the Universitat Autònoma de Barcelona and while Nicolau was visiting IMUB at the Universitat de Barcelona. It is a pleasure to thank both institutions for their support.

References

[Carleson 1958] L. Carleson, "An interpolation problem for bounded analytic functions", *Amer. J. Math.* **80** (1958), 921–930. MR 22 #8129 Zbl 0085.06504

[Carleson 1962] L. Carleson, "Interpolations by bounded analytic functions and the corona problem", *Ann. of Math.* (2) **76** (1962), 547–559. MR 25 #5186 Zbl 0112.29702

[Frostman 1935] O. Frostman, "Potentiel d'équilibre et capacité des ensembles avec quelques applications á la théorie des fonctions", *Medd. Lund. Univ. Math. Sem.* **3** (1935). JFM 61.1262.02

[Garnett 1981] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics **96**, Academic Press, New York, 1981. MR 83g:30037 Zbl 0469.30024

[Garnett and Nicolau 1996] J. Garnett and A. Nicolau, "Interpolating Blaschke products generate H^∞", *Pacific J. Math.* **173**:2 (1996), 501–510. MR 97f:30050 Zbl 0871.30031

[Havin and Nikol'skiĭ 1994] V. P. Havin and N. K. Nikol'skiĭ (editors), *Linear and complex analysis, problem book 3, part I*, edited by V. P. Havin and N. K. Nikol'skiĭ, Lecture Notes in Mathematics **1573**, Springer, Berlin, 1994. MR 96c:00001a Zbl 0893.30037

[Jones 1981] P. W. Jones, "Ratios of interpolating Blaschke products", *Pacific J. Math.* **95**:2 (1981), 311–321. MR 82m:30032 Zbl 0479.30021

[Lyubarskii and Malinnikova 2001] Y. Lyubarskii and E. Malinnikova, "On approximation of subharmonic functions", *J. Anal. Math.* **83** (2001), 121–149. MR 2002b:30043 Zbl 0981.31002

[Marshall 1976] D. E. Marshall, "Blaschke products generate H^∞", *Bull. Amer. Math. Soc.* **82**:3 (1976), 494–496. MR 53 #5877 Zbl 0327.30029

[Marshall and Stray 1996] D. E. Marshall and A. Stray, "Interpolating Blaschke products", *Pacific J. Math.* **173**:2 (1996), 491–499. MR 97c:30042 Zbl 0855.30028

[Mortini and Nicolau 2004] R. Mortini and A. Nicolau, "Frostman shifts of inner functions", *J. Anal. Math.* **92** (2004), 285–326. MR 2005e:30088 Zbl 1064.30029

[Nicolau and Suárez ≥ 2006] A. Nicolau and D. Suárez, "Approximation by invertible functions of H^∞", *Math. Scandinavica*. To appear.

[Nikol'skiĭ 1986] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften **273**, Springer, Berlin, 1986. MR 87i:47042 Zbl 0587.47036

[Seip 2004] K. Seip, *Interpolation and sampling in spaces of analytic functions*, University Lecture Series **33**, American Math. Society, Providence, RI, 2004. MR 2005c:30038 Zbl 1057.30036

Received October 4, 2005.

GEIR ARNE HJELLE
DEPARTMENT OF MATHEMATICAL SCIENCES
NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
7491 TRONDHEIM
NORWAY
Current address:
Department of Mathematics
Washington University
St. Louis, MO 63130
United States
http://www.math.wustl.edu/~hjelle/
hjelle@math.wustl.edu

ARTUR NICOLAU
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
08193 BELLATERRA, BARCELONA
SPAIN
http://mat.uab.es/~artur/
artur@mat.uab.es

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

FOLIATIONS ON SUPERMANIFOLDS AND CHARACTERISTIC CLASSES

CAMILLE LAURENT-GENGOUX

We construct secondary characteristic classes of regular superfoliations on smooth supermanifolds. We interpret these secondary characteristic classes as characteristic classes of flat foliated connections.

1. Introduction

Smooth supermanifolds are becoming an increasingly important subject in mathematical physics [Leites 1980]; superfoliations should then be a central object. In this paper, we construct secondary characteristic classes of regular superfoliations with trivialized normal bundle, in the spirit of [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1986]. We show that the role of $\mathrm{Vect}(n)$ (the Lie algebra of formal vector fields with n variables) from the classical theory is played by the even part $\mathrm{Vect}(n,m)_0$ of the super-Lie algebra $\mathrm{Vect}(n,m)$ of formal supervector fields with n even variables and m odd variables. More precisely, when on a supermanifold $\mathcal{M}$ we are given a superfoliation $\mathcal{F}$ of codimension $n+\varepsilon m$ with trivialized normal bundle, we will associate to any element H in the Chevalley–Eilenberg cohomology $H^*\big(\mathrm{Vect}(n,m)_0\big)$ a secondary characteristic class $\varphi_{\mathcal{M},\mathcal{F}}(H) \in H^*(M)$, where M is the base manifold of the supermanifold $\mathcal{M}$.

Theorem A. *For any supermanifold $\mathcal{M}$ foliated by a superfoliation $\mathcal{F}$ of codimension $n+\varepsilon m$ with trivialized normal bundle, there exists a natural homomorphism $\varphi_{\mathcal{M},\mathcal{F}}$ from $H^*\big(\mathrm{Vect}(n,m)_0\big)$ to $H^*(M)$ such that*:

(1) *If $\Phi : \mathcal{N} \to \mathcal{M}$ is a submersion of supermanifolds and $\varphi : N \to M$ is the map induced by Φ on their base manifolds M and N, then*

$$\varphi_{\mathcal{N},\Phi^*\mathcal{F}} = \varphi^* \circ \varphi_{\mathcal{M},\mathcal{F}},$$

where $\Phi^*\mathcal{F}$ is the pull-back of the superfoliation $\mathcal{F}$ via Φ and $\varphi^* : H^*(M) \to H^*(N)$ is the pull-back through φ.

MSC2000: primary 14M30; secondary 53C10.
Keywords: superfoliation, supergeometry, secondary characteristic classes.

(2) *If $\mathcal{M}$ is an ordinary smooth manifold endowed with a foliation F of codimension n with trivialized normal bundle, then φ_F reduces to the well-known homomorphism of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld* [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1973].

The reader should not be surprised that the de Rham cohomology of $\mathcal{M}$ is not used: by a result of Batchelor [Leites 1980], this cohomology is indeed isomorphic to the cohomology of the base manifold. Note that the even part $\mathrm{Vect}(n, m)_0$ appears instead of $\mathrm{Vect}(n, m)$ itself. The theory is indeed more interesting like this, since, for example, $H^*\big(\mathrm{Vect}(n, m)\big)$ has only two nontrivial generators for $n < m$ (see [Fuchs 1973]). In [Koszul 1988], Godbillon–Vey classes for superfoliations of codimension $0+\varepsilon m$ are constructed on supermanifolds of superdimension $n + \varepsilon m$ as maps from $H^*\big(\mathrm{Vect}(0, m)\big)$ to $H^*(\mathcal{M})$. We will see that these classes vanish for any class $H \in H^*\big(\mathrm{Vect}(0, m)\big)$ whose restriction to the cohomology $H^*\big(\mathrm{Vect}(0, m)_0\big)$ of the even part is 0. Furthermore, we will show in Section 4D that the classes constructed in [Koszul 1988] are among those built in this article.

Given a vector bundle $E \to M$ and a foliation F, a *foliated connection* means a connection on E defined (smoothly) over each leaf of the foliation F. A foliated connection is said to be *flat* if it is flat on each leaf, and *trivial* if E is a trivial vector bundle and F a foliation with trivialized normal bundle. For simplicity, we say *flat trivial foliated connection* to name the entire collection of a foliation with trivialized normal bundle, a vector bundle, and a flat foliated connection. There is a canonical way to construct, from a flat foliated connection, a superfoliation with trivialized normal bundle on a supermanifold. Therefore, our theory of secondary characteristic classes of superfoliations also gives a theory of secondary characteristic classes of flat foliated connections.

Furthermore, to a superfoliation $\mathcal{F}$ of codimension $n + \varepsilon m$ with trivialized normal bundle on a supermanifold $\mathcal{M}$ we shall associate a flat trivial foliated connection, that is, a foliation $F_{\mathcal{F}}$ of codimension n with trivialized normal bundle on the base manifold M, a trivial vector bundle $E_{\mathcal{F}}$ of dimension m over M, and a flat foliated connection $\nabla^{\mathcal{F}}$ of this bundle. It should be noted that this construction is not the inverse of the previous one. However, we shall see that our theory of secondary characteristic classes, because of our preceding construction, gives in fact a theory of secondary characteristic classes of flat foliated connections. More precisely, we show:

Theorem B. *The homomorphism $\varphi_{\mathcal{F},\mathcal{M}} : H^*\big(\mathrm{Vect}(n, m)_0\big) \to H^*(M)$ is completely determined by the flat trivial foliated connection $\big(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}}\big)$.*

The paper is organized as follows. Section 2 is devoted to constructions and properties of superfoliations on supermanifolds. More precisely, in Section 2A we introduce several useful maps; in Section 2B we recall the definition of a superfoliation; in Section 2C we exhibit relations between superfoliations and flat foliated connections; in Section 2D we show how to replace a superfoliation with a superfoliation with trivialized normal bundle.

Section 3 is devoted to the study of the Chevalley–Eilenberg cohomology of the Lie algebra $\mathrm{Vect}(n,m)_0$. We introduce this cohomology in Section 3A, then present in Section 3B technical results that will play a fundamental role in the construction of secondary characteristic classes. We show in Section 3C how to compute the cohomology of $\mathrm{Vect}(n,m)_0$ with the help of the Weil algebra.

We construct secondary characteristic classes of superfoliations with trivialized normal bundle in Section 4. Secondary characteristic classes of a superfoliation are defined in Section 4A by constructing a homomorphism of differential graded algebras from the complex $\bigwedge \mathrm{Vect}(n,m)_0$ of the Lie algebra $\mathrm{Vect}(n,m)_0$ to the de Rham complex $\Omega(M)$ of the manifold M. We give examples in Section 4B, and prove Theorem A (in fact, a more precise statement) in Section 4D.

We relate in Section 5A the previously constructed secondary characteristic classes to the secondary characteristic classes of the (ordinary) foliation $F_{\mathcal{F}}$ on the base manifold. We prove Theorem B in Section 5B.

Most result of this paper have been announced, but not proved, in [Laurent-Gengoux 2004].

2. Superfoliations

2A. ***The algebra of superdifferential forms.*** For any supervector space $\mathcal{V}$, denote by $\mathcal{V}_0$ its even part. For convenience, write DGA for “differential graded algebra”. For a DGA (A, d_A), denote by $[a] \in H^*(A)$ the cohomology class of an element $a \in \operatorname{Ker} d_A$. Throughout this article, $\mathcal{M}$ is a smooth supermanifold of superdimension $p+\varepsilon q$ (p and q being nonnegative integers), and M is its p-dimensional base manifold. In the sequel, we say “dimension” for “superdimension”, and “codimension” for “supercodimension”

We now recall some basic results about supermanifolds, and introduce some maps that we will need in the sequel.

Denote by $\mathbb{O}(\mathcal{M})$ the superalgebra of superfunctions on $\mathcal{M}$, and by $\mathcal{I}(\mathcal{M})$ the superalgebra of nilpotent superfunctions; denote by $\mathrm{Re}(f)$ the smooth function on M defined by the canonical projection $\mathbb{O}(\mathcal{M}) \to \mathbb{O}(\mathcal{M})/\mathcal{I}(\mathcal{M}) \simeq \mathscr{C}^\infty(M, \mathbb{R})$.

Let $\big(\Omega(\mathcal{M}), \wedge\big)$ be the superalgebra of differential forms on $\mathcal{M}$. The notion of parity of a superdifferential form can be ambiguous; in this paper the parity is the one that endows the superalgebra $\Omega(\mathcal{M})$ with a structure of a supercommutative superalgebra. More precisely, in a local chart $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$, the 1-form dx_i is odd, the 1-form $d\theta_j$ is even, the 0-form x_i is even, and the 0-form θ_j is odd.

The parity is then extended by multiplication: for any $\omega_1, \omega_2 \in \Omega(\mathcal{M})$, the parity of $\omega_1 \wedge \omega_2$ is the sum of the parity of ω_1 with the parity of ω_2 (modulo 2). For example, $\theta_j\,dx_i$ is even, $x_i^2 d\theta_j$ is even, and $dx_i \wedge d\theta_j$ is odd. Note that, with this definition, a 1-form on an ordinary differential manifold is an odd element. Let $d_{\mathcal{M}} : \Omega^*(\mathcal{M}) \to \Omega^{*+1}(\mathcal{M})$ be the de Rham differential.

Let $E_{\mathcal{M}}$ be a q-dimensional vector bundle over M whose sheaf of sections is $\mathcal{I}(\mathcal{M})/\mathcal{I}(\mathcal{M})^2$. The supermanifold $\mathcal{M}(M, E)$ whose superfunctions are the sections of $\bigwedge^* E_{\mathcal{M}}$ is isomorphic to $\mathcal{M}$ (as a supermanifold, see [Leites 1980]). A local chart $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ is said to be *attached* to $E_{\mathcal{M}}$ if the odd superfunctions $\theta_1, \dots, \theta_q$ are identified with sections $s_1, \dots, s_q$ of $E_{\mathcal{M}}$. Note that in this case the supervector fields $\partial/\partial\theta_1, \dots, \partial/\partial\theta_q$ can be identified with sections of the dual vector bundle $E^*_{\mathcal{M}}$, and the 1-forms $d\theta_1, \dots, d\theta_q$ can be identified with sections of $E_{\mathcal{M}}$ again. More precisely, there is a natural bijection $\mathcal{S}$ from the vector bundle whose sections are 1-forms in $\Omega^1(\mathcal{M})$ of the form $\sum_{j=1}^q f_j(x_1, \dots, x_p)\,d\theta_j$ to the vector bundle $E_{\mathcal{M}}$ given, in a local chart attached to $E_{\mathcal{M}}$, by

$$\mathcal{S}\Big(\sum_{j=1}^{q} f_j(x_1, \dots, x_p)\,d\theta_j\Big) = \sum_{j=1}^{q} f_j(x_1, \dots, x_p)\,s_j,$$

with $f_1(x_1, \dots, x_p), \dots, f_q(x_1, \dots, x_p)$ smooth functions on M. It is straightforward to check that $\mathcal{S}$ does not depend on the local chart attached to $E_{\mathcal{M}}$.

We recall the definition of three maps:

• The first map is a well-known (see [Leites 1980; Tuynman 2004]) canonical DGA homomorphism p from $\big(\Omega^*(\mathcal{M}), \wedge, d_{\mathcal{M}}\big)$ onto $\big(\Omega^*(M), \wedge, d_M\big)$, constructed in [Leites 1980]. In a local chart $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$, it is defined on the generators dx_i, $d\theta_j$, f (f being a superfunction) of $\Omega(\mathcal{M})$ by

$$p(dx_i) = dx_i, \qquad p(d\theta_i) = 0, \qquad p(f) = \mathrm{Re}(f).$$

A proof of the next lemma can be found in [Leites 1980; Tuynman 2004].

Lemma 2.1. *The map p from $\big(\Omega^*(\mathcal{M}), \wedge, d_{\mathcal{M}}\big)$ onto $\big(\Omega^*(M), \wedge, d_M\big)$ is well defined* (*that is*, *independent of the chart*) *and a* DGA *homomorphism. The kernel of p contains odd $2k$-forms and even $(2k+1)$-forms.*

Since p is a DGA homomorphism, it induces a homomorphism $\hat{p}$ from $H^*(\mathcal{M})$ to $H^*(M)$. According to a theorem of Batchelor (see [Tuynman 2004, Theorem 8.2]), $\hat{p} : H^*(\mathcal{M}) \to H^*(M)$ is indeed an isomorphism.

• The second map, ρ, is a map from $\Omega^1(\mathcal{M})$ to $\Gamma(E_{\mathcal{M}})$, the space of sections of $E_{\mathcal{M}}$. It is defined in a local chart by

$$\rho\Big(\sum_{i=1}^{p} f_i\,dx_i + \sum_{j=1}^{q} g_i\,d\theta_i\Big) = \mathcal{S}\Big(\sum_{j=1}^{q} \mathrm{Re}(g_j)\,d\theta_j\Big) = \sum_{j=1}^{q} \mathrm{Re}(g_j)\,s_j. \tag{2-1}$$

Lemma 2.2. *The map ρ is well defined (that is, it does not depend on the chart attached to $E_{\mathcal{M}}$). For any odd 1-form ω, we have $\rho(\omega) = 0$. For any nilpotent function $f \in \mathcal{I}(\mathcal{M})$ and any $\omega \in \Omega^1(M)$, we have $\rho(f\omega) = 0$.*

• The third map that we need, Π, is a map from the super-Lie algebra $\mathrm{Vect}(\mathcal{M})$ of supervector fields on $\mathcal{M}$ to the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M. For any odd supervector field $\mathcal{X} \in \mathrm{Vect}(\mathcal{M})$, we simply set $\Pi(\mathcal{X}) = 0$. For any even supervector field $\mathcal{X} \in \mathrm{Vect}(\mathcal{M})_0$, we set $\Pi(\mathcal{X}) = X$, where X is the derivation of $C^\infty(M, \mathbb{R}) \simeq \mathcal{O}(\mathcal{M})/\mathcal{I}(\mathcal{M})$ induced by the even derivation $\mathcal{X}$ of $\mathcal{O}(\mathcal{M})$.

In a local chart attached to $E_{\mathcal{M}}$, Π is given by

$$\Pi\Big(\sum_{i=1}^{p} f_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{q} g_j \frac{\partial}{\partial x_j}\Big) = \sum_{i=1}^{p} \mathrm{Re}(f_i) \frac{\partial}{\partial x_i}.$$

Lemma 2.3. *The restriction of Π to the subalgebra* $\mathrm{Vect}(\mathcal{M})_0$ *of even supervector fields is a Lie algebra homomorphism.*

The next properties can be checked locally in a chart; we leave the computations to the reader. Denote by $\iota_{\mathcal{Y}}\eta$ the contraction of a supervector field by a 1-form in $\Omega(\mathcal{M})$. For any 2-form $\eta \in \Omega^2(\mathcal{M})$ and even supervector field $\mathcal{Y}$ such that $\Pi(\mathcal{Y}) = 0$,

$$\rho(\iota_{\mathcal{Y}}\eta) = 0. \tag{2-2}$$

For any even 1-form $\alpha \in \Omega^1(\mathcal{M})$ with $\rho(\alpha) = 0$ and any even supervector field $\mathcal{Y}$,

$$\rho(\iota_{\mathcal{Y}}\, d\alpha) = 0. \tag{2-3}$$

For any even supervector field $\mathcal{X} \in \mathrm{Vect}(\mathcal{M})_0$ and odd 1-form $\omega \in \Omega^1(\mathcal{M})$,

$$\mathrm{Re}(\iota_{\mathcal{X}}\omega) = \iota_{\Pi(X)}\, p(\omega). \tag{2-4}$$

2B. *Definition of a superfoliation.* We recall the definition of a superfoliation of codimension $n + \varepsilon m$ [Leites 1980; Monterde et al. 1997; Tuynman 2004]. First, a *distribution* of codimension $n+\varepsilon m$ is a sub-supervector bundle of $T\mathcal{M}$ of dimension $(p-n) + \varepsilon(q-m)$.

Definition 2.4. A *superfoliation* $\mathcal{F}$ of codimension $n + \varepsilon m$ is a distribution $\mathcal{D}_{\mathcal{F}}$ of dimension $(p-n) + \varepsilon(q-m)$ whose sections are closed under the bracket of supervector fields.

Remark 2.5. Note that in the literature the terminology is not fixed: in [Leites 1980] the name "foliation" is used, while in [Tuynman 2004] it is called "integrable distribution". We choose "superfoliation" to avoid confusions when dealing simultaneously with superfoliations on a supermanifold and (ordinary) foliations on its base manifold.

A supermanifold is said to be *foliated* when it is endowed with a superfoliation of codimension $n + \varepsilon m$, for some $n, m \in \mathbb{N}$. A supervector field $\mathscr{X}$ is said to be *tangent to the leaves* if and only if it is a section of the distribution that defines $\mathscr{F}$. We denote by $\mathscr{X}_{\mathscr{F}}$ the superalgebra of supervector fields tangent to the leaves of $\mathscr{F}$.

For a superfoliation $\mathscr{F}$, denote by $\Omega^*_{\mathscr{F}}$ the subalgebra of $\Omega(\mathscr{M})$ of k-forms ω such that $\iota_{\mathscr{Y}_1} \dots \iota_{\mathscr{Y}_k} \omega = 0$ for any supervector fields $\mathscr{Y}_1, \dots, \mathscr{Y}_k$ tangent to the leaves of $\mathscr{F}$. Of course,

$$d_{\mathscr{M}} \Omega^*_{\mathscr{F}} \subset \Omega^*_{\mathscr{F}}. \tag{2-5}$$

In particular, $\Omega^1_{\mathscr{F}}$ is the space of 1-forms $\omega \in \Omega^1(\mathscr{M})$ such that $\omega(\mathscr{X}) = 0$ for any supervector field $\mathscr{X}$ tangent to the leaves.

A superfunction is said to be *constant on the leaves* if $\iota_{\mathscr{X}} d_{\mathscr{M}} f = 0$ for any supervector field $\mathscr{X}$ tangent to the leaves. We denote by $\mathbb{O}_{\mathscr{F}}$ the superalgebra of superfunctions constant on the leaves of $\mathscr{F}$.

Here is a basic example: On the supermanifold $\mathbb{R}^{p,q}$, denote by $x_1, \dots, x_p$ (respectively, $\theta_1, \dots, \theta_q$) the even (respectively, odd) variables. The *elementary superfoliation* $\mathbb{R}^{p,q}_{n,m}$ is the superfoliation given by the distribution generated by the supervector fields $\partial/\partial x_{n+1}, \dots, \partial/\partial x_p, \partial/\partial \theta_{m+1}, \dots, \partial/\partial \theta_q$.

An important particular case is when the superfoliation is defined by a distribution that admits a trivialized normal bundle.

Definition 2.6. A superfoliation $\mathscr{F}$ of codimension $n + \varepsilon m$ is said to *have a trivialized normal bundle* if the distribution $\mathscr{D}_{\mathscr{F}}$ that defines $\mathscr{F}$ is generated by a free family of n odd and m even 1-forms $b_1, \dots, b_{n+m}$; that is,

$$\mathscr{D}_{\mathscr{F}} = \big\{ \mathscr{X} \in \mathrm{Vect}(\mathscr{M}) \;\big|\; \iota_{\mathscr{X}} b_1 = \dots = \iota_{\mathscr{X}} b_{n+m} = 0 \big\}.$$

We say that the 1-forms $b_1, \dots, b_{n+m}$ *define* the superfoliation $\mathscr{F}$.

Example 2.7. The elementary superfoliation $\mathbb{R}^{p,q}_{n,m}$ is defined by the odd 1-forms $dx_1, \dots, dx_n$ together with the even 1-forms $d\theta_1, \dots, d\theta_m$.

If the superfoliation $\mathscr{F}$ with trivialized normal bundle is defined by a free family $b_1, \dots, b_{n+m}$, then (see [Monterde et al. 1997]) there must exist 1-forms b_i^j, $i, j \in \{1, \dots, n+m\}$, satisfying

$$d_{\mathscr{M}} b_i = \sum_{i=1}^{n+m} b_j \wedge b_i^j. \tag{2-6}$$

Moreover, we can assume that the parity of b_i^j is the sum of the parity of b_i with the parity of b_j.

Remark 2.8. For any superfoliation $\mathcal{F}$ with trivialized normal bundle, defined by 1-forms $b_1, \dots, b_{n+m}$ (with $b_1, \dots, b_n$ odd, and $b_{n+1}, \dots, b_{n+m}$ even), there exist supervector fields $X_1, \dots, X_{n+m}$ (with $X_1, \dots, X_n$ even, and $X_{n+1}, \dots, X_{n+m}$ odd) such that $b_i(X_j) = \delta_i^j$ (where δ_i^j is the Kronecker symbol). The family $X_1, \dots, X_{n+m}$ is free (with respect to the $\mathcal{O}(\mathcal{M})$-module structure). The supertangent bundle $T\mathcal{M}$ is the direct sum of the distribution generated by $X_1, \dots, X_{n+m}$ with the distribution $\mathcal{D}_{\mathcal{F}}$ that defines $\mathcal{F}$. This justifies calling it "with trivialized normal bundle".

We now define morphisms between foliated supermanifolds, and pull-backs of superfoliations.

Definition 2.9. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be supermanifolds, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be superfoliations of codimension $n + \varepsilon m$ on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. A submersion $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is said to be a *morphism of superfoliations* if and only if

$$\mathcal{O}(\mathcal{M}_1)\, \Phi^*\Omega^1_{\mathcal{F}_2} = \Omega^1_{\mathcal{F}_1}, \tag{2-7}$$

where $\mathcal{O}(\mathcal{M}_1)\, \Phi^*\Omega^1_{\mathcal{F}_2}$ denotes the $\mathcal{O}(\mathcal{M}_1)$-module generated by $\Phi^*\Omega^1_{\mathcal{F}_2}$

In other words, (2-7) means that any 1-form in $\Omega_{\mathcal{F}_1}$ is a linear combination, with coefficients in $\mathcal{O}(\mathcal{M}_1)$, of pull-backs by Φ of forms from $\Omega_{\mathcal{F}_2}$.

Note that, if $\mathcal{F}_1$ has a trivialized normal bundle, so has $\mathcal{F}_2$. If $b_1, \dots, b_{n+m}$ are the 1-forms defining $\mathcal{F}_1$, then the superfoliation $\mathcal{F}_2$ is defined by the 1-forms $\Phi^* b_1, \dots, \Phi^* b_{n+m}$.

The *category of foliated supermanifolds* is the category whose objects are superfoliations and whose arrows are morphism of foliated manifolds. For any $n, m \in \mathbb{N}$, the superfoliations of codimension $n + \varepsilon m$ make up a subcategory, while the superfoliations of codimension $n + \varepsilon m$ with trivialized normal bundle form again a subcategory of the latter category.

An *isomorphism* of foliated manifolds is an invertible morphism of foliated manifolds.

The local structure of a superfoliation is always the same, as proved in [Hill and Simanca 1991] (see also [Tuynman 2004, Chapter V.4]):

Theorem 2.10 [Hill and Simanca 1991; Monterde et al. 1997]. *Any superfoliation of codimension $n + \varepsilon m$ on a supermanifold of dimension $p + \varepsilon q$ is locally isomorphic to the elementary superfoliation $\mathbb{R}^{p,q}_{n,m}$.*

Let $\mathcal{N}$ and $\mathcal{M}$ be supermanifolds, and let $\mathcal{F}$ be a superfoliation of codimension $n + \varepsilon m$ on $\mathcal{M}$. Given any submersion $\Phi : \mathcal{N} \to \mathcal{M}$, the *pull-back* $\Phi^*\mathcal{F}$ is the unique superfoliation satisfying

$$\mathcal{O}(\mathcal{N})\, \Phi^*\Omega^1_{\mathcal{F}} = \Omega^1_{\Phi^*\mathcal{F}}.$$

This definition needs some justification. For any open set $U \subset M$, we denote by $\mathcal{U}$ the supermanifold defined by $\mathcal{O}(\mathcal{M})|_U$ and we say that $\mathcal{U}$ is an open subset of $\mathcal{M}$. Let $r + \varepsilon s$, with $r \geq p$ and $s \geq q$, be the dimension of $\mathcal{N}$. By Theorem 2.10, in local coordinates $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ the superfoliation $\mathcal{F}$ restricted to $\mathcal{U}$ is defined by $dx_1, \dots, dx_n, d\theta_1, \dots, d\theta_m$. Since Φ is a submersion, there are local coordinates $(x_1, \dots, x_r, \theta_1, \dots, \theta_s)$ on some open subset $\mathcal{V} \subset \mathcal{N}$ such that Φ is given by

$$\Phi(x_1, \dots, x_r, \theta_1, \dots, \theta_s) = (x_1, \dots, x_p, \theta_1, \dots, \theta_q).$$

The pull-back $\Phi^*\mathcal{F}$ of $\mathcal{F}$ via Φ is the superfoliation whose restriction to $\mathcal{V}$ is defined by the 1-forms $dx_1, \dots, dx_n, d\theta_1, \dots, d\theta_m \in \Omega^1(\mathcal{V})$. This justifies the existence and uniqueness of the pull-back of a superfoliation via a submersion.

2C. ***Geometric constructions associated to a superfoliation.*** Let N be an (ordinary) smooth manifold and F a foliation of codimension n on N, defined by a distribution D_F. Let $E \to N$ be a vector bundle over N, and let $X_F = \Gamma(D_F)$ be the algebra of vector fields tangent to F. A *foliated connection* is a bilinear map from $X_F \otimes \Gamma(E)$ to $\Gamma(E)$, denoted by $(X, s) \mapsto \nabla_X s$, that satisfies the usual axioms of a connection; that is, for all $X \in X_F,\ f \in C^\infty(M),\ s \in \Gamma(E)$, we have

$$\nabla_{fX} s = f \nabla_X s \qquad \text{and} \qquad \nabla_X fs = f \nabla_X s + (X \cdot f)s.$$

A foliated connection is said to be *flat* if ∇ is flat on each leaf of F; that is, if for all $X, Y \in X_F,\ s \in \Gamma(E)$ the identity

$$\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s = 0$$

holds. A foliated connection is said to be *trivial* if E is a trivial vector bundle on M and F is a foliation with trivialized normal bundle on M.

In the following, we will say simply "foliated connection" (respectively, "trivial flat foliated connection") for the collection of a foliation F, a vector bundle E, and a foliated connection ∇. We denote by (M, F, E, ∇) a foliated connection (respectively, a trivial flat foliated connection).

2C(a). *From superfoliations to flat foliated connections.* To a superfoliation on $\mathcal{M}$ we associate a foliation $F_{\mathcal{F}}$ on the base manifold M, a vector bundle $E_{\mathcal{F}}$ on M, and a flat foliated connection $\nabla^{\mathcal{F}}$ on the latter. In the particular case of a superfoliation of codimension $0 + \varepsilon m$ on a supermanifold of dimension $n + \varepsilon m$, this construction is identical to the construction given in [Koszul 1988, Lemme 2.1].

Definition 2.11. Given a superfoliation $\mathcal{F}$ of codimension $n + \varepsilon m$, we denote by $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ the flat foliated connection defined in what follows.

It is easy to check that $p(\Omega^*_{\mathbb{R}^{p,q}_{n,m}}) = \Omega^*_{\mathbb{R}^p_n}$, where by $\mathbb{R}^p_n$ we mean the foliation $\mathbb{R}^{p,0}_{n,0}$ of codimension n on $\mathbb{R}^p$. In other words, $\mathbb{R}^p_n$ is the foliation of codimension n on $\mathbb{R}^p$ defined by the 1-forms $dx_{n+1}, \dots, dx_p$. Theorem 2.10 immediately implies:

Lemma 2.12. *The distribution whose sections are $X_{F_{\mathscr{F}}} = \Pi(\mathscr{X}_{\mathscr{F}}) \subset \Gamma(TM)$ defines a foliation $F_{\mathscr{F}}$ of codimension n on M. Moreover, $\Omega^*_{F_{\mathscr{F}}} = p(\Omega^*_{\mathscr{F}})$.*

Remark 2.13. If $\mathscr{F}$ is a superfoliation with trivialized normal bundle, defined by 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m$, then $F_{\mathscr{F}}$ is a foliation with trivialized normal bundle defined by the 1-forms $p(\omega_1), \dots, p(\omega_n) \in \Omega^1(M)$.

Now, we define $E_{\mathscr{F}}$ to be the vector subbundle of $E_{\mathcal{M}}$ whose sections $\Gamma(E_{\mathscr{F}})$ are the sections of $E_{\mathcal{M}}$ of the form $\rho(\omega)$ with $\omega \in \Omega^1_{\mathscr{F}}$, where $\Omega^1_{\mathscr{F}}$ is the space of 1-forms such that $\iota_{\mathscr{X}}\omega = 0$ for any supervector field $\mathscr{X} \in \mathscr{X}_{\mathscr{F}}$ tangent to the leaves of $\mathscr{F}$, and ρ was defined in (2-1). In the case of $\mathbb{R}^{p,q}_{n,m}$, $E_{\mathscr{F}}$ is generated by $\mathscr{S}(d_{\mathbb{R}^{p,q}}\theta_1), \dots, \mathscr{S}(d_{\mathbb{R}^{p,q}}\theta_m)$. Then, Theorem 2.10 implies that $E_{\mathscr{F}}$ is an m-dimensional vector bundle on M.

Remark 2.14. If $\mathscr{F}$ is a superfoliation with trivialized normal bundle, defined by 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m$, then $E_{\mathscr{F}}$ is a trivial vector space, and a trivialization is given by the sections $\rho(a_1), \dots, \rho(a_m) \in \Gamma(E_{\mathscr{F}})$.

For any section s of $E_{\mathscr{F}}$, let $\omega \in \Omega^1_{\mathscr{F}}$ be an even element satisfying $\rho(\omega) = s$. For any vector field $Y \in X_{F_{\mathscr{F}}}$ tangent to the foliation $F_{\mathscr{F}}$, let $\mathscr{Y} \in (\mathscr{X}_{\mathscr{F}})_0$ be an even supervector field satisfying $\Pi(\mathscr{Y}) = Y$. We define a foliated connection by

$$\nabla^{\mathscr{F}}_Y s = \rho(L_{\mathscr{Y}}\omega). \tag{2-8}$$

Lemma 2.15. *The foliated connection $\nabla^{\mathscr{F}}_Y s$ is well defined and flat.*

Proof. Since $L_{\mathscr{Y}} = d_{\mathcal{M}}\iota_{\mathscr{Y}} + \iota_{\mathscr{Y}} d_{\mathcal{M}}$, the identity

$$\rho(L_{\mathscr{Y}}\omega) = \rho(\iota_{\mathscr{Y}} d_{\mathcal{M}}\omega)$$

holds. In order to check that $\nabla^{\mathscr{F}}_Y s = \rho(L_{\mathscr{Y}}\omega)$ is a well-defined section of $E_{\mathscr{F}}$, notice these facts:

- Let $\mathscr{Z} \in \mathscr{X}_{\mathscr{F}}$ be a supervector field tangent to the leaves of $\mathscr{F}$ such that $\Pi(\mathscr{Z}) = 0$. Equation (2-2) implies that $\rho(L_{\mathscr{Z}}\omega) = \rho(\iota_{\mathscr{Z}} d_{\mathcal{M}}\omega)$ is equal to 0.
- Let $\alpha \in \Omega^1_{\mathscr{F}}$ be an even 1-form such that $\rho(\alpha) = 0$. Equation (2-3) implies that $\rho(L_Y\alpha) = 0$.

We now verify that indeed $\nabla^{\mathscr{F}}$ is a section of $E_{\mathscr{F}}$. If $\omega \in \Omega^1_{\mathscr{F}}$, then $L_{\mathscr{Y}}\omega$ is an element of $\Omega^1_{\mathscr{F}}$ as well, since

$$\iota_{\mathscr{X}} L_{\mathscr{Y}}\omega = \iota_{[\mathscr{X},\mathscr{Y}]}\omega - \iota_{\mathscr{Y}} L_{\mathscr{X}}\omega = 0$$

for all $X \in \mathscr{X}_{\mathscr{F}}$. The section $\nabla^{\mathscr{F}}_Y s$ of $E_{\mathcal{M}}$ given by $\nabla^{\mathscr{F}}_Y s = \rho(L_{\mathscr{Y}}\omega)$ is thus a section of $E_{\mathscr{F}}$ again.

Now, we check that the curvature is zero. For this, we show that there exists a local trivialization of $E_{\mathscr{F}}$ by sections $s_1, \dots, s_m$ that are horizontal (that is, $\nabla^{\mathscr{F}}_Y s_i = 0$

for all $Y \in X_{F_{\mathcal{F}}}$). For any $x \in M$, let U be a neighborhood of x and $\varphi_U : \mathcal{U} \to \mathbb{R}^{p,q}_{n,m}$ an isomorphism of foliated manifolds, where $\mathcal{U}$ is the supermanifold over U defined by restricting $\mathcal{M}$ to U. Since the superfunctions $\theta_1, \dots, \theta_m$ are constant on the leaves of $\mathbb{R}^{n,m}_{p,q}$, the odd superfunctions $f_1 = \varphi_U^* \theta_1, \dots, f_m = \varphi_U^* \theta_m \in \Omega(\mathcal{M})$ are constant on the leaves of $\mathcal{F}$. Moreover, the sections $s_1 = \rho(d_{\mathcal{M}} f_1), \dots, s_n = \rho(d_{\mathcal{M}} f_m)$ define a local trivialization of $E_{\mathcal{F}}$. For any vector field $Y \in X_{F_{\mathcal{F}}}$ and any even supervector field $\mathcal{Y} \in (\mathcal{X}_{\mathcal{F}})_0$ with $\Pi(\mathcal{Y}) = Y$, we have

$$\nabla^{\mathcal{F}}_Y s_i = \rho(L_{\mathcal{Y}} d_{\mathcal{M}} f_i) = \rho(d_{\mathcal{M}} \iota_{\mathcal{Y}} d_{\mathcal{M}} f_i) = \rho(0) = 0.$$

The existence of a local horizontal trivialization of $E_{\mathcal{F}}$ implies that the connection $\nabla^{\mathcal{F}}$ is flat. □

We summarize:

Proposition 2.16. *Let $\mathcal{F}$ be a superfoliation of codimension $n + \varepsilon m$.*

(1) $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ *is a flat foliated connection.*

(2) *If the superfoliation $\mathcal{F}$ has a trivialized normal bundle, then the flat foliated connection $(M, F_{\mathcal{F}}, E_{\mathcal{F}}, \nabla^{\mathcal{F}})$ is trivial.*

Proof. Statement (1) is a consequence of Lemma 2.12, while (2) is a consequence of Remarks 2.13 and 2.14 □

2C(b). *From flat foliated connections to superfoliations.* Let (N, F, E, ∇) be a flat foliated connection. Consider the supermanifold $\mathcal{M}(N, E)$ whose superfunctions are the sections of $\bigwedge E$. For any vector field $X \in \mathrm{Vect}(N)$, ∇_X is an even derivation of the superalgebra of superfunctions $\Gamma\big(\bigwedge E\big)$. These even derivations can be considered as even vector fields of the supermanifold $\mathcal{M}(N, E)$. Moreover, since $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$, these even derivations form a Lie algebra. The $\Gamma\big(\bigwedge E\big)$-module generated by the supervector fields $\{\nabla_X \mid X \in X_F\}$ is a distribution that defines a superfoliation of codimension $n + \varepsilon m$ on the supermanifold $\mathcal{M}(N, E)$.

This provides a canonical way to associate a superfoliation of codimension $n + \varepsilon m$ to a manifold N, a foliation F of codimension n, a vector-bundle E of dimension m, and a flat foliated connection ∇.

Definition 2.17. We denote by $\mathcal{F}(N, F, E, \nabla)$ the superfoliation of codimension $n + \varepsilon m$ on the supermanifold $\mathcal{M}(N, E)$, defined above.

A foliated connection (N, F, E, ∇) is said to be *trivial* if F has a trivialized normal bundle and the vector bundle E is trivial. The next proposition is straightforward and we leave it to the reader.

Proposition 2.18. (1) *If (N, F, E, ∇) is a trivial flat foliated connection, then the superfoliation $\mathcal{F}(N, F, E, \nabla)$ has a trivialized normal bundle.*

(2) *For any foliated connection* (M, F, E, ∇), *we have*

$$\big(M, F, E, \nabla\big) = \big(M, F_{\mathcal{F}(M,F,E,\nabla)}, E_{\mathcal{F}(M,F,E,\nabla)}, \nabla^{\mathcal{F}(M,F,E,\nabla)}\big). \tag{2-9}$$

Remark 2.19. By (2-9), the construction of Section 2C(a) is the left inverse of the construction of Section 2C(b). However, it is not a right inverse. More precisely, we show now that the construction of Section 2C(a) is not injective.

Consider the supercircle $S^{1,3}$ defined by a trivial vector bundle $E_3 = S^1 \times \mathbb{R}^3 \to S^1$ of dimension 3 over S^1. Denote by $x \in S^1$ the even parameter and by $\theta_1, \theta_2, \theta_3$ the odd parameters associated to three canonical sections s_1, s_2, s_3 of $E_3 = S^1 \times \mathbb{R}^3 \to S^1$.

Take the superfoliations $\mathcal{F}_1$ and $\mathcal{F}_2$ defined by the distributions of codimension $0 + \varepsilon 3$ generated by the distributions $\mathcal{O}(\mathcal{M})\, \partial/\partial x$ and $\mathcal{O}(\mathcal{M})\big(\partial/\partial x + \theta_1\theta_2\theta_3\, \partial/\partial\theta_1\big)$. These two superfoliations define the same foliation on M, the foliation with only one leaf: S^1 itself. They define the same vector bundle: E_3 itself. They both also define the same connection, namely, the connection given by

$$\nabla_{a(x)\frac{\partial}{\partial x}} s = a(x)\frac{ds}{dx}$$

for all $a(x) \in C^\infty(S^1)$ and $s \in \Gamma(E_3)$. However, these superfoliations are not isomorphic, because their superalgebras of superfunctions constant on leaves are not isomorphic.

For the superfoliation $\mathcal{F}_1$, the superfunctions $\theta_1, \theta_2, \theta_3$ are constant on leaves, and the algebra of superfunctions that are constant on the leaves of $\mathcal{F}_1$ is therefore isomorphic to $\bigwedge \mathbb{R}^3$. In particular, the vector space of odd superfunctions that are constant on the leaves of $\mathcal{F}_1$ has dimension 4.

For the superfoliation $\mathcal{F}_2$, any odd superfunction f can be written

$$f = \sum_{i=1}^{3} f_i(x)\theta_i + g(x)\theta_1\theta_2\theta_3$$

with $f_1, f_2, f_3, g \in \mathscr{C}^\infty(S^1, \mathbb{R})$. If f is an odd superfunction constant on the leaves of $\mathcal{F}_2$, then

$$\Big(\frac{\partial}{\partial x} + \theta_1\theta_2\theta_3\frac{\partial}{\partial\theta_1}\Big) \cdot f = 0 \quad \text{and} \quad \sum_{i=1}^{3} \frac{df_i(x)}{dx}\theta_i + \Big(\frac{dg(x)}{dx} + f_1(x)\Big)\theta_1\theta_2\theta_3 = 0.$$

This implies that $df_1/dx = df_2/dx = df_3/dx = 0$ and $dg/dx + f_1(x) = 0$. Hence, for any $i \in \{1, 2, 3\}$, we obtain $f_i(x) = a_i$ for some constant $a_i \in \mathbb{R}$. But the equation $dg/dx + a_1 = 0$ has no periodic solution unless $a_1 = 0$. As a consequence, the vector space of odd superfunctions that are constant on the leaves of $\mathcal{F}_2$ has only dimension 3. The superfoliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are therefore not isomorphic.

2D. ***Superfoliations with trivialized normal bundle.*** We describe a canonical process to replace a superfoliation on $\mathcal{M}$ by a superfoliation with trivialized normal bundle on a $GL(n, m)$-bundle $\mathcal{P}_{\mathcal{F}}$ over $\mathcal{M}$. For a definition of the super-Lie group $GL(n, m)$, see [Leites 1980; Tuynman 2004].

The tangent bundle of the superfoliation $\mathcal{F}$ is a supervector subbundle of $T\mathcal{M}$, of dimension $(p-n)+\varepsilon(q-m)$. The normal bundle $\mathcal{G}$ of $\mathcal{F}$ is a supervector subbundle of $T^*\mathcal{M}$ of dimension $n+\varepsilon m$. Let $\mathcal{P}_{\mathcal{F}}$ be the frame bundle of $\mathcal{G}$; this is a $GL(n, m)$-bundle over $\mathcal{M}$. Denote by $\pi : \mathcal{P}_{\mathcal{F}} \to \mathcal{M}$ the canonical projection on $\mathcal{M}$.

By the construction of a frame bundle, there is a canonical inclusion of $\mathcal{P}_{\mathcal{F}}$ into $\mathcal{G}+\mathcal{G}[1] \subset (T^*\mathcal{M})^{\oplus n} \oplus T^*\mathcal{M}[1]^{\oplus m}$, where $\mathcal{G}[1]$ and $T^*\mathcal{M}[1]$ are the supervector bundles obtained by reversing parities on the fibers. Let α be the canonical odd 1-form of $T^*\mathcal{M}$ and β the canonical even 1-form of $T^*\mathcal{M}[1]$. Since we have not been able to locate these 1-forms in the existing literature, we introduce them now. Consider local coordinates on $T^*\mathcal{M}$

$$(x_1, \dots, x_p, \theta_1, \dots, \theta_q, y_1, \dots, y_p, \eta_1, \dots, \eta_q),$$

where $x_1, \dots, x_p, \theta_1, \dots, \theta_q$ are local coordinates on $\mathcal{M}$ and $y_1, \dots, y_p, \eta_1, \dots, \eta_q$ are the even and odd coordinates on the cotangent bundle of $\mathcal{M}$ corresponding to the basis dual to $\partial/\partial x_1, \dots, \partial/\partial x_p, \partial/\partial\theta_1, \dots, \partial/\partial\theta_q$ of $T\mathcal{M}$. We define α as

$$\alpha = \sum_{i=1}^{p} y_i \, dx_i - \sum_{j=1}^{p} \eta_j \, d\theta_j.$$

It is routine to check that α does not depend on the local coordinates $x_1, \dots, x_p$, $\theta_1, \dots, \theta_q$ on $\mathcal{M}$. The 1-form β is defined by the same formula, where $y_1, \dots, y_p$ are now considered to be odd variables and $\eta_1, \dots, \eta_q$ are considered to be even variables.

For $i = 1, \dots, n$, denote by $f_i : \mathcal{P}_{\mathcal{F}} \to T^*\mathcal{M}$ the i-th projection on $T^*\mathcal{M}$ and, for $j = 1, \dots, m$, denote by $g_j : \mathcal{P}_{\mathcal{F}} \to T^*\mathcal{M}$ the j-th projection on $T^*\mathcal{M}[1]$. Define a family of 1-forms on $\mathcal{P}_{\mathcal{F}}$ as

$$\omega_i = f_i^*\alpha, \quad i = 1, \dots, n, \qquad \text{and} \qquad a_j = g_j^*\beta, \quad j = 1, \dots, m.$$

Proposition 2.20. *The pull-back $\pi^*\mathcal{F}$ of the superfoliation $\mathcal{F}$ by π is a superfoliation on $\mathcal{P}_{\mathcal{F}}$ of codimension $n+\varepsilon m$ with trivialized normal bundle, defined by the 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m$.*

Proof. According to Theorem 2.10, we just have to check that Proposition 2.20 holds in the case of the superfoliation $\mathbb{R}^{p,q}_{n,m}$. Let $(x_1, \dots, x_p, \theta_1, \dots, \theta_q)$ be the (global) coordinates of $\mathbb{R}^{p,q}_{n,m}$ and y_i^j, η_k^j, ζ_i^l, z_k^l, for $i, j \in \{1, \dots, n\}$ and $k, l \in \{1, \dots, m\}$, and with $\det(y_i^k) \neq 0$ and $\det(z_k^l) \neq 0$, be the (global) coordinates of $GL(n, m)$. These coordinates define a system of coordinates on the supermanifold

$\mathcal{P}_{\mathbb{R}^{p,q}_{n,m}} \simeq \mathbb{R}^{p,q}_{n,m} \times GL(n,m)$. In the latter system, we have

$$\omega_i = \sum_{k=1}^{n} y^i_k \, dx_k - \sum_{l=1}^{m} \theta^i_l \, d\theta_l \qquad \text{and} \qquad a_j = \sum_{k=1}^{n} \zeta^j_k \, dx_k - \sum_{l=1}^{m} z^j_l \, d\theta_l.$$

Since $\det(y^k_i) \neq 0$ and $\det(z^l_k) \neq 0$, the 1-forms $\omega_1, \dots, \omega_n, a_1, \dots, a_m \in \Omega^1(\mathcal{P}_{\mathbb{R}^{p,q}_{n,m}})$ and the 1-forms $dx_1, \dots, dx_n, d\theta_1, \dots, d\theta_m \in \Omega^1(\mathcal{P}_{\mathbb{R}^{p,q}_{n,m}})$ define the same superfoliation on $\mathcal{P}_{\mathbb{R}^{p,q}_{n,m}}$. This superfoliation is clearly $\pi^*\mathcal{F}$. This completes the proof. □

Remark 2.21. Proposition 2.20 provides a canonical way to replace a superfoliation $\mathcal{F}$, which has no trivialized normal bundle, by a superfoliation $\pi^*\mathcal{F}$ that has a trivialized normal bundle. This construction is functorial, in the sense that, if $\mathcal{F}_1$ is a superfoliation on $\mathcal{M}_1$, then $\mathcal{F}_2$ is a superfoliation on $\mathcal{M}_2$, and, if $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a morphism of foliated supermanifolds, then Φ induces a submersion $\Phi_{\mathcal{P}} : \mathcal{P}_{\mathcal{F}_1} \to \mathcal{P}_{\mathcal{F}_2}$ that is a morphism of foliated supermanifolds with respect to the superfoliations $\pi^*\mathcal{F}_1$ and $\pi^*\mathcal{F}_2$. In other words, the application $\mathcal{F} \mapsto \mathcal{P}_{\mathcal{F}}$ defines a functor from the category of foliated supermanifolds of codimension $n+\varepsilon m$ to the category of foliated supermanifolds of codimension $n+\varepsilon m$ with trivialized normal bundle.

3. The cohomology of Vect$(n, m)_0$

For any super-Lie algebra $\mathfrak{g}$, let $X_1, \dots, X_g$ be a basis of $\mathfrak{g}$ and $a_1, \dots, a_g$ the dual basis, with $a_i(X_i) = (-1)^{|X_i|}$, where $|X_i|$ is the parity of X_i, and $a_i(X_j) = 0$ for $i \neq j$. If for any $i, j, k \in \{1, \dots, g\}$ there exists some $C^k_{i,j} \in \mathbb{R}$ with

$$[X_i, X_j] = \sum_{k=1}^{g} C^k_{i,j} X_k,$$

then the Chevalley–Eilenberg differential is given by (see [Fuchs 1986])

$$\delta a_k = \tfrac{1}{2} \sum_{i,j=1}^{g} (-1)^{|X_i|} C^k_{i,j} a_i \wedge a_j. \tag{3-1}$$

Since we work with infinite-dimensional Lie algebras, we have to replace the Chevalley–Eilenberg cohomology by an infinite-dimensional generalization, but the signs in Equation (3-1) will remain valid.

3A. ***The Chevalley–Eilenberg complex of* Vect(n, m) *and* Vect$(n, m)_0$.** To construct the theory of secondary characteristic classes of superfoliations, we need some prerequisites about the cohomology of a certain infinite-dimensional Lie algebra of supervector fields; this construction can be found, for instance, in [Fuchs 1986].

Take the superalgebra of *superpolynomials*

$$\mathbb{R}[x_1,\dots,x_n,\theta_1,\dots,\theta_m]=\mathbb{R}[x_1,\dots,x_n]\otimes\bigwedge\mathbb{R}^m.$$

The degree of a superpolynomial $x_1^{i_1}\dots x_n^{i_n}\theta_1^{j_1}\dots\theta_m^{j_m}$, with $i_1,\dots,i_n\in\mathbb{N}$ and $j_1,\dots,j_m\in\{0,1\}$, is defined to be $i_1+\dots+i_n+j_1+\dots+j_m$. This turns $\mathbb{R}[x_1,\dots,x_n,\theta_1,\dots,\theta_m]$ into an $\mathbb{N}$-graded superalgebra.

The graded super-Lie algebra

$$\mathrm{Vect}(n,m)=\bigoplus_{i=-1}^{+\infty}\mathrm{Vect}(n,m)^i$$

of *formal supervector fields* is defined to be the super-Lie algebra of superderivations of $\mathbb{R}[x_1,\dots,x_n,\theta_1,\dots,\theta_m]$. Elements of $\mathrm{Vect}(n,m)^i$ are said to be of *weight* i. We denote $\mathrm{Vect}(n,0)$ simply by $\mathrm{Vect}(n)$. We denote the even part of $\mathrm{Vect}(n,m)$ by $\mathrm{Vect}(n,m)_0=\bigoplus_{i=-1}^{+\infty}\mathrm{Vect}(n,m)_0^i$.

Let $\bigwedge\mathrm{Vect}(n,m)^*$ be the DGA of multilinear superalternating forms for the projective topology of $\mathrm{Vect}(n,m)$; let $\bigwedge\mathrm{Vect}(n,m)_0^*$ be the DGA of multilinear superalternating forms for the projective topology of $\mathrm{Vect}(n,m)_0^*$; see, for instance, [Fuchs 1986]. "Projective" means, in short, that an element of $\mathrm{Vect}(n,m)^*$ (respectively, $\mathrm{Vect}(n,m)_0^*$) is a linear form on $\mathrm{Vect}(n,m)$ that vanishes on all the spaces $\mathrm{Vect}(n,m)^i$ (respectively, $\mathrm{Vect}(n,m)_0^i$) but for finitely many $i\in\mathbb{N}\cap\{-1\}$. The spaces $\bigwedge\mathrm{Vect}(n,m)^*$ and $\bigwedge\mathrm{Vect}(n,m)_0^*$ are the exterior products of $\mathrm{Vect}(n,m)^*$ and $\mathrm{Vect}(n,m)_0^*$, respectively.

Denote by ∂ be the Chevalley–Eilenberg differential on $\bigwedge\mathrm{Vect}(n,m)^*$, and by ∂_0 the Chevalley–Eilenberg differential on $\bigwedge\mathrm{Vect}(n,m)_0^*$. The cohomology of the DGA $\big(\bigwedge\mathrm{Vect}(n,m)_0^*,\ \partial_0\big)$ is called the (*Chevalley–Eilenberg*) *cohomology of* $\mathrm{Vect}(n,m)_0$, and is denoted by $H^*\big(\mathrm{Vect}(n,m)_0\big)$. Maybe the name "Chevalley–Eilenberg" is not absolutely correct, since we do not consider the complex of all skew-symmetric forms, but there is no risk of confusion here.

Let $\mathscr{E}$ be the ideal (with respect to the product $\wedge$) generated by the even elements of $\mathrm{Vect}(n,m)^*$, that is, by the continuous linear forms on $\mathrm{Vect}(n,m)$ that identically vanish on $\mathrm{Vect}(n,m)_0$. The next lemma will be useful.

Lemma 3.1. *The* DGA $\bigwedge\mathrm{Vect}(n,m)^*/\mathscr{E}$ *and* $\bigwedge\mathrm{Vect}(n,m)_0^*$ *are isomorphic.*

Remark 3.2. Note that $\mathrm{Vect}(n,m)$ is sometimes denoted by $W(n,m)$. We prefer the notation $\mathrm{Vect}(n,m)$ in order to avoid confusion with the Weil algebra.

3B. ***Two technical lemmas about the complex*** $\big(\bigwedge\mathrm{Vect}(n,m)^*,\ \delta\big)$. To prove Proposition 4.2, we need to fix some notation and give the two technical Lemmata 3.5 and 3.6 about the DGA $\big(\bigwedge\mathrm{Vect}(n,m),\ \partial\big)$.

The family

$$\text{(3-2)}\qquad \begin{cases} \dfrac{1}{i_1!\dots i_n!} x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial x_i},\\ \dfrac{1}{i_1!\dots i_n!} x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial \theta_j}\end{cases}$$

(with $i_1,\dots,i_n\in\mathbb{N},\ j_1,\dots,j_m\in\{0,1\},\ i=1,\dots,n$, and $j=1,\dots,m$) is a basis of $\mathrm{Vect}(n,m)$. Denote elements of its dual basis by $\big(1/(i_1!\dots i_n!)\,x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\,\partial/\partial x_i\big)^*$ and $\big(1/(i_1!\dots i_n!)\,x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\,\partial/\partial \theta_j\big)^*$.

Remark that, if we denote by $(x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\,\partial/\partial x_i\big)^*$ and $(x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\,\partial/\partial \theta_j\big)^*$ the elements of the basis dual to the basis given by $x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}$ $\partial/\partial x_i$ and $x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\,\partial/\partial \theta_j$, then we have

$$\text{(3-3)}\quad \begin{cases} \Big(\dfrac{1}{i_1!\dots i_n!} x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial x_i}\Big)^* = i_1!\dots i_n!\Big(x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial x_i}\Big)^*\\ \Big(\dfrac{1}{i_1!\dots i_n!} x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial \theta_j}\Big)^* = i_1!\dots i_n!\Big(x_1^{i_1}\dots x_n^{i_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial \theta_j}\Big)^*\end{cases}$$

For convenience, we introduce two notations.

Definition 3.3. For all $i\in\{1,\dots,n+m\}$, we define $|i|$ to be 0 for $i\in\{1,\dots,n\}$ and be 1 for $i\in\{n+1,\dots n+m\}$.

Definition 3.4. We define $h_i,\ i=1,\dots,n+m,$ to be x_i if $|i|=0$ and be θ_{i-n} if $|i|=1$.

For example, we denote by $h_{i_1}\dots h_{i_s}h_{n+j_1}\dots h_{n+j_t}\,\partial/\partial h_1\in\mathrm{Vect}(n,m)$ the element that we used to denote by $x_{i_1}\dots x_{i_s}\,\theta_{j_1}\dots\theta_{j_t}\,\partial/\partial x_1$, where $i_1,\dots,i_s\in\{1,\dots,n\}$ and $j_1,\dots,j_t\in\{1,\dots,m\}$.

Define the *weight* of $(h_{i_1}\dots h_{i_k}\,\partial/\partial h_i)^*$ to be $k-1$ for any $i_1,\dots,i_k\in\{1,\dots,n+m\}$. Define Λ_k to be the subalgebra of $\bigwedge\mathrm{Vect}(n,m)_0^*$ generated (with respect to $\wedge$) by elements of weights $-1,0,1,\dots,k$. One has $\partial\Lambda_{k-1}\subset\Lambda_k$.

We now compute the Chevalley–Eilenberg differential in the basis dual to the basis in (3-2). For this purpose, we define $c_j^{i_1,\dots,i_k}$ for any $i_1,\dots,i_k,\ j\in\{1,\dots,n+m\}$ by

$$c_j^{i_1,\dots,i_k}=\prod_{i=1}^{n}K\big(i,[i_1,\dots,i_k]\big)!\,\Big(h_{i_1}\dots h_{i_k}\frac{\partial}{\partial h_j}\Big)^*,$$

where $K\big(i,[i_1,\dots,i_k]\big)$ is the number of integers equal to i in the list $[i_1,\dots,i_k]$. If $|j|=1$ and j appears more than twice in the list $[i_1,\dots,i_k]$, then, of course,

$$\text{(3-4)}\qquad c_j^{i_1,\dots,i_k}=0.$$

Otherwise, we have

$$c_j^{i_1,\dots,i_k} = \varepsilon\left(\frac{1}{k_1!\dots k_n!}x_1^{k_1}\dots x_n^{k_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\frac{\partial}{\partial h_j}\right)^*$$

for some $\varepsilon \in \{-1, 1\}$, where $k_1 = K\big(1, [i_1, \dots, i_k]\big), \dots, k_n = K\big(n, [i_1, \dots, i_k]\big)$ and where, for any $i \in \{1, \dots, m\}$, $j_i = 1$ if and only if $n+i$ appears in the list $[i_1, \dots, i_k]$.

For any $i_1, \dots, i_k, j \in \{1, \dots, n+m\}$, we define $P_j^{i_1,\dots,i_k} \in \Lambda_{k-1}$ by:

$$\partial c_j^{i_1,\dots,i_k} = P_j^{i_1,\dots,i_k} + \sum_{l=1}^{n+m}(-1)^{|l|}\left(\frac{\partial}{\partial h_l}\right)^* \wedge c_j^{l,i_1,\dots,i_k}. \tag{3-5}$$

Lemma 3.5. *For any* $i_1, \dots, i_k, j \in \{1, \dots, n+m\}$, $P_j^{i_1,\dots,i_k}$ *is an element of* Λ_{k-1}.

Proof. Denote by $\Sigma_{l,k}$ the set of shuffles of the sets $\{1, \dots, l\}$ and $\{l+1, \dots, k\}$. We have

$$\partial c_j^{i_1,\dots,i_k} = \sum_{l=1}^{n+m}\sum_{\sigma\in\Sigma_{l,k}} D(i_1, \dots, i_k, l, j, \sigma)\, c_l^{i_{\sigma(l+1)},\dots,i_{\sigma(k)}} \wedge c_j^{l,i_{\sigma(1)},\dots,i_{\sigma(l)}}, \tag{3-6}$$

for some $D(i_1, \dots, i_k, l, j, \sigma) \in \mathbb{R}$.

From the relations

$$\left[\frac{\partial}{\partial x_i}, \frac{1}{k_1!\dots(k_i+1)!\dots k_n!}x_1^{k_1}\dots x_i^{k_i+1}\dots x_n^{k_n}\,\theta_1^{j_1}\dots\theta_m^{j_m}\frac{\partial}{\partial h_j}\right] = \frac{1}{k_1!\dots k_i!\dots k_n!}x_1^{k_1}\dots x_i^{k_i}\dots x_n^{k_n}\theta_1^{j_1}\dots\theta_m^{j_m}\frac{\partial}{\partial h_j}, \tag{3-7}$$

$$\left[\frac{\partial}{\partial \theta_l}, \frac{\theta_l}{k_1!\dots k_n!}x_1^{k_1}\dots x_n^{k_n}\,\theta_1^{j_1}\dots\theta_{l-1}^{j_{l-1}}\theta_{l+1}^{j_{l+1}}\dots\theta_m^{j_m}\frac{\partial}{\partial h_j}\right] = \frac{1}{k_1!\dots k_n!}x_1^{k_1}\dots x_n^{k_n}\theta_1^{j_1}\dots\theta_{l-1}^{j_{l-1}}\theta_{l+1}^{j_{l+1}}\dots\theta_m^{j_m}\frac{\partial}{\partial h_j}, \tag{3-8}$$

we obtain, using (3-1), (3-4), and the definition of ∂, that

$$\partial c_j^{i_1,\dots,i_k} - \sum_{l=1}^{n+m}(-1)^{|l|}\left(\frac{\partial}{\partial h_l}\right)^* \wedge c_j^{l,i_1,\dots,i_k}$$

is a linear combinations of products of elements of weight $0, \dots, k-1$ that do not involve any terms of weight -1 or k. In other words, it is an element of Λ_{k-1}. □

Lemma 3.6. *For any* $i_1, \dots, i_k, l \in \{1, \dots, n+m\}$, *define 2-forms* $Q_j^{l,i_1,\dots,i_k} \in \Lambda_k$ *by*

$$Q_j^{l,i_1,\dots,i_k} = P_j^{l,i_1,\dots,i_k} - \sum_{l'=1}^{n+m}(-1)^{|l|+|l'|}\left(h_l\frac{\partial}{\partial h_{l'}}\right)^* \wedge c_j^{l',i_1,\dots,i_k}. \tag{3-9}$$

The following identity holds:

$$\partial P_j^{i_1,\dots,i_k} = \sum_{l=1}^{n+m} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge Q_j^{l,i_1,\dots,i_k}. \tag{3-10}$$

Proof. The identity

$$\partial c_i = \sum_{j=1}^{n+m} (-1)^{|j|} c_j \wedge c_i^j \tag{3-11}$$

holds for any $i \in \{1,\dots,n+m\}$. Applying ∂ to (3-5) and using (3-11), we obtain

$$\partial P_j^{i_1,\dots,i_k} + \partial \sum_{l=1}^{n+m} (-1)^{|l|} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge c_j^{l,i_1,\dots,i_k} = 0,$$

$$\begin{aligned}\partial P_j^{i_1,\dots,i_k} + \sum_{l=1}^{n+m}\sum_{l'=1}^{n+m} (-1)^{|l'|+|l|} \Big(\frac{\partial}{\partial h_{l'}}\Big)^* \wedge \Big(h_{l'}\frac{\partial}{\partial h_l}\Big)^* \wedge c_j^{l,i_1,\dots,i_k},\\ - \sum_{l=1}^{n+m} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge \partial c_j^{l,i_1,\dots,i_k} = 0,\end{aligned}$$

$$\begin{aligned}\partial P_j^{i_1,\dots,i_k} + \sum_{l=1}^{n+m}\sum_{l'=1}^{n+m} (-1)^{|l'|+|l|} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge \Big(h_l\frac{\partial}{\partial h_{l'}}\Big)^* \wedge c_j^{l',i_1,\dots,i_k} - \sum_{l=1}^{n+m} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge P_j^{l,i_1,\dots,i_k}\\ - \sum_{l'=1}^{n+m}\sum_{l=1}^{n+m} (-1)^{|l'|} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge \Big(\frac{\partial}{\partial h_{l'}}\Big)^* \wedge c_j^{l',l,i_1,\dots,i_k} = 0.\end{aligned}$$

Now, for any $l', l \in \{1,\dots,n+m\}$, we have

$$\begin{cases} \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge \Big(\frac{\partial}{\partial h_{l'}}\Big)^* = (-1)^{(|l|+1)(|l'|+1)} \Big(\frac{\partial}{\partial h_{l'}}\Big)^* \wedge \Big(\frac{\partial}{\partial h_l}\Big)^* \\ c_j^{l',l,i_1,\dots,i_k} = (-1)^{|l'||l|} c_j^{l,l',i_1,\dots,i_k} \end{cases}$$

Therefore, we have

$$(-1)^{|l'|}\Big(\frac{\partial}{\partial h_l}\Big)^* \wedge \Big(\frac{\partial}{\partial h_{l'}}\Big)^* \wedge c_j^{l',l,i_1,\dots,i_k} = -(-1)^{|l|}\Big(\frac{\partial}{\partial h_{l'}}\Big)^* \wedge \Big(\frac{\partial}{\partial h_l}\Big)^* \wedge c_j^{l,l',i_1,\dots,i_k}$$

and

$$\sum_{l'=1}^{n+m}\sum_{l=1}^{n+m} (-1)^{|l'|}\Big(\frac{\partial}{\partial h_l}\Big)^* \wedge \Big(\frac{\partial}{\partial h_{l'}}\Big)^* \wedge c_j^{l',l,i_1,\dots,i_k} = 0.$$

The lemma follows immediately. □

3C. *Generators of the cohomology of* Vect$(n, m)_0$. We introduce a DGA whose cohomology is $H^*\big(\mathrm{Vect}(n,m)_0\big)$ but that is easier to work with than $\mathrm{Vect}(n,m)_0$ itself.

Let gl_n (respectively, gl_m) be the Lie algebra of linear endomorphism of $\mathbb{R}^n$ (respectively, $\mathbb{R}^m$). Let $(a_{i,j})_{i,j\in\{1,\dots,n\}}$ (respectively, $(d_{k,l})_{k,l\in\{1,\dots,m\}}$) be the canonical basis of gl_n (respectively, gl_m).

The elements $\mathrm{Vect}(n,m)_0^0$ of weight 0 in $\mathrm{Vect}(n,m)_0$ form a Lie subalgebra of $\mathrm{Vect}(n,m)_0$, isomorphic to $gl_n \oplus gl_m$ through the isomorphism φ with

$$\varphi\Big(x_i \frac{\partial}{\partial x_j}\Big) = a_{i,j} \qquad \text{and} \qquad \varphi\Big(\theta_k \frac{\partial}{\partial \theta_l}\Big) = d_{k,l}. \tag{3-12}$$

Let p_0 be the projection from $\mathrm{Vect}(n,m)_0$ to $\mathrm{Vect}(n,m)_0^0$, with kernel

$$\mathrm{Ker}\, p_0 = \mathrm{Vect}(n,m)_0^{-1} \oplus \bigoplus_{i=1}^{\infty} \mathrm{Vect}(n,m)_0^i.$$

Let $\alpha : (gl_n \oplus gl_m)^* \to \mathrm{Vect}(n,m)_0^*$ be the linear map $\alpha = p_0^* \circ \varphi^*$, where p_0^* and φ^* are the dual maps of p_0 and φ.

The map α defines a $gl_n \oplus gl_m$-connection (see [Guillemin and Sternberg 1999, Chapter 3]) of the DGA $\bigwedge \mathrm{Vect}(n,m)_0^*$ that induces a DGA homomorphism (denoted again by α) from the Weil algebra $W(gl_n \oplus gl_m) = S\big((gl_n \oplus gl_m)^*\big) \otimes \bigwedge (gl_n \oplus gl_m)^*$ to $\bigwedge \mathrm{Vect}(n,m)_0^*$.

Let $\mathcal{K}$ be the kernel of α. The cohomology of the DGA $W(gl_n \oplus gl_m)/\mathcal{K}$ is denoted by $H^*\big(W(gl_n \oplus gl_m)/\mathcal{K}\big)$. Write $\tilde{\alpha}$ for the DGA homomorphism from $W(gl_n \oplus gl_m)/\mathcal{K}$ to $\bigwedge \mathrm{Vect}(n,m)_0^*$ induced by α. Let $\hat{\alpha} : H^*\big(W(gl_n \oplus gl_m)/\mathcal{K}\big) \to H^*\big(\mathrm{Vect}(n,m)_0\big)$ be the map induced by $\tilde{\alpha}$ in cohomology.

Theorem 3.7. *The map $\hat{\alpha}$ is an isomorphism between $H^*\big(W(gl_n \oplus gl_m)/\mathcal{K}\big)$ and $H^*\big(\mathrm{Vect}(n,m)_0\big)$.*

The proof is provided in Appendix A.

4. Secondary characteristic classes of superfoliations

We construct, for any superfoliation $\mathcal{F}$ on $\mathcal{M}$ of codimension $n+\varepsilon m$ with trivialized normal bundle, a map from $H^*\big(\mathrm{Vect}(n,m)_0\big)$ to the de Rham cohomology of the base manifold $H^*(M)$.

We use the conventions introduced in Defintions 3.3 and 3.4. Throughout this section, we often say “homomorphism” for “DGA homomorphism”.

4A. DGA *homomorphism defining a superfoliation.*

Definition 4.1. We say that a DGA homomorphism

$$\omega : \big(\bigwedge \mathrm{Vect}(n,m)^*, \partial\big) \to \big(\Omega^*(\mathcal{M}), d_{\mathcal{M}}\big)$$

defines the superfoliation $\mathcal{F}$ of codimension $n+\varepsilon m$ with trivialized normal bundle if the 1-forms $\omega\big((\partial/\partial h_i)^*\big)$, $i = 1, \ldots, n+m$, define the superfoliation $\mathcal{F}$.

Consider a superfoliation $\mathcal{F}$ with trivialized normal bundle on $\mathcal{M}$, defined by 1-forms $b_1, \ldots, b_{n+m}$, where $b_1, \ldots, b_n$ are odd 1-forms and $b_{n+1}, \ldots, b_{n+m}$ are even 1-forms. Take 1-forms $(b_i^j)_{i,j\in\{1,\dots,n+m\}}$ such that

$$d_{\mathcal{M}} b_i = \sum_{j=1}^{n} b_j \wedge b_i^j - \sum_{j=n+1}^{n+m} b_j \wedge b_i^j. \tag{4-1}$$

According to (2-6), such 1-forms b_i^j always exist. The choice of signs in (4-1) is explained by:

Proposition 4.2. *For any superfoliation $\mathcal{F}$ with trivialized normal bundle on $\mathcal{M}$, there exists a* DGA *homomorphism ω from $\left(\bigwedge \mathrm{Vect}(n,m)^*, \partial\right)$ to $\left(\Omega(\mathcal{M}), d_{\mathcal{M}}\right)$ that defines the superfoliation $\mathcal{F}$. Moreover, we can assume that*

$$\omega\big((\partial/\partial h_i)^*\big) = b_i, \qquad i = 1, \dots, n+m \tag{4-2}$$

$$\omega\big((h_i\, \partial/\partial h_j)^*\big) = b_j^i, \qquad i, j = 1, \dots, n+m. \tag{4-3}$$

First, we need the following lemma, proved in Appendix B.

Lemma 4.3. *Let $d_1, \dots, d_n$ be odd 1-forms and $d_{n+1}, \dots, d_{n+m}$ even 1-forms, forming a free family. Let d^i, $i = 1, \dots, n+m$, be 2-forms on $\mathcal{M}$ such that*

$$\sum_{i=1}^{n+m} d_i \wedge d^i = 0$$

There exist homogeneous 1-forms $d^{i,l}$ of parity $|i| + |l| + 1$ (modulo 2) such that

$$d^{i,l} = -(-1)^{(|i|+1)(|l|+1)} d^{l,i} \qquad \text{and} \qquad \sum_{i=1}^{n+m} d_i \wedge d^{i,l} = d^l.$$

Proof of Proposition 4.2. Define $\varepsilon(i_1, \dots, i_k, \sigma) \in \{-1, 1\}$, for any permutation σ of $\{1, \dots, k\}$, by

$$h_{i_{\sigma(1)}} \dots h_{i_{\sigma(k)}} = \varepsilon(i_1, \dots, i_k, \sigma)\, h_{i_1} \dots h_{i_k}.$$

We will construct by induction a homomorphism $\omega : \Lambda_k \to \Omega(\mathcal{M})$ such that, for any $\alpha \in \Lambda_{k-1}$,

$$d_{\mathcal{M}} \omega(\alpha) = \omega\big(\partial(\alpha)\big). \tag{4-4}$$

Equations (4-2) and (4-3) define such a map ω for $k = 0$. Equations (3-11) and (4-1) imply that the condition (4-4) holds for $k = 0$. We have therefore constructed ω for $k = 0$.

We now assume that ω can be constructed for some $k \in \mathbb{N}$ and construct it for $k + 1$. Applying ω to (3-5), we obtain

$$d_{\mathcal{M}} \omega(c_j^{i_1,\dots,i_k}) = \omega(P_j^{i_1,\dots,i_k}) + \sum_{l=1}^{n+m} (-1)^{|l|} \omega\big((\partial/\partial h_l)^*\big) \wedge \omega(c_j^{l,i_1,\dots,i_k}).$$

Denote $\omega(c_j^{i_1,\dots,i_n})$ by $b_j^{i_1,\dots,i_n}$. In particular, $b_l = \omega\big((\partial/\partial x_l)^*\big)$. By applying $d_{\mathcal{M}}$ to both sides of the previous expression and using Equation (4-2), we obtain

$$d_{\mathcal{M}}\,\omega(P_j^{i_1,\dots,i_k}) + d_{\mathcal{M}}\Big(\sum_{l=1}^{n+m}(-1)^{|l|}\,b_l\wedge b_j^{l,i_1,\dots,i_k}\Big) = 0.$$

By (4-1), we have

$$d_{\mathcal{M}}\omega(P_j^{i_1,\dots,i_k}) + \sum_{i=1}^{n+m} b_i\wedge\Big(\sum_{l=1}^{n+m}(-1)^{|i|+|l|}b_l^i\wedge b_j^{l,i_1,\dots,i_k} - d_{\mathcal{M}}b_j^{i,i_1,\dots,i_k}\Big) = 0.$$

Since $P_j^{i_1,\dots,i_k}\in\Lambda_{k-1}$, we have $\omega(\partial P_j^{i_1,\dots,i_k}) = d_{\mathcal{M}}\,\omega(P_j^{i_1,\dots,i_k})$. Equation (3-10) implies that

$$\sum_{i=1}^{n+m} b_i\wedge\Big(\omega(Q_j^{i,i_1,\dots,i_k}) + \sum_{l=1}^{n+m}(-1)^{|l|+|i|}\,b_l^i\wedge b_j^{l,i_1,\dots,i_k} - d_{\mathcal{M}}\,b_j^{i,i_1,\dots,i_k}\Big) = 0.$$

From (3-9) it follows that

$$\sum_{i=1}^{n+m} b_i\wedge\big(\omega(P_j^{i,i_1,\dots,i_k}) - d_{\mathcal{M}}\,c_j^{i,i_1,\dots,i_k}\big) = 0.$$

By Lemma 4.3, there exist 1-forms $b_l^{i,l,i_1,\dots,i_k}$ satisfying

$$\text{(4-5)}\qquad \begin{cases} d_{\mathcal{M}}\,b_j^{i,i_1,\dots,i_k} - \omega\big(P_j^{i,i_1,\dots,i_k}\big) = \displaystyle\sum_{l=1}^{n+m}(-1)^{|l|}\,b_l\wedge b_j^{l,i,i_1,\dots,i_k}\\ (-1)^{|i|}\,b_j^{i,l,i_1,\dots,i_k} = -(-1)^{(|l|+1)(|i|+1)}(-1)^{|l|}\,b_j^{l,i,i_1,\dots,i_k}. \end{cases}$$

This last equation can be rewritten

$$\text{(4-6)}\qquad b_j^{i,l,i_1,\dots,i_k} = (-1)^{|i||l|}\,b_j^{l,i,i_1,\dots,i_k}.$$

Equation (4-5) and the relations

$$\begin{cases} b_j^{i_{\sigma(1)},\dots,i_{\sigma(k)},i_{\sigma(k+1)}} = \varepsilon(i_1,\dots,i_{k+1},\sigma)\,b_j^{i_1,\dots,i_k,i_{k+1}}\\ P_j^{i_{\sigma(1)},\dots,i_{\sigma(k)},i_{\sigma(k+1)}} = \varepsilon(i_1,\dots,i_{k+1},\sigma)\,P_j^{i_1,\dots,i_k,i_{k+1}}. \end{cases}$$

imply that, for any permutation σ of $\{1,\dots,k{+}1\}$,

$$\text{(4-7)}\qquad b_j^{i,i_{\sigma(1)},\dots,i_{\sigma(k+1)}} = \varepsilon(i_1,\dots,i_{k+1},\sigma)\,b_j^{i,i_1,\dots,i_{k+1}}.$$

Equations (4-6) and (4-7) imply that, for any permutation σ of $\{1,\dots,k{+}2\}$ and any $i_1,\dots,i_{k+1},i_{k+2}\in\{1,\dots,n{+}m\}$, we have

$$\text{(4-8)}\qquad b_j^{i_{\sigma(1)},\dots,i_{\sigma(k+1)},i_{\sigma(k+2)}} = \varepsilon(i_1,\dots,i_k,i_{k+2},\sigma)\,b_j^{i_1,\dots,i_{k+1},i_{k+2}}.$$

We can then define ω on elements of weight $k+1$ by $\omega(c_j^{i_1,\dots,i_{k+2}}) = b_j^{i_1,\dots,i_{k+2}}$. Equation (4-8) implies that

$$\omega(c_j^{i_{\sigma(1)},\dots,i_{\sigma(k+2)}}) = \varepsilon(i_1,\dots,i_{k+2},\sigma)\,\omega(c_j^{i_1,\dots,i_{k+2}}),$$

and hence ω is well defined. The map ω can be uniquely extended to a DGA homomorphism from Λ_{k+1} to $\Omega(\mathcal{M})$. According to (3-5) and (4-5), this homomorphism satisfies $d_{\mathcal{M}}\omega(\alpha) = \omega\big(\partial(\alpha)\big)$ for any $\alpha \in \Lambda_k$. This completes the proof. □

4B. ***Construction of secondary characteristic classes.*** Assume that

$$\omega : \bigwedge \mathrm{Vect}(n,m)_0^* \to \Omega(\mathcal{M})$$

is a DGA homomorphism defining a superfoliation $\mathcal{F}$ of codimension $n+\varepsilon m$ with trivialized normal bundle. Set $\beta_\omega = p\circ\omega$, where p is the canonical projection from $\Omega^*(\mathcal{M})$ to $\Omega^*(M)$. By Lemma 2.1, the kernel of p contains all odd $2k$-forms and even $(2k{+}1)$-forms. As a consequence, $\omega(\mathcal{E})$ is contained in the kernel of β_ω. The homomorphism β_ω induces a homomorphism from $\bigwedge \mathrm{Vect}(n,m)^*/\mathcal{E}$ to $\Omega^*(M)$. By Lemma 3.1, $\bigwedge \mathrm{Vect}(n,m)^*/\mathcal{E} \simeq \bigwedge \mathrm{Vect}(n,m)_0^*$, and therefore β_ω induces a homomorphism β'_ω from $\bigwedge \mathrm{Vect}(n,m)_0^*$ to $\Omega^*(M)$.

Definition 4.4. Let $\varphi_{\mathcal{M},\mathcal{F}}$ be the homomorphism from $H^*\big(\mathrm{Vect}(n,m)_0\big)$ to $H^*(M)$ induced in cohomology by the DGA homomorphism

$$\beta'_\omega : \bigwedge \mathrm{Vect}(n,m)_0^* \to \Omega^*(M).$$

This definition is justified by:

Proposition 4.5. *The homomorphism $\varphi_{\mathcal{M},\mathcal{F}} : H^*\big(\mathrm{Vect}(n,m)_0\big) \to H^*(M)$ is independent of the choice of homomorphism $\omega : \bigwedge \mathrm{Vect}(n,m)^* \to \Omega(\mathcal{M})$ defining the superfoliation $\mathcal{F}$.*

First, we need a lemma. Let ω^1 and ω^2 be two homomorphisms defining the superfoliation $\mathcal{F}$, and let $\varphi^1_{\mathcal{M},\mathcal{F}}, \varphi^2_{\mathcal{M},\mathcal{F}}$ be the homomorphisms $\bigwedge \mathrm{Vect}(n,m)^* \to \Omega(\mathcal{M})$ constructed from ω_1 and ω_2 as in Definition 4.4.

Lemma 4.6. *If $p\circ\omega^1\big((x_i\,\partial/\partial x_j)^*\big) = p\circ\omega^2\big((x_i\,\partial/\partial x_j)^*\big)$ and $p\circ\omega^1\big((\theta_k\,\partial/\partial\theta_l)^*\big) = p\circ\omega^2\big((\theta_k\,\partial/\partial\theta_l)^*\big) \in \Omega(M)$ for all $i,j \in \{1,\dots,n\}$ and $k,l \in \{1,\dots,m\}$, then*

$$\varphi^1_{\mathcal{M},\mathcal{F}} = \varphi^2_{\mathcal{M},\mathcal{F}}.$$

Proof. A DGA homomorphism β from the Weil algebra $W(\mathfrak{g}) = S(\mathfrak{g}^*)\otimes\bigwedge\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ to a given DGA (A, d_A) depends only on the restriction of β to $1\otimes\bigwedge^1\mathfrak{g}^*$, see [Guillemin and Sternberg 1999]. In particular, the homomorphisms $\beta_{\omega^a}\circ\alpha : W(gl_n\oplus gl_m) \to \Omega(M)$ for $a = 1,2$ depend only on the

1-forms $p \circ \omega\big((x_i\,\partial/\partial x_j)^*\big)$ and $p \circ \omega\big((\theta_k\,\partial/\partial\theta_l)^*\big)$, where $i, j \in \{1, \dots, n\}$ and $k, l \in \{1, \dots, m\}$. Therefore,

$$\beta_{\omega^1} \circ \alpha = \beta_{\omega^2} \circ \alpha \qquad \text{and} \qquad \beta'_{\omega^1} \circ \tilde{\alpha} = \beta'_{\omega^2} \circ \tilde{\alpha}.$$

For $a = 1, 2$, the diagram

$$\begin{tikzcd} \dfrac{W(gl_n + gl_m)}{\mathcal{K}} \arrow[r, "\tilde{\alpha}"] \arrow[dr, "\beta'_{\omega^a} \circ \tilde{\alpha}"'] & \bigwedge \mathrm{Vect}(n, m)_0^* \arrow[d, "\beta'_{\omega^a}"] \\ & \Omega(M) \end{tikzcd}$$

is commutative. Moreover, according to Theorem 3.7, $\tilde{\alpha}$ induces an isomorphism in cohomology. This completes the proof. $\square$

Proof of Proposition 4.5. Set $a_i = \omega^1\big((\partial/\partial h_i)^*\big)$ and $b_i = \omega^2\big((\partial/\partial h_i)^*\big)$. Set also $a_i^j = \omega^1\big((h_j \partial/\partial h_i)^*\big)$ and $b_i^j = \omega^2\big((h_j \partial/\partial h_i)^*\big)$.

For any superfunction f on the supermanifold $\mathcal{M} \times \mathbb{R}$, denote by $f|_t$ its restriction to $M \times \{t\}$. Denote by i_t the maps from $\mathcal{M}$ to $\mathcal{M} \times \mathbb{R}$ induced by $f \to f|_t$. Let

$$\mathrm{pr} : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$$

be the natural projection.

On the supermanifold $\mathcal{M} \times \mathbb{R}$, consider the superfoliation $\mathcal{F}'$ that is the pull-back of $\mathcal{F}$ by the projection pr. There exist 1-forms $c_1, \dots, c_{n+m}$ defining the superfoliation $\mathcal{F}'$ and 1-forms $(c_i^k)_{i,k \in \{1,\dots,n+m\}}$ satisfying the identity

$$d_{\mathcal{M}} c_i = \sum_{k=1}^{n+m} (-1)^{|k|} c_k \wedge c_i^k$$

and the following two properties:

- The 1-forms $c_i,\ i \in \{1, \dots, n{+}m\}$ (respectively, $c_i^k,\ i, k \in \{1, \dots, n{+}m\}$) restricted to $\mathcal{M} \times (-\infty, 1/4]$ are equal to $\mathrm{pr}^* a_i,\ i = 1, \dots, n{+}m$ (respectively, $\mathrm{pr}^* a_k^i,\ i, k \in \{1, \dots, n{+}m\}$).
- The 1-forms $c_i,\ i \in \{1, \dots, n{+}m\}$ (respectively, $c_i^k,\ i, k \in \{1, \dots, n{+}m\}$) restricted to $\mathcal{M} \times (3/4, \infty)$ are equal to $\mathrm{pr}^* b_i,\ i = 1, \dots, n{+}m$ (respectively, $\mathrm{pr}^* b_k^i,\ i, k \in \{1, \dots, n{+}m\}$).

By Proposition 4.2, there exists a homomorphism $\omega_3 : \bigwedge \mathrm{Vect}(n, m)^* \to \Omega(\mathcal{M} \times \mathbb{R})$ defining $\mathcal{F}'$, such that $\omega_3\big((h_i \partial/\partial h_j)^*\big) = c_j^i$ and $\omega_3\big((\partial/\partial h_i)^*\big) = c_i$. Then, the

The editors acknowledge the use of the diagrams package by Paul Taylor.

following diagram is commutative:

$$\begin{tikzcd}
 & & \Omega(M) & \\
\bigwedge\big(\mathrm{Vect}(n,m)_0\big) \arrow[r, "\tilde{\alpha}"] & \dfrac{W(gl_n \oplus gl_m)}{\mathcal{K}} \arrow[ur, "p \circ \omega^1 \circ \tilde{\alpha}"] \arrow[rr, "p \circ \omega_3 \circ \tilde{\alpha}"] \arrow[dr, "p \circ \omega^2 \circ \tilde{\alpha}"'] & & \Omega(M \times \mathbb{R}). \arrow[ul, "i_1^*"'] \arrow[dl, "i_2^*"] \\
 & & \Omega(M) &
\end{tikzcd}$$

Therefore, we also have the commutative diagram

$$\begin{tikzcd}
 & & H^*(M) & \\
H^*\big(\mathrm{Vect}(n,m)_0\big) \arrow[r, "\hat{\alpha}"] & H^*\Big(\dfrac{W(gl_n \oplus gl_m)}{\mathcal{K}}\Big) \arrow[ur, "\varphi^1_{\mathcal{M},\mathcal{F}} \circ \hat{\alpha}"] \arrow[rr, "\varphi_{\mathcal{M}\times\mathbb{R},\mathcal{F}'} \circ \hat{\alpha}"] \arrow[dr, "\varphi^2_{\mathcal{M},\mathcal{F}} \circ \hat{\alpha}"'] & & H^*(M \times \mathbb{R}), \arrow[ul, "i_1^*"'] \arrow[dl, "i_2^*"] \\
 & & H^*(M) &
\end{tikzcd}$$

where $\varphi_{\mathcal{M}\times\mathbb{R},\mathcal{F}'} : H^*\big(\mathrm{Vect}(n,m)\big)$ is constructed as in Definition 4.4, with the help of the DGA homomorphism ω_3 defining $\mathcal{F}'$.

Since by Theorem 3.7 $\hat{\alpha}$ is a bijection, and $i_1^* : H^*(\mathcal{M} \times \mathbb{R}) \to H^*(\mathcal{M})$ and $i_2^* : H^*(\mathcal{M} \times \mathbb{R}) \to H^*(\mathcal{M})$ are isomorphisms with $i_1^* \circ (i_2^*)^{-1} = \mathrm{Id}$, we have

$$\varphi^1_{\mathcal{M},\mathcal{F}} = \varphi^2_{\mathcal{M},\mathcal{F}}. \qquad \square$$

We now study how $\varphi_{\mathcal{M},\mathcal{F}}$ behaves with respect to morphisms of superfoliations.

Proposition 4.7. *Let $\mathcal{F}$ (respectively, $\mathcal{G}$) be a superfoliation with trivialized normal bundle on $\mathcal{M}$ (respectively, $\mathcal{N}$). Let $\Phi : \mathcal{N} \to \mathcal{M}$ be a morphism of foliated supermanifolds, and $\varphi : N \to M$ the map induced by Φ, from the base manifold N of $\mathcal{N}$ to the base manifold M of $\mathcal{N}$. Let $\varphi^* : H^*(M) \to H^*(N)$ be the pull-pack by φ. Under these hypotheses, we have*

$$\varphi_{\mathcal{N},\mathcal{G}} = \varphi^* \circ \varphi_{\mathcal{M},\mathcal{F}}. \tag{4-9}$$

In other words, for any supermanifold $\mathcal{N}$ and any submersion $\Phi : \mathcal{N} \to \mathcal{M}$, we have $\varphi_{\mathcal{N},\Phi^*\mathcal{F}} = \varphi^* \circ \varphi_{\mathcal{M},\mathcal{F}}$, where $\Phi^*\mathcal{F}$ is the pull-back via Φ of the superfoliation $\mathcal{F}$.

Proof. If $\omega : \bigwedge \mathrm{Vect}(n,m)^* \to \Omega(\mathcal{M})$ is a DGA homomorphism defining $\mathcal{F}$, then $\Phi^* \circ \omega : \bigwedge \mathrm{Vect}(n,m)^* \to \Omega(\mathcal{N})$ is a homomorphism of DGAs defining $\mathcal{G}$. Let $p_N : \Omega(\mathcal{N}) \to \Omega(N)$ and $p_M : \Omega(\mathcal{N}) \to \Omega(N)$ be the canonical projections of the supermanifolds $\mathcal{N}$ and $\mathcal{M}$ to their base manifolds, as defined in Lemma 2.1. We have $p_N \circ \Phi^* = \varphi^* \circ p_M$, which implies that

$$p_N \circ \Phi^* \circ \omega = \varphi^* \circ p_M \circ \omega.$$

Hence, $\beta_{\Phi^* \circ \omega} = \varphi^* \circ \beta_\omega$, and Equation (4-9) follows. □

4C. *Examples.*

4C(a). *Superfoliations of codimension* $0+\varepsilon 1$. If $\mathcal{F}$ is a superfoliation of codimension $0+\varepsilon 1$ defined by an even 1-form $a \in \Omega^1(\mathcal{M})$, then there exists an odd 1-form $b \in \Omega^1(M)$ with $d_{\mathcal{M}} a = -a \wedge b$.

By Proposition A.11, $H^k\big(\mathrm{Vect}(0, 1)_0\big) = 0$ if $k \notin \{0, 1\}$, and $H^1\big(\mathrm{Vect}(0, 1)_0\big) \simeq \mathbb{R}$. Moreover, $H^1\big(\mathrm{Vect}(0, 1)_0\big)$ is generated by $H = \big[(\theta\, \partial/\partial\theta)^*\big]$. By Proposition 4.2, there exists a homomorphism $\omega : \bigwedge \mathrm{Vect}(0, 1)^* \to \Omega(\mathcal{M})$ such that $\omega\big((\theta\, \partial/\partial\theta)^*\big) = b$. By construction, we have $\varphi_{\mathcal{M}\mathcal{F}}(H) = [p(b)]$. See [Laurent-Gengoux 2004] for additional details.

Example 4.8. Let $E \to S^1$ be a trivial 1-dimensional vector bundle. The supermanifold $S^{1,1} = \mathcal{M}(S^1, E)$, with base manifold S^1 and superalgebra of superfunctions $\Omega(\mathcal{M}) = \Gamma\big(\bigwedge^* E\big)$, is called a *supercircle*. Let $x \in S^1$ be the even parameter and θ the odd parameter.

Let $t \in \mathbb{R}^*$ be a real number different from zero, and $\mathcal{F}_t$ the superfoliation of codimension $0 + 1\varepsilon$ defined by the 1-form $b_t = d\theta + t\theta\, dx$. Note that

$$d_{S^{1,1}} b_t = t\, dx \wedge d\theta = (d\theta + t\theta\, dx) \wedge t\, dx.$$

By the preceding discussion, we have

$$\varphi_{S^{1,1}, \mathcal{F}_t}(H) = -[p(t\, dx)] = -t\,[dx].$$

For $t \neq 0$, we obtain a nonzero element of $H^1(S^1)$. We have thus constructed a supermanifold $\mathcal{M}$ and a superfoliation $\mathcal{F}$ with $\varphi_{\mathcal{M},\mathcal{F}}$ not zero.

4C(b). *Superfoliations of codimension* $1+\varepsilon 1$. Let $\mathcal{F}$ be a superfoliation of codimension $1+\varepsilon 1$ given by an odd 1-form $\omega \in \Omega^1(\mathcal{M})$ and an even 1-form $a \in \Omega^1(\mathcal{M})$.

There exist $b, c \in \Omega^1(\mathcal{M})$ with $d_{\mathcal{M}} a = \omega \wedge c - a \wedge b$. By Proposition 4.2, there is a DGA homomorphism $\omega_{\mathcal{F}} : \bigwedge \mathrm{Vect}(1, 1)^* \to \Omega(\mathcal{M})$, defining the superfoliation $\mathcal{F}$, with $\omega_{\mathcal{F}}\big((\partial/\partial x)^*\big) = \omega$ and $\omega_{\mathcal{F}}\big((\theta\, \partial/\partial\theta)^*\big) = b$.

By Proposition A.12, $H^k\big(\mathrm{Vect}(n, m)_0\big)$ vanishes if $k \neq 3$, and $H^3\big(\mathrm{Vect}(n, m)_0\big)$ is generated the classes H_1, H_2, H_3 described by Equations (A-13), (A-14), and (A-15).

Proposition 4.9. *Let $\mathcal{F}$ be a superfoliation of codimension $1+\varepsilon 1$ defined by the odd 1-form ω and the even 1-form a. Suppose b is an odd 1-form and c an even 1-form on $\mathcal{M}$ satisfying $d_{\mathcal{M}} a = \omega \wedge c - a \wedge b$.*

(1) *There exist* $\alpha, \beta, \xi \in \Omega^1(M)$ *on the base manifold* M *such that*

$$\begin{cases} d_M p(\omega) = p(\omega) \wedge \alpha, \\ d_M \alpha = p(\omega) \wedge \beta, \\ d_M p(b) = p(\omega) \wedge \xi. \end{cases} \tag{4-10}$$

(2) *We have*

$$\begin{cases} \varphi_{\mathcal{M},\mathcal{F}}(H_1) = \big[\beta \wedge \alpha \wedge p(\omega)\big], \\ \varphi_{\mathcal{M},\mathcal{F}}(H_2) = \big[\xi \wedge \alpha \wedge p(\omega)\big], \\ \varphi_{\mathcal{M},\mathcal{F}}(H_3) = \big[\xi \wedge p(b) \wedge p(\omega)\big]. \end{cases} \tag{4-11}$$

Proof. (1) Define α, β, ξ by

$$\begin{cases} \alpha = p \circ \omega_{\mathcal{F}}\big((x\, \partial/\partial x)^*\big) \\ \beta = p \circ \omega_{\mathcal{F}}\big((\tfrac{1}{2}x^2\, \partial/\partial x)^*\big) \\ \xi = p \circ \omega_{\mathcal{F}}\big((x\theta\, \partial/\partial \theta)^*\big) \end{cases} \tag{4-12}$$

Applying $p \circ \omega_{\mathcal{F}}$ to the identity

$$\partial\Big(\frac{\partial}{\partial x}\Big)^* = \Big(\frac{\partial}{\partial x}\Big)^* \wedge \Big(x\frac{\partial}{\partial x}\Big)^* - \Big(\frac{\partial}{\partial \theta}\Big)^* \wedge \Big(\theta\frac{\partial}{\partial x}\Big)^*,$$

and using Lemma 2.1, we obtain

$$d_M\big(p(\omega)\big) = p(\omega) \wedge \alpha.$$

Applying $p \circ \omega_{\mathcal{F}}$ to the identity

$$\partial\Big(x\frac{\partial}{\partial x}\Big)^* = \Big(\frac{\partial}{\partial x}\Big)^* \wedge \Big(\frac{x^2}{2}\frac{\partial}{\partial x}\Big)^* - \Big(\frac{\partial}{\partial \theta}\Big)^* \wedge \Big(\theta x\frac{\partial}{\partial x}\Big)^* - \Big(x\frac{\partial}{\partial \theta}\Big)^* \wedge \Big(\theta\frac{\partial}{\partial x}\Big)^*,$$

and using Lemma 2.1, we get

$$d_M(\alpha) = p(\omega) \wedge \beta.$$

Applying $p \circ \omega_{\mathcal{F}}$ to the identity

$$\partial\Big(\theta\frac{\partial}{\partial \theta}\Big)^* = \Big(\frac{\partial}{\partial x}\Big)^* \wedge \Big(x\theta\frac{\partial}{\partial \theta}\Big)^*$$

and using Lemma 2.1, we obtain

$$d_M\big(p(b)\big) = p(\omega) \wedge \xi.$$

Thus, there exist 1-forms α, β, ξ satisfying (4-10).

(2) We must now check that the cohomology classes

$$\big[\beta\wedge\alpha\wedge p(\omega)\big],\ \big[\xi\wedge\alpha\wedge p(\omega)\big],\ \big[\xi\wedge p(b)\wedge p(\omega)\big]\in H^3(M)$$

do not depend on the chosen 1-forms α,β,ξ that satisfy (4-10).

The 1-form $[\beta\wedge\alpha\wedge p(\omega)]$ is the Godbillon–Vey class of the codimension-1 foliation $F_{\mathcal{F}}$ constructed from $\mathcal{F}$ on the base manifold in Section 2C(a). This $F_{\mathcal{F}}$ is defined by $p(\omega)$. The fact that $[\beta\wedge\alpha\wedge p(\omega)]$ does not depend on the choice of α and β was proved in [Godbillon and Vey 1971].

From the identity $\xi\wedge\alpha\wedge p(\omega)=-(d_M p(b))\wedge\alpha=\xi\wedge d_M p(\omega)$ follows that $[\xi\wedge\alpha\wedge p(\omega)]$ does not depend on the 1-forms α,β,ξ; while from the identity $\xi\wedge p(b)\wedge p(\omega)=-(d_M p(b))\wedge p(b)$ follows that $[\xi\wedge\alpha\wedge p(\omega)]$ does not depend on the 1-forms α,β,ξ. This completes the proof. □

Example 4.10. Let M be a manifold and F a foliation of codimension 1 on M, defined by $\omega\in\Omega^1(M)$ with $d_M\omega=\omega\wedge\alpha$. Assume moreover that the Godbillon–Vey class $-[\alpha\wedge d_M\alpha]\in H^3(M)$ is not zero.

Let $E\to M$ be the trivial 1-dimensional bundle $E=\mathbb{R}\times M$. Consider the supermanifold $\mathcal{M}$ with $\mathcal{O}(\mathcal{M})=\Gamma\big(\bigwedge E\big)$, and denote by θ the unique odd parameter corresponding to some constant section of E. There is a canonical embedding $I:\Omega^1(M)\to\Omega^1(\mathcal{M})$ given by the pull-back of the canonical projection $\mathcal{M}\to M$. Of course, $p\circ I=\mathrm{Id}_{\Omega^1(M)}$.

Define a superfoliation of codimension $1+\varepsilon 1$ by the odd 1-form $I(\omega)$ and the even 1-form $a=d\theta+\theta\, I(\alpha)$. We check that these 2-forms define a superfoliation. We have $d_{\mathcal{M}}I(\omega)=I(\omega)\wedge I(\alpha)$ and

$$d_{\mathcal{M}}a=d\theta\wedge I(\alpha)+\theta\, d_{\mathcal{M}}I(\alpha)=\big(d\theta+\theta\, I(\alpha)\big)\wedge I(\alpha)+\theta\, I(d_M\alpha). \tag{4-13}$$

Since $d_M\alpha=\omega\wedge\xi$ for some $\xi\in\Omega^1(M)$ [Godbillon and Vey 1971], we have

$$d_{\mathcal{M}}a=a\wedge I(\alpha)+I(\omega)\wedge\theta\, I(\xi).$$

Therefore, the pair $(a,I(\omega))$ defines a superfoliation of codimension $1+\varepsilon 1$.

Now, from (4-13) we see that $b=\alpha$ and $p(b)=\alpha$. By (4-11), we obtain

$$\varphi_{\mathcal{M},\mathcal{F}}(H_1)=\varphi_{\mathcal{M},\mathcal{F}}(H_2)=\varphi_{\mathcal{M},\mathcal{F}}(H_3)=-[\alpha\wedge d_M\alpha]\neq 0.$$

Example 4.11. We describe a superfoliation of codimension $1+\varepsilon 1$ with trivialized normal bundle such that $\varphi_{\mathcal{M},\mathcal{H}}(H_1)=0$ and $\varphi_{\mathcal{M},\mathcal{H}}(H_3)\neq 0$.

Consider the supermanifold given by the trivial 1-vector bundle E over the 3-dimensional torus $T^3\simeq(S^1)^3$. Let $x,y,z\in S^1$ be the coordinates of T^3, and let θ be the odd parameter corresponding to the constant section of E. Take $f(x)$, $g(x)$ two smooth functions on S^1 with $\int_{S^1}W(f,g)\neq 0$, where W is the Wronskian.

We leave the reader to check that the odd 1-form $\omega = dx$ and the even 1-form

$$a = d\theta + \theta(f(x)dy + g(x)dz)$$

define a superfoliation of codimension $1+\varepsilon 1$. In this case, we can choose $b = f(x)\,dy + g(x)\,dz$, and it is routine to check that

$$\varphi_{\mathcal{M},\mathcal{H}}(H_3) = \big[W(f,g)\,dx \wedge dy \wedge dz\big]$$

Since $[W(f,g)\,dx \wedge dy \wedge dz] = \big(\int_{S^1} W(f,g)\big)\,[dx \wedge dy \wedge dz]$, it follows that $\varphi_{\mathcal{M},\mathcal{H}}(H_3)$ is a nonzero class in de Rham cohomology. Since $p(\omega) = dx$ is a closed 1-form, $\varphi_{\mathcal{M},\mathcal{H}}(H_1) = 0$.

4C(c). *Superfoliations of codimension* $0+\varepsilon m$. According to Proposition A.10, we have $H^*\big(\mathrm{Vect}(0,m)_0\big) \simeq H^*(gl_m)$, where $H^*(gl_m)$ is the Chevalley–Eilenberg cohomology of the Lie algebra gl_m. In such a case, therefore, $\varphi_{\mathcal{M},\mathcal{F}}$ is a map from $H^*(gl_m)$ to $H^*(M)$. Moreover, if ω is a DGA homomorphism defining the superfoliation $\mathcal{F}$, then $\varphi_{\mathcal{M},\mathcal{F}}$ is given by

$$d_k^l \to p\big(\omega\big((\theta_k\,\partial/\partial\theta_l)^*\big)\big) \qquad \text{for any } k, l = 1, \ldots, m. \tag{4-14}$$

Example 4.12. We compute this homomorphism in a particular case. The semidirect product $gl_m \ltimes \mathbb{R}^m$ of the Lie algebra gl_m with $\mathbb{R}^m$ can be considered as a super-Lie algebra with even part gl_m and odd part $\mathbb{R}^m$. Let $\mathcal{G}$ be the super-Lie group associated to this Lie algebra by Lie's third theorem (which is true for super-Lie algebras [Tuynman 2004]). Since $gl_m \subset gl_m \ltimes \mathbb{R}^m$, the Lie group GL_n acts on the left on $\mathcal{G}$, and a superfoliation of codimension $0+\varepsilon m$ is given on $\mathcal{G}$ by this left action.

This superfoliation is defined by the left-invariant forms $\bar{a}_1, \ldots, \bar{a}_m$ associated to the canonical basis $a_1, \ldots, a_m \in (\mathbb{R}^m)^*$. Moreover, for any $k \in \{1, \ldots, m\}$, we have

$$d_{\mathcal{G}}\bar{a}_k = \sum_{l=1}^{m} \bar{a}_l \wedge \bar{d}_k^l$$

where $\bar{d}_k^l \in T^*(\mathcal{G})$ are the left invariant 1-forms on $\mathcal{G}$ associated to the canonical basis d_k^l of $gl(m)^*$.

By construction, therefore, there exists a homomorphism ω defining the superfoliation, such that

$$\omega\big((\theta_k\,\partial/\partial\theta_l)^*\big) = \bar{d}_k^l \qquad \text{for any } k, l \in \{1, \ldots, m\}.$$

By Equation (4-14), we obtain that, for any $H \in H^*(gl_m)$,

$$\varphi_{\mathcal{M},\mathcal{F}}(H) = \bar{H},$$

where $\bar{H}$ is the class of the left-invariant form on GL_m that corresponds to H. In other words, $\varphi_{\mathcal{M},\mathcal{F}}$ is equal to the natural homomorphism $H^*(gl_m) \to H^*(GL_m)$.

It is well known that $H^*(GL_m) \simeq H^*(O(m))$, where $O(m)$ is the orthogonal group, and that $H^3(O(m)) \simeq \mathbb{R}$ and $H^3(gl_m) \simeq \mathbb{R}$ for $m \geq 3$. We leave the reader to check that the homomorphism $H^3(gl_m) \to H^3(GL_m)$ is not trivial. Therefore, we have proved the existence of nontrivial secondary characteristic classes for superfoliations of codimension $0 + \varepsilon m$ with $m \geq 3$.

4D. *Conclusion.* We summarize the results of the preceding sections.

Theorem 4.13. *For any supermanifold $\mathcal{M}$ foliated by a superfoliation $\mathcal{F}$ of codimension $n + \varepsilon m$ with trivialized normal bundle, there exists a map $\varphi_{\mathcal{M},\mathcal{F}}$ from $H^*\big(\mathrm{Vect}(n, m)_0\big)$ to $H^*(M)$ such that*:

(1) *$\varphi_{\mathcal{M},\mathcal{F}}$ is a functor from the category of supermanifolds endowed with a superfoliation with trivialized normal bundle to the category of algebra homomorphisms from $H^*\big(\mathrm{Vect}(n, m)_0\big)$ to an algebra A*;

(2) *for $(n, m) = (0, 1)$, or $(n, m) = (1, 1)$, or $n = 0$ and $m \geq 3$, there exists a supermanifold $\mathcal{M}$ and a superfoliation $\mathcal{F}$ of codimension $n + \varepsilon m$ such that $\varphi_{\mathcal{M},\mathcal{F}}$ is not the zero map*;

(3) *if $\mathcal{M}$ is an ordinary smooth manifold endowed with a foliation F of codimension n with trivialized normal bundle, then φ_F reduces to the usual homomorphism of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld* [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1986].

Proof. By "category of algebra homomorphisms from $H^*\big(\mathrm{Vect}(n, m)_0\big)$ to an algebra A" we mean the category whose objects are algebra homomorphism from $H^*\big(\mathrm{Vect}(n, m)_0\big)$ to an algebra A, and whose arrows between objects

$$\varphi_A : H^*\big(\mathrm{Vect}(n, m)_0\big) \to A \qquad \text{and} \qquad \varphi_B : H^*\big(\mathrm{Vect}(n, m)_0\big) \to B$$

are homomorphisms $\varphi : A \to B$ such that $\varphi_B = \varphi \circ \varphi_A$. Conclusion (1) is now a paraphrase of Proposition 4.7.

Statement (2) follows from Examples 4.8, 4.10, and 4.12. Note that a more precise statement will be given in Remark 5.3.

It is proved in [Fuchs 1986, Section 3.2.B (page 231)] that, when we are given a foliation F of codimension n with trivialized normal bundle, the "classical" map of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld is constructed by the passing to cohomology of a DGA homomorphism ω from $\big(\bigwedge \mathrm{Vect}(n)^*, \partial\big)$ to $\big(\Omega(M), d_M\big)$, with the foliation F being defined by the 1-forms $\omega\big((\partial/\partial x_i)^*\big)$. This proves (3). □

The functoriality of this construction allows us to say that the assignment $\mathcal{F} \mapsto \varphi_{\mathcal{F}}(H)$, for any $H \in H^*\big(\mathrm{Vect}(n, m)_0\big)$, defines a *secondary characteristic class of superfoliations with trivialized normal bundle*.

We have defined in Proposition 2.18 a way to construct from any flat trivial foliated connection (M, F, E, ∇) a supermanifold $\mathcal{M}(M, E)$ whose base manifold is M, and a superfoliation $\mathcal{F}(M, F, E, \nabla)$ on $\mathcal{M}$ with trivialized normal bundle. We call *secondary characteristic classes of flat trivial foliated connections* the secondary characteristic classes of this superfoliation. More precisely, we have associated a homomorphism from $H^*\big(\mathrm{Vect}(n, m)_0\big)$ to $H^*(M)$ to any flat trivial foliated connection on M.

Remark 4.14. One may ask whether, for any $H \in H^*\big(\mathrm{Vect}(n, m)_0\big)$, there is a superfoliation $\mathcal{F}$ and a supermanifold $\mathcal{M}$ with $\varphi_{\mathcal{M},\mathcal{F}}(H) \neq 0$. This question remains open even in the case of foliations on smooth manifolds, and therefore we cannot hope to find a simple answer.

Remark 4.15. According to Section 2D, if the superfoliation $\mathcal{F}$ of codimension $n + \varepsilon m$ does not have a trivialized normal bundle, then we can replace it by a superfoliation $\mathrm{pr}^*(\mathcal{F})$ with trivialized normal bundle, over some $GL(n, m)$-principal bundle $\mathcal{P}_{\mathcal{F}} \to \mathcal{M}$. The map $\varphi_{\mathcal{P}_{\mathcal{F}}, \mathrm{pr}^*\mathcal{F}}$ takes values in the cohomology of the base manifold of $\mathcal{P}_{\mathcal{F}}$, which is a $GL(n) \times GL(m)$-principal bundle over M. Therefore, for any $H \in H^*\big(\mathrm{Vect}(n, m)\big)$, we can construct secondary characteristic classes of superfoliation of codimension $n + \varepsilon m$ (not necessarily with trivialized normal bundle), but these characteristic classes have values in the cohomology of some $GL(n) \times GL(m)$-principal bundle over the base manifold M of $\mathcal{M}$. By Remark 2.21, this construction is functorial, that is, it behaves well with respect to pullbacks of superfoliations.

We now link this construction to the cohomology of $H\big(\mathrm{Vect}(n, m)\big)$ and the Godbillon–Vey classes constructed in [Koszul 1988].

The homomorphism ω constructed in Proposition 4.2 induces a homomorphism $\varphi_\omega : H^*\big(\mathrm{Vect}(n, m)\big) \to H^*(\mathcal{M})$, where $H^*\big(\mathrm{Vect}(n, m)\big)$ is the cohomology of the superalgebra $\mathrm{Vect}(n, m)$. The following diagram is commutative:

$$\begin{tikzcd} H^*\big(\mathrm{Vect}(n, m)\big) \arrow[r, "\varphi_\omega"'] \arrow[d, "J"] & H^*(\mathcal{M}) \arrow[d, "\hat{p}"] \\ H^*\big(\mathrm{Vect}(n, m)_0\big) \arrow[r, "\varphi_{\mathcal{M},\mathcal{F}}"] & H^*(M), \end{tikzcd}$$

with $J : H^*\big(\mathrm{Vect}(n, m)_0\big) \to H^*\big(\mathrm{Vect}(n, m)\big)$ given by the inclusion $\mathrm{Vect}(n, m)_0 \to \mathrm{Vect}(n, m)$, and $\hat{p} : H^*(\mathcal{M}) \to H^*(M)$ induced by p.

According to Batchelor's theorem, $\hat{p}$ is indeed an isomorphism, and thus $\varphi_\omega = \hat{p}^{-1} \circ \varphi_{\mathcal{M},\mathcal{F}} \circ J$ is independent of ω. To emphasize that this homomorphism does not depend on ω, we denote it by $\psi_{\mathcal{M},\mathcal{F}}$. We cannot say that $\psi_{\mathcal{M},\mathcal{F}}$ gives new secondary characteristic classes, since

$$\psi_{\mathcal{M},\mathcal{F}} = \hat{p} \circ \varphi_{\mathcal{M},\mathcal{F}} \circ J. \tag{4-15}$$

In [Koszul 1988], a homomorphism from $H^*\big(\mathrm{Vect}(0, m)\big)$ to $H^*(M)$ is associated to any superfoliation of codimension $n + \varepsilon m$ with trivialized normal bundle, on a supermanifold of dimension $n + \varepsilon m$. It is easy to check that, by construction, this homomorphism coincides with $\psi_{\mathcal{M},\mathcal{F}}$. It defines classes of superfoliations of dimension $0 + \varepsilon m$, called "Godbillon–Vey classes" by the author. By Equation (4-15), these Godbillon–Vey classes are among the classes we built in this article.

From Proposition A.11 and [Koszul 1988, Corollaire 1.2], it follows that $J : H^1\big(\mathrm{Vect}(0, 1)\big) \to H^1\big(\mathrm{Vect}(0, 1)_0\big)$ is an isomorphism. This implies that the class constructed in Section 4C(a) is equal to the class constructed in [Koszul 1988, Exemple 1].

We summarize:

Proposition 4.16. (1) *For any superfoliation $\mathcal{F}$ of codimension $n + \varepsilon m$ with trivialized normal bundle, the following diagram is commutative*:

$$\begin{tikzcd} H^*\big(\mathrm{Vect}(n, m)\big) \arrow[r, "\psi_{\mathcal{M},\mathcal{F}}"'] \arrow[d, "J"] & H^*(\mathcal{M}) \arrow[d, "\hat{p}"] \\ H^*\big(\mathrm{Vect}(n, m)_0\big) \arrow[r, "\varphi_{\mathcal{M},\mathcal{F}}"] & H^*(M). \end{tikzcd}$$

(2) *The Godbillon–Vey classes constructed in* [Koszul 1988] *are among the secondary characteristic classes of superfoliation constructed above.*

(3) *In particular, for foliations of codimension* $0 + \varepsilon 1$, *the secondary class constructed in Section 4C(a) coincides with the class constructed in* [Koszul 1988, Exemple 1].

5. Foliated flat vector bundles

5A. ***Secondary characteristic classes on the base manifold.*** Let $F_{\mathcal{F}}$ be the codimension-n foliation induced by $\mathcal{F}$ on M, as in Lemma 2.12. Since F has a trivialized normal bundle, according to Theorem 4.13 to any $H \in H^*\big(\mathrm{Vect}(n)\big)$ is associated an element $K \in H^*(M)$ by the "classical" construction of Bernstein, Bott, Fuchs, Haefliger and Rosenfeld [Bernstein and Rosenfeld 1973; Bott and Haefliger 1972; Fuchs 1986]. We would like to investigate the relation between this construction and our construction.

There is a natural inclusion i from $\mathrm{Vect}(n)^*$ into $\mathrm{Vect}(n, m)_0^*$, obtained by considering an element $(x_1^{i_1} \dots x_n^{i_n}\, \partial/\partial x_a)^* \in \mathrm{Vect}(n)^*$ as an element of $\mathrm{Vect}(n, m)_0^*$.

Lemma 5.1. *The inclusion i is a* DGA *homomorphism*, *and induces a map from $H^*\big(\mathrm{Vect}(n)\big)$ to $H^*\big(\mathrm{Vect}(n, m)_0\big)$.*

Proof. The family

$$\text{(5-1)}\qquad \begin{cases} x_1^{i_1}\dots x_n^{i_n}\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial x_i}, & \begin{array}{l} i_1,\dots,i_n\in\mathbb{N},\\ j_1,\dots,j_m\in\{0,1\} \text{ with } \sum_{k=1}^{n} j_k \text{ even},\\ \text{and } i\in\{1,\dots,n\};\end{array}\\ \text{and} & \\ x_1^{i_1}\dots x_n^{i_n}\theta_1^{j_1}\dots\theta_m^{j_m}\dfrac{\partial}{\partial \theta_j}, & \begin{array}{l} i_1,\dots,i_n\in\mathbb{N},\\ j_1,\dots,j_m\in\{0,1\} \text{ with } \sum_{k=1}^{m} j_k \text{ odd},\\ \text{and } j\in\{1,\dots,m\},\end{array}\end{cases}$$

forms a basis of $\mathrm{Vect}(n,m)_0$.

The *θ-degree* $(h_{i_1}\dots h_{i_k}\,\partial/\partial h_i)\in\mathrm{Vect}(n,m)_0$ is the number of integers greater or equal to $n+1$ in the list $i_1,\dots,i_k,i$. For example, the θ-degree of $x_1\theta_1\,\partial/\partial\theta_2$ or $x_3\theta_2\theta_1\,\partial/\partial x_2$ is 2, and the θ-degree of $x_1^3\,\partial/\partial x_1$ is 0.

If we enumerate the basis as described in (5-1), then the structure constant $\Gamma_{i,j}^k$ is equal to zero if the index k corresponds to an element of θ-degree 0 and one of the indices i or j corresponds to an element of nonzero θ-degree. Moreover, if i, j, k correspond to elements of the basis (5-1) with a vanishing θ-degree, then the structure constant $\Gamma_{i,j}^k$ is equal to the corresponding structure constant in $\mathrm{Vect}(n)$. By the definition of the Chevalley–Eilenberg differential, this implies that $i\big(\bigwedge\mathrm{Vect}(n)^*\big)$ is stable under ∂_0, and that i is a DGA homomorphism. □

Proposition 5.2. *Take $H\in H^*\big(\mathrm{Vect}(n)^*\big)$ and $H'=i(H)\in H^*\big(\mathrm{Vect}(n,m)_0\big)$. One has*

$$\varphi_{M,F}(H)=\varphi_{\mathcal{M},\mathcal{F}}(H').$$

Proof. If $\omega:\bigwedge\mathrm{Vect}(n,m)\to\Omega(\mathcal{M})$ is a DGA homomorphism that defines $\mathcal{F}$, then $p\circ\omega\circ i:\bigwedge\mathrm{Vect}(n)\to\Omega(M)$ is a DGA homomorphism that defines F. The diagram

$$\begin{tikzcd} \bigwedge\big(\mathrm{Vect}(n)\big)^* \arrow[r, "i"] \arrow[dr, "p\circ\omega\circ i"'] & \bigwedge\big(\mathrm{Vect}(n,m)_0\big)^* \arrow[d, "p\circ\omega"] \\ & \Omega(M) \end{tikzcd}$$

is commutative. As a consequence, we have the commutative diagram

$$\begin{tikzcd} H^*\big(\mathrm{Vect}(n)\big) \arrow[r] \arrow[dr, "\varphi_{M,F}"'] & H^*\big(\mathrm{Vect}(n,m)_0\big) \arrow[d, "\varphi_{\mathcal{M},\mathcal{F}}"] \\ & H^*(M). \end{tikzcd}$$

□

Remark 5.3. The nontrivial secondary classes in Section 4C are not secondary classes of the induced foliation on the base manifold. We could therefore replace (2) in Theorem 4.13 by a more precise statement:

Let S be a supplement of $H^*\big(\mathrm{Vect}(n)\big)$ in $H^*\big(\mathrm{Vect}(n,m)_0\big)$. For $(n,m)=(0,1)$, or $(n,m)=(1,1)$, or $n=0$ and $m\geq 3$, there exists a supermanifold $\mathcal{M}$ and a superfoliation $\mathcal{F}$ of codimension $n+\varepsilon m$ such that the restriction of $\varphi_{\mathcal{M},\mathcal{F}}$ to S is not the zero map.

5B. ***Secondary characteristic classes of foliated connections.*** Let $\mathcal{F}$ be a superfoliation of codimension $n+\varepsilon m$ with trivialized normal bundle, defined by odd 1-forms $\omega_1,\dots,\omega_n$ and even 1-forms $a_1,\dots,a_m$. It is convenient to rename these forms $b_1,\dots,b_{n+m}$, where $b_i=\omega_i$ for all $i\in\{1,\dots,n\}$ and $b_{n+j}=a_j$ for all $j\in\{1,\dots,m\}$. In fact, we will in general use both notations at the same time. Let $\big(M,F_{\mathcal{F}},E_{\mathcal{F}},\nabla^{\mathcal{F}}\big)$ be the trivial flat foliated connection associated to the superfoliation $\mathcal{F}$, as constructed in Section 2C.

Theorem 5.4. *The homomorphism $\varphi_{\mathcal{M},\mathcal{F}}: H^*\big(\mathrm{Vect}(n,m)_0\big)\to H^*(M)$ is completely determined by the flat trivial foliated connection $\big(M,F_{\mathcal{F}},E_{\mathcal{F}},\nabla^{\mathcal{F}}\big)$.*

In other words, the theory of secondary characteristic classes of superfoliations can indeed be considered as a theory of secondary characteristic classes of flat foliated connections.

Proof. Let $\big(b_i^j\big)_{i,j\in\{1,\dots,n+m\}}$ be homogeneous 1-forms satisfying the relation

$$d_{\mathcal{M}}b_i=(-1)^{|j|}\sum_{j=1}^{n+m}b_j\wedge b_i^j \tag{5-2}$$

We divide the proof into four steps.

Step 1: *$\varphi_{\mathcal{M},\mathcal{F}}$ depends only on $(p(b_i^j))_{i,j=1,\dots,n}$ and $(p(b_k^l))_{k,l\in\{n+1,\dots,n+m\}}$.*
By Lemma 4.6, the secondary characteristic classes depend only on the 1-forms $(p(b_i^j))_{i,j\in\{1,\dots,n\}}$ and $(p(b_k^l))_{i,j\in\{n+1,\dots,n+m\}}$. Moreover, any family of 1-forms $(\tilde{b}_i^j)_{i,j\in\{1,\dots,n+m\}}$ such that (5-2) holds for some 1-forms $\tilde{b}_1,\dots,\tilde{b}_{n+m}$ that define the same superfoliation $\mathcal{F}$ will define the same homomorphism $\varphi_{\mathcal{M},\mathcal{F}}$ by Proposition 4.5.

Step 2: *$\varphi_{\mathcal{M},\mathcal{F}}$ depends only on $F_{\mathcal{F}}$ and $(p(b_k^l))_{k,l\in\{n+1,\dots,n+m\}}$.*
After applying p to (5-2) and taking $i\in\{1,\dots,n\}$, we see that the 1-forms $(p(b_i^j))_{i,j\in\{1,\dots,n\}}$ satisfy

$$d_M\big(p(\omega_i)\big)=\sum_{j=1}^{n}p(\omega_j)\wedge p(b_i^j).$$

By Remark 2.13, the 1-forms $p(\omega_i),\ i=1,\dots,n,$ define the foliation $F_{\mathcal{F}}$. For any other choice of 1-forms $\tilde{\omega}_1,\dots,\tilde{\omega}_n\in\Omega^1(M)$ and any 1-forms $\tilde{c}_i^j\in\Omega^1(M),\ i,j\in\{1,\dots,n\},$ with $d_M\tilde{\omega}_i=\sum_{j=1}^n\tilde{\omega}_j\wedge c_i^j$, there exist 1-forms $\tilde{b}_1,\dots,\tilde{b}_{n+m}\in\Omega^1(\mathcal{M})$

defining $\mathcal{F}$, and 1-forms $\tilde{b}_i^j \in \Omega_1(\mathcal{M})$, $i, j \in \{1, \dots, n+m\}$, satisfying (5-2) such that, for all $i, j \in \{1, \dots, n\}$, the identity $p(b_i^j) = c_i^j$ holds. As a consequence, $\varphi_{\mathcal{M},\mathcal{F}}$ is entirely determined by $F_{\mathcal{F}}$ and $(p(b_k^l))_{k,l\in\{n+1,\dots,n+m\}}$

Step 3: *$\varphi_{\mathcal{M},\mathcal{F}}$ depends only on $F_{\mathcal{F}}$ and on the restriction of $(p(b_k^l))_{k,l\in\{n+1,\dots,n+m\}}$ to the tangent space of the leaves of $F_{\mathcal{F}}$.*
For $k \in \{n+1, \dots, n+m\}$, Equation (5-2) can be rewritten

$$d_{\mathcal{M}} a_{k-n} = \sum_{i=1}^{n} \omega_i \wedge b_k^i - \sum_{l=n+1}^{n+m} a_l \wedge b_k^l.$$

As consequence, for any even superfunctions $f_{k,j}^l$ with $k, j \in \{1, \dots, m\}$ and $l \in \{1, \dots, n\}$,

$$d_{\mathcal{M}} a_{k-n} = \sum_{i=1}^{n} \omega_i \wedge \Big(b_k^i - \sum_{c=1}^{m} f_i^{k,c} a_c\Big) - \sum_{l=n+1}^{n+m} a_l \wedge \Big(b_k^l - \sum_{c=1}^{n} f_c^{k,l} \omega_c\Big).$$

We have therefore

$$d_{\mathcal{M}} a_{k-n} = \sum_{i=1}^{n} \omega_i \wedge \tilde{b}_k^i - \sum_{l=n+1}^{n+m} a_l \wedge \tilde{b}_k^l,$$

where

$$\tilde{b}_k^i = b_k^i - \sum_{c=n+1}^{n+m} f_i^{k,c} a_c \qquad \text{and} \qquad \tilde{b}_k^l = b_k^l - \sum_{c=1}^{n} f_c^{k,l} \omega_c. \tag{5-3}$$

Consequently, we can add to the 1-forms $(b_k^l)_{k,l\in\{n+1,\dots,n+m\}}$ any linear combination of the 1-forms $\omega_1, \dots, \omega_n$ without modifying $\varphi_{\mathcal{M},\mathcal{F}}$.

Applying p to (5-3), we obtain

$$p(\tilde{b}_l^k) = p(b_l^k) - \sum_{c=1}^{n} \mathrm{Re}(f_c^{k,l})\, p(\omega_c) \tag{5-4}$$

Therefore, we can add to the 1-forms $(p(b_k^l))_{k,l=n+1,\dots,n+m}$ any linear combination of the 1-forms $p(\omega_1), \dots, p(\omega_n)$ without modifying $\varphi_{\mathcal{M},\mathcal{F}}$. This implies that $\varphi_{\mathcal{M},\mathcal{F}}$ depends only on the restriction of $(p(b_k^l))_{k,l\in\{n+1,\dots,n+m\}}$ to the tangent spaces of the leaves of $F_{\mathcal{F}}$.

Step 4: *The restriction of $(p(b_k^l))_{k,l\in\{n+1,\dots,n+m\}}$ to the tangent space of the leaves of $F_{\mathcal{F}}$ depends only on $E_{\mathcal{F}}$ and $\nabla^{\mathcal{F}}$.*
Let X be a vector field tangent to the leaves of $F_{\mathcal{F}}$, and $\mathcal{X} \in (\mathcal{X}_{\mathcal{F}})_0$ an even supervector field tangent to $\mathcal{F}$ such that $\Pi(\mathcal{X}) = X$. One has, for any $k \in \{n+1, \dots, n+m\}$,

$$\begin{aligned}\nabla_X^{\mathcal{F}}\big(\rho(a_{k-n})\big) = \rho\big(L_{\mathcal{X}} a_{k-n}\big) &= \rho\big(\iota_{\mathcal{X}} d_{\mathcal{M}} a_{k-n}\big) = \rho\Big(\iota_{\mathcal{X}} \sum_{j=1}^{n+m} (-1)^{|j|} b_j \wedge b_k^j\Big) \\ &= \rho\Big(\iota_{\mathcal{X}} \sum_{j=1}^{n} \omega_j \wedge b_k^j - \sum_{l=n+1}^{n+m} a_j \wedge b_k^l\Big).\end{aligned}$$

By the definition of the map ρ, one has $\rho\big((\iota_{\mathscr{X}}\omega_j)b^j_{n+i}\big)=0$ for any $j\in\{1,\ldots,n+m\}$ and $\rho\big(\iota_{\mathscr{X}}(\omega_j\wedge b^j_{n+i})\big)=0$ for any $j\in\{1,\ldots,n\}$. Therefore,

$$\nabla^{\mathscr{F}}_X\big(\rho(a_{k-n})\big)=-\rho\Big(\sum_{j=1}^{m}(\iota_{\mathscr{X}}a_j\wedge b^{n+j}_{n+i})\Big).$$

By Equation (2-1), we obtain

$$\nabla^{\mathscr{F}}_X\big(\rho(a_{k-n})\big)=-\sum_{l=n+1}^{n+m}\big(\mathrm{Re}(\iota_{\mathscr{X}}b^l_k)\big)\,\rho(a_j).$$

By (2-4), the identity $\mathrm{Re}(\iota_{\mathscr{X}}b^l_k)=\iota_X p(b^l_k)$ holds, and thus

$$\nabla^{\mathscr{F}}_X\big(\rho(a_{k-n})\big)=-\sum_{l=n+1}^{n+m}\big(\iota_X p(b^l_k)\big)\,\rho(a_l).$$

The restrictions of the 1-forms $\big(p(b^{n+j}_{n+i})\big)_{i,j\in\{1,\ldots,m\}}$ to the tangent space of each leaf of $F_{\mathscr{F}}$ are therefore completely determined by the connection $\nabla^{\mathscr{F}}$. This completes the proof. □

Remark 5.5. We have associated secondary characteristic classes to a flat trivial foliated vector bundle. In particular, we have associated characteristic classes to any $SO(n)$-bundle over a manifold M, endowed with a flat foliated connection in the sense of [Kamber and Tondeur 1975]. In [Kamber and Tondeur 1974] or [Kamber and Tondeur 1975], characteristic classes are also associated to such objects. It should be interesting to investigate the relation with this construction, in particular with the map (4.4) from [Kamber and Tondeur 1974].

Remark 5.6. Before ending this section, we have to point out the relation between this approach and the theory of Γ-structures [Bott and Haefliger 1972]. A foliation on a supermanifold $\mathcal{M}$ with base manifold M can be defined by a family $\{f_i\}_{i\in I}$ of local submersions onto $\mathbb{R}^{n,m}$, defined on an open covering $\{U_i\}_{i\in I}$ of M. For any two submersions f_i and f_j, there exists φ^j_i in $\mathrm{Diff}(\mathbb{R}^{n,m})$ such that $f_i=\varphi^j_i\circ f_j$, where $\mathrm{Diff}(\mathbb{R}^{n,m})$ is the pseudogroup of local diffeomorphisms of the supermanifold $\mathbb{R}^{n,m}$. This is (almost) the definition of a $\mathrm{Vect}(n,m)$-structure from [Bott and Haefliger 1972]; the only difference is that Γ is not a subspace of local diffeomorphisms of some vector space. As a consequence, it should be possible to obtain again most constructions of the present paper by generalizing the results of [Bott and Haefliger 1972] to this case; note that the Lie algebra of $\mathrm{Diff}(\mathbb{R}^{n,m})$ is precisely $\mathrm{Vect}(n,m)_0$.

Appendix A. The cohomology of $\mathrm{Vect}(n,m)_0$

We prove Theorem 3.7 and, as an application, compute $H^*\big(\mathrm{Vect}(n,m)_0\big)$ in some particular cases. First, we will need some technical results about representations of the Lie algebra $gl_n \oplus gl_m$. The methods of this section are mainly inspired by [Astashkevich and Fuchs 1993; Fuchs 1986].

For any vector space V and any $k \in \mathbb{N}$, we denote by $V^{\otimes k}$ the tensor-product of k copies of V. For any family of vector spaces $V_1, \dots, V_k$, we denote by $\bigotimes_{i=1}^k V_i$ the tensor product $V_1 \otimes V_2 \otimes \dots \otimes V_k$. We denote by $S^k(V)$ the space of elements of degree k in the symmetric algebra of V.

A1. ***Some results about representations of $gl_n \oplus gl_m$.*** Let V and W be vector spaces of dimension n and m, respectively. Let gl_n and gl_m be the Lie algebras of linear endomorphism of V and W, respectively. The Lie algebras gl_n and gl_m act on V and W, respectively, by $g \cdot v = g(v)$ and $h \cdot w = h(w)$, where $v \in V,\ g \in gl_n$, $w \in W,\ h \in gl_m$. Moreover, the dual spaces V^* and W^* are gl_n- and gl_m-modules, respectively, with actions

$$g \cdot v' = (-g)^*(v') \qquad \text{and} \qquad h \cdot w' = (-h)^*(w),$$

where g^* and h^* are the dual endomorphisms and $v' \in V^*,\ w' \in W^*$. These actions extend naturally to an action of the Lie algebra $gl_n \oplus gl_m$ on

$$E^{k,l}_{p,q} = V^{\otimes k} \otimes (V^*)^{\otimes l} \otimes W^{\otimes p} \otimes (W^*)^{\otimes q}.$$

We give some lemmas about the representations of $gl_n \oplus gl_m$. It is well known (see [Fuchs 1986, Theorem 2.1.4] or [Howe 1989]) that, for any vector space V,

$$\begin{cases} \big(V^{\otimes k} \otimes (V^*)^{\otimes l}\big)^{gl_n} = 0 & \text{if } k \neq l, \\ \big(V^{\otimes k} \otimes (V^*)^{\otimes k}\big)^{gl_n} = \bigoplus_{\sigma \in \Sigma_k} a_\sigma \sum_{i_1=1}^{n} \dots \sum_{i_k=1}^{n} x_{i_1} \otimes \dots \otimes x_{i_n} \otimes y_{i_{\sigma(1)}} \otimes \dots \otimes y_{i_{\sigma(n)}}, \end{cases}$$

where $a_\sigma \in \mathbb{R},\ x_1, \dots, x_n \in V$ is a basis of V, and $y_1, \dots, y_n \in V^*$ is the dual basis. The following is an obvious generalization of this result:

Lemma A.1. *Let $\{x_i \mid i = 1, \dots, n\}$ be a basis of V and $\{y_i \in V^* \mid i = 1, \dots, n\}$ the dual basis. Let $\{\zeta_i \mid i = 1, \dots m\}$ be a basis of W and $\{\eta_i \in W^* \mid i = 1, \dots, m\}$ the dual basis.*

(1) If $k \neq l$ or $p \neq q$, then $\big(E^{k,l}_{p,q}\big)^{gl_n \oplus gl_m} = 0$

(2) If $k = l$ and $p = q$, the space $\big(E^{k,k}_{p,p}\big)^{gl_n \oplus gl_m}$ is generated by the elements

$$g_{\sigma,\tau} = \sum_{s_1=1}^{n} \dots \sum_{s_k=1}^{n} \sum_{s'_1=1}^{m} \dots \sum_{s'_p=1}^{m} x_{s_1} \otimes \dots \otimes x_{s_k} \otimes y_{s_{\sigma(1)}} \otimes \dots \otimes y_{s_{\sigma(k)}} \otimes \zeta_{s'_1} \otimes \dots \otimes \zeta_{s'_p} \otimes \eta_{s'_{\tau(1)}} \otimes \dots \otimes \eta_{s'_{\tau(p)}},$$

where σ is a permutation of $\{1, \dots, k\}$ and τ a permutation of $\{1, \dots, p\}$.

Let H be a $gl_n \oplus gl_m$-submodule of $V^{\otimes k} \otimes (V^*)^{\otimes m} \otimes W^{\otimes p} \otimes (W^*)^{\otimes l}$. Let G be the $gl_n \oplus gl_m$-module

$$G := E^{k,l}_{p,q}/H.$$

Such a $gl_n \oplus gl_m$-module is said to be of *type I*.

Lemma A.2. *Let π be the projection of $E^{k,l}_{p,q}$ onto G. One has*:

$$G^{gl_n \oplus gl_m} = \pi\big(E^{k,l}_{p,q}\big)^{gl_n \oplus gl_m}. \tag{A-1}$$

Proof. Both spaces reduce to 0 if $k \neq l$ or if $p \neq q$. If $k = l$ and $p = q$, the center of $gl_n \oplus gl_m$ acts trivially, and then both G and $E^{k,l}_{p,q}$ are $sl_n \oplus sl_m$-modules satisfying

$$G^{sl_n \oplus sl_m} = G^{gl_n \oplus gl_m}, \qquad \big(E^{k,k}_{p,p}\big)^{sl_n \oplus sl_m} = \big(E^{k,k}_{p,p}\big)^{gl_n \oplus gl_m}. \tag{A-2}$$

Since $sl_n \oplus sl_m$ is a semisimple Lie algebra, any finite-dimensional $sl_n \oplus sl_m$-module is the direct sum of the $sl_n \oplus sl_m$-submodule of invariants with the $sl_n \oplus sl_m$-submodule of coinvariants. In particular, we have the decompositions

$$\begin{cases} G = (sl_n \oplus sl_m)\,.\,G \oplus G^{sl_n \oplus sl_m} \\ E^{k,k}_{p,p} = (sl_n \oplus sl_m)\,.\,E^{k,k}_{p,p} \oplus \big(E^{k,k}_{p,p}\big)^{sl_n \oplus sl_m} \end{cases}$$

Since π is a Lie algebra morphism,

$$\begin{cases} \pi\big(\big(E^{k,k}_{p,p}\big)^{sl_n \times sl_m}\big) \subset G^{sl_n \oplus sl_m} \\ \pi\big((sl_n \oplus sl_m)\,.\,E^{k,k}_{p,p}\big) \subset (sl_n \oplus sl_m)\,.\,G \end{cases}$$

Since π is onto, we must have

$$\begin{cases} \pi\big(\big(E^{k,k}_{p,p}\big)^{sl_n \oplus sl_m}\big) = G^{sl_n \oplus sl_m} \\ \big(sl_n \oplus sl_m\big)\,.\,\pi\big(E^{k,k}_{p,p}\big) = (sl_n \oplus sl_m)\,.\,G \end{cases}$$

The conclusion thus follows from (A-2). □

We now introduce the $gl_n \oplus gl_m$-modules that we will study. (In the sequel, we will often simply say “module” for a $gl_n \oplus gl_m$-module.)

Definition A.3. Consider a family $L_1 = \{(\mu_a, \nu_a) \mid a = 1, \dots, K\}$ of pairs of nonnegative integers such that $\mu_a + \nu_a \geq 2$ and ν_a is even. Consider a family $L_2 = \{(p_b, q_b) \mid b = 1 \dots, K'\}$ of pairs of nonnegative integers such that $p_b + q_b \geq 2$ and q_b is odd. Let $L \in \mathbb{N}$ and let $\mathscr{L} = L_1 \sqcup L_2 \sqcup \{L\}$. We associate to $\mathscr{L}$ the $gl_n \oplus gl_m$-module

$$G_{\mathscr{L}} = \bigwedge_{a=1}^{K}\big(S^{\mu_a}(V) \otimes \bigwedge^{\nu_a} W \otimes V^*\big) \wedge \bigwedge_{b=1}^{K'}\big(S^{p_b}(V) \otimes \bigwedge^{q_b} W \otimes W^*\big) \wedge \bigwedge^{L} V^*.$$

Such a module is called a *module of type II*.

Our goal for now is to describe $(G_{\mathcal{L}})^{gl_n \oplus gl_m}$; see Section A2 for motivation.

We describe a natural projection

$$\pi_{\mathcal{L}} : Q \to G_F$$

where $Q = E^{\mathfrak{m},\, K+L}_{\mathfrak{n},\, K'}$, with $\mathfrak{m} = \sum_{a=1}^{K} \mu_a + \sum_{b=1}^{K'} p_b$ and $\mathfrak{n} = \sum_{a=1}^{K} \nu_a + \sum_{b=1}^{K'} q_b$.

First, we can construct an isomorphism ψ of $gl_n \oplus gl_m$-module from Q to the module

$$Q_{\mathcal{L}} = \bigotimes_{a=1}^{K} \bigl(V^{\otimes \mu_a} \otimes W^{\otimes \mu_a} \otimes V^*\bigr) \otimes \bigotimes_{b=1}^{K'} \bigl(V^{\otimes p_b} \otimes W^{\otimes q_b} \otimes W^*\bigr) \otimes (V^*)^{\otimes L}$$

obtained by permuting terms in the tensor product defining Q, according to the rule: allow permutation of the tensor product of two elements if and only if they do not belong to the same vector space. With this restriction, there is only one natural isomorphism ψ from Q to $Q_{\mathcal{L}}$.

Second, there is a natural projection

$$\pi' : \bigotimes_{a=1}^{K} \bigl(V^{\otimes \mu_a} \otimes W^{\otimes \nu_a} \otimes V^*\bigr) \otimes \Bigl(\bigotimes_{b=1}^{K'} V^{\otimes p_b} \otimes W^{\otimes q_b} \otimes W^*\Bigr) \otimes (V^*)^{\otimes L} \longrightarrow G_{\mathcal{L}}$$

constructed by taking the appropriate symmerizations and skew-symmetrizations. Define $\pi_{\mathcal{L}}$ as $\pi_{\mathcal{L}} = \pi' \circ \psi$.

Proposition A.4. (1) *If $\sum_{a=1}^{K} \mu_a + \sum_{b=1}^{K'} p_b \neq K + L$ or if $\sum_{a=1}^{K} \nu_a + \sum_{b=1}^{K'} q_b \neq K'$, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.*

(2) *If there exists $a \in \{1, \dots, K\}$ with $\nu_a \neq 0$, or if there exists $b \in \{1, \dots, K'\}$ with $q_b \neq 1$, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.*

Proof. Since $G_{\mathcal{L}}$ is a module of type I, Lemma A.1(1) and Lemma A.2 immediately imply statement (1). The identity $\sum_{a=1}^{K} \nu_a + \sum_{b=1}^{K'} q_b = K'$ implies statement (2), since q_b is odd for $b = 1, \dots, K$, and ν_a is even for $a = 1, \dots, K$, and since all these numbers are nonnegative. □

From now on we assume that $q_1 = \cdots = q_{K'} = 1$, $\nu_1 = \cdots = \nu_K = 0$, and

$$\sum_{a=1}^{K} \mu_a + \sum_{b=1}^{K'} p_b = K + L. \tag{A-3}$$

Write $H = \sum_{a=1}^{K} \mu_a$.

The map $\pi_{\mathcal{M}}$ can now be easily described: for all $s_1, \dots, s_{K+L} \in \{1, \dots, n\}$, $t_1, \dots, t_{K'} \in \{1, \dots, m\}$, $s'_1, \dots, s'_{K'} \in \{1, \dots, m\}$, and $t'_1, \dots, t'_{K'} \in \{1, \dots, m\}$, we have

$$\begin{aligned}
\text{(A-4)}\quad &\pi_{\mathscr{L}}\big(x_{s_1}\otimes\cdots\cdot x_{s_{K+L}}\cdot t_{s_1}\otimes\cdots\cdot y_{t'_{K'}}\zeta_{s'_1}\cdot\cdots\cdot\zeta_{s'_{K'}}\cdot\eta_{t'_1}\cdot\cdots\cdot\eta_{t'_{K'}}\big)\\
&= \textstyle\bigwedge_{i=0}^{K} R\big(s_{\mu_1+\cdots+\mu_{i-1}+1},\ldots,s_{\mu_1+\cdots+\mu_i}\big)\\
&\qquad\qquad \big(x_{s_{\mu_1+\cdots+\mu_i+1}}\ldots x_{s_{\mu_1+\cdots+\mu_{i+1}}}\cdot y_{t_i}\big)\\
&\wedge \textstyle\bigwedge_{j=0}^{K'} R\big(s_{H+p_1+\cdots+p_{j-1}+1},\ldots,s_{H+p_1+\cdots+p_j}\big)\\
&\qquad\qquad \big(x_{s_{H+p_1+\cdots+p_{j-1}+1}}\ldots x_{s_{H+p_1+\cdots+p_j}}\cdot\zeta_{s'_j}\cdot\eta_{t'_j}\big)\\
&\wedge \textstyle\bigwedge_{k=K+1}^{K+L} y_{t'_k},
\end{aligned}$$

where, for all $h, a_1, \ldots, a_h \in \mathbb{N}$, $R(a_1, \ldots, a_h)$ is an integer that appears in the symmetrization map $\bigotimes^h V \to S^*(V)$. We leave the reader to check that indeed

$$R(a_1,\ldots,a_h)=\prod_{i=1}^{n} K(i,[a_1,\ldots,a_h])!\,. \tag{A-5}$$

Two indices $i, j \in \{1, \ldots, K+L\}$ are said *symmetric with respect to* $\mathscr{L}$ if there exists $k \in \{0, \ldots, K-1\}$ such that $i, j \in \big\{\sum_{a=1}^{k} \nu_a, \ldots, \sum_{a=1}^{k+1} \nu_a\big\}$, or if there exists $k \in \{0, \ldots, K'-1\}$ such that $i, j \in \big\{\sum_{a=1}^{k} p_b, \ldots, \sum_{a=1}^{k+1} p_b\big\}$. Intuitively, i and j are symmetric with respect to $\mathscr{L}$ if the projection $\pi_{\mathscr{L}}$ maps the i-th and the j-th terms of the tensor product $V^{\otimes(K+L)}$ involved in the definition of Q to the same symmetric algebra $S^{\mu_a}(V)$ or to the same symmetric algebra $S^{p_b}(V)$.

Two indices $i, j \in \{1, \ldots, K+L\}$ are said *antisymmetric with respect to* $\mathscr{L}$ if $i \geq K+1$ and $j \geq K+1$. Intuitively, i and j are antisymmetric with respect to $\mathscr{L}$ if the projection $\pi_{\mathscr{L}}$ maps the i-th and the j-th terms of the tensor product $(V^*)^{\otimes(K+L)}$ involved in the definition of Q to the exterior algebra $\bigwedge^L V^*$.

Lemma A.5. *Let σ be a permutation of $\{1, \ldots, K+L\}$. If there are two indices $i, j \in \{1, \ldots, K+L\}$ symmetric with respect to $\mathscr{L}$ such that $\sigma(i), \sigma(j)$ are antisymmetric with respect to $\mathscr{L}$, then $g_{\sigma,\tau} = 0$.*

Proof. Consider two indices $i, j \in \{1, \ldots, K+L\}$ symmetric with respect to $\mathscr{L}$ such that $\sigma(i), \sigma(j)$ are antisymmetric with respect to $\mathscr{L}$. For any $k, l \in \{1, \ldots, n\}$, the terms in (A-4) corresponding to $s_i = k$ and $s_j = l$ and the terms corresponding to $s_i = l$ and $s_j = k$ appear with opposite signs. By Equation (A-4), $\pi_{\mathscr{L}}(g_{\sigma,\tau})$ must therefore vanish. □

Lemma A.6. *If $\mu_a \neq 2$ for some $a \in \{1, \ldots, K\}$, or if $p_b \neq 1$ for some $b \in \{1, \ldots, K\}$, then $G_{\mathscr{L}}^{gl_m \oplus gl_m} = 0$.*

Proof. If $\mu_a \geq 3$ for some a in $\{1, \ldots, K\}$ or if $p_b \geq 2$ for some b in $\{1, \ldots, K'\}$, then by (A-3) we must have $L > K + K'$. By a simple argument of cardinality, this implies that, for any permutation σ of $\{1, \ldots, K+L\}$, there exist $i', j' \in \{K+1, \ldots, K+L\}$ such that $\sigma^{-1}(i'), \sigma^{-1}(j')$ are symmetric with respect to $\mathscr{L}$. Therefore, by Lemma A.5, $\pi_{\mathscr{L}}(g_{\sigma,\tau}) = 0$. □

Assume now that $\mu_1 = \cdots = \mu_K = 2$ and $p_1 = \cdots = p_{K'} = 1$. In this case,

$$G_{\mathcal{L}} = \bigwedge^K \big(S^2(V) \otimes V^*\big) \wedge \bigwedge^{K'} \big(V \otimes W \otimes W^*\big) \wedge \bigwedge^{K+K'} V^*.$$

Let $\mathcal{A}$ be the exterior algebra over the space $S^2(V) \otimes V^* \oplus V \otimes W \otimes W^* \oplus V^*$.

Lemma A.7. *If σ and τ are permutations of $\{1, \dots, K+L\}$ and $\{1, \dots, K'\}$, respectively, then $\pi_{\mathcal{L}}(g_{\sigma,\tau})$ is an element of the algebra generated (with respect to the product $\wedge$) by elements of the form*

$$\sum_{c=1}^{n} \gamma_c^i \, y_c \wedge x_c x_i \,.\, y_j \in \big(S^2(V) \otimes V^*\big) \otimes V^*, \qquad i, j \in \{1, \dots, n\},$$

$$\sum_{c=1}^{n} y_c \wedge x_c \,.\, \zeta_k \,.\, \eta_l \in \big(V \otimes W \otimes W^*\big) \wedge V^*, \qquad k, l \in \{1, \dots, m\},$$

where, for any $i, c = 1, \dots, n$, γ_c^i is defined by $\gamma_c^i = 1$ for $i \neq c$ and by $\gamma_i^i = 2$.

Proof. Let σ be a permutation of $\{1, \dots, 2K+K'\}$. If there is an $i \in \{1, \dots, K\}$ such that $\sigma(2i-1) \geq K$ and $\sigma(2i+1) \geq K$, then Lemma A.5 implies that $\pi_{\mathcal{L}}(g_{\sigma,\tau}) = 0$. Assume now that such an $i \in \{1, \dots, K\}$ does not exist for σ; then, for all $i \in \{1, \dots, K\}$, one of the two integers $\sigma(2i-1), \sigma(2i)$ is greater than K, and one is smaller or equal to K. We define a permutation σ' of $\{1, \dots, K\}$ by $\sigma'(i) = \min\{\sigma(2i-1), \sigma(2i)\}$.

By Lemma A.1(2), we have

$$\pi_{\mathcal{L}}(g_{\sigma,\tau}) = \sum_{s_1,\dots,s_{2K+K'}=1}^{n} \; \sum_{s'_1,\dots,s'_{K'}=1}^{m} \bigwedge\nolimits_{i=1}^{K} R(s_{2i-1}, s_{2i})(x_{s_{2i-1}} x_{s_{2i}} \,.\, y_{s_{\sigma(i)}}) \wedge \bigwedge\nolimits_{j=1}^{K'} x_{s_{2K+j}} \zeta_{s'_j} . \eta_{s'_{\tau(j)}} \wedge \bigwedge\nolimits_{k=K+1}^{2K+K'} y_{s_{\sigma(k)}}.$$

From Equation (A-5), it follows that $R(s_{2i-1}, s_{2i}) = \gamma_{s_{2i-1}}^{s_{2i}}$. Therefore,

$$\pi_{\mathcal{L}}(g_{\sigma,\tau}) = \varepsilon \sum_{s_1,\dots,s_K=1}^{n} \; \sum_{s'_1,\dots,s'_{K'}=1}^{m} \bigwedge\nolimits_{i=1}^{K} \Big(\sum_{c=1}^{n} y_c \wedge \gamma_c^{s_i} x_c x_{s_i} \,.\, y_{s_{\sigma'(i)}} \big) \Big) \wedge \bigwedge\nolimits_{j=1}^{K'} \Big(y_c \wedge \sum_{c=1}^{n} x_c \,.\, \zeta_{t_i} \,.\, \eta_{t_{\tau(i)}} \Big)$$

for some $\varepsilon \in \{-1, +1\}$. $\square$

We recapitulate:

Theorem A.8. *Let $G_{\mathcal{L}}$ be a $gl_n \oplus gl_m$-module of type II* (as in Definition A.3).

- *If $\sum_{a=1}^{K} \mu_a + \sum_{b=1}^{K} p_b \neq K + L$, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.*
- *If one of the even integers $\{\nu_a\}_{a=1,\dots,K}$ is not 0, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.*
- *If one of the odd integers $\{q_b\}_{b=1,\dots,K'}$ is not 1, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.*
- *If one of the integers $\{\mu_a\}_{a=1,\dots,K}$ is not 2, then $G_{\mathcal{L}}^{gl_n \oplus gl_m} = 0$.*

- *If one of the integers* $\{p_b\}_{b=1,\dots,K'}$ *is not* 1, *then* $G_{\mathcal{L}}^{gl_n\oplus gl_m}=0$.
- *If* $\mu_a=2$ *and* $\nu_a=0$ *for* $a=1,\dots,n$, *and if* $p_b=q_b=1$ *for* $b=1,\dots,m$, *then the space* $G_{\mathcal{L}}^{gl_n\oplus gl_m}$ *is contained in the subalgebra of* $\mathcal{A}$ *generated by the elements*

$$\begin{cases}\sum_{c=1}^{n} y_c\wedge(\gamma_c^i x_c.x_i.y_j), & i,j\in\{1,\dots,n\},\\ \sum_{c=1}^{n} y_c\wedge x_c.\zeta_k.\eta_l, & k,l\in\{1,\dots,m\}.\end{cases}$$

Later, we will also need the following:

Lemma A.9 [Fuchs 1986]. *For any* $gl_n\oplus gl_m$*-module* E *of finite dimension*,

$$H^*\big(gl_n\oplus gl_m,E\big)=H^*\big(gl_n\oplus gl_m,E^{gl_n\oplus gl_m}\big).$$

Actually, $H^*\big(gl_n\oplus gl_m,E\big)=H^*\big(gl_n\oplus gl_m,E^{gl_n\oplus gl_m}\big)=H^*\big(gl_n\oplus gl_m,\mathbb{R}\big)\otimes E^{gl_n\oplus gl_m}$.

A2. *The cohomology of* $\mathrm{Vect}(n,m)_0$ *and the Weil algebra.* Now we are able to prove Theorem 3.7. Let $\mathcal{K}$ be the kernel of the DGA homomorphism $\tilde\alpha: W(gl_n\oplus gl_m)\to\bigwedge\mathrm{Vect}(n,m)_0^*$, and denote again by $\tilde\alpha$ the induced DGA homomorphism from $W(gl_n\oplus gl_m)/\mathcal{K}$ to $\bigwedge\mathrm{Vect}(n,m)_0^*$. We restate the theorem:

Theorem 3.7. *Let* $\hat\alpha$ *be the map from* $H^*\big(W(gl_n\oplus gl_m)/\mathcal{K}\big)$ *to* $H^*\big(\mathrm{Vect}(n,m)_0\big)$ *induced by* $\tilde\alpha$. *The map* $\hat\alpha$ *is an isomorphism.*

Proof. We first describe $\tilde\alpha: W(gl_n\oplus gl_m)\to\bigwedge\mathrm{Vect}(n,m)_0^*$ precisely. By (3-12), the DGA homomorphism $\tilde\alpha$ from $W(gl_n\oplus gl_m)$ to $\bigwedge\mathrm{Vect}(n,m)_0^*$ is given by:

$$1\otimes a_{i,j}\mapsto(x_i\,\partial/\partial x_j)^*,\qquad 1\otimes d_{i,j}\mapsto(\theta_i\,\partial/\partial\theta_j)^*,\tag{A-6}$$

where $a_{i,j}$, for $i,j\in\{1,\dots,n\}$, and $d_{i,j}$, for $i,j\in\{1,\dots,m\}$, are bases that are dual to the canonical bases of gl_n and gl_m, respectively.

One can easily check that

$$\begin{cases}\partial\Big(x_i\dfrac{\partial}{\partial x_j}\Big)^*-\sum_{k=1}^{n}\Big(x_i\dfrac{\partial}{\partial x_k}\Big)^*\wedge\Big(x_k\dfrac{\partial}{\partial x_j}\Big)^*=\sum_{c=1}^{n}\Big(\dfrac{\partial}{\partial x_c}\Big)^*\wedge\Big(\dfrac{x_cx_i}{\gamma_c^i}\dfrac{\partial}{\partial x_j}\Big)^*,\\ \partial\Big(\theta_i\dfrac{\partial}{\partial\theta_j}\Big)^*-\sum_{k=1}^{m}\Big(\theta_i\dfrac{\partial}{\partial\theta_k}\Big)^*\wedge\Big(\theta_k\dfrac{\partial}{\partial\theta_j}\Big)^*=\sum_{c=1}^{n}\Big(\dfrac{\partial}{\partial x_c}\Big)^*\wedge\Big(x_c\theta_i\dfrac{\partial}{\partial\theta_j}\Big)^*.\end{cases}\tag{A-7}$$

Since $\tilde\alpha$ is a DGA homomorphism, by the definition of the differential of a Weil algebra [Guillemin and Sternberg 1999], we have

$$\begin{aligned}\tilde\alpha(a_{i,j}\otimes1)&=\sum_{c=1}^{n}\Big(\frac{\partial}{\partial x_c}\Big)^*\wedge\Big(\frac{x_cx_i}{\gamma_c^i}\frac{\partial}{\partial x_j}\Big)^*,\\ \tilde\alpha(d_{i,j}\otimes1)&=\sum_{c=1}^{n}\Big(\frac{\partial}{\partial x_c}\Big)^*\wedge\Big(x_c\theta_i\frac{\partial}{\partial\theta_j}\Big)^*.\end{aligned}\tag{A-8}$$

We compute the cohomology of $\mathrm{Vect}(n, m)_0$ with the help of the Hochschild–Serre spectral sequence [Fuchs 1986] associated to the sub-Lie algebra of elements of weight zero. Its second term $E^2_{i,j}$ is

$$E^2_{i,j} = H^i\big(gl_n \oplus gl_m,\ \textstyle\bigwedge^j \bigoplus_{k\neq 0} F_k\big),$$

where F_k is the space of elements of weight k in $\big(\mathrm{Vect}(n, m)_0\big)^*$. By Lemma A.9, $E^2_{i,j}$ reduces to

$$E^2_{i,j} = H^i\Big(gl_n \oplus gl_m,\ \Big(\textstyle\bigwedge^j \bigoplus_{k\neq 0} F_k\Big)^{gl(n)\oplus gl_m}\Big).$$

The $gl_n \oplus gl_m$-module $\bigwedge^j \bigoplus_{k\neq 0} F_k$ is a direct sum of modules of type II. For example,

$$F_{-1} \simeq W^* \qquad \text{and} \qquad F_1 = \big(S^2(V) \otimes V^*\big) \oplus \big(V \otimes W \otimes W^*\big).$$

More generally, denoting by $\lfloor x\rfloor$ the integer part of $x \in \mathbb{R}$,

$$F_k \simeq \bigoplus_{i=0}^{\lfloor (k+1)/2\rfloor} \big(V^{\otimes k+1-2i} \otimes W^{\otimes 2i} \otimes V^*\big) \oplus \bigoplus_{j=0}^{\lfloor k/2\rfloor} \big(V^{\otimes k+1-(2j+1)} \otimes W^{\otimes (2j+1)} \otimes W^*\big).$$

Therefore, the modules $\bigwedge^{j_1} F_{i_1} \wedge \cdots \wedge \bigwedge^{j_k} F_{i_k}$, where $i_1, \ldots, i_k \in \{-1, 1, 2, 3, \ldots\}$ and $j_1, \ldots, j_k \in \mathbb{N}$, are direct sums of modules of type II.

The $gl_n \oplus gl_m$-module

$$\textstyle\bigwedge^j F_{-1} \otimes \bigwedge^j F_1$$

is a module of type II with $\mu_a = 2$ and $\nu_a = 0$ for $a = 1, \ldots, j$, with $p_b = 1$ and $q_b = 1$ for $b = 1, \ldots, j$, and with $L = j$. By Theorem A.8, the modules $\bigwedge^{j_1} F_{i_1} \wedge \cdots \wedge \bigwedge^{j_k} F_{i_k}$ have no nontrivial space of invariants except those of the form $\bigwedge^j F_{-1} \wedge \bigwedge^j F_1$. Therefore,

$$\begin{cases} \big(\bigwedge^{2j} \bigoplus_{k\neq 0} F_k\big)^{gl_n\oplus gl_m} = \big(\bigwedge^j F_{-1} \otimes \bigwedge^j F_1\big)^{gl_n\oplus gl_m}, \\ \big(\bigwedge^{2j+1} \bigoplus_{k\neq 0} F_k\big)^{gl_n\oplus gl_m} = 0. \end{cases}$$

By Theorem A.8(6), we obtain that the $gl_n \oplus gl_m$-invariant elements of $\bigwedge^j \bigoplus_{k\neq 0} F_k$ are elements of the subalgebra of $\bigwedge \mathrm{Vect}(n, m)^*_0$ generated by

$$\begin{cases} \displaystyle\sum_{c=1}^{n} \gamma^i_c \Big(\frac{\partial}{\partial x_c}\Big)^* \wedge \Big(x_c x_i \frac{\partial}{\partial x_j}\Big)^*, & i, j \in \{1, \ldots, n\}, \\ \displaystyle\sum_{c=1}^{m} \Big(\frac{\partial}{\partial x_c}\Big)^* \wedge \Big(x_c \theta_k \frac{\partial}{\partial \theta_l}\Big)^*, & k, l \in \{1, \ldots, m\}. \end{cases}$$

From (3-3), the identity $\gamma_c^i(x_c x_i\,\partial/\partial x_j)^* = ((x_c x_i/\gamma_c^i)\;\partial/\partial x_j)^*$ holds. Hence, the generators can be written

$$\begin{cases} \sum_{c=1}^{n}\Big(\frac{\partial}{\partial x_c}\Big)^* \wedge \Big(\frac{x_c x_i}{\gamma_c^i}\frac{\partial}{\partial x_j}\Big)^*, & i,j\in\{1,\dots,n\},\\ \sum_{c=1}^{m}\Big(\frac{\partial}{\partial x_c}\Big)^* \wedge \Big(x_c\theta_k\frac{\partial}{\partial \theta_l}\Big)^*, & k,l\in\{1,\dots,m\}.\end{cases} \tag{A-9}$$

Let $\mathcal{I} = \tilde{\alpha}\big(W(gl_n \oplus gl_m)\big)$ be the sub-DGA of $\bigwedge^*\big(\mathrm{Vect}(n,m)_0\big)$ that is the image of the complex $W(gl_n \oplus gl_m)$ through $\tilde{\alpha}$. By (A-8) and (A-9), we have

$$\big(\textstyle\bigwedge^j \bigoplus_{k\neq 0} F_k\big)^{gl_n\oplus gl_m} \subset \mathcal{I}. \tag{A-10}$$

Moreover, by Equation (A-6) we have the identity

$$\mathcal{I}\cap F_0 = F_0 \simeq gl_n \oplus gl_m. \tag{A-11}$$

The cohomology of $\mathcal{I}$ can be computed with the help of a spectral sequence whose second term is

$$\tilde{E}^2_{i,j} = H^i\Big(\mathcal{I}\cap F_0,\ \mathcal{I}\cap\big(\textstyle\bigwedge^j \bigoplus_{k\neq 0} F_k\big)\Big).$$

By (A-11), $\tilde{E}^2_{i,j} = H^i\big(gl_n\oplus gl_m,\ \mathcal{I}\cap\big(\bigwedge^j\bigoplus_{k\neq 0}F_k\big)\big)$. From Lemma A.9 we obtain $\tilde{E}^2_{i,j} = H^i\big(gl_n\oplus gl_m,\ \big(\mathcal{I}\cap\big(\bigwedge^j\bigoplus_{k\neq 0}F_k\big)\big)^{gl_n\oplus gl_m}\big)$. By (A-10), the identity $\tilde{E}^2_{i,j} = E^2_{i,j}$ holds.

Therefore, the cohomology of $\bigwedge \mathrm{Vect}(n,m)_0^*$ is equal to the cohomology of the subcomplex $\mathcal{I} = \tilde{\alpha}\big(W(gl_n\oplus gl_m)\big) \simeq W(gl_n\oplus gl_m)/\mathcal{K}$. This completes the proof of Theorem 3.7. □

As an application, we compute the cohomology of $H^*\big(\mathrm{Vect}(n,m)_0\big)$ in some particular cases:

Proposition A.10. *The cohomology of* $\mathrm{Vect}(0,m)_0$ *is isomorphic to the Chevalley–Eilenberg cohomology of the Lie algebra* gl_m.

Proof. The DGA homomorphism from $W(gl_m)$ to $\bigwedge\mathrm{Vect}(0,m)_0^*$ is:

$$1\otimes d_{i,j}\mapsto(\theta_i\,\partial/\partial\theta_j)^* \qquad\text{and}\qquad d_{i,j}\otimes 1\mapsto 0.$$

The kernel of this homomorphism is $S\big(gl(m)\big)\otimes 1$. The cohomology of

$$W(gl_m)/S\big(gl(m)\big)\otimes 1 = \textstyle\bigwedge gl_m$$

is the Chevalley–Eilenberg cohomology of the Lie algebra gl_m. □

In particular:

Proposition A.11. *We have isomorphisms* $H^1\big(\mathrm{Vect}(0,1)_0\big) \simeq H^0\big(\mathrm{Vect}(0,1)_0\big) \simeq \mathbb{R}$, *and* $H^1\big(\mathrm{Vect}(0,1)_0\big) = 0$. *Moreover*, $H^1(\mathrm{Vect}(0,1)_0)$ *is generated by* $(\theta\,\partial/\partial\theta)^*$.

We also determine the cohomology of $\mathrm{Vect}(1,1)_0$:

Proposition A.12. *The cohomology of* $\mathrm{Vect}(1,1)_0$ *is given by* $H^n\big(\mathrm{Vect}(1,1)_0\big) = 0$ *if* $n \neq 0, 3$, *and by* $H^0\big(\mathrm{Vect}(1,1)_0\big) = \mathbb{R}$ *and* $H^3\big(\mathrm{Vect}(1,1)_0\big) = \mathbb{R}^3$. *Moreover, the latter is generated by the classes described in* (A-13), (A-14), *and* (A-15).

Proof. We denote by a and d the generators $a_{1,1}$ and $d_{1,1}$ of $gl_1 \oplus gl_1$. The DGA homomorphism from $W(gl_1 \oplus gl_1)$ to $\bigwedge \mathrm{Vect}(1,1)_0^*$ is given by

$$\text{(A-12)} \qquad \begin{cases} 1 \otimes a \mapsto \Big(x\dfrac{\partial}{\partial x}\Big)^*, \\ 1 \otimes d \mapsto \Big(\theta\dfrac{\partial}{\partial\theta}\Big)^*, \\ a \otimes 1 \mapsto 2\Big(\dfrac{\partial}{\partial x}\Big)^* \wedge \Big(x^2\dfrac{\partial}{\partial x}\Big)^* = \Big(\dfrac{\partial}{\partial x}\Big)^* \wedge \Big(\dfrac{x^2}{2}\dfrac{\partial}{\partial x}\Big)^*, \\ d \otimes 1 \mapsto \Big(\dfrac{\partial}{\partial x}\Big)^* \wedge \Big(x\theta\dfrac{\partial}{\partial\theta}\Big)^*. \end{cases}$$

The kernel $\mathcal{K}$ of this application is generated by $S^2(gl_1 \oplus gl_1) \otimes 1$. This implies that the cohomology vanishes in all degrees different from 3. We can now compute the cohomology of $W(gl_1 \oplus gl_1)/\mathcal{K}$. It is easy to check that $H^3\big(\mathrm{Vect}(1,1)_0\big) = \mathbb{R}^3$. Generators of $H^3\big(\mathrm{Vect}(1,1)_0\big)$ are given by

$$\text{(A-13)} \qquad H_1 = \Big[\Big(\frac{x^2}{2}\frac{\partial}{\partial x}\Big)^* \wedge \Big(x\frac{\partial}{\partial x}\Big)^* \wedge \Big(\frac{\partial}{\partial x}\Big)^*\Big],$$

$$\text{(A-14)} \qquad H_2 = \Big[\Big(x\theta\frac{\partial}{\partial\theta}\Big)^* \wedge \Big(x\frac{\partial}{\partial x}\Big)^* \wedge \Big(\frac{\partial}{\partial x}\Big)^*\Big],$$

$$\text{(A-15)} \qquad H_3 = \Big[\Big(x\theta\frac{\partial}{\partial\theta}\Big)^* \wedge \Big(\theta\frac{\partial}{\partial\theta}\Big)^* \wedge \Big(\frac{\partial}{\partial x}\Big)^*\Big]. \qquad \square$$

Appendix B. Proof of Lemma 4.3

Lemma 4.3. *Let* $d_1, \dots, d_n$ *be odd* 1*-forms and* $d_{n+1}, \dots, d_{n+m}$ *even* 1*-forms, forming a free family. If* d^i, $i \in \{1, \dots, n+m\}$, *are* 2*-forms on* $\mathcal{M}$ *such that, for any* $j \in \{1, \dots, n+m\}$,

$$\text{(B-1)} \qquad \sum_{i=1}^{n+m} d_i \wedge d^i = 0,$$

then there exist homogeneous 1*-forms* $d^{i,j}$, *of parity* $|i| + |l| + 1$, *with*

$$\text{(B-2)} \qquad d^{i,l} = -(-1)^{(|i|+1)(|l|+1)} d^{l,i} \qquad \text{and} \qquad \sum_{i=1}^{n+m} d_i \wedge d^{i,l} = d^l.$$

Proof. There are partitions of unity on $\mathcal{M}$, that is, for any open covering $\{U_s\}_{s\in S}$ of the base manifold, there exist even superfunctions $\{\varphi_s \in \Omega(\mathcal{M})\}_{s\in S}$ with support in U_s and such that $\sum_{s\in S} \varphi_s = 1_{\mathcal{M}}$, where $1_{\mathcal{M}}$ is the unit of $\mathcal{O}(\mathcal{M})$. As consequence, if there is an open covering $\{U_s\}_{s\in S}$, of the base manifold such that (B-2) has local solutions on U_s for any $s \in S$, then (B-2) has a global solution. We therefore only have to prove that (B-2) has solutions locally.

For this, consider 1-forms $d_{n+m+1}, \ldots, d_{p+q}$ such that $d_1, \ldots, d_{p+q}$ is a trivialization of $\Omega^1(\mathcal{M}) \simeq T\mathcal{M}^*$. For any $i \in \{1, \ldots, p+q\}$, define $|i|$ to be 1 if d_i is even, and 0 otherwise. (This definition generalizes the previous definition of $|i|$ given in Definition 3.3.)

For any $l \in \{1, \ldots, n+m\}$, there exist superfunctions $F^{r,s;i}$, $G^{t,u;i}$, and $H^{v,w;i}$, with $r, s, t \in \{1, \ldots, n+m\}$, $u, v, w \in \{n+m+1, \ldots, p+q\}$, $i \in \{1, \ldots, n+m\}$, such that

$$\text{(B-3)}\quad \begin{cases} d^l = \displaystyle\sum_{r,s=1}^{n+m} d_r \wedge d_s \wedge F^{r,s;l} + \sum_{t=1}^{n+m} \sum_{u=n+m+1}^{p+q} d_t \wedge d_u \wedge G^{t,u;l} \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\sum_{v,w=n+m+1}^{p+q} d_v \wedge d_w \wedge H^{v,w;l}, \\ F^{r,s;l} = (-1)^{(|r|+1)(|s|+1)} F^{s,r;l}, \\ H^{v,w;l} = (-1)^{(|v|+1)(|w|+1)} H^{w,v;l}. \end{cases}$$

For convenience, we have chosen to multiply a 1-form with a superfunction on the right, which is unusual but will simplify the signs. Equation (B-1) gives

$$\sum_{l=1}^{n+m} \sum_{r,s=1}^{n+m} d_l \wedge d_r \wedge d_s \wedge F^{r,s;l} + \sum_{l=1}^{n+m} \sum_{t=1}^{n+m} \sum_{u=n+m+1}^{p+q} d_l \wedge d_t \wedge d_u \wedge G^{t,u;l} \\ + \sum_{l=1}^{n+m} \sum_{v,w=n+m+1}^{p+q} d_l \wedge d_v \wedge d_w \wedge H^{v,w;l} = 0.$$

This is equivalent to the three conditions

$$\text{(B-4)}\quad \begin{cases} F^{r,s;l} + (-1)^{(|l|+1)(|s|+|r|)} F^{s,l;r} + (-1)^{(|s|+1)(|r|+|l|)} F^{l,r;s} = 0 \\ G^{t,u;l} = -(-1)^{(|l|+1)(|t|+1)} G^{l,u;t} \\ H^{v,w;l} = 0 \end{cases}$$

Define $(d^{i;l})_{i,l=1,\ldots,n+m}$ by

$$\text{(B-5)}\quad d^{i;l} = \tfrac{4}{3} \sum_{s=1}^{n+m} d_s \wedge \left(F^{i,s;l} + \tfrac{1}{2}(-1)^{|s|(|i|+1)+|i|(|l|+1)} d_s \wedge F^{i,l;s}\right) \\ + \sum_{u=n+m+1}^{p+q} d_u \wedge G^{i,u;l}.$$

We check that (B-4) implies that (B-2) can be satisfied:

Step 1: *Checking that* $d^{i;l} = (-1)^{(|i|+1)(|l|+1)} d^{l;i}$.

From $G^{t,u;i} = (-1)^{(|i|+1)(|l|+1)} G^{i,u;l}$, we obtain

(B-6) $$\sum_{u=n+m+1}^{p+q} d_u \wedge G^{i,u;l} = (-1)^{(|i|+1)(|l|+1)} \sum_{u=n+m+1}^{p+q} d_t \wedge G^{l,u;i}.$$

Moreover, from (B-3) and (B-4), we obtain

(B-7) $$F^{l,s;i} + (-1)^{(|s|+1)(|l|+|i|)} F^{i,l;s} + (-1)^{(|l|+1)(|i|+1)} F^{i,s;l} = 0.$$

It is then straightforward to check that

$$F^{l,s;i} + \tfrac{1}{2}(-1)^{(|s|+1)(|l|+|i|)} F^{l,i;s} = -(-1)^{(|l|+1)(|i|+1)}\big(F^{i,s;l} + \tfrac{1}{2}(-1)^{(|s|+1)(|l|+|i|)} F^{i,l;s}\big).$$

This and (B-6) imply that

(B-8) $$d^{i;l} = -(-1)^{(|i|+1)(|l|+1)} d^{l;i}.$$

Step 2: *Checking that* $\sum_{i=1}^{n+m} d_i \wedge d^{i;l} = d^l$.

We compute $\sum_{i,s=1}^{n+m} (-1)^{(|s|+1)(|l|+|i|)} d_i \wedge d_s F^{i,l;s}$. It is equal to

$$\tfrac{1}{2}\sum_{i,s=1}^{n+m} (-1)^{(|s|+1)(|l|+|i|)} d_i \wedge d_s F^{i,l;s} + \tfrac{1}{2}\sum_{i,s=1}^{n+m} (-1)^{(|s|+1)(|l|+|i|)} d_s \wedge d_i \wedge F^{s,l;i}.$$

From the identity $(-1)^{(|s|+1)(|l|+|i|)}(-1)^{(|s|+1)(|i|+1)} = (-1)^{(|l|+1)(|s|+1)}$, we deduce

$$\begin{aligned}\sum_{i,s=1}^{n+m} (-1)^{|s|(|i|+1)+|i|(|l|+1)} d_i \wedge d_s \wedge F^{i,l;s} &= \tfrac{1}{2} d_i \wedge d_s \wedge \big((-1)^{(|s|+1)(|l|+|i|)} F^{i,l;s} + \tfrac{1}{2}(-1)^{(|l|+1)(|s|+1)} F^{s,l;i}\big)\\ &= \tfrac{1}{2} d_i \wedge d_s \wedge \big((-1)^{(|s|+1)(|l|+|i|)} F^{i,l;s} + (-1)^{(|l|+1)(|i|+1)} F^{l,s;i}\big).\end{aligned}$$

From (B-7), we obtain

(B-9) $$\sum_{i,s=1}^{n+m} (-1)^{(|s|+1)(|l|+|i|)} d_i \wedge d_s F^{i,l;s} = -\tfrac{1}{4}\sum_{i,s=1}^{n+m} d_i \wedge d_s \wedge F^{i,s;l}.$$

The result now follows immediately from (B-5) and (B-9). □

References

[Astashkevich and Fuchs 1993] A. B. Astashkevich and D. B. Fuchs, "On the cohomology of the Lie superalgebra $W(m|n)$", pp. 1–13 in *Unconventional Lie algebras*, edited by D. Fuchs, Adv. Soviet Math. **17**, Amer. Math. Soc., Providence, RI, 1993. MR 94k:17003 Zbl 0801.17021

[Bernstein and Rosenfeld 1973] I. N. Bernšteĭn and B. I. Rosenfel'd, "Homogeneous spaces of infinite-dimensional Lie algebras and the characteristic classes of foliations", *Uspehi Mat. Nauk* **28**:4(172) (1973), 103–138. MR 54 #3714 Zbl 0289.57011

[Bott and Haefliger 1972] R. Bott and A. Haefliger, "On characteristic classes of Γ-foliations", *Bull. Amer. Math. Soc.* **78** (1972), 1039–1044. MR 46 #6370 Zbl 0262.57010

[Fuchs 1973] D. B. Fuchs, "Characteristic classes of foliations", *Uspehi Mat. Nauk* **28**:2(170) (1973), 3–17. MR 54 #3716 Zbl 0274.57005

[Fuchs 1986] D. B. Fuchs, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986. MR 88b:17001 Zbl 0667.17005

[Godbillon and Vey 1971] C. Godbillon and J. Vey, "Un invariant des feuilletages de codimension 1", *C. R. Acad. Sci. Paris Sér. A-B* **273** (1971), A92–A95. MR 44 #1046 Zbl 0215.24604

[Guillemin and Sternberg 1999] V. W. Guillemin and S. Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer, Berlin, 1999. MR 2001i:53140 Zbl 0934.55007

[Hill and Simanca 1991] C. D. Hill and S. R. Simanca, "The super complex Frobenius theorem", *Ann. Polon. Math.* **55** (1991), 139–155. MR 93e:32009 Zbl 0756.5800

[Howe 1989] R. Howe, "Remarks on classical invariant theory", *Trans. Amer. Math. Soc.* **313**:2 (1989), 539–570. MR 90h:22015a Zbl 0674.15021

[Kamber and Tondeur 1974] F. W. Kamber and P. Tondeur, "Characteristic invariants of foliated bundles", *Manuscripta Math.* **11** (1974), 51–89. MR 48 #12556 Zbl 0267.57012

[Kamber and Tondeur 1975] F. W. Kamber and P. Tondeur, *Foliated bundles and characteristic classes*, Springer, Berlin, 1975. MR 53 #6587 Zbl 0308.57011

[Koszul 1988] J.-L. Koszul, "Les superalgèbres de Lie $\mathbf{W}(n)$ et leurs représentations", pp. 161–171 in *Géométrie différentielle* (Paris, 1986), edited by D. Bernard and Y. Choquet-Bruhat, Travaux en Cours **33**, Hermann, Paris, 1988. MR 90d:17006 Zbl 0652.17014

[Laurent-Gengoux 2004] C. Laurent-Gengoux, "Secondary characteristic classes of superfoliations", *C. R. Math. Acad. Sci. Paris* **339**:8 (2004), 567–572. MR 2005i:58006 Zbl 1065.58004

[Leites 1980] D. A. Leĭtes, "Introduction to the theory of supermanifolds", *Uspekhi Mat. Nauk* **35**:1(211) (1980), 3–57, 255. MR 81j:58003 Zbl 0462.58002

[Monterde et al. 1997] J. Monterde, J. Muñoz-Masqué, and O. A. Sánchez-Valenzuela, "Geometric properties of involutive distributions on graded manifolds", *Indag. Math.* (*N.S.*) **8**:2 (1997), 217–246. MR 99c:58007 Zbl 0887.58008

[Tuynman 2004] G. M. Tuynman, *Supermanifolds and supergroups: basic theory*, Mathematics and its Applications **570**, Kluwer Academic Publ., Dordrecht, 2004. MR 2005k:58007 Zbl 1083.58001

Received February 3, 2005.

CAMILLE LAURENT-GENGOUX
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE POITIERS
TÉLÉPORT 2, BOULEVARD MARIE ET PIERRE CURIE BP 30179
86962 FUTUROSCOPE CHASSENEUIL CEDEX
FRANCE

laurent@math.univ-poitiers.fr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

CURVATURE OF SPECIAL ALMOST HERMITIAN MANIFOLDS

FRANCISCO MARTÍN CABRERA AND ANDREW SWANN

We study the curvature of almost Hermitian manifolds and their special analogues via intrinsic torsion and representation theory. By deriving different formulae for the skew-symmetric part of the $*$-Ricci curvature, we find that some of these contributions are dependent on the approach used and, for the almost Hermitian case, we obtain tables that differ from those of Falcitelli, Farinola, and Salamon. We show how the exterior algebra may be used to explain some of these variations.

1. Introduction

Tricerri and Vanhecke [1981] gave a complete decomposition of the Riemannian curvature tensor R of an almost Hermitian manifold

$$\big(M, I, \langle\,\cdot\,,\cdot\,\rangle\big)$$

into irreducible $\mathrm{U}(n)$-components. These divide naturally into two groups, one forming the space $\mathcal{K} = \mathcal{K}(\mathfrak{u}(n))$ of algebraic curvature tensors for a Kähler manifold, and the other being its orthogonal complement $\mathcal{K}^\perp$.

Falcitelli et al. [1994] showed that the components of R in $\mathcal{K}^\perp$ are linearly determined by the covariant derivative $\nabla\xi$, where ∇ is the Levi-Civita connection and ξ is the intrinsic torsion of the $\mathrm{U}(n)$-structure on M. Gray and Hervella [1980] showed that, in general dimensions, ξ may be split into four components $\xi_1, \dots, \xi_4$ under the action of $\mathrm{U}(n)$. By using the minimal $\mathrm{U}(n)$-connection $\widetilde{\nabla} = \nabla + \xi$ of M, Falcitelli et al. display some tables showing whether the tensors $\widetilde{\nabla}\xi_i$ and $\xi_i \odot \xi_j$ contribute to the components of R in $\mathcal{K}^\perp$. This provides a unified approach to many of the curvature results obtained in [Gray 1976a].

The present paper is motivated by the interest in extending the above-mentioned results to special almost Hermitian manifolds. These are defined as almost Hermitian manifolds $\big(M, I, \langle\,\cdot\,,\cdot\,\rangle\big)$ equipped with a complex volume form

$$\Psi = \psi_+ + i\psi_-.$$

MSC2000: primary 53C55; secondary 53C10, 53C15.
Keywords: almost Hermitian, special almost Hermitian, intrinsic torsion, curvature tensor, G-connection.

Equivalently, they are manifolds with structure group $\mathrm{SU}(n)$. A detailed study of the intrinsic torsion $\eta+\xi$ of such manifolds was made in [Martín Cabrera 2005], extending results of Chiossi and Salamon [2002]; here, ξ is the intrinsic $\mathrm{U}(n)$-torsion, as above, and η is essentially a 1-form. There is much current interest in $\mathrm{SU}(n)$-structures, partly as generalisations of Calabi–Yau manifolds [Grantcharov et al. 2003; Banos 2002] and partly because of the role played by torsion connections with $\mathrm{SU}(n)$ holonomy in string theory [Papadopoulos 1999; Gutowski et al. 2003].

For $\mathrm{SU}(n)$ structures, the algebraic curvature tensors lie in $\mathcal{K}(\mathfrak{su}(n))$ and are automatically Ricci-flat. Therefore, one may compute the Ricci curvature Ric, and indeed the $*$-Ricci curvature Ric^*, in terms of the intrinsic $\mathrm{SU}(n)$-torsion $\eta+\xi$. This enables us to find information about those $\mathrm{SU}(n)$-components of the Riemannian curvature R that are determined by the tensors Ric and Ric^*. Some of these components are contained in $\mathcal{K}^\perp$, and others are contained in $\mathcal{K}$. This will allow us, on the one hand, to get more concrete information about some components of R contained in $\mathcal{K}^\perp$ and, on the other hand, to enlarge the tables of Falcitelli et al. with columns related to some components contained in $\mathcal{K}$.

In working out these contributions, we arrived at various alternative formulae for certain curvature components purely in terms of the intrinsic $\mathrm{U}(n)$-torsion ξ. This leads to some table entries that are different from those obtained by Falcitelli et al. To try to account for this, we consider the identity $d^2=0$ in the exterior algebra. Applying this to the Kähler 2-form ω and considering a particular component leads indeed to a nontrivial relation between the tensors contributing to the curvature. One may view the relation $d^2\omega=0$ as one way of taking into account some of the information that the Levi-Civita connection connection $\nabla=\widetilde{\nabla}-\xi$ is torsion-free.

The paper is organized as follows. In Section 2 we present some preliminary material: definitions, results, notation, etc. Then, in Section 3 we derive some formulae relating the curvature and the intrinsic torsion. As an immediate application, we give an alternative proof of the result of Gray [1976b] that any nearly Kähler manifold of dimension 6 that is not Kähler is an Einstein manifold. We then proceed to computing the contributions of different components of the intrinsic torsion and its covariant derivative to the Ricci, $*$-Ricci and Riemannian curvatures. Because of representation theory, this behaves differently in dimensions 4 and 6 than in higher dimensions: in dimension 6, ξ splits into more $\mathrm{SU}(3)$-components; in dimension 4, the space of curvature tensors is decomposed more finely under the action of $\mathrm{SU}(2)$. This motivates us to display results and tables in two separate sections: Section 4 for high dimensions, $2n\geqslant 8$, and Section 5 for dimensions 6 and 4. Finally, in Section 6 we discuss identities derived from the exterior algebra.

Note. We will often use decompositions of tensor products without providing details, since such information can be readily obtained from available software.

2. Preliminaries

An *almost Hermitian manifold* is a $2n$-dimensional manifold M, $n > 0$, with a $\mathrm{U}(n)$-structure. This means that M is equipped with a Riemannian metric $\langle \cdot , \cdot \rangle$ and an orthogonal almost complex structure I. Each fibre $T_m M$ of the tangent bundle can be considered as a complex vector space by defining $ix = Ix$. We will write $T_m M_{\mathbb{C}}$ when we are regarding $T_m M$ as such a space.

We define a Hermitian scalar product $\langle \cdot , \cdot \rangle_{\mathbb{C}} = \langle \cdot , \cdot \rangle + i\omega(\cdot , \cdot)$, where ω is the Kähler form given by $\omega(x, y) = \langle x, Iy\rangle$. The real tangent bundle TM is identified with the cotangent bundle T^*M by the map $x \mapsto \langle \cdot , x\rangle = x$. Similarly, the conjugate complex vector space $\overline{T_m M_{\mathbb{C}}}$ is identified with the dual complex space $T^*_m M_{\mathbb{C}}$ by the map $x \mapsto \langle \cdot , x\rangle_{\mathbb{C}} = x_{\mathbb{C}}$. It follows immediately that $x_{\mathbb{C}} = x + iIx$.

If we consider the spaces $\bigwedge^p T^*_m M_{\mathbb{C}}$ of skew-symmetric complex forms, one can check that $x_{\mathbb{C}} \wedge y_{\mathbb{C}} = (x+iIx)\wedge(y+iIy)$. There are natural extensions of the scalar products $\langle \cdot , \cdot \rangle$ and $\langle \cdot , \cdot \rangle_{\mathbb{C}}$ to $\bigwedge^p T^*_m M$ and $\bigwedge^p T^*_m M_{\mathbb{C}}$, defined respectively by

$$\langle a, b\rangle = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{2n} a(e_{i_1}, \dots , e_{i_p})\, b(e_{i_1}, \dots , e_{i_p}),$$

$$\langle a_{\mathbb{C}}, b_{\mathbb{C}}\rangle_{\mathbb{C}} = \frac{1}{p!} \sum_{i_1,\dots,i_p=1}^{n} a_{\mathbb{C}}(u_{i_1}, \dots , u_{i_p})\, \overline{b_{\mathbb{C}}(u_{i_1}, \dots , u_{i_p})},$$

where $e_1, \dots , e_{2n}$ is an orthonormal basis for real vectors, and $u_1, \dots , u_n$ is a unitary basis for complex vectors.

The following conventions will be used in this paper. If b is a $(0, s)$-tensor, we write

$$I_{(i)} b(X_1, \dots , X_i, \dots , X_s) = -b(X_1, \dots , IX_i, \dots , X_s),$$
$$Ib(X_1, \dots , X_s) = (-1)^s b(IX_1, \dots , IX_s).$$

Tricerri and Vanhecke [1981] gave a complete decomposition of the Riemannian curvature tensor R of an almost Hermitian manifold $\big(M, I, \langle \cdot , \cdot \rangle\big)$ into irreducible $\mathrm{U}(n)$-components. As indicated above, some of these components, constituting a $\mathrm{U}(n)$-space denoted by $\mathcal{K} = \mathcal{K}(\mathfrak{u}(n))$, are the only components that can occur when M is a Kähler manifold. In this text we will follow the notation used in [Falcitelli et al. 1994] for such components. Likewise, we will adopt the formalism used in [Salamon 1989] and [Falcitelli et al. 1994] for irreducible $\mathrm{U}(n)$-modules. Thus, for $n \geqslant 2$,

$$\mathcal{K} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2,$$

where $\mathcal{C}_3 \cong [\sigma_0^{2,2}]$, $\mathcal{K}_1 \cong \mathbb{R}$, $\mathcal{K}_2 \cong [\lambda_0^{1,1}]$, and $+$ denotes direct sum. We recall that $\lambda_0^{p,q}$ is a complex irreducible $\mathrm{U}(n)$-module coming from the (p, q)-part of the complex exterior algebra, and that its corresponding dominant weight in standard coordinates is given by $(1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$, where 1 and -1 are repeated p and q times, respectively. By analogy with the exterior algebra, there are also irreducible $\mathrm{U}(n)$-modules $\sigma_0^{p,q}$, with dominant weights $(p, 0, \dots, 0, -q)$ coming from the symmetric algebra. The notation $[\![V]\!]$ stands for the real vector space underlying a complex vector space V, and $[W]$ denotes a real vector space that admits W as its complexification.

Moreover, let Ric and Ric* be the Ricci and $*$-Ricci curvatures, defined by

$$\mathrm{Ric}(X, Y) = \langle R_{X,e_i}Y,\ e_i\rangle, \qquad \mathrm{Ric}^*(X, Y) = \langle R_{X,e_i}IY,\ Ie_i\rangle,$$

where $R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, and the summation convention is used.

The components of the curvature R in $\mathcal{K}_1$ and $\mathcal{K}_2$ are determined by, respectively, the trace and the trace-free components of $\mathrm{Ric}_H + 3\,\mathrm{Ric}^*_H$ (see [Tricerri and Vanhecke 1981]), where b_H indicates the Hermitian part of a bilinear form b, that is, the part satisfying $b_H(IX, IY) = b_H(X, Y)$. Note that Ric^*_H coincides with the symmetric part of Ric*.

The remaining components of R, not included in $\mathcal{K}$, are contained in a $\mathrm{U}(n)$-space denoted by $\mathcal{K}^\perp$. For $n \geqslant 4$, one has [Falcitelli et al. 1994]:

$$\mathcal{K}^\perp = \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,$$

where $\mathcal{K}_{-1} \cong \mathbb{R}$, $\mathcal{K}_{-2} \cong [\lambda_0^{1,1}]$, $\mathcal{C}_4 \cong [\lambda_0^{2,2}]$, $\mathcal{C}_5 \cong [\![U]\!]$, $\mathcal{C}_6 \cong [\![\lambda^{2,0}]\!]$, $\mathcal{C}_7 \cong [\![V]\!]$, and $\mathcal{C}_8 \cong [\![\sigma^{2,0}]\!]$. The irreducible $\mathrm{U}(n)$-modules U and V have dominant weights $(2, 2, 0, \dots, 0)$ and $(2, 1, 0, \dots, 0, -1)$. For $n = 3$, the decomposition of $\mathcal{K}^\perp$ is formed by the same summands but omitting $\mathcal{C}_4$. Finally, when $n = 2$ we have to omit $\mathcal{K}_{-2}$, $\mathcal{C}_4$, and $\mathcal{C}_7$.

We are dealing with G-structures where G is a subgroup of the linear group $\mathrm{GL}(m, \mathbb{R})$. If M possesses a G-structure, then there always exists a G-connection defined on M. Moreover, if $(M^m, \langle\cdot,\cdot\rangle)$ is an orientable m-dimensional Riemannian manifold and G a closed and connected subgroup of $\mathrm{SO}(m)$, then there exists a unique metric G-connection $\widetilde{\nabla}$ such that $\xi_x = \widetilde{\nabla}_x - \nabla_x$ takes its values in $\mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ denotes the orthogonal complement in $\mathfrak{so}(m)$ of the Lie algebra $\mathfrak{g}$ of G, and ∇ is the Levi-Civita connection [Salamon 1989; Cleyton and Swann 2004]. The tensor ξ is the *intrinsic torsion* of the G-structure, and $\widetilde{\nabla}$ is called the *minimal G-connection*.

For $\mathrm{U}(n)$-structures, the minimal $\mathrm{U}(n)$-connection is given by $\widetilde{\nabla} = \nabla + \xi$, with

$$\xi_X Y = -\tfrac{1}{2} I(\nabla_X I)Y, \tag{2–1}$$

see [Falcitelli et al. 1994]. Since $\mathrm{U}(n)$ stabilizes the Kähler form ω, it follows that $\widetilde{\nabla}\omega = 0$. Moreover, $\xi_X(IY) + I(\xi_X Y) = 0$ implies $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(n)^\perp$. Thus, one can identify the $\mathrm{U}(n)$-components of ξ with those of $\nabla\omega$:

(1) if $n=1$, $\xi \in T^*M \otimes \mathfrak{u}(1)^\perp = \{0\}$;

(2) if $n=2$, $\xi \in T^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{W}_2 + \mathcal{W}_4$;

(3) if $n \geqslant 3$, $\xi \in T^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4$.

The summands $\mathcal{W}_i$ are the irreducible $\mathrm{U}(n)$-modules given by Gray and Hervella [1980], so $\mathcal{W}_1 \cong [\![\lambda^{3,0}]\!]$, $\mathcal{W}_2 \cong [\![A]\!]$, $\mathcal{W}_3 \cong [\![\lambda_0^{2,1}]\!]$, and $\mathcal{W}_4 \cong [\![\lambda^{1,0}]\!]$, where $A \subset \lambda^{1,0} \otimes \lambda^{2,0}$ is the irreducible $\mathrm{U}(n)$-module with dominant weight $(2, 1, 0, \dots, 0)$. In the following, ξ_i will denote the component in $\mathcal{W}_i$ of the torsion tensor ξ.

Falcitelli et al. [1994] proved that the components of R in $\mathcal{K}^\perp$ are linearly determined by the covariant derivative $\nabla\xi$ with respect to the Levi-Civita connection ∇. To prove this result, they consider the space $\mathcal{R} = \mathcal{K} + \mathcal{K}^\perp$ of curvature tensors (we recall that $\mathcal{R}$ is the kernel of the mapping $\odot^2\big(\bigwedge^2 T_m^*M\big) \to \bigwedge^4 T_m^*M$ defined by wedging 2-forms together). Then, they deduce that the orthogonal projection $\pi^\perp = (\pi_2 \circ \pi_1)\,|_{\mathcal{R}} : \mathcal{R} \to \mathcal{K}^\perp$ can be expressed as the restriction to $\mathcal{R}$ of the composition map $\pi_2 \circ \pi_1$, where $\pi_1 : \bigwedge^2 T_m^*M \otimes \bigwedge^2 T_m^*M \to \bigwedge^2 T_m^*M \otimes \mathfrak{u}(n)^\perp$ is the orthogonal projection, and $\pi_2 : \bigwedge^2 T_m^*M \otimes \mathfrak{u}(n)^\perp \to \mathcal{K}^\perp$ is a certain $\mathrm{U}(n)$-equivariant homomorphism. Since we have the identity [Falcitelli et al. 1994]

$$\begin{aligned}\pi_1(R)(X, Y, Z, W)& \\ &= \langle (\nabla_X I\xi)_Y\, IZ,\ W\rangle - \langle (\nabla_Y I\xi)_X\, IZ,\ W\rangle \\ &= \langle (\nabla_X \xi)_Y\, Z,\ W\rangle - \langle (\nabla_Y \xi)_X\, Z,\ W\rangle + 2\langle \xi_X \xi_Y Z,\ W\rangle - 2\langle \xi_Y \xi_X Z,\ W\rangle,\end{aligned}$$

with the third and fourth summands in $\bigwedge^2 T_m^*M \otimes \mathfrak{u}(n)$, and since π_2 is $\mathrm{U}(n)$-equivariant, it follows that the components of $\pi^\perp(R)$ in $\mathcal{K}^\perp$ are linear functions of the components of $\nabla\xi$. Now, taking the $\mathrm{U}(n)$-connection $\widetilde{\nabla} = \nabla + \xi$ into account, one obtains

$$\begin{aligned}\pi_1(R)(X, Y, Z, W)& \\ &= \langle (\widetilde{\nabla}_X \xi)_Y Z, W\rangle - \langle (\widetilde{\nabla}_Y \xi)_X Z, W\rangle + \langle \xi_{\xi_X Y} Z, W\rangle - \langle \xi_{\xi_Y X} Z, W\rangle.\end{aligned} \tag{2–2}$$

From this equation and by considering the image $\pi_2 \circ \pi_1(R)$, Falcitelli et al. give some tables that show whether the tensors $\widetilde{\nabla}\xi_i$ and $\xi_i \odot \xi_j$ contribute to the components of R in $\mathcal{K}^\perp$.

Here, we also consider manifolds equipped with an $\mathrm{SU}(n)$-structure. Such manifolds are called *special almost Hermitian manifolds*. These are almost Hermitian manifolds $\big(M, I, \langle\cdot,\cdot\rangle\big)$ equipped with a complex volume form $\Psi = \psi_+ + i\psi_-$ such that $\langle \Psi, \Psi\rangle_{\mathbb{C}} = 1$. Note that $I_{(i)}\psi_+ = \psi_-$. See [Martín Cabrera 2005] for details and more exhaustive information, or [Bryant 1999; Joyce 2000; Hitchin 1997].

For a special almost Hermitian $2n$-manifold M, we have the intrinsic torsion $\eta+\xi \in T^*M\otimes\mathbb{R}\omega + T^*M\otimes\mathfrak{u}(n)^\perp = T^*M\otimes\mathfrak{su}(n)^\perp$ and the minimal $\mathrm{SU}(n)$-connection $\overline{\nabla} = \nabla+\eta+\xi$. Since $\overline{\nabla}$ is metric and $\eta\in T^*M\otimes\mathbb{R}\omega$, we have $\langle Y, \eta_X Z\rangle = \hat{\eta}(X)\omega(Y,Z)$, where $\hat{\eta}$ is a 1-form. Hence,

$$\eta_X Y = \hat{\eta}(X) I Y.$$

In [Martín Cabrera 2005] it is shown that the 1-form $\hat{\eta}$ is given by

$$-I\hat{\eta} = \frac{1}{2^{n-1}n} *(*d\psi_+\wedge\psi_+ + *d\psi_-\wedge\psi_-) - \frac{1}{2n} I d^*\omega,$$

where $*$ is the Hodge star operator and d^* the coderivative. This formula simplifies for $n\geqslant 3$, since then $*d\psi_+\wedge\psi_+ = *d\psi_-\wedge\psi_-$, and one sees that $nI\hat{\eta} - \frac{1}{2}Id^*\omega$ is essentially the coefficient of Ψ in the $(n,1)$-part of $d\Psi$. The other part of the intrinsic torsion $\xi\in T^*M\otimes\mathfrak{u}(n)^\perp$ is still given by equation (2–1).

The tensors ω, ψ_+, and ψ_- are stabilised by the $\mathrm{SU}(n)$-action, and we have $\overline{\nabla}\omega = 0$, $\overline{\nabla}\psi_+ = 0$, and $\overline{\nabla}\psi_- = 0$. Moreover, one can check that $\eta\omega = 0$ and obtain $\nabla\omega = -\xi\omega \in T^*M\otimes\mathfrak{u}(n)^\perp$. In general, the above-mentioned $\mathrm{U}(n)$-spaces $\mathcal{W}_i$ are also irreducible as $\mathrm{SU}(n)$-spaces. The only exceptions are $\mathcal{W}_1$ and $\mathcal{W}_2$ when $n=3$. In fact, for that case, we have the following decompositions into irreducible $\mathrm{SU}(3)$-components:

$$\mathcal{W}_i = \mathcal{W}_i^+ + \mathcal{W}_i^-, \qquad i = 1,2,$$

where the spaces $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$ consist of those tensors $a\in\mathcal{W}_i\subseteq T^*M\otimes\bigwedge^2 T^*M$ such that the bilinear form $r(a)$, defined by $2r(a)(x,y) = \langle x\lrcorner\psi_+,\ y\lrcorner a\rangle$, is, respectively, symmetric or skew-symmetric, see [Martín Cabrera 2005; Chiossi and Salamon 2002]. The components of the tensor ξ in $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$, $i=1,2$, will be denoted by ξ_i^+ and ξ_i^-. Writing $\eta\in\mathcal{W}_5\cong T^*M$, the intrinsic $\mathrm{SU}(n)$-torsion $\xi+\eta$ is contained in $\big(T^*M\otimes\mathfrak{u}(n)^\perp\big)+\mathcal{W}_5$. The space $\mathcal{W}_5$ is always $\mathrm{SU}(n)$-irreducible.

From the equations $\overline{\nabla}\psi_+ = 0$ and $\overline{\nabla}\psi_- = 0$, we have $\nabla\psi_+ = -\xi\psi_+ - \eta\psi_+$ and $\nabla\psi_- = -\xi\psi_- - \eta\psi_-$. Moreover, for $n\geqslant 2$, it is shown in [Martín Cabrera 2005] that

$$\begin{gathered}\xi_X\psi_+,\ \xi_X\psi_- \in [\![\lambda^{n-2,0}]\!]\wedge\omega,\\ \eta_X\psi_+ = n\,\hat{\eta}(X)\psi_- \qquad\text{and}\qquad \eta_X\psi_- = -n\,\hat{\eta}(X)\psi_+.\end{gathered} \tag{2–3}$$

When considering curvature, note that the module $\mathcal{C}_3 = \mathcal{K}(\mathfrak{su}(n))$ in $\mathcal{K}$ consists of the algebraic curvature tensors for a metric with holonomy algebra $\mathfrak{su}(n)$.

3. Some curvature formulae

For special almost Hermitian $2n$-manifolds, results and tables given in [Falcitelli et al. 1994] are still valid with respect to the tensors $\widetilde{\nabla}\xi_i$ and $\xi_i\odot\xi_j$. Here, $\widetilde{\nabla} = \overline{\nabla}-\eta$ is the minimal $\mathrm{U}(n)$-connection, with $\overline{\nabla}$ denoting the minimal $\mathrm{SU}(n)$-connection.

For $\mathrm{SU}(n)$-structures, the additional information coming from η will allow us to compute the components of R in $\mathcal{K}_1$ and $\mathcal{K}_2$ in terms of the intrinsic torsion $\eta+\xi$. To achieve this, we compute the difference between the Ricci and the $*$-Ricci curvatures. In the first instance, we only need the almost Hermitian structure.

Lemma 3.1. *If M is an almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal $\mathrm{U}(n)$-connection $\widetilde{\nabla} = \nabla + \xi$, then*

$$\begin{aligned}\mathrm{Ric}^*(X,Y) - \mathrm{Ric}(X,Y)\\ &= 2\langle (\nabla_{e_i} I\xi)_X IY,\ e_i\rangle - 2\langle (\nabla_X I\xi)_{e_i} IY,\ e_i\rangle,\\ &= 2\langle (\widetilde{\nabla}_{e_i}\xi)_X Y,\ e_i\rangle - 2\langle (\widetilde{\nabla}_X\xi)_{e_i} Y,\ e_i\rangle + 2\langle \xi_{\xi_{e_i}X} Y,\ e_i\rangle - 2\langle \xi_{\xi_X e_i} Y,\ e_i\rangle.\end{aligned}$$

Proof. It is straightforward to check that

$$\mathrm{Ric}^*(X,Y) - \mathrm{Ric}(X,Y) = -(R_{X,e_i}\omega)(IY, e_i). \tag{3–1}$$

However, the so-called Ricci formula [Besse 1987, p. 26] implies that

$$-(R_{X,e_i}\omega)(IY, e_i) = \tilde{\mathbf{a}}(\nabla^2\omega)_{X,e_i}(IY, e_i), \tag{3–2}$$

where $\tilde{\mathbf{a}} : T^*M \otimes T^*M \otimes \bigwedge^2 T^*M \to \bigwedge^2 T^*M \otimes \bigwedge^2 T^*M$ is the alternation map.

The required identities follow from equations (3–1) and (3–2), by taking into account that $\widetilde{\nabla}\omega = 0$. □

The components of R in $\mathcal{K}_{-1}$ and $\mathcal{K}_{-2}$ are determined by the trace and trace-free parts of $\mathrm{Ric}^*_H - \mathrm{Ric}_H$. Similarly, the $\mathscr{C}_6$-component of R is determined by the skew-symmetric (or anti-Hermitian) part Ric^*_{AH} of Ric^*. Moreover, the anti-Hermitian part Ric_{AH} of the Ricci curvature, which satisfies $\mathrm{Ric}_{AH}(IX, IY) = -\mathrm{Ric}_{AH}(X,Y)$, determines the component of R in $\mathscr{C}_8$. These assertions motivate the expressions contained in the next lemma.

Lemma 3.2. *If M is an almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal $\mathrm{U}(n)$-connection $\widetilde{\nabla} = \nabla + \xi$, then*

$$\begin{aligned}(\mathrm{Ric}^*_H - \mathrm{Ric}_H)(X,Y) &= \langle (\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle - \langle (\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle\\ &\quad + \langle (\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle - \langle (\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle + \langle \xi_{\xi_{e_i}X} Y, e_i\rangle\\ &\quad - \langle \xi_{\xi_X e_i} Y, e_i\rangle + \langle \xi_{\xi_{e_i}IX} IY, e_i\rangle - \langle \xi_{\xi_{IX}e_i} IY, e_i\rangle,\end{aligned} \tag{3–3}$$

$$\begin{aligned}2\,\mathrm{Ric}^*_{AH}(X,Y) &= \langle (\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle - \langle (\widetilde{\nabla}_{e_i}\xi)_Y X, e_i\rangle - \langle (\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle\\ &\quad + \langle (\widetilde{\nabla}_{e_i}\xi)_{IY} IX, e_i\rangle - \langle (\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle + \langle (\widetilde{\nabla}_Y\xi)_{e_i} X, e_i\rangle\\ &\quad + \langle (\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle - \langle (\widetilde{\nabla}_{IY}\xi)_{e_i} IX, e_i\rangle + \langle \xi_{\xi_X e_i} Y, e_i\rangle\\ &\quad - \langle \xi_{\xi_Y e_i} X, e_i\rangle - \langle \xi_{\xi_{IX}e_i} IY, e_i\rangle + \langle \xi_{\xi_{IY}e_i} IX, e_i\rangle,\end{aligned} \tag{3–4}$$

$$
\begin{aligned}
(3\text{–}5)\quad 2\operatorname{Ric}_{AH}(X,Y) = &-\langle(\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle + \langle(\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle - \langle(\widetilde{\nabla}_{e_i}\xi)_Y X, e_i\rangle \\
&+ \langle(\widetilde{\nabla}_Y\xi)_{e_i} X, e_i\rangle + \langle(\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle - \langle(\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle \\
&+ \langle(\widetilde{\nabla}_{e_i}\xi)_{IY} IX, e_i\rangle - \langle(\widetilde{\nabla}_{IY}\xi)_{e_i} IX, e_i\rangle - \langle\xi_{\xi_{e_i}X} Y, e_i\rangle \\
&+ \langle\xi_{\xi_X e_i} Y, e_i\rangle - \langle\xi_{\xi_{e_i}Y} X, e_i\rangle + \langle\xi_{\xi_Y e_i} X, e_i\rangle + \langle\xi_{\xi_{e_i}IX} IY, e_i\rangle \\
&- \langle\xi_{\xi_{IX}e_i} IY, e_i\rangle + \langle\xi_{\xi_{e_i}IY} IX, e_i\rangle - \langle\xi_{\xi_{IY}e_i} IX, e_i\rangle.
\end{aligned}
$$

Proof. This follows from Lemma 3.1 together with $\langle\xi_{\xi_{e_i}X} Y, e_i\rangle = \langle\xi_{\xi_{e_i}Y} X, e_i\rangle$. □

Up to this point, we have not said anything particular to $\mathrm{SU}(n)$-structures. We now give a first result that uses the complex volume form Ψ.

Lemma 3.3. *If M is a special almost Hermitian $2n$-manifold, $n \geqslant 2$, with complex volume form $\Psi = \psi_+ + i\psi_-$ and minimal $\mathrm{SU}(n)$-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$, then*

$$(3\text{–}6)\quad \operatorname{Ric}^*(X,Y) = -n\, d\hat{\eta}(X, IY) - \langle \xi_X e_i,\ \xi_{IY} I e_i\rangle,$$

$$
\begin{aligned}
(3\text{–}7)\quad \operatorname{Ric}(X,Y) = &-n\, d\hat{\eta}(X, IY) - \langle \xi_X e_i,\ \xi_{IY} I e_i\rangle - 2\langle(\widetilde{\nabla}_{e_i}\xi)_X Y,\ e_i\rangle \\
&+ 2\langle(\widetilde{\nabla}_X\xi)_{e_i} Y,\ e_i\rangle - 2\langle\xi_{\xi_{e_i}X} Y,\ e_i\rangle + 2\langle\xi_{\xi_X e_i} Y,\ e_i\rangle.
\end{aligned}
$$

Proof. Start by noticing that $\langle R_{X,Y}\psi_+,\ \psi_-\rangle = -2^{n-2}\langle R_{X,Y} I e_i,\ e_i\rangle$. By the first Bianchi identity, we have

$$(3\text{–}8)\qquad \langle R_{X,Y}\psi_+,\ \psi_-\rangle = -2^{n-1}\operatorname{Ric}^*(X, IY).$$

On the other hand, using the Ricci formula $-R_{X,Y}\psi_+ = \tilde{\mathbf{a}}(\nabla^2\psi_+)_{(X,Y)}$ and taking $\overline{\nabla} = \nabla + \eta + \xi$ into account, we obtain

$$
\begin{aligned}
-R_{X,Y}\psi_+ = &\, n\, d\hat{\eta}(X,Y)\psi_- + n\,\hat{\eta}(X)(\xi_Y\psi_-) - n\,\hat{\eta}(Y)(\xi_X\psi_-) \\
&+ Y \lrcorner \left(\nabla_X(\xi\psi_+)\right) - X \lrcorner \left(\nabla_Y(\xi\psi_+)\right).
\end{aligned}
$$

Using the inclusions of (2–3), we have $\langle\xi_X\psi_+, \psi_-\rangle = 0$, $\langle\xi_X\psi_-, \psi_-\rangle = 0$, and $\langle Y \lrcorner \left(\nabla_X(\xi\psi_+)\right), \psi_-\rangle = -\langle\xi_X(\xi_Y\psi_+), \psi_-\rangle$. This gives

$$(3\text{–}9)\qquad \langle R_{X,Y}\psi_+,\ \psi_-\rangle = -n\,2^{n-1} d\hat{\eta}(X,Y) - 2^{n-1}\langle\xi_X e_i, \xi_Y I e_i\rangle.$$

Using equations (3–8), (3–9), and Lemma 3.1, we obtain the required identities for Ric^* and Ric. □

Theorem 3.4. *If M is a special almost Hermitian $2n$-manifold, $n \geqslant 2$, that is Kähler, then $\operatorname{Ric}^* = \operatorname{Ric}$, and*

(1) *if $d\hat{\eta} = \lambda\omega$, for some $\lambda \in \mathbb{R} \setminus \{0\}$, then the manifold is Einstein; or*

(2) *if the 1-form $\hat{\eta}$ is closed, then the manifold is Ricci-flat.*

Proof. This is an immediate consequence of the previous lemma. □

Gray proved that any nearly Kähler (type $\mathcal{W}_1$) connected 6-manifold that is not Kähler is Einstein. Here we give an alternative proof.

Theorem 3.5 [Gray 1976b]. *If M is a special almost Hermitian connected 6-manifold of type $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5$ that is not of type $\mathcal{W}_5$, then M is an Einstein manifold such that* $\mathrm{Ric} = 5\,\mathrm{Ric}^* = 5\alpha\langle\cdot\,,\cdot\rangle$, *where* $\alpha = (w_1^+)^2 + (w_1^-)^2$ *with* $\nabla\omega = w_1^+\psi_+ + w_1^-\psi_-$.

Proof. We already know that $\alpha = (w_1^+)^2 + (w_1^-)^2$ is a positive constant and the 1-form $\hat{\eta}$ is closed (see [Martín Cabrera 2005, Theorem 3.7]). Since $\nabla\omega = -\xi\omega$ and $\nabla\omega = w_1^+\psi_+ + w_1^-\psi_-$, we have

$$2\langle Y,\ \xi_X Z\rangle = w_1^-\psi_+(X, Y, Z) - w_1^+\psi_-(X, Y, Z).$$

Therefore, using

$$\begin{aligned} \langle X\lrcorner\psi_+,\ Y\lrcorner\psi_+\rangle &= \langle X\lrcorner\psi_-,\ Y\lrcorner\psi_-\rangle = 2\langle X, Y\rangle, \\ \langle X\lrcorner\psi_+,\ Y\lrcorner\psi_-\rangle &= -2\omega(X, Y), \end{aligned}$$

we get

$$\langle\xi_X e_i,\ \xi_Y e_i\rangle = \langle e_j,\ \xi_X e_i\rangle\langle e_j,\ \xi_Y e_i\rangle = \alpha\langle X, Y\rangle. \tag{3–10}$$

Moreover, since $\xi \in \mathcal{W}_1^+ + \mathcal{W}_1^-$ and $\widetilde{\nabla}$ is a $\mathrm{U}(3)$-connection, the $(0,3)$-tensors $\langle\cdot\,,\ \xi.\cdot\rangle$ and $\langle\cdot\,,\ (\widetilde{\nabla}_X\xi).\cdot\rangle$ are skew symmetric [Gray and Hervella 1980]. Thus, from (3–7), we get

$$\mathrm{Ric}(X, Y) = 5\langle\xi_X e_i,\ \xi_Y e_i\rangle = 5\alpha\langle X, Y\rangle.$$

We recall that $\langle Y,\ \xi_{IX} IZ\rangle = -\langle Y,\ \xi_X Z\rangle$ for $\xi \in \mathcal{W}_1$, and note that the contractions $\langle(\widetilde{\nabla}_X\xi)_{e_i}Y,\ e_i\rangle$ and $\langle(\widetilde{\nabla}_{e_i}\xi)_X Y,\ e_i\rangle$ both vanish. In fact, the last term is a skew-symmetric 2-form, and the remaining summands in the expression for Ric are symmetric. □

Remark 3.6. Theorem 3.5 can be extended to connected almost Hermitian 6-manifolds which are nearly Kähler but not Kähler. In fact, one can define a complex volume form on an open neighbourhood U of a point where $\nabla\omega \neq 0$, by using the $(3, 0)$-component of this tensor. Then, U is a special almost Hermitian 6-manifold of type $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5$. Therefore, $\mathrm{Ric} = 5\,\mathrm{Ric}^* = 5\alpha\langle\cdot\,,\cdot\rangle$ on U. Since the manifold is connected, it follows that $\mathrm{Ric} = 5\alpha\langle\cdot\,,\cdot\rangle$ everywhere.

The expressions (3–6) and (3–7) for Ric^* and Ric allow us to compute $3\,\mathrm{Ric}^*_H + \mathrm{Ric}_H$ and study the contributions of the intrinsic torsion of the $\mathrm{SU}(n)$-structure to the components of R in $\mathcal{K}_1$ and $\mathcal{K}_2$.

Lemma 3.7. *If M is a special almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal* $\mathrm{SU}(n)$*-connection* $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$, *then*

$$
\begin{aligned}
(3\text{–}11)\quad & (3\,\mathrm{Ric}^*_H + \mathrm{Ric}_H)(X, Y) \\
&= -2n\, d\hat{\eta}(X, IY) + 2n\, d\hat{\eta}(IX, Y) - \langle (\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle \\
&\qquad + \langle (\widetilde{\nabla}_X \xi)_{e_i} Y, e_i\rangle - \langle (\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle + \langle (\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle \\
&\qquad - \langle \xi_{\xi_{e_i}X} Y, e_i\rangle + \langle \xi_{\xi_X e_i} Y, e_i\rangle - \langle \xi_{\xi_{e_i}IX} IY, e_i\rangle \\
&\qquad + \langle \xi_{\xi_{IX}e_i} IY, e_i\rangle - 2\langle \xi_X e_i, \xi_{IY} Ie_i\rangle - 2\langle \xi_Y e_i, \xi_{IX} Ie_i\rangle.
\end{aligned}
$$

Finally, we record an alternative to equation (3–4):

Lemma 3.8. *If M is an almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal* $\mathrm{U}(n)$*-connection* $\widetilde{\nabla} = \nabla + \xi$, *then*

$$
\begin{aligned}
(3\text{–}12)\qquad \mathrm{Ric}^*_{AH}(X, Y) &= \langle (\widetilde{\nabla}_{e_i}\xi)_{Ie_i} IX, Y\rangle - \langle \xi_{I\xi_{e_i}e_i} IX, Y\rangle \\
&= -\langle (\nabla_{e_i} I\xi)_{Ie_i} X, Y\rangle.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
-2\,\mathrm{Ric}^*&(X, IY) - 2\,\mathrm{Ric}^*(IX, Y) \\
&= \langle R_{e_i, Ie_i} X, Y\rangle - \langle R_{e_i, Ie_i} IX, IY\rangle \\
&= 4\langle \widetilde{\nabla}_{e_i}\xi_{Ie_i} X, Y\rangle - 4\langle \xi_{Ie_i}\widetilde{\nabla}_{e_i} X, Y\rangle - 4\langle \xi_{\widetilde{\nabla}_{e_i} Ie_i} X, Y\rangle + 4\langle \xi_{\xi_{e_i} Ie_i} X, Y\rangle,
\end{aligned}
$$

from which the lemma follows. □

4. High dimensions

In this section, we consider special almost Hermitian manifolds of dimension higher than or equal to 8. For such manifolds, the decomposition into $\mathrm{SU}(n)$-irreducible modules of the space of curvature tensors $\mathcal{R}$ is the same as that coming from the action of $\mathrm{U}(n)$. Thus,

$$
\mathcal{R} = \mathcal{K} + \mathcal{K}^{\perp} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,
$$

where all $\mathcal{K}_i$ and $\mathcal{C}_j$ are also $\mathrm{SU}(n)$-irreducible spaces. Our aim here is to see whether different components of the intrinsic torsion of the $\mathrm{SU}(n)$-structure contribute to the components of the curvature.

We start by studying such contributions to the $\mathrm{SU}(n)$-components of the Ricci and $*$-Ricci curvatures. For $n \geqslant 3$, the spaces $\mathcal{R}ic$ and $\mathcal{R}ic^*$ of such tensors admit the following decompositions into $\mathrm{SU}(n)$-irreducible modules

$$
\mathcal{R}ic = \mathbb{R}\langle\,\cdot\,,\cdot\,\rangle + [\lambda_0^{1,1}] + [\![\sigma^{2,0}]\!], \qquad \mathcal{R}ic^* = \mathbb{R}\langle\,\cdot\,,\cdot\,\rangle + [\lambda_0^{1,1}] + [\![\lambda^{2,0}]\!].
$$

	Ric* (3–6)			Ric (3–7)		
$2n \geqslant 8$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$\llbracket\lambda^{2,0}\rrbracket$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$\llbracket\sigma^{2,0}\rrbracket$
$d\hat{\eta}$	✓	✓	✓	✓	✓	
$\overline{\nabla}\xi_1,\ \eta\xi_1$						
$\overline{\nabla}\xi_2,\ \eta\xi_2$						✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$					✓	
$\overline{\nabla}\xi_4,\ \eta\xi_4$				✓	✓	✓
$\xi_1 \otimes \xi_1$	✓	✓		✓	✓	
$\xi_2 \otimes \xi_2$	✓	✓		✓	✓	
$\xi_3 \otimes \xi_3$	✓	✓		✓	✓	✓
$\xi_4 \otimes \xi_4$	✓	✓		✓	✓	✓
$\xi_1 \odot \xi_2$		✓			✓	
$\xi_1 \odot \xi_3$			✓			✓
$\xi_1 \odot \xi_4$			✓			
$\xi_2 \odot \xi_3$			✓			✓
$\xi_2 \odot \xi_4$			✓			✓
$\xi_3 \odot \xi_4$		✓			✓	

Table 1. Ricci curvatures, $2n \geqslant 8$.

Taking into account the symmetry properties and types of the Gray–Hervella components ξ_i of ξ, we obtain:

Theorem 4.1. *Let M be a special almost Hermitian $2n$-manifold, $2n \geqslant 8$, with minimal* $\mathrm{SU}(n)$*-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$. The tensors $d\hat{\eta}$, $\overline{\nabla}\xi$, and $\xi_i \odot \xi_j$ contribute to the components of the $*$-Ricci curvature* Ric^* *via equation* (3–6) *and to the Ricci curvature* Ric *via equation* (3–7) *if and only if there is a tick in the corresponding place in Table 1.*

Using in addition that $\langle \xi_{\xi_X e_i} Y,\ e_i \rangle = -\langle \xi_X e_i,\ \xi_{e_i} Y \rangle$, we get part (1) of the next theorem. Part (2) is proved in [Falcitelli et al. 1994].

Theorem 4.2. *If M is a special almost Hermitian $2n$-manifold, $2n \geqslant 8$, with minimal* $\mathrm{SU}(n)$*-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$, then*

(1) *Using equations* (3–3), (3–4), (3–5), *and* (3–11), *each of the tensors $\overline{\nabla}\xi_i$, $\eta\xi_i$, and $\xi_i \odot \xi_j$ contributes to the components of R in $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_6$, and $\mathcal{C}_8$ if and only if there is a tick in the corresponding place in Table 2.*

	(3–11)		(3–3)		(3–4)	(3–12)	(3–5)	[Falcitelli et al.]		
$2n \geqslant 8$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_{-1}$	$\mathcal{K}_{-2}$	$\mathcal{C}_6$	$\mathcal{C}_6$	$\mathcal{C}_8$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_7$
$d\hat{\eta}$	✓	✓								
$\overline{\nabla}\xi_1,\ \eta\xi_1$					✓	✓				
$\overline{\nabla}\xi_2,\ \eta\xi_2$					✓	✓	✓		✓	✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$		✓		✓		✓		✓		✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$	✓	✓	✓	✓	✓	✓	✓			
$\xi_1 \otimes \xi_1$	✓	✓	✓	✓				✓*		
$\xi_2 \otimes \xi_2$	✓	✓	✓	✓				✓		
$\xi_3 \otimes \xi_3$	✓	✓					✓			✓
$\xi_4 \otimes \xi_4$	✓	✓					✓			
$\xi_1 \odot \xi_2$		✓		✓				✓		
$\xi_1 \odot \xi_3$					✓		✓		✓	✓
$\xi_1 \odot \xi_4$					✓	✓				
$\xi_2 \odot \xi_3$					✓		✓		✓	✓
$\xi_2 \odot \xi_4$					✓	✓	✓		✓	✓
$\xi_3 \odot \xi_4$		✓				✓				✓

*absent when $2n = 8$

Table 2. Curvature complementary to $\mathcal{C}_3 = \mathcal{K}(\mathfrak{su}(n))$, $2n \geqslant 8$.

(2) *Taking the image* $\pi_2 \circ \pi_1(R)$ *into account, where* $\pi_1(R)$ *is given by* (2–2), *each of the tensors* $\overline{\nabla}\xi_i$, $\eta\xi_i$, *and* $\xi_i \odot \xi_j$ *contributes to the components of* R *in* $\mathcal{C}_4$, $\mathcal{C}_5$, *and* $\mathcal{C}_7$ *if and only if there is a tick in the corresponding place in Table 2.*

For part (1), we emphasize that the columns for $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_6$, and $\mathcal{C}_8$ are obtained by a different method than that in [Falcitelli et al. 1994], and that for $\mathcal{C}_6$ this even leads to a different result. In particular, we claim that the tensors $\widetilde{\nabla}\xi_3$ and $\xi_3 \odot \xi_4$ do not contribute to the $\mathcal{C}_6$-component of R, but that $\widetilde{\nabla}\xi_1$ and $\eta\xi_1$ do. Thus, the contributions of the different tensors to the distinct components of R depend on the choice of the current expression that we use; different expressions may lead to different behaviour in the contributions. For the $\mathcal{C}_6$-component of R, we get a third formula from equation (3–12), which we also list in Table 2. A partial explanation for these different results will be given in Section 6. Note that the entries for $\mathcal{C}_6$ in Table 2 only involve the intrinsic $\mathrm{U}(n)$-torsion. The $[\![\lambda^{2,0}]\!]$-column of Table 1 provides yet another description of the $\mathcal{C}_6$-component using the $\mathrm{SU}(n)$-structure.

5. Low dimensions

We consider in turn special almost Hermitian manifolds of dimension 6 and 4.

Six dimensions. The decomposition of the space of curvature tensors $\mathcal{R}$ into irreducible SU(3)-modules has the same subspaces as for U(3). Thus,

$$\mathcal{R} = \mathcal{K} + \mathcal{K}^{\perp} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,$$

with $\mathcal{K}_i$ and $\mathcal{C}_j$ all SU(3)-irreducible. As we noted above, the summand $\mathcal{C}_4$ is absent in this dimension. On the other hand, the U(3)-intrinsic torsion splits under SU(3) as $\xi = \xi_1^+ + \xi_1^- + \xi_2^+ + \xi_2^- + \xi_3 + \xi_4$, where $\xi_i = \xi_i^+ + \xi_i^-$, $i = 1, 2$. This was briefly described in Section 2, and more detailed information is contained in [Chiossi and Salamon 2002] and [Martín Cabrera 2005].

The next result concerns the contributions of the components of ξ to the components of the Ricci and the $*$-Ricci curvatures, and then to the curvature components complementary to $\mathcal{C}_3$.

Theorem 5.1. *Let M be a special almost Hermitian 6-manifold with* SU(3)-*connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$. The tensors $d\hat{\eta}$, $\overline{\nabla}\zeta$, $\eta\zeta$, and $\zeta \odot \vartheta$, for $\zeta, \vartheta = \xi_1^+, \xi_1^-, \xi_2^+, \xi_2^-, \xi_3, \xi_4$, contribute to the components of* Ric^* *and* Ric *if and only if there is a tick in the corresponding place in Table 3.*

The corresponding contributions to the curvature components $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_6$, and $\mathcal{C}_8$, via equations (3–3), (3–4), (3–5), *and* (3–11), *and to the components $\mathcal{C}_5$ and $\mathcal{C}_7$ via $\pi_2 \circ \pi_1(R)$ are given in Table 4* (cf. [Falcitelli et al. 1994]).

Four dimensions. The U(2)-decomposition of the space of curvature tensors $\mathcal{R}$ is given by

$$\mathcal{R} = \mathcal{K} + \mathcal{K}^{\perp} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_{-1} + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_8.$$

When we consider the SU(2)-action, only the modules $\mathcal{C}_3$, $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_{-1}$ remain irreducible. To describe the decompositions of $\mathcal{C}_5$ and $\mathcal{C}_6$ into SU(2)-irreducible modules, we will make use of tensors defined by

$$\chi(a, b) = 6\, a \odot b - a \wedge b,$$

for all $a, b \in \bigwedge^2 T^*M$, where $\odot$ denotes the symmetric product given by $2\, a \odot b = a \otimes b + b \otimes a$. The relevant decompositions are now given by

(1) $\mathcal{C}_5 = \mathcal{C}_5^{++} + \mathcal{C}_5^{--} + \mathcal{C}_5^{+-}$, where $\mathcal{C}_5^{++} = \mathbb{R}\chi(\psi_+, \psi_+)$, $\mathcal{C}_5^{--} = \mathbb{R}\chi(\psi_-, \psi_-)$, and $\mathcal{C}_5^{+-} = \mathbb{R}\chi(\psi_+, \psi_-)$,

(2) $\mathcal{C}_6 = \mathcal{C}_6^{+} + \mathcal{C}_6^{-}$, where $\mathcal{C}_6^{+} = \mathbb{R}\chi(\psi_+, \omega)$ and $\mathcal{C}_6^{-} = \mathbb{R}\chi(\psi_-, \omega)$.

For the intrinsic torsion, the U(2)-decomposition of ξ is given by

$$\xi = \xi_2 + \xi_4 \in \mathcal{W} = \mathcal{W}_2 + \mathcal{W}_4.$$

	Ric* (3–6)			Ric (3–7)		
$2n=6$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\lambda^{2,0}]\!]$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\sigma^{2,0}]\!]$
$d\hat{\eta}$	✓	✓	✓	✓	✓	
$\overline{\nabla}\xi_1^{\pm},\ \eta\xi_1^{\pm}$						
$\overline{\nabla}\xi_2^{\pm},\ \eta\xi_2^{\pm}$						✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$					✓	
$\overline{\nabla}\xi_4,\ \eta\xi_4$				✓	✓	✓
$\xi_1^{\pm}\otimes\xi_1^{\pm}$	✓			✓		
$\xi_2^{\pm}\otimes\xi_2^{\pm}$	✓	✓		✓	✓	
$\xi_3\otimes\xi_3$	✓	✓		✓	✓	✓
$\xi_4\otimes\xi_4$	✓	✓		✓	✓	✓
$\xi_1^{+}\odot\xi_1^{-}$						
$\xi_1^{\pm}\odot\xi_2^{\pm}$		✓			✓	
$\xi_1^{\pm}\odot\xi_2^{\mp}$						
$\xi_1^{\pm}\odot\xi_3$						✓
$\xi_1^{\pm}\odot\xi_4$			✓			
$\xi_2^{+}\odot\xi_2^{-}$		✓			✓	
$\xi_2^{\pm}\odot\xi_3$			✓			✓
$\xi_2^{\pm}\odot\xi_4$			✓			✓
$\xi_3\odot\xi_4$		✓			✓	

Table 3. Ricci curvatures, $2n=6$.

Under SU(2), we have $\mathcal{W}_2 \cong \mathcal{W}_4 \cong T^*M$, which, as we will see, gives rise to different choices of decompositions of ξ.

For an SU(2)-structure, we have $\nabla\omega \in \mathcal{W} = T^*M \otimes \psi_+ + T^*M \otimes \psi_-$. Consequently, $\nabla\omega = \xi_+ \otimes \psi_+ + \xi_- \otimes \psi_-$, where ξ_+ and ξ_- are 1-forms. Moreover,

$$2\langle Y, \xi_X Z\rangle = -\xi_+(X)\psi_-(Y, Z) + \xi_-(X)\psi_+(Y, Z),$$

so $\xi = \xi_+ + \xi_-$, where

$$2\langle Y, (\xi_+)_X Z\rangle = -\xi_+(X)\psi_-(Y, Z), \qquad 2\langle Y, (\xi_-)_X Z\rangle = \xi_-(X)\psi_+(Y, Z).$$

The two decompositions of ξ are related as follows:

$$\xi \in \mathcal{W}_2 \text{ if and only if } \xi_+ = I\xi_-; \qquad \xi \in \mathcal{W}_4 \text{ if and only if } \xi_+ = -I\xi_-.$$

	(3–11)		(3–3)		(3–4)	(3–5)	[Falcitelli et al.]	
$2n = 6$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_{-1}$	$\mathcal{K}_{-2}$	$\mathcal{C}_6$	$\mathcal{C}_8$	$\mathcal{C}_5$	$\mathcal{C}_7$
$d\hat{\eta}$	✓	✓						
$\overline{\nabla}\xi_1^{\pm},\ \eta\xi_1^{\pm}$					✓			
$\overline{\nabla}\xi_2^{\pm},\ \eta\xi_2^{\pm}$					✓	✓	✓	✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$		✓		✓				✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$	✓	✓	✓	✓	✓	✓		
$\xi_1^{\pm} \otimes \xi_1^{\pm}$	✓		✓					
$\xi_2^{\pm} \otimes \xi_2^{\pm}$	✓	✓	✓	✓				
$\xi_3 \otimes \xi_3$	✓	✓				✓		✓
$\xi_4 \otimes \xi_4$	✓	✓				✓		
$\xi_1^{+} \odot \xi_1^{-}$								
$\xi_1^{\pm} \odot \xi_2^{\pm}$		✓		✓				
$\xi_1^{\pm} \odot \xi_2^{\mp}$								
$\xi_1^{\pm} \odot \xi_3$						✓	✓	
$\xi_1^{\pm} \odot \xi_4$					✓			
$\xi_2^{+} \odot \xi_2^{-}$		✓		✓				
$\xi_2^{\pm} \odot \xi_3$					✓	✓	✓	✓
$\xi_2^{\pm} \odot \xi_4$					✓	✓	✓	✓
$\xi_3 \odot \xi_4$		✓						✓

Table 4. Curvature complementary to $\mathcal{C}_3 = \mathcal{K}(\mathfrak{su}(n))$, $2n = 6$.

The next theorem deals with the contributions of the components of the intrinsic torsion to the tensors Ric* and Ric. First, in dimension four, $\mathcal{R}ic^*$ decomposes under SU(2) as

$$\mathcal{R}ic^* = \mathbb{R}\langle\,\cdot\,,\cdot\,\rangle + [\lambda_0^{1,1}] + \mathbb{R}\psi_+ + \mathbb{R}\psi_-.$$

Theorem 5.2. *Let M be a special almost Hermitian 4-manifold with minimal* SU(2)*-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$. The curvature contributions corresponding to Theorems 4.1 and 4.2 via the decompositions $\xi = \xi_2 + \xi_4$ and $\xi = \xi_+ + \xi_-$ are given in Tables 5 and 6.*

Proof. The absence of $\mathcal{K}_{-2}$ in the decomposition of $\mathcal{R}$ comes from the fact that

$$\left(\mathrm{Ric}^*_H - \mathrm{Ric}_H\right)(X, Y) = \beta\,\langle X, Y\rangle, \tag{5–1}$$

	Ric* (3–6)				Ric (3–7)		
$2n=4$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$\mathbb{R}\psi_+$	$\mathbb{R}\psi_-$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\sigma^{2,0}]\!]$
$d\hat{\eta}$	✓	✓	✓	✓	✓	✓	
$\overline{\nabla}\xi_2,\ \eta\xi_2$							✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$					✓		✓
$\xi_2\otimes\xi_2$	✓	✓			✓	✓	
$\xi_4\otimes\xi_4$	✓	✓			✓	✓	✓
$\xi_2\odot\xi_4$						✓	✓
$\overline{\nabla}\xi_+,\ \eta\xi_+$					✓		✓
$\overline{\nabla}\xi_-,\ \eta\xi_-$					✓		✓
$\xi_+\otimes\xi_+$					✓		✓
$\xi_-\otimes\xi_-$					✓		✓
$\xi_+\odot\xi_-$	✓	✓			✓	✓	✓

Table 5. Ricci curvatures, $2n=4$.

where $\beta = \langle(\widetilde{\nabla}_{e_i}\xi)_{e_j}e_j,\ e_i\rangle + \langle\xi_{\xi_{e_i}e_j}e_j,\ e_i\rangle$. Therefore, by (3–4), we have

$$\text{(5–2)}\quad \left(3\,\mathrm{Ric}^*_H + \mathrm{Ric}_H\right)(X,Y) = -\beta\langle X,Y\rangle - 4d\hat{\eta}(X,IY) + 4d\hat{\eta}(IX,Y) \\ -2\langle\xi_X e_i,\ \xi_{IY}Ie_i\rangle - 2\langle\xi_Y e_i,\ \xi_{IX}Ie_i\rangle.$$

Using equations (5–1) and (5–2), the tables follow. □

Remark 5.3. We list some direct consequences of the results and tables presented here and in Section 4:

(1) if $\xi\in\mathcal{W}_3$, the components of R in $\mathcal{K}_{-1}$, $\mathcal{C}_5$, and $\mathcal{C}_6$ vanish;

(2) if $\xi\in\mathcal{W}_3+\mathcal{W}_4$ and $d\hat{\eta}$ is Hermitian, the components of R in $\mathcal{C}_5$ and $\mathcal{C}_6$ vanish;

(3) if $\xi\in\mathcal{W}_1+\mathcal{W}_2$ and $d\hat{\eta}$ is Hermitian, the component of R in $\mathcal{C}_6$ vanishes;

(4) if $n=2$ and $d\hat{\eta}$ is Hermitian, then the component of R in $\mathcal{C}_6$ vanishes.

There are more consequences of this sort, but they have already been pointed out in [Falcitelli et al. 1994].

Remark 5.4. For special almost Hermitian 2-manifolds, we have the following identity, obtained in [Martín Cabrera 2005]:

$$K(\psi_+,\psi_-) = d\hat{\eta}(\psi_+,\psi_-) = d\eta_+(\psi_+) + d\eta_-(\psi_-) - \eta_+^2 - \eta_-^2,$$

where K denotes the sectional curvature and $\hat{\eta} = \eta_+\psi_- - \eta_-\psi_+$.

$2n=4$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_{-1}$	$\mathcal{C}_6^+$	$\mathcal{C}_6^-$	$\mathcal{C}_8$	$\mathcal{C}_5^{++}$	$\mathcal{C}_5^{--}$	$\mathcal{C}_5^{+-}$
$d\hat{\eta}$	✓	✓							
$\overline{\nabla}\xi_2,\ \eta\xi_2$				✓	✓	✓	✓	✓	✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$	✓		✓	✓	✓	✓			
$\xi_2\otimes\xi_2$	✓	✓	✓						
$\xi_4\otimes\xi_4$	✓	✓				✓			
$\xi_2\odot\xi_4$				✓	✓	✓	✓	✓	✓
$\overline{\nabla}\xi_+,\ \eta\xi_+$	✓		✓		✓	✓	✓	✓	✓
$\overline{\nabla}\xi_-,\ \eta\xi_-$	✓		✓	✓		✓	✓	✓	✓
$\xi_+\otimes\xi_+$	✓		✓			✓		✓	✓
$\xi_-\otimes\xi_-$	✓		✓			✓	✓		✓
$\xi_+\odot\xi_-$	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 6. Curvature complementary to $\mathcal{C}_3=\mathcal{K}(\mathfrak{su}(n))$, $2n=4$.

6. Identities from exterior algebra

As remarked in Section 4, one may see different contributions to the module $\mathcal{C}_6\cong[\![\lambda^{2,0}]\!]$ by using different computations of the curvature. This is because of nontrivial identities that relate the components of $\widetilde{\nabla}\xi_i$ and $\xi_j\odot\xi_k$. Such an identity for the $[\![\lambda^{2,0}]\!]$-components may be obtained by comparing equations (3–4) and (3–12). However, we claim that this information may also be obtained from the exterior algebra of a $\mathrm{U}(n)$-manifold.

Consider the Kähler 2-form ω. Being a differential form, it satisfies $d^2\omega=0$. However, since the Levi-Civita connection ∇ is torsion-free, we may compute $d^2\omega$ using ∇. Writing $\nabla=\widetilde{\nabla}-\xi$ and using that $\widetilde{\nabla}\omega=0$, we first have

$$\tfrac{1}{2}d\omega(Y,Z,W)=\langle\xi_Y Z, IW\rangle+\langle\xi_W Y, IZ\rangle+\langle\xi_Z W, IY\rangle.$$

Now, $d^2\omega=\mathbf{a}(\widetilde{\nabla}d\omega)-\mathbf{a}(\xi\,d\omega)$, where $\mathbf{a}: T^*M\otimes\bigwedge^3 T^*M\to\bigwedge^4 T^*M$ is the alternation map. One computes that these two terms are the expressions obtained by summing, respectively, $\varepsilon\langle(\widetilde{\nabla}_X\xi)_Y Z, IW\rangle$ and $\varepsilon\langle\xi_{\xi_X Y}Z, IW\rangle$ over all permutations of (X,Y,Z,W), where ε is the sign of the permutation.

We have

$$\textstyle\bigwedge^4 T^*M=[\![\lambda^{4,0}]\!]+[\![\lambda^{3,1}]\!]+[\![\lambda^{2,0}]\!]\omega+[\lambda_0^{2,2}]+[\lambda_0^{1,1}]\omega+\mathbb{R}\omega^2,$$

so, in order to compute the $[\![\lambda^{2,0}]\!]$-component of $d^2\omega$, we contract with ω on the first two arguments and then take the projection to $[\![\lambda^{2,0}]\!]$, which is the (-1)-eigenspace of I acting on 2-forms. Using the symmetries of the components of ξ, one obtains that the $[\![\lambda^{2,0}]\!]$-component of $d^2\omega$ is

$$
\begin{aligned}
\text{(6–1)}\quad 0 = {} & 3\langle(\widetilde{\nabla}_{e_i}\xi_1)_{e_i}X,\, Y\rangle - \langle(\widetilde{\nabla}_{e_i}\xi_3)_{e_i}X,\, Y\rangle + (n-2)\langle(\widetilde{\nabla}_{e_i}\xi_4)_{e_i}X,\, Y\rangle \\
& + \langle(\xi_3)_X e_i,\, (\xi_1)_{e_i}Y\rangle - \langle(\xi_3)_Y e_i,\, (\xi_1)_{e_i}X\rangle \\
& + \langle(\xi_3)_X e_i,\, (\xi_2)_{e_i}Y\rangle - \langle(\xi_3)_Y e_i,\, (\xi_2)_{e_i}X\rangle \\
& - \tfrac{n-5}{n-1}\langle(\xi_1)_{\xi_{4e_i}e_i}X,\, Y\rangle - \tfrac{n-2}{n-1}\langle(\xi_2)_{\xi_{4e_i}e_i}X,\, Y\rangle + \langle(\xi_3)_{\xi_{4e_i}e_i}X,\, Y\rangle.
\end{aligned}
$$

We conclude that, in general dimensions, there is a nontrivial linear relation between the $[\![\lambda^{2,0}]\!]$-components of $\widetilde{\nabla}\xi_1$, $\widetilde{\nabla}\xi_3$, $\widetilde{\nabla}\xi_4$, $\xi_1\odot\xi_3$, $\xi_1\odot\xi_4$, $\xi_2\odot\xi_3$, $\xi_2\odot\xi_4$, and $\xi_3\odot\xi_4$. By 'nontrivial' we mean that no coefficient is zero, so this relation may be used to write any of the terms as a linear combination of the others. Interestingly, when $2n = 10$ this relation does not involve $\xi_1\odot\xi_4$.

This is sufficient to explain the difference between the ticks in the $\mathscr{C}_6$ column in [Falcitelli et al. 1994] and those we obtained from equation (3–4). An extra coincidence in the coefficients explains the differences between our results from (3–4) and (3–12).

One may try to apply the above approach to the other modules that $\bigwedge^4 T^*M$ has in common with the space of curvature tensors, namely $[\lambda_0^{2,2}]$, $[\lambda_0^{1,1}]\omega$, and $\mathbb{R}\omega^2$. However, this is not so rewarding, because of the higher multiplicities that these modules have in the relevant decompositions. Indeed, $\mathscr{C}_6 \cong [\![\lambda^{2,0}]\!]$ is distinguished by occurring only with multiplicity one or zero in the modules for $\widetilde{\nabla}\xi_i$ and $\xi_i\otimes\xi_j$.

In [Falcitelli et al. 1994] it is pointed out that, if $\xi\in\mathscr{W}_4$, the components of R in $\mathscr{C}_4$, $\mathscr{C}_5$, $\mathscr{C}_6$, and $\mathscr{C}_7$ vanish. We indicate how equation (6–1) gives an alternative proof of this result, for $n > 2$. By using Tables 2 and 4, the vanishing of the components in $\mathscr{C}_4$, $\mathscr{C}_5$, and $\mathscr{C}_7$ is in fact immediate. On the other hand, equations (3–4) and (6–1) give the vanishing of the component in $\mathscr{C}_6$.

Finally, a comparison of Tables 1 and 2 reveals another relation on special almost Hermitian manifolds: the $[\![\lambda^{2,0}]\!]$-part of $d\hat{\eta}$ carries all the information from the corresponding components of $\widetilde{\nabla}\xi_i$ modulo the $[\![\lambda^{2,0}]\!]$-parts of $\xi_1\odot\xi_3$, $\xi_1\odot\xi_4$, $\xi_2\odot\xi_3$, and $\xi_2\odot\xi_4$. This relation is obtainable by considering the $(n,2)$-part of the equation $d^2\Psi = 0$, where Ψ is the complex volume, see [Martín Cabrera 2005].

Acknowledgments

This work is supported by a grant from the MEC (Spain), project MTM2004-2644. Andrew Swann thanks the Department of Fundamental Mathematics at the University of La Laguna for kind hospitality during the initial stages of this work.

Francisco Martín Cabrera thanks the Department of Mathematics and Computer Science at the University of Southern Denmark for kind hospitality while working on this project.

References

[Banos 2002] B. Banos, "Nondegenerate Monge–Ampère structures in dimension 6", *Lett. Math. Phys.* **62**:1 (2002), 1–15. MR 2004a:32037 Zbl 1033.32014

[Besse 1987] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik **10**, Springer, Berlin, 1987. MR 88f:53087 Zbl 0613.53001

[Bryant 1999] R. L. Bryant, "Some examples of special Lagrangian tori", *Adv. Theor. Math. Phys.* **3**:1 (1999), 83–90. MR 2000f:32033 Zbl 0980.32006

[Chiossi and Salamon 2002] S. Chiossi and S. Salamon, "The intrinsic torsion of SU(3) and G_2 structures", pp. 115–133 in *Differential geometry* (Valencia, 2001), edited by O. Gil-Medrano and V. Miquel, World Sci. Publishing, River Edge, NJ, 2002. MR 2003g:53030 Zbl 1024.53018

[Cleyton and Swann 2004] R. Cleyton and A. Swann, "Einstein metrics via intrinsic or parallel torsion", *Math. Z.* **247**:3 (2004), 513–528. MR 2005i:53054 Zbl 1069.53041

[Falcitelli et al. 1994] M. Falcitelli, A. Farinola, and S. Salamon, "Almost-Hermitian geometry", *Differential Geom. Appl.* **4**:3 (1994), 259–282. MR 95i:53047 Zbl 0813.53044

[Grantcharov et al. 2003] D. Grantcharov, G. Grantcharov, and Y. Poon, "Calabi–Yau connections with torsion on toric bundles", preprint, 2003. math.DG/0306207

[Gray 1976a] A. Gray, "Curvature identities for Hermitian and almost Hermitian manifolds", *Tôhoku Math. J.* (2) **28**:4 (1976), 601–612. MR 55 #9005 Zbl 0351.53040

[Gray 1976b] A. Gray, "The structure of nearly Kähler manifolds", *Math. Ann.* **223**:3 (1976), 233–248. MR 54 #6010 Zbl 0345.53019

[Gray and Hervella 1980] A. Gray and L. M. Hervella, "The sixteen classes of almost Hermitian manifolds and their linear invariants", *Ann. Mat. Pura Appl.* (4) **123** (1980), 35–58. MR 81m:53045 Zbl 0444.53032

[Gutowski et al. 2003] J. Gutowski, S. Ivanov, and G. Papadopoulos, "Deformations of generalized calibrations and compact non-Kähler manifolds with vanishing first Chern class", *Asian J. Math.* **7**:1 (2003), 39–79. MR 2004j:53069 Zbl 1089.53034

[Hitchin 1997] N. J. Hitchin, "The moduli space of special Lagrangian submanifolds", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **25**:3-4 (1997), 503–515. MR 2000c:32075 Zbl 1015.32022

[Joyce 2000] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford, 2000. MR 2001k:53093 Zbl 1027.53052

[Martín Cabrera 2005] F. Martín Cabrera, "Special almost Hermitian geometry", *J. Geom. Phys.* **55**:4 (2005), 450–470. MR 2006g:53063 Zbl 02213141

[Papadopoulos 1999] G. Papadopoulos, "Brane solitons and hypercomplex structures", pp. 299–312 in *Quaternionic structures in mathematics and physics* (Rome, 1999), edited by S. Marchiafava et al., Univ. Studi Roma "La Sapienza", Rome, 1999. MR 2002h:53081 Zbl 1060.53055

[Salamon 1989] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Math. Series **201**, Longman Scientific & Technical, Harlow, 1989. MR 90g:53058 Zbl 0685.53001

[Tricerri and Vanhecke 1981] F. Tricerri and L. Vanhecke, "Curvature tensors on almost Hermitian manifolds", *Trans. Amer. Math. Soc.* **267**:2 (1981), 365–397. MR 82j:53071 Zbl 0484.53014

Received January 10, 2005.

FRANCISCO MARTÍN CABRERA
DEPARTMENT OF FUNDAMENTAL MATHEMATICS
UNIVERSITY OF LA LAGUNA
38200 LA LAGUNA
TENERIFE
SPAIN
fmartin@ull.es

ANDREW SWANN
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF SOUTHERN DENMARK
CAMPUSVEJ 55
DK-5230 ODENSE M
DENMARK
swann@imada.sdu.dk

PACIFIC JOURNAL OF MATHEMATICS
Vol. 228, No. 1, 2006

A GENERALIZATION OF GAUCHMAN'S RIGIDITY THEOREM

HONG-WEI XU, WANG FANG AND FEI XIANG

Dedicated to Professor Chaohao Gu on the occasion of his 80th birthday.

We generalize the well-known Gauchman theorem for closed minimal submanifolds in a unit sphere, and prove that if M is an n-dimensional closed submanifold of parallel mean curvature in S^{n+p} and if $\sigma(u) \le \frac{1}{3}$ for any unit vector $u \in TM$, where $\sigma(u) = \|h(u,u)\|^2$, and h is the second fundamental form of M, then either $\sigma(u) \equiv H^2$ and M is a totally umbilical sphere, or $\sigma(u) \equiv \frac{1}{3}$. Moreover, we give a geometrical classification of closed submanifolds with parallel mean curvature satisfying $\sigma(u) \equiv \frac{1}{3}$.

1. Introduction and statement of results

Let $S^m(r)$ be the m-dimensional sphere of radius r, with $S^m = S^m(1)$. By M we will always denote an n-dimensional connected and closed Riemannian manifold isometrically immersed in some S^{n+p}. We will be interested in the case when M has *parallel mean curvature*, meaning that the mean curvature vector ξ on M forms a parallel vector field in the normal bundle over M. (When ξ vanishes identically, M is a minimal submanifold; M is a hypersurface of constant mean curvature if $p = 1$ and the norm of ξ is constant.)

Our investigation contributes to the theory of geometrical invariants and structures of Riemannian manifolds and submanifolds, an important problem in global differential geometry. After the pioneering rigidity theorem for closed minimal submanifolds in a sphere due to Simons [1968], Lawson [1969], and Chern, do Carmo and Kobayashi [Chern et al. 1970], A. M. Li and J. M. Li [1992] improved Simons' pinching constant to $\max\{n/(2-1/p), 2n/3\}$.

Extending this rigidity result to submanifolds of parallel mean curvature in a sphere, we have the theorem below, first proved by Okumura [1965] and Yau [1974; 1975], then by Xu [1991], and finally by Alenca and do Carmo [1994] in codimension 1 and independently by Xu [1993; 1995] in codimension p.

MSC2000: 53C40, 53C42.
Keywords: closed submanifolds, rigidity theorem, parallel mean curvature.
Research supported by the Chinese NSF, Grant No. 10231010; the Trans-Century Training Programme Foundation for Talents by the Ministry of Education of China; and the Natural Science Foundation of Zhejiang Province, Grant No. 101037.

Theorem 1.1. *For given $H \geq 0$ and positive integers $n \geq 2$ and p, set*

$$C(n, p, H) = \begin{cases} \alpha(n, H) & \text{if } p = 1 \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\big(\alpha(n, H), \frac{1}{3}(2n + 5nH^2)\big) & \text{if } p \geq 3 \text{ or } p = 2 \text{ and } H = 0, \end{cases}$$

where

$$\alpha(n, H) = n + \frac{n^3 H^2}{2(n-1)} - \frac{n(n-2)H}{2(n-1)} \sqrt{n^2 H^2 + 4(n-1)}.$$

If M^n is a closed submanifold in the standard unit sphere S^{n+p} of parallel mean curvature vector of norm H, and if the squared norm S of the second fundamental form satisfies

$$S \leq C(n, p, H),$$

then M is congruent to one of the following:

(1) $S^n_H := S^n\Big(\frac{1}{\sqrt{1+H^2}}\Big)$;

(2) *the isoparametric hypersurface* $S^{n-1}\Big(\frac{1}{\sqrt{1+\lambda^2}}\Big) \times S^1\Big(\frac{\lambda}{\sqrt{1+\lambda^2}}\Big)$ *in* $S^{n+1}(1)$, *where*

$$\lambda = \frac{nH + \sqrt{n^2 H^2 + 4(n-1)}}{2(n-1)};$$

(3) *one of the Clifford minimal hypersurfaces* $S^k\Big(\sqrt{\frac{k}{n}}\,\Big) \times S^{n-k}\Big(\sqrt{\frac{n-k}{n}}\,\Big)$ *in* S^{n+1}, *for* $k = 1, \ldots, n-1$;

(4) *the Clifford torus* $S^1(r_1) \times S^1(r_2)$ *in* $S^3(r)$ *with constant mean curvature* H_0, *where* $0 \leq H_0 \leq H$,

$$r_1, r_2 = \frac{1}{\sqrt{2(1+H^2) \pm 2H_0(1+H^2)^{1/2}}} \quad \text{and} \quad r = \frac{1}{\sqrt{1+H^2-H_0^2}};$$

(5) *the Veronese surface in* $S^4_H = S^4\Big(\frac{1}{\sqrt{1+H^2}}\Big)$.

Taking $H = 0$, we have:

Corollary 1.2 [Chern et al. 1970; An-Min and Jimin 1992]. *If M^n is a closed minimal submanifold in the standard unit sphere S^{n+p}, and if*

$$S \leq \max\Big(\frac{n}{2-1/p}, \frac{2}{3}n\Big),$$

then M is congruent to one of the following:

(1) S^n;

(2) *one of the Clifford minimal hypersurfaces* $S^k\Big(\sqrt{\frac{k}{n}}\,\Big) \times S^{n-k}\Big(\sqrt{\frac{n-k}{n}}\,\Big)$ *in* S^{n+1}, *for* $k = 1, \ldots, n-1$;

(3) *the Veronese surface in* S^4.

Since $\min_{H\geq 0} \alpha(n, H) = 2\sqrt{n-1}$, we get from Theorem 1.1:

Corollary 1.3. *Let M^n be a closed submanifold with parallel mean curvature in S^{n+p}. Suppose that $H \neq 0$ and that*

$$S \leq \begin{cases} 2\sqrt{n-1} & \text{if } p \leq 2 \text{ or } p \geq 3 \text{ and } n \geq 8, \\ \frac{2}{3}n & \text{if } p \geq 3 \text{ and } n \leq 7. \end{cases}$$

Then M is either a totally umbilical sphere in S^{n+p}, a Clifford isoparametric hypersurface in an $(n+1)$-dimensional sphere, or the Veronese surface in S^4_H.

Gauchman [1986] proved that if M is an n-dimensional closed minimal submanifold in S^{n+p} and if $\sigma(u) \leq \frac{1}{3}$ for any unit vector $u \in TM$, where $\sigma(u) = \|h(u, u)\|^2$ for h the second fundamental form of M, then either $\sigma(u) \equiv 0$ and M is a totally geodesic sphere, or $\sigma(u) \equiv \frac{1}{3}$. Moreover, he gave a geometrical classification of closed minimal submanifolds satisfying $\sigma(u) \equiv \frac{1}{3}$.

A natural question is how to generalize this striking rigidity result to the case where M is an n-dimensional closed submanifold of parallel mean curvature in S^{n+p}. In this paper we provide such a generalization. To state our main result precisely, we start with some explicit examples of submanifolds with parallel mean curvature in a sphere, which extend Gauchman's examples for the minimal cases [Gauchman 1986; Sakamoto 1977].

Example 1.4. Let $S^q(r)$ be a q-dimensional sphere of radius r in $\mathbb{R}^{q+1}$, and let $1 \leq k \leq n-1$. We embed $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$ in $S^{n+1}(1)$ as follows. Let $u \in S^k(1/\sqrt{2})$ and $v \in S^{n-k}(1/\sqrt{2})$ be vectors of length $1/\sqrt{2}$ in $\mathbb{R}^{k+1}$ and $\mathbb{R}^{n-k+1}$, respectively. We can consider (u, v) as a unit vector in $\mathbb{R}^{n+2} = \mathbb{R}^{k+1} \times \mathbb{R}^{n-k+1}$. It is easy to see that $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$ is a submanifold in $S^{n+1}(1)$ of parallel mean curvature

$$H = \left|\frac{2k-n}{n}\right|.$$

In particular, M is minimal if $n = 2k$. The exact same construction yields an embedding of $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$ in $S^{n+2}(1)$.

Example 1.5. Denote by RP^2, CP^2, QP^2, and $CayP^2$ the projective plane over the real numbers, complex numbers, quaternions and octonions, and by $\psi_1 : RP^2 \to S^4(1)$, $\psi_2 : CP^2 \to S^7(1)$, $\psi_3 : QP^2 \to S^{13}(1)$ and $\psi_4 : CayP^2 \to S^{25}(1)$ the corresponding isometric embeddings. Let $\psi_1' : S^2(\sqrt{3}) \to S^4(1)$ be the isometric immersion defined by $\psi_1' = \psi_1 \circ \pi$, where $\pi : S^2(\sqrt{3}) \to RP^2$ is the canonical projection.

For $n \geq 2$, $m \geq 0$, let $S^n(1)$ be the great sphere in $S^{n+m}(1)$ given by

$$S^n(1) = \big\{(x_1, \ldots, x_{n+m+1}) \in S^{n+m}(1) \mid x_{n+2} = \cdots = x_{n+m+1} = 0\big\},$$

and $\tau_{n,m} : S^n(1) \to S^{n+m}(1)$ the inclusion. We set

$$\begin{aligned}
\phi_{1,p} &= \tau_{4,p-2} \circ \psi_1 : RP^2 \to S^{2+p}, && p \geq 2,\\
\phi_{2,p} &= \tau_{7,p-3} \circ \psi_2 : CP^2 \to S^{4+p}, && p \geq 3,\\
\phi_{3,p} &= \tau_{13,p-5} \circ \psi_3 : QP^2 \to S^{8+p}, && p \geq 5,\\
\phi_{4,p} &= \tau_{25,p-9} \circ \psi_4 : CayP^2 \to S^{16+p}, && p \geq 9,\\
\phi'_{1,p} &= \tau_{4,p-2} \circ \psi'_1 : S^2(\sqrt{3}) \to S^{2+p} && p \geq 2.
\end{aligned}$$

Then $\phi_{i,p}$ is an isometric minimal embedding and $\phi'_{1,p}$ is an isometric minimal immersion.

Denote by UM the unit tangent bundle of M. Define

$$C(p, H) = \begin{cases} 1 & \text{for } p = 1 \text{ or } p = 2 \text{ and } H \neq 0;\\ \frac{1}{3} & \text{for } p \geq 3 \text{ or } p = 2 \text{ and } H = 0. \end{cases}$$

Main Theorem 1.6. *Let M be an n-dimensional compact submanifold of the unit sphere S^{n+p}, with parallel mean curvature vector field of norm H. If*

$$\sigma(u) \leq C(p, H) \quad \textit{for any } u \in UM,$$

we are in one of the following cases:

(1) *M is the totally umbilical sphere $S^n_H = S^n\Big(\dfrac{1}{\sqrt{1+H^2}}\Big)$*;

(2) *M is one of the embeddings $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, with $k = 1, 2, \ldots, n$ and $k \neq \frac{1}{2}n$*;

(3) *the isometric immersion of M in S^{n+p} is either the totally umbilical sphere $S^n(\sqrt{3}/2) \to S^{n+p}$, or one of the embeddings $\phi_{i,p}$, $i = 1, 2, 3, 4$, or the immersion $\phi'_{1,p}$.*

The case $H = 0$ goes back to Gauchman [1986, p. 781].

2. Preliminaries

We make the following conventions on the range of indices:

$$1 \leq A, B, C \leq n + p, \quad 1 \leq i, j, k, l, m \leq n < \alpha, \beta, \gamma, \delta \leq n + p.$$

Choose a local orthonormal frame field $\{e_A\}$ on S^{n+p} such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of S^{n+p} respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\, \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}, \quad h = \sum_{\alpha,i,j} h^\alpha_{ij}\, \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha,i} h^\alpha_{ii}\, e_\alpha,$$

$$R_{ijkl} = \delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk} + \sum_{\alpha}\left(h^{\alpha}_{ik}\,h^{\alpha}_{jl} - h^{\alpha}_{il}\,h^{\alpha}_{jk}\right), \tag{1}$$

$$R_{\alpha\beta kl} = \sum_{i}\left(h^{\alpha}_{ik}\,h^{\beta}_{il} - h^{\alpha}_{il}\,h^{\beta}_{ik}\right),$$

where h, ξ, R_{ijkl} and $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of M. We set

$$S = \|h\|^2, \quad H = \|\xi\|, \quad H_\alpha = (h^{\alpha}_{ij})_{n\times n}.$$

Denoting the first and second covariant derivatives of h^{α}_{ij} by h^{α}_{ijk} and h^{α}_{ijkl} respectively, we have

$$\sum_k h^{\alpha}_{ijk}\,\omega_k = dh^{\alpha}_{ij} + \sum_k h^{\alpha}_{kj}\,\omega_{ik} + \sum_k h^{\alpha}_{ik}\,\omega_{jk} + \sum_\beta h^{\beta}_{ij}\,\omega_{\alpha\beta}, \tag{2}$$

$$\sum_l h^{\alpha}_{ijkl}\,\omega_l = dh^{\alpha}_{ijk} + \sum_l h^{\alpha}_{ljk}\,\omega_{il} + \sum_l h^{\alpha}_{ilk}\,\omega_{jl} + \sum_l h^{\alpha}_{ijl}\,\omega_{kl} + \sum_\beta h^{\beta}_{ijk}\,\omega_{\alpha\beta}. \tag{3}$$

The Laplacian of h is defined by $\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{ijkk}$. Following [Yau 1974; 1975], we have

$$\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{kkij} + \sum_{k,m} h^{\alpha}_{km}\,R_{mijk} + \sum_{k,m} h^{\alpha}_{mi}\,R_{mkjk} + \sum_{k,\beta} h^{\beta}_{ki}\,R_{\alpha\beta kj}.$$

From now on we assume that M is a submanifold of parallel mean curvature in S^{n+p}. Choose e_{n+1} such that e_{n+1} is parallel to ξ, $\operatorname{tr} H_{n+1} = nH$ and $\operatorname{tr} H_\beta = 0$, where $n+2 \le \beta \le n+p$. Again by the same work of Yau, we have

$$\Delta h^{n+1}_{ij} = \sum_{k,m} h^{n+1}_{km}\,R_{mijk} + \sum_{k,m} h^{n+1}_{im}\,R_{mkjk},$$

$$\Delta h^{\beta}_{ij} = \sum_{k,m} h^{\beta}_{mk}\,R_{mijk} + \sum_{k,m} h^{\beta}_{im}\,R_{mkjk} + \sum_{k,\alpha\neq n+1} h^{\alpha}_{ki}\,R_{\alpha\beta jk}, \quad \beta \neq n+1.$$

Since the Laplacian formulas for the special orthonormal frame field as above are not apply to our case, we will give the following Laplacian formula which holds for any orthonormal frame fields.

Proposition 2.1. *Let M be an n-dimensional submanifold of parallel mean curvature in S^{n+p}. Then*

$$\Delta h^{\alpha}_{ij} = \sum_{k,m} h^{\alpha}_{km}\,R_{mijk} + \sum_{k,m} h^{\alpha}_{mi}\,R_{mkjk} + \sum_{k,\beta} h^{\beta}_{ki}\,R_{\beta\alpha jk}, \tag{4}$$

$$\sum_\alpha R_{\alpha\beta kl}\,(\operatorname{tr} H_\alpha) = 0. \tag{5}$$

Proof. Putting $c_\alpha = (1/n)\ \mathrm{tr}\, H_\alpha$, we have $\xi = \sum c_\alpha\, e_\alpha$. Since ξ is parallel in the normal bundle over M, we have

$$\begin{aligned}0 = \nabla^\perp_X \xi &= \sum_\alpha X(c_\alpha)\, e_\alpha + \sum_\alpha c_\alpha\, \nabla^\perp_X e_\alpha \\ &= \sum_\alpha X(c_\alpha)\, e_\alpha + \sum_\alpha c_\alpha \Big(\sum_\beta \omega_{\beta\alpha}(X)\, e_\beta\Big) = \sum_\alpha \Big(X(c_\alpha) + \sum_\beta c_\beta\ \omega_{\alpha\beta}(X)\Big)\, e_\alpha\end{aligned}$$

for any tangent vector field X on M. It follows that

$$dc_\alpha + \sum_\beta c_\beta\, \omega_{\alpha\beta} = 0 \quad \text{for any } \alpha. \tag{6}$$

To prove (4), it is sufficient to show that $\sum_k h^\alpha_{kkij} = 0$ for any α, i, j. By (2), we get

$$\sum_{i,k} h^\alpha_{iik}\, \omega_k = d\Big(\sum_i h^\alpha_{ii}\Big) + 2\sum_{i,k} h^\alpha_{ik}\, \omega_{ik} + \sum_{\beta,i} h^\beta_{ii}\, \omega_{\alpha\beta} = n\Big(dc_\alpha + \sum_\beta c_\beta\, \omega_{\alpha\beta}\Big) = 0.$$

Therefore, $\sum_i h^\alpha_{iik} = 0$ for all k, α. Together with (3), this implies

$$\sum_{i,l} h^\alpha_{iikl}\, \omega_l = d\Big(\sum_i h^\alpha_{iik}\Big) + 2\sum_{i,l} h^\alpha_{ilk}\, \omega_{il} + \sum_{i,l} h^\alpha_{iil}\, \omega_{kl} + \sum_{i,\beta} h^\beta_{iik}\, \omega_{\alpha\beta} = 0.$$

Hence $\sum_i h^\alpha_{iikl} = 0$ for all k, l, α.

Taking the exterior derivative of (6) we get

$$\begin{aligned}0 &= d^2 c_\alpha + d\Big(\sum_\beta c_\beta\, \omega_{\alpha\beta}\Big) \\ &= \sum_\beta dc_\beta \wedge \omega_{\alpha\beta} + \sum_\beta c_\beta \Big(-\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \tfrac{1}{2}\sum_{k,l} R_{\alpha\beta kl}\, \omega_k \wedge \omega_l\Big) \\ &= \sum_\beta \Big(dc_\beta + \sum_\gamma c_\gamma\, \omega_{\beta\gamma}\Big) \wedge \omega_{\alpha\beta} + \tfrac{1}{2}\sum_{\beta,k,l} c_\beta\, R_{\alpha\beta kl}\ \omega_k \wedge \omega_l \\ &= \tfrac{1}{2}\sum_{\beta,k,l} c_\beta\, R_{\alpha\beta kl}\ \omega_k \wedge \omega_l.\end{aligned}$$

Thus $\sum_\beta R_{\alpha\beta kl}\, (\mathrm{tr}\, H_\beta) = 0$ for all α, k, l, as desired. □

3. Maximal directions

Let $x \in M$. A vector $u \in UM_x$ is called a *maximal direction* at x if $\sigma(u) = \max_{v \in UM_x} \sigma(v)$.

Choose an orthonormal frame $\{e_1, \dots, e_{n+p}\}$ at x such that restricted to M, the vectors $e_1, \dots, e_n$ are tangent to M. Assume that e_1 is a maximal direction at x, $\sigma(e_1) \neq 0$, and $e_{n+1} = h(e_1, e_1)/\|h(e_1, e_1)\|$. Choose e_{n+2} such that

$$e_{n+2} = \frac{\xi - \langle \xi, e_{n+1}\rangle e_{n+1}}{\|\xi - \langle \xi, e_{n+1}\rangle e_{n+1}\|}$$

if ξ is not parallel to e_{n+1}. By our choices of e_{n+1} and e_{n+2}, we have

(7) $$h^{\alpha}_{11} = 0 \quad \text{if } \alpha \neq n+1 \qquad \text{and} \qquad c_\alpha = 0 \quad \text{if } \alpha \neq n+1, n+2.$$

Since e_1 is a maximal direction, we have at the point x for any $t \in \mathbb{R}$

(8) $$\|h(e_1 + te_i, e_1 + te_i)\|^2 \leq (1+t^2)^2 \, (h^{n+1}_{11})^2.$$

Expanding in terms of t, we obtain

$$4\,t\,h^{n+1}_{11}\,h^{n+1}_{1i} + O(t^2) \leq 0.$$

It follows that

(9) $$h^{n+1}_{1i} = 0, \quad i = 2, \ldots, n.$$

It is easy to see that e_1 is also an eigenvector of the Weingarten transformation A^{n+1}. Therefore, we can choose an adapted frame at $x \in M$ such that in addition to (7) and (9),

(10) $$h^{n+1}_{ij} = 0, \quad i \neq j.$$

Once more expanding (8) in terms of t, we obtain

$$-2t^2\Big(h^{n+1}_{11}(h^{n+1}_{11} - h^{n+1}_{ii}) - 2\sum_{\alpha \neq n+1}(h^{\alpha}_{1i})^2\Big) + O(t^3) \leq 0.$$

It follows that

(11) $$2\sum_{\alpha \neq n+1}(h^{\alpha}_{1i})^2 \leq h^{n+1}_{11}(h^{n+1}_{11} - h^{n+1}_{ii}) \quad \text{for } i = 2, \ldots, n.$$

Define a tensor field $T = (T_{ijkl})$ on M by

$$T_{ijkl} = \sum_{\alpha} h^{\alpha}_{ij}\,h^{\alpha}_{kl}.$$

It is obvious that $\sigma(u) = T(u, u, u, u)$.

Lemma 3.1. *Let u be a maximal direction at $x \in M$. Assume that $\sigma(u) \neq 0$. Let $e_1, \ldots, e_{n+p}$ be an adapted frame at x such that*

$$e_1 = u, \quad e_{n+1} = \frac{h(e_1, e_1)}{\|h(e_1, e_1)\|},$$

$h^{n+1}_{ij} = 0$ for $i \neq j$, and $e_{n+2} = (\xi - \langle \xi,\ e_{n+1}\rangle e_{n+1})/\|\xi - \langle \xi,\ e_{n+1}\rangle e_{n+1}\|$ if ξ is not parallel to e_{n+1}. At the point x,

(i) *if $p = 1$, or $p = 2$ and $H \neq 0$, then*

(12) $$\tfrac{1}{2}(\Delta T)_{1111} \geq h^{n+1}_{11}\Big(n\big(h^{n+1}_{11} + c_{n+1}(h^{n+1}_{11})^2 - c_{n+1}\big) - h^{n+1}_{11}\sum_{k}(h^{n+1}_{kk})^2\Big);$$

(ii) *if $p \geq 3$, or $p = 2$ and $H = 0$, then*

$$\tag{13} \begin{aligned} &\tfrac{1}{2}(\Delta T)_{1111} \\ &\geq h_{11}^{n+1}\Big(n(h_{11}^{n+1} + 3c_{n+1}(h_{11}^{n+1})^2 - c_{n+1} - (h_{11}^{n+1})^3) - 2h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2\Big), \end{aligned}$$

and equality holds if and only if

$$\tag{14} (h_{11}^{n+1} - h_{kk}^{n+1})\Big(h_{11}^{n+1}(h_{11}^{n+1} - h_{kk}^{n+1}) - 2\sum_{\alpha\neq n+1}(h_{1k}^{\alpha})^2\Big) = 0$$

and $h_{11k}^{\alpha} = 0$, for all k and α.

Proof. We have

$$\tag{15} \tfrac{1}{2}(\Delta T)_{1111} = h_{11}^{n+1}\Delta h_{11}^{n+1} + \sum_{i,\alpha}(h_{11i}^{\alpha})^2.$$

From Proposition 2.1 and equations (7) and (10), we have

$$\begin{aligned} \Delta h_{11}^{n+1} &= \sum_{k,m} h_{km}^{n+1} R_{m11k} + \sum_{k,m} h_{m1}^{n+1} R_{mk1k} + \sum_{k,\alpha} h_{1k}^{\alpha} R_{\alpha n+11k} \\ &= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1}) R_{1k1k} + \sum_{k,\alpha} h_{1k}^{\alpha}\Big(\sum_l (h_{l1}^{\alpha}h_{lk}^{n+1} - h_{lk}^{\alpha}h_{l1}^{n+1})\Big) \\ &= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1})\Big(1 - (\delta_{1k})^2 + \sum_\alpha (h_{11}^{\alpha}h_{kk}^{\alpha} - (h_{1k}^{\alpha})^2)\Big) \\ &\qquad\qquad + \sum_{k,\alpha}(h_{1k}^{\alpha})^2(h_{kk}^{n+1} - h_{11}^{n+1}) \\ &= \sum_k (h_{11}^{n+1} - h_{kk}^{n+1}) + \sum_k (h_{11}^{n+1} - h_{kk}^{n+1}) h_{11}^{n+1}h_{kk}^{n+1} - 2\sum_{k,\alpha}(h_{1k}^{\alpha})^2(h_{11}^{n+1} - h_{kk}^{n+1}) \\ &= n\big(h_{11}^{n+1} + c_{n+1}(h_{11}^{n+1})^2 - c_{n+1}\big) - h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2 \\ &\qquad\qquad - 2\sum_{k,\alpha\neq n+1}(h_{1k}^{\alpha})^2(h_{11}^{n+1} - h_{kk}^{n+1}). \end{aligned}$$

If $p=1$, the last term above vanishes. If $p=2$ and $H\neq 0$, we have $R_{(n+1)(n+2)kl}=0$ for any k, l, by (5) and (7); hence the last term above vanishes again. If $p \geq 3$, or if or $p = 2$ and $H = 0$, we obtain by (11)

$$\begin{aligned} \Delta h_{11}^{n+1} &\geq n\big(h_{11}^{n+1} + c_{n+1}(h_{11}^{n+1})^2 - c_{n+1}\big) - h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2 - \sum_k h_{11}^{n+1}(h_{11}^{n+1} - h_{kk}^{n+1})^2 \\ &= n\big(h_{11}^{n+1} + 3c_{n+1}(h_{11}^{n+1})^2 - c_{n+1} - (h_{11}^{n+1})^3\big) - 2h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2. \end{aligned}$$

Substituting this into (15), we obtain

$$\tfrac{1}{2}(\Delta T)_{1111} \geq h_{11}^{n+1}\Big(n(h_{11}^{n+1} + c_{n+1}(h_{11}^{n+1})^2 - c_{n+1}) - h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2\Big)$$

if $p = 1$ or $p = 2$ and $H \neq 0$, and

$$\tfrac{1}{2}(\Delta T)_{1111} \geq h_{11}^{n+1}\Big(n(h_{11}^{n+1} + 3c_{n+1}(h_{11}^{n+1})^2 - c_{n+1} - (h_{11}^{n+1})^3) - 2h_{11}^{n+1}\sum_k (h_{kk}^{n+1})^2\Big)$$

if $p \geq 3$ or $p = 2$ and $H = 0$. □

Lemma 3.2. *Let $\{e_1, \dots, e_{n+p}\}$ be an adapted frame at $x \in M$ as in Lemma 3.1. Suppose that*

$$\sigma(u) \leq \begin{cases} 1 & \text{if } p = 1 \text{ or } p = 2 \text{ and } H \neq 0, \\ \frac{1}{3} & \text{if } p \geq 3 \text{ or } p = 2 \text{ and } H = 0, \end{cases}$$

for all $u \in UM$. Then $(\Delta T)_{1111} \geq 0$. If equality holds, i.e., if $(\Delta T)_{1111} = 0$, then

$$h_{11}^{n+1} = |h_{22}^{n+1}| = \cdots = |h_{nn}^{n+1}|. \tag{16}$$

Proof. Since e_1 is a maximal direction at $x \in M$,

$$-h_{11}^{n+1} \leq h_{kk}^{n+1} \leq h_{11}^{n+1}, \quad k = 2, \dots, n. \tag{17}$$

It is clear that the convex function $f(h_{22}^{n+1}, \dots, h_{nn}^{n+1}) = \sum_{k=2}^{n} (h_{kk}^{n+1})^2$ subject to the constraint (17) attains its maximal value when

$$|h_{22}^{n+1}| = \cdots = |h_{nn}^{n+1}| = h_{11}^{n+1}.$$

Therefore, by inequalities (12) and (13),

$$\tfrac{1}{2}(\Delta T)_{1111} \geq \begin{cases} nh_{11}^{n+1}(h_{11}^{n+1} - c_{n+1})(1 - \sigma(e_1)) & \text{if } p = 1 \text{ or } p = 2 \text{ and } H \neq 0, \\ nh_{11}^{n+1}(h_{11}^{n+1} - c_{n+1})(1 - 3\sigma(e_1)) & \text{if } p \geq 3 \text{ or } p = 2 \text{ and } H = 0, \end{cases}$$

where $c_{n+1} = (1/n)\sum_{i=1}^{n} h_{ii}^{n+1} \leq h_{11}^{n+1}$. □

Let $L(x)$ be a function on M defined by $L(x) = \max_{u \in UM_x} \sigma(u)$. By a similar argument as in [Gauchman 1986], we get:

Lemma 3.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. If*

$$\sigma(u) \leq \begin{cases} 1, & \text{for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0 \\ \frac{1}{3}, & \text{for } p \geq 3, \text{ or } p = 2 \text{ and } H{=}0, \end{cases}$$

for all $u \in UM$, then $L(x)$ is a constant function on M.

4. Rigidity of submanifolds of parallel mean curvature

This section is devoted to the proof of the Main Theorem 1.6, through a series of intermediate results.

Lemma 4.1. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. Suppose that*

$$\sigma(u) < \begin{cases} 1 & \text{if } p = 1 \text{ or } p = 2 \text{ and } H \neq 0, \\ \frac{1}{3} & \text{if } p \geq 3 \text{ or } p = 2 \text{ and } H = 0, \end{cases}$$

for all $u \in UM$. Then M is the totally umbilical sphere S^n_H.

Proof. Let e_1 be a maximal direction at $x \in M$. Assume $\sigma(e_1) \neq 0$. By Lemmas 3.2 and 3.3, we have $(\Delta T)_{1111} = 0$ on M. From the proof of Lemma 3.2, we see that

$$h^{n+1}_{11} = c_{n+1}.$$

Thus the average value of the $\{h^{n+1}_{ii}\}^n_{i=1}$ equals their maximum. This possibility occurs if and only if

$$h^{n+1}_{11} = \cdots = h^{n+1}_{nn}.$$

This and (11) yield $h^\alpha_{1i} = 0$, for $\alpha \neq n+1$ and $i = 2, \ldots, n$. Since each of the vectors e_i, for $i = 1, \ldots, n$, is a maximal direction, we have

$$h^\alpha_{ij} = 0 \quad \text{for } i, j = 1, 2, \ldots, n \text{ and } i \neq j.$$

From $\|h(e_i, e_i)\|^2 \leq (h^{n+1}_{11})^2$, we obtain

$$h^\alpha_{ii} = 0 \quad \text{for } \alpha \neq n+1 \text{ and } i = 1, 2, \ldots, n.$$

The last three displayed equations say that M is a totally umbilical sphere. □

For convenience, we establish a convention on indices $a, b, \ldots, r, s, \ldots$:

$$1 \leq a, b, c, d \leq k < r, s, t, w \leq n,$$

where k is a fixed integer in the range $1, \ldots, n$.

Here is the rigidity theorem for hypersurfaces with constant mean curvature in a sphere:

Theorem 4.2. *Let M be an n-dimensional compact hypersurface with constant mean curvature in a unit sphere $S^{n+1}(1)$.*

(i) *If $\sigma(u) < 1$ for any $u \in UM$, then M is the totally umbilical sphere S^n_H.*

(ii) *If $\max_{u \in UM} \sigma(u) = 1$, M is one of the embeddings $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, with $k = 1, 2, \ldots, n$.*

Proof. Assertion (i) follows from Lemma 4.1. We prove (ii). As in the proof of Lemma 4.1, $(\Delta T)_{1111} = 0$. By (16), we may assume after a suitable renumbering of $e_1, \ldots, e_n$ that

$$h^{n+1}_{aa} = -h^{n+1}_{rr} = 1 \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n.$$

By Lemma 3.1, h^{n+1}_{11k} vanishes for $k = 1, \dots, n$. It follows that $h^{n+1}_{iik} = 0$. By polarization, h^{n+1}_{ijk} vanishes for all i, j, k. By (2) and (10), we have

$$0 = \sum_l h^{n+1}_{il}\omega_{lj} + \sum_l h^{n+1}_{lj}\omega_{li} = (h^{n+1}_{ii} - h^{n+1}_{jj})\omega_{ij}.$$

Hence, $\omega_{ar} = 0$. It follows that the two distributions defined by $\omega_1 = \cdots = \omega_k = 0$ and $\omega_{k+1} = \cdots = \omega_n = 0$ are integrable and give a local decomposition of M. Then every point of M has a neighborhood U which is a Riemannian product $V_1 \times V_2$ with $\dim V_1 = k$ and $\dim V_2 = n - k$. The curvatures of V_1 and V_2 are

$$\begin{aligned} R_{abcd} &= 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \quad \text{for } 1 \le a, b, c, d \le k, \\ R_{rstw} &= 2(\delta_{rt}\delta_{sw} - \delta_{rw}\delta_{st}) \quad \text{for } k+1 \le r, s, t, w \le n. \end{aligned}$$

Thus V_1 and V_2 are spaces of constant curvature 2. The compactness of M allows us to complete the proof. □

For the case of codimension two:

Theorem 4.3. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+2}(1)$, $H \neq 0$.*

(i) *If $\sigma(u) < 1$ for any $u \in UM$, then M is the totally umbilical sphere S^n_H.*

(ii) *If $\max_{u\in UM} \sigma(u) = 1$, M is one of the embeddings $S^k(1/\sqrt{2}) \times S^{n-k}(1/\sqrt{2})$, with $k = 1, \dots, n$, $k \neq \frac{1}{2}n$.*

Proof. Assertion (i) follows from Lemma 4.1. We prove (ii). As in the proof of Lemma 4.1, $(\Delta T)_{1111} = 0$. By (16), we have

$$h^{n+1}_{aa} = -h^{n+1}_{rr} = 1 \quad \text{for } a = 1, \dots, k \text{ and } r = k+1, \dots, n.$$

From (7) and (11) we obtain $h^{n+2}_{1a} = 0$ for $a = 1, \dots, k$. Since each of vectors e_i, for $i = 1, \dots, n$, is a maximal direction, we get

$$h^{n+2}_{ab} = 0 \quad \text{for } a, b = 1, \dots, k.$$

Similarly,

$$h^{n+2}_{rs} = 0 \quad \text{for } r, s = k+1, \dots, n.$$

As in the proof of Lemma 3.1, we have $R_{(n+1)(n+2)kl} = 0$. Hence

$$h^{n+2}_{kl}(h^{n+1}_{kk} - h^{n+1}_{ll}) = 0,$$

which implies $h^{n+2}_{ar} = 0$ for $a = 1, \dots, k$ and $r = k+1, \dots, n$. Thus

$$h^{n+2}_{ij} = 0 \quad \text{for } i, j = 1, \dots, n. \tag{18}$$

By a similar argument as in the proof of Theorem 4.2, we have $h^{n+1}_{ijk}=0$ for all i, j, k. By (2), (10) and (18), we have

$$0=\sum_l h^{n+1}_{il}\omega_{lj}+\sum_l h^{n+1}_{jl}\omega_{li}=(h^{n+1}_{ii}-h^{n+1}_{jj})\omega_{ij}.$$

Therefore, $\omega_{ar}=0$. Then M is a locally Riemannian product $V_1\times V_2$, with dim $V_1=k$ and dim $V_2=n-k$. The curvature of V_1 is

$$\begin{aligned}R_{abcd}&=\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}+\sum_{\alpha=n+1}^{n+2}(h^\alpha_{ac}h^\alpha_{bd}-h^\alpha_{ad}h^\alpha_{bc})\\&=\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}+h^{n+1}_{ac}h^{n+1}_{bd}-h^{n+1}_{ad}h^{n+1}_{bc}=2(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})\end{aligned}$$

(see (1)), where the second equality follows from (18). A similar argument applies to V_2. In conclusion, V_1 and V_2 are spaces of constant curvature 2. The compactness of M allows us to complete the proof. □

Remark 4.4. In assertion (ii) of Theorem 4.3, we exclude the case of $n=2m$ even and $k=m$, in that it results in $H=0$, contradicting the theorem's assumption.

Let F be the real numbers, the complex numbers, or the quaternions, and let d be the dimension of F as a real vector space (1, 2, or 4). Let FP^m denote the projective space over F, $M^n(c)$ the n-dimensional Riemannian manifold with constant curvature c.

Lemma 4.5 [Sakamoto 1977]. *Let $f: M^n\to S^{n+p}(\bar c)$ be an isotropic immersion of parallel second fundamental tensor. Except for the totally umbilical case, f is a composition of a minimal isotropic immersion $\eta: M^n\to S^{n+q}(\tilde c)$ $(q\le p)$ of parallel second fundamental tensor, and a totally umbilical $\tau: S^{n+q}(\tilde c)\to S^{n+p}(\bar c)$, where $n=md$ and M must be one of $S^n(c)$, FP^m and $CayP^2$. Assume that the isotropic constants of f and η are λ and μ respectively. Then*

$$c=\frac{m}{2(m+1)}\tilde c,\quad q=\frac{(m-1)(md+2)}{2},\quad \mu^2=\frac{m-1}{m+1}\tilde c,$$

where $m=n$ if $M=S^n(c)$ and $m=2$ if $M=CayP^2$.

Lemma 4.6. *Let $f: M^n\to S^{n+p}(1)$ be a λ-isotropic immersion of parallel second fundamental tensor. If $\lambda^2\le(m-1)/(m+1)$, then f is totally umbilical, or minimal with $\lambda^2=(m-1)/(m+1)$.*

Proof. Assume that f is not totally umbilical. Following Lemma 4.5, f can be considered as composition of a minimal μ-isotropic immersion $\eta: M^n\to S^{n+q}(\tilde c)$ and a totally umbilical sphere $\tau: S^{n+q}(\tilde c)\to S^{n+p}(1)$, where μ and $\tilde c$ satisfy

$$\mu^2=\frac{m-1}{m+1}\tilde c.$$

On the other hand, if H is the mean curvature of immersion f, it is easy to see

$$\mu^2 + H^2 = \lambda^2, \quad \tilde{c} = \bar{\rightarrow} c + H^2.$$

Substituting into the preceding equation, we get

$$\lambda^2 - \frac{m-1}{m+1} = \frac{2m}{m+1} H^2 \geq 0. \tag{19}$$

The assumption $\lambda^2 \leq (m-1)/(m+1)$ and (19) together give

$$\lambda^2 = \frac{m-1}{m+1} \quad \text{and} \quad H = 0. \qquad \square$$

Theorem 4.7. *Let M be an n-dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$. Assume that $p \geq 3$, or $p = 2$ and $H = 0$.*

(i) *If $\sigma(u) < \frac{1}{3}$ for any $u \in UM$, then M is the totally umbilical sphere S^n_H.*

(ii) *If $\max_{u \in UM} \sigma(u) = \frac{1}{3}$, then $\sigma(u) \equiv \frac{1}{3}$ on UM, and the isometric immersion of M into S^{n+p} is either the totally umbilical sphere $S^n(\sqrt{3}/2) \to S^{n+p}(1)$, one of the embeddings $\phi_{i,p}$, $i = 1, 2, 3, 4$, or one of the immersions $\phi'_{1,p}$ described above.*

Proof. We need only consider the case $\max_{v \in UM_x} \sigma(v) = \sigma(u)$. As in the proof of Lemma 4.1, we obtain $(\Delta T)_{1111} = 0$. By (16), we have, after a suitable renumbering of $e_1, \ldots, e_n$,

$$h^{n+1}_{aa} = -h^{n+1}_{rr} = \frac{\sqrt{3}}{3} \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n. \tag{20}$$

Since $\|h(e_a, e_a)\|^2 \leq \frac{1}{3}$ and $\|h(e_r, e_r)\|^2 \leq \frac{1}{3}$, we obtain

$$h^\alpha_{aa} = h^\alpha_{rr} = 0 \quad \text{for } \alpha \neq n+1,\ a = 1, \ldots, k \text{ and } r = k+1, \ldots, n. \tag{21}$$

Still from (11),

$$h^\alpha_{ab} = h^\alpha_{rs} = 0 \quad \text{for } \alpha \neq n+1,\ a, b = 1, \ldots, k \text{ and } r, s = k+1, \ldots, n. \tag{22}$$

By (14), $\sum_{\alpha \neq n+1} (h^\alpha_{1r})^2 = \frac{1}{3}$. Since each vector e_i, for $i = 1, \ldots, n$, is a maximal direction,

$$\sum_{\alpha \neq n+1} (h^\alpha_{ar})^2 = \tfrac{1}{3} \quad \text{for } a = 1, \ldots, k \text{ and } r = k+1, \ldots, n. \tag{23}$$

For $x^2, \ldots, x^n$ and $t \in \mathbb{R}$, using (20)–(23) and (7)–(10), expanding the inequality

$$\left\| h\left(e_1 + t \sum_{i=2}^n x^i e_i,\ e_1 + t \sum_{i=2}^n x^i e_i\right) \right\|^2 \leq \left(1 + t^2 \sum_{i=2}^n (x^i)^2\right)^2 (h^{n+1}_{11})^2 \tag{24}$$

in terms of t, we obtain

$$4t^2\sum_\alpha\sum_{r,s} h^\alpha_{1r}h^\alpha_{1s}x^r x^s + O(t^3) \le 0.$$

It follows that $\sum_\alpha h^\alpha_{1r}h^\alpha_{1s} = 0$ if $r \ne s$. Since each vector e_i is a maximal direction, we have

$$\sum_\alpha h^\alpha_{ar}h^\alpha_{as} = 0 \quad \text{if } r \ne s, \qquad \sum_\alpha h^\alpha_{ar}h^\alpha_{br} = 0 \quad \text{if } a \ne b.$$

Once more expand (24) to obtain

$$2t^3\sum (h^\alpha_{1r}h^\alpha_{bs} + h^\alpha_{1s}h^\alpha_{br})x^a x^r x^s + O(t^4) \le 0.$$

It follows that

$$\sum_\alpha (h^\alpha_{ar}h^\alpha_{bs} + h^\alpha_{as}h^\alpha_{br}) = 0 \quad \text{if } a \ne b \text{ or } r \ne s. \tag{25}$$

Using (10) and (20)–(25), we obtain by direct computation that $\sigma(u) \equiv \frac{1}{3}$ for any $u \in UM$. It is easy to see that $h^\alpha_{ijk} = 0$ for all α, i, j, k. Therefore, M is a $(\sqrt{3}/3)$-isotropic submanifold in a unit sphere of parallel second fundamental tensor. By Lemmas 4.5 and 4.6 we know that M is either totally umbilical or minimal. This, together with a [Gauchman 1986, Theorem 3], completes the proof. □

Theorems 4.2, 4.3 and 4.7 together imply the Main Theorem 1.6.

References

[Alencar and do Carmo 1994] H. Alencar and M. do Carmo, "Hypersurfaces with constant mean curvature in spheres", *Proc. Amer. Math. Soc.* **120** (1994), 1223–1229. MR 94f:53108 Zbl 0802.53017

[An-Min and Jimin 1992] L. An-Min and L. Jimin, "An intrinsic rigidity theorem for minimal submanifolds in a sphere", *Arch. Math.* (*Basel*) **58**:6 (1992), 582–594. MR 93b:53050 Zbl 0767.53042

[Chern et al. 1970] S. S. Chern, M. do Carmo, and S. Kobayashi, "Minimal submanifolds of constant length", in *Functional analysis and related fields*, edited by F. E. Browder, Springer, New York, 1970.

[Gauchman 1986] H. Gauchman, "Minimal submanifolds of a sphere with bounded second fundamental form", *Trans. Amer. Math. Soc.* **298**:2 (1986), 779–791. MR 87i:53085 Zbl 0608.53056

[Lawson 1969] H. B. Lawson, Jr., "Local rigidity theorems for minimal hypersurfaces", *Ann. of Math.* (2) **89** (1969), 187–197. MR 38 #6505 Zbl 0174.24901

[O'Neill 1965] B. O'Neill, "Isotropic and Kähler immersions", *Canad. J. Math.* **17** (1965), 907–915. MR 32 #1654 Zbl 0171.20503

[Sakamoto 1977] K. Sakamoto, "Planar geodesic immersions", *Tôhoku Math. J.* (2) **29**:1 (1977), 25–56. MR 57 #10657 Zbl 0357.53035

[Simons 1968] J. Simons, "Minimal varieties in riemannian manifolds", *Ann. of Math.* (2) **88** (1968), 62–105. MR 38 #1617 Zbl 0181.49702

[Xu 1991] H. W. Xu, "A Simons-type pinching constant and the problem of isometric immersion", *Chinese Ann. Math. Ser. A* **12**:3 (1991), 261–269. In Chinese. MR 92h:53077 Zbl 0767.53038

[Xu 1993] H. W. Xu, "A rigidity theorem for submanifolds with parallel mean curvature in a sphere", *Arch. Math.* (*Basel*) **61**:5 (1993), 489–496. MR 94m:53084 Zbl 0785.53049

[Xu 1995] H. W. Xu, "On closed minimal submanifolds in pinched Riemannian manifolds", *Trans. Amer. Math. Soc.* **347**:5 (1995), 1743–1751. MR 95h:53088 Zbl 0830.53048

[Yau 1974] S. T. Yau, "Submanifolds with constant mean curvature", *Amer. J. Math.* **96**:2 (1974), 346–366. MR 51 #6670 Zbl 0304.53041

[Yau 1975] S. T. Yau, "Submanifolds with constant mean curvature, II", *Amer. J. Math.* **97**:1 (1975), 76–100. MR 51 #6670 Zbl 0304.53042

Received February 12, 2005.

HONG-WEI XU
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU 310027
THE PEOPLE'S REPUBLIC OF CHINA
xuhw@cms.zju.edu.cn

WANG FANG
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU 310027
THE PEOPLE'S REPUBLIC OF CHINA
fangw@cms.zju.edu.cn

FEI XIANG
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU 310027
THE PEOPLE'S REPUBLIC OF CHINA
xiangf@cms.zju.edu.cn

Guidelines for Authors

Authors may submit manuscripts at pjm.math.berkeley.edu/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text (“the curve looks like this:”). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as “Place Figure 1 here”. The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal’s preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 228 No. 1 November 2006

0030-8730(200611)228:1;1-F