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We give a generalization of random matrix ensembles, which includes all classical ensembles. We derive the joint-density function of the generalized ensemble by one simple formula, giving a direct and unified way to compute the density functions for all classical ensembles and various kinds of new ensembles. An integration formula associated with the generalized ensembles is given. We propose a taxonomy of generalized ensembles encompassing all classical ensembles and some new ones not considered before.

1. Introduction

One of the most fundamental problems in the theory of random matrices is to derive the joint-density functions for the eigenvalues (or, equivalently, the measures associated with the eigenvalue distributions) of various types of matrix ensembles. Mehta [1991] summarized the classical analysis methods by which the density functions for various types of ensembles were derived case by case; but a systematic method to compute the density functions was desired.

The first achievement in this direction was made by Dyson [1970], who introduced an idea of expressing various kinds of circular ensembles in terms of symmetric spaces with invariant probability measures. From then on, guided by Dyson's idea, many authors observed new random matrix ensembles in terms of Cartan's classification of Riemannian symmetric spaces, and obtained the joint-density functions for such ensembles by using the integration formula on symmetric space (see, for example, [Altland and Zirnbauer 1997; Caselle 1994; Caselle 1996; Dueñez 2004; Ivanov 2001; Titov et al. 2001; Zirnbauer 1996]).

We briefly mention the recent work of Dueñez [2004]. He explored the random matrix ensembles that correspond to infinite families of compact irreducible Riemannian symmetric spaces of type I, including circular orthogonal and symplectic ensembles, and various kinds of Jacobi ensembles. Using an integration formula

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associated with the KAK decomposition of compact groups, he obtained the induced measure on the space of eigenvalues associated to the underlying symmetric space, and then derived the eigenvalue distribution of the corresponding random matrix ensemble. These methods of deriving the eigenvalue distributions of random matrix ensembles by means of Riemannian symmetric spaces were summarized in the excellent article [Caselle and Magnea 2004].

In this paper we provide a generalization of random matrix ensembles, which includes all classical ensembles, and then give a unified way to derive — with one simple formula — the joint-density function for the eigenvalue distribution. The proof of this formula makes no use of an integration formula. In fact, the corresponding integration formula can be derived from this formula as a corollary. We also show how these generalized ensembles encompass all classical ensembles and some new ensembles that were not considered before.

Let $\sigma : G \times X \to X$ be a smooth action of a Lie group G on a Riemannian manifold X that preserves the induced Riemannian measure dx. Let $p(x)$ be a G-invariant smooth function on X, and consider the measure $p(x)\,dx$ on X, which is not necessarily a finite measure. We choose a closed submanifold Y of X consisting of representation points for almost all G-orbits in X. The Riemannian structure on X induces a Riemannian measure dy on Y. If K is the closed subgroup of G that fixes all points in Y, then σ reduces to a map $\varphi : G/K \times Y \to X$. Suppose there is a G-invariant measure $d\mu$ on G/K and that $\dim(G/K \times Y) = \dim X$; it can then be proved that the pull-back measure $\varphi^*(p(x)\,dx)$ of $p(x)\,dx$ is of the form $\varphi^*(p(x)\,dx) = d\mu\,d\nu$ for some measure $d\nu$ on Y, the latter being the measure associated with the eigenvalue distribution. The measure $d\nu$ can be expressed as $d\nu(y) = \mathcal{P}(y)\,dy$ for some function $\mathcal{P}(y)$ on Y, this being the joint-density function. If we write $\mathcal{P}(y)$ as $\mathcal{P}(y) = p(y)\,J(y)$, then, under some orthogonality condition (that is, $T_yY \perp T_yO_y$ for almost all $y \in Y$), we can compute the factor $J(y)$ by

$$J(y) = C\left|\det \Psi_y\right|, \tag{1–1}$$

where C is a constant that can be computed explicitly. This formula is the main result of this paper, and the density function $\mathcal{P}(y)$ and the eigenvalue distribution $d\nu$ are determined by it. Here, the map $\Psi_y : \mathfrak{l} \to T_yO_y$ is defined as

$$\Psi_y(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_{\exp t\xi}(y),$$

where $\mathfrak{l}$ is a linear subspace of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, with $\mathfrak{k}$ being the Lie algebra of K. We call the system $(G, \sigma, X, p(x)\,dx, Y, dy)$ a *generalized random matrix ensemble*. The measure $d\nu$ on Y is called a *generalized eigenvalue distribution* and the function $\mathcal{P}(y)$ is a *generalized joint density function*.

Using (1–1), one can derive the joint-density function for the Gaussian ensemble, chiral ensemble, new transfer matrix ensemble, circular ensemble, Jacobi ensemble, as well as some other new generalized ensembles. The precise deriving process will be the content of a sequel paper [An et al. 2005]. We should point out that the proof of formula (1–1) is not difficult, but this formula is very effective and useful; the derivation of all concrete examples in [An et al. 2005], including all classical random matrix ensembles, will be based on it.

Once the eigenvalue distribution $d\nu$ is derived from formula (1–1), under a covering condition we can get the associated integration formula. The Weyl integration formula for compact Lie groups, the Harish-Chandra integration formula for complex semisimple Lie groups and real reductive groups, the integration formulae on Riemannian symmetric spaces of noncompact and compact types that appeared in [Helgason 2000], as well as their Lie algebra versions, are all particular cases of it (see [An et al. 2005]).

We sketch the contents of this paper. In Section 2 we develop some geometrical preliminaries on the geometry of G-space, which will be required to establish the generalized ensembles. After presenting four conditions — that is, the invariance condition, the transversality condition, the dimension condition, and the orthogonality condition — on which the definition of generalized ensembles will be based, we prove in Theorem 2.5 a primary form of formula (1–1).

Section 3 is devoted to integration over G-spaces, which will be needed when we derive the integration formula associated with a generalized random matrix ensemble. Based on the four conditions from Section 2 and a covering condition, we prove an integration formula in Theorem 3.3, converting the integration over a G-space to, first, integrating over each G-orbit, and then integrating over the orbit space. Two criteria on when the covering condition holds are also given.

Prepared by Section 2 and Section 3, in Section 4 we give the precise definition of a generalized random matrix ensemble, as well as of the associated generalized eigenvalue distribution and generalized joint-density function. In Theorem 4.1 is presented formula (1–1), from which the associated eigenvalue distribution measure and density function will be derived in [An et al. 2005] for various concrete examples of the generalized ensemble, in a unified way.

In Section 5 we discuss a number of classes of generalized ensembles: the linear ensemble, the nonlinear noncompact ensemble, the compact ensemble, the group and algebra ensembles, as well as the pseudogroup and pseudoalgebra ensembles. Gaussian and chiral ensembles are included under linear ensembles; new transfer matrix ensembles are included under nonlinear noncompact ensembles; circular and Jacobi ensembles are included under compact ensembles. Some new ensembles not considered before are also mentioned.

2. Geometry of G-spaces

We develop some geometrical preliminaries needed for our theory of generalized random matrix ensembles.

We start with measures on manifolds. Let M be an n-dimensional smooth manifold. A measure dx on M is called *smooth* (or *quasi-smooth*) if on any local coordinate chart $(U; x_1, \ldots, x_n)$ of M, dx has the form $dx = f(x)\, dx_1 \ldots dx_n$, where f is a smooth function on U with $f > 0$ (or $f \geq 0$), and $dx_1 \ldots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$. Note that the smooth measures on M are unique up to multiplication with a positive smooth function on M, so the concept of a *set of measure zero* makes sense independently of the choice of smooth measure.

Let M, N be two n-dimensional smooth manifolds, and let $\varphi : M \to N$ be a smooth map. If dy is a smooth (or quasi-smooth) measure on N, expressed locally as $dy = f(y)\, dy_1 \ldots dy_n$, we can define its *pull-back* $\varphi^*(dy)$ locally as

$$\varphi^*(dy) = f(\varphi(x)) \left|\det\left(\frac{\partial y}{\partial x}\right)\right| dx_1 \ldots dx_n. \tag{2–1}$$

It is easy to check that the local definitions are compatible when different coordinate charts are chosen, and that $\varphi^*(dy)$ is a quasi-smooth measure on M. Even if dy is smooth, we cannot expect $\varphi^*(dy)$ to be smooth in general, since φ may have critical points; but if φ is a local diffeomorphism and dy is smooth, then $\varphi^*(dy)$ is smooth.

If M, N are Riemannian manifolds and dx, dy are the associated Riemannian measures, then we can express the pull-back measure $\varphi^*(dy)$ globally. To do this, first we need some comments on the "determinant" of a linear map between two different inner-product vector spaces of the same dimension. Suppose V, W are two n-dimensional vector spaces with inner products. For n vectors $v_1, \ldots, v_n \in V$, set $a_{ij} = \langle v_i, v_j \rangle$ for $1 \leq i, j \leq n$, and define

$$\mathrm{Vol}(v_1, \ldots, v_n) = \sqrt{\det(a_{ij})}.$$

Note that if $v_1, \ldots, v_n$ is an orthogonal basis, then $\mathrm{Vol}(v_1, \ldots, v_n) = |v_1| \ldots |v_n|$. For vectors in W, define the same things. Supposing $A : V \to W$ is a linear map, define

$$|\det A| = \frac{\mathrm{Vol}(Av_1, \ldots, Av_n)}{\mathrm{Vol}(v_1, \ldots, v_n)}, \tag{2–2}$$

where $v_1, \ldots, v_n$ is a basis of V. It is easy to check that the definition is independent of the choice of the basis $v_1, \ldots, v_n$. In the special case when $v_1, \ldots, v_n$ is an orthogonal basis of V and $Av_1, \ldots, Av_n$ are mutually orthogonal, we have

$$|\det A| = \frac{|Av_1| \ldots |Av_n|}{|v_1| \ldots |v_n|}. \tag{2–3}$$

We can expect only the norm $|\det A|$ of the determinant to be well defined, since the sign $\pm$ depends on a choice of orientations for V and W.

Proposition 2.1. *Suppose M, N are two n-dimensional Riemannian manifolds with associated Riemannian measures dx, dy, respectively. If $\varphi : M \to N$ is a smooth map, then*

$$\varphi^*(dy) = \left|\det(d\varphi)_x\right| dx. \tag{2–4}$$

Proof. Suppose that in local-coordinate charts the Riemannian metrics on M and N are $ds^2 = \sum_{ij} g_{ij}(x)\, dx_i\, dx_j$ and $d\tilde{s}^2 = \sum_{ij} \tilde{g}_{ij}(y)\, dy_i\, dy_j$, respectively, with $g_{ij}(x) = \langle \partial/\partial x_i,\ \partial/\partial x_j\rangle$ and $\tilde{g}_{ij}(y) = \langle \partial/\partial y_i,\ \partial/\partial y_j\rangle$. By definition, the Riemannian measures dx, dy are

$$dx = \sqrt{\det(g_{ij}(x))}\, dx_1 \dots dx_n \qquad \text{and} \qquad dy = \sqrt{\det(\tilde{g}_{ij}(y))}\, dy_1 \dots dy_n.$$

We have:

$$\begin{aligned}
|\det(d\varphi)_x|^2 &= \frac{\operatorname{Vol}\Big((d\varphi)_x\Big(\dfrac{\partial}{\partial x_1}\Big), \dots, (d\varphi)_x\Big(\dfrac{\partial}{\partial x_n}\Big)\Big)^2}{\operatorname{Vol}\Big(\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}\Big)^2} \\
&= \frac{\det\Big\langle \sum_k \dfrac{\partial y_k}{\partial x_i}\Big(\dfrac{\partial}{\partial y_k}\Big)_{\varphi(x)},\ \sum_l \dfrac{\partial y_l}{\partial x_j}\Big(\dfrac{\partial}{\partial y_l}\Big)_{\varphi(x)}\Big\rangle}{\det\Big\langle \dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j}\Big\rangle} \\
&= \frac{\det\Big(\sum_{kl} \dfrac{\partial y_k}{\partial x_i}\dfrac{\partial y_l}{\partial x_j}\tilde{g}_{kl}\big(\varphi(x)\big)\Big)}{\det(g_{ij}(x))} = \frac{\det\Big(\Big(\dfrac{\partial y_k}{\partial x_i}\Big)^t\big(\tilde{g}_{kl}(\varphi(x))\big)\Big(\dfrac{\partial y_l}{\partial x_j}\Big)\Big)}{\det(g_{ij}(x))} \\
&= \frac{\det^2\Big(\dfrac{\partial y}{\partial x}\Big)\det\big(\tilde{g}_{ij}(\varphi(x))\big)}{\det(g_{ij}(x))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\varphi^*(dy) &= \sqrt{\det\big(\tilde{g}_{ij}(\varphi(x))\big)}\,\left|\det\Big(\frac{\partial y}{\partial x}\Big)\right| dx_1 \dots dx_n \\
&= \left|\det(d\varphi)_x\right| \sqrt{\det(g_{ij}(x))}\, dx_1 \dots dx_n = \left|\det(d\varphi)_x\right| dx. \qquad \square
\end{aligned}$$

We now come to the main geometric problems that will concern us in the following sections. Let G be a Lie group that acts on an n-dimensional smooth manifold X. The action is denoted by $\sigma : G \times X \to X$ and we write $\sigma_g(x) = \sigma(g, x)$. Our first goal is, roughly speaking, to choose a representation point in each G-orbit $O_x = \{\sigma_g(x) \mid g \in G\}$, depending smoothly on the orbit. In general, this aim can be only partially achieved. Hence, suppose we have a closed submanifold Y of X, consisting of chosen representation points of the orbits, such that Y intersects "almost all" orbits transversally. More precisely, supposing there are closed zero-measure subsets $X_z \subset X$ and $Y_z \subset Y$, set $X' = X \setminus X_z$ and $Y' = Y \setminus Y_z$, and assume that

(a) $X' = \bigcup_{y \in Y'} O_y$ (*invariance condition*).

(b) $T_y X = T_y O_y \oplus T_y Y$ for all $y \in Y'$ (*transversality condition*).

It is clear that (a) implies that $Y' = Y \cap X'$ and $Y_z = Y \cap X_z$. Notice that X' and Y' are open and dense submanifolds of X and Y, respectively. So, for all $y \in Y'$, we have $T_y X' = T_y X$ and $T_y Y' = T_y Y$.

Set $K = \{g \in G \mid \sigma_g(y) = y, \text{ for all } y \in Y\}$; it is a closed subgroup of G. For $g \in G$, write $[g] = gK$ in G/K. The map $\sigma : G \times X \to X$ reduces to a map

$$\varphi : G/K \times Y \to X \qquad \text{with} \qquad \varphi([g], y) = \sigma_g(y).$$

By restriction, the latter induces a map $G/K \times Y' \to X'$, also denoted by φ. From assumption (a), $\varphi : G/K \times Y' \to X'$ is surjective. For $x \in X$, let $G_x = \{g \in G \mid \sigma_g(x) = x\}$ be the isotropy subgroup of x. Then $K \subset G_y$ for all $y \in Y$. Let dx, dy be smooth measures on X and Y, respectively. Suppose that dx is G-invariant. In what follows we also assume that

(c) $\dim G_y = \dim K$ for all $y \in Y'$ (*dimension condition*).

This means that G_y and K have the same Lie algebra for all $y \in Y'$, and that the only difference between G_y and K is that G_y may have more components than K. Then, for some $y \in Y'$, we have

$$\begin{aligned} \dim X = \dim T_y X &= \dim T_y Y + \dim T_y O_y \\ &= \dim Y + \dim G - \dim G_y = \dim Y + \dim G - \dim K. \end{aligned}$$

So φ is a map between manifolds of the same dimension, and thus the pull-back $\varphi^*(dx)$ of dx makes sense. If there is a G-invariant smooth measure $d\mu$ on G/K, then the product measure $d\mu\, dy$ on $G/K \times Y$ is smooth, and so

$$\varphi^*(dx) = J([g], y)\, d\mu\, dy \tag{2–5}$$

for some $J \in C^\infty(G/K \times Y)$ with $J \geq 0$.

Remark. The G-invariant smooth measure $d\mu$ on G/K exists if and only if $\Delta_G|_K = \Delta_K$, where Δ_G and Δ_K are the modular functions on G and K, respectively; see, for example, [Knapp 2002, Section 8.3]. For the concrete examples in the following sections, this condition always holds.

Proposition 2.2. *The smooth function $J \in C^\infty(G/K \times Y)$ is independent of the first variable $[g] \in G/K$. So we can rewrite (2–5) as*

$$\varphi^*(dx) = J(y)\, d\mu\, dy, \tag{2–6}$$

where $J \in C^\infty(Y)$ with $J \geq 0$.

Proof. If we denote by l_h the natural action of $h \in G$ on G/K, then one can easily verify that $\sigma_h \circ \varphi = \varphi \circ (l_h \times \mathrm{id})$. By the G-invariance of dx and $d\mu$, we have

$$\begin{aligned} J\big([g], y\big)\, d\mu\, dy = \varphi^*(dx) &= \varphi^* \circ \sigma_h^*(dx) = (l_h \times \mathrm{id})^* \circ \varphi^*(dx) \\ &= (l_h \times \mathrm{id})^* \big(J\big([g], y\big)\, d\mu\, dy\big) = J\big(h[g], y\big)\big(l_h^*(d\mu) \times \mathrm{id}^*(dy)\big) \\ &= J\big([hg], y\big)\, d\mu\, dy. \end{aligned}$$

So $J\big([g], y\big) = J\big([hg], y\big)$ for all $g, h \in G$, which means that J is independent of the first variable. □

Corollary 2.3. *There exists a quasi-smooth measure $d\nu$ on Y such that*

$$\varphi^*(dx) = d\mu\, d\nu. \tag{2–7}$$

The measure $d\nu$ is given by

$$d\nu(y) = J(y)\, dy. \tag{2–8}$$

The factor $J(y)$ can also be given for more general smooth measures $u(x)\, dx$ and $v(y)\, dy$ on X and Y. A direct calculation yields:

Proposition 2.4. *Suppose conditions (a), (b), and (c) hold. If we replace the measures with*

$$dx' = u(x)\, dx, \qquad dy' = v(y)\, dy, \qquad \text{and} \qquad d\mu' = \lambda\, d\mu,$$

where u, v are positive smooth functions on X, Y, respectively, and if u is G-invariant and λ is a positive constant, then $J(y)$ changes to

$$J'(y) = \frac{u(y)}{\lambda v(y)} J(y).$$

Now we suppose that there is a Riemannian structure on X such that dx and dy are the induced Riemannian measures on X and Y, respectively. We also assume that the following condition holds:

(d) $T_y Y \perp T_y O_y$ for all $y \in Y'$ (*orthogonality condition*).

In this case, the next theorem computes the factor $J(y)$ in a simple way.

Let $\mathfrak{l}$ be a linear subspace of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$, where $\mathfrak{k}$ is the Lie algebra of K. If $\pi : G \to G/K$ is the natural projection, then

$$(d\pi)_e|_{\mathfrak{l}} : \mathfrak{l} \to T_{[e]}(G/K)$$

is an isomorphism. If we endow G/K with a Riemannian structure such that its associated Riemannian measure is $d\mu$, then it also induces an inner product on $T_{[e]}(G/K)$. For $y \in Y$, we define a linear map $\Psi_y : \mathfrak{l} \to T_y O_y$ by

$$\Psi_y(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_{\exp t\xi}(y) \quad \text{for all } \xi \in \mathfrak{l}. \tag{2–9}$$

If $y \in Y'$, then $\dim \mathfrak{l} = \dim T_y O_y$. We choose an inner product on $\mathfrak{l}$ and endow $T_y O_y$ with the inner product induced from the Riemannian structure on X. The "determinants" $|\det \Psi_y|$ and $\left|\det\big((d\pi)_e|_{\mathfrak{l}}\big)\right|$ now make sense.

Theorem 2.5. *Under these assumptions, we have*

$$J(y) = C\left|\det \Psi_y\right| \qquad \textit{for all } y \in Y', \tag{2–10}$$

where C is the constant $C = \left|\det\big((d\pi)_e|_{\mathfrak{l}}\big)\right|^{-1}$.

Proof. By the transversality condition (b), the tangent map of φ at the point $([e], y)$,

$$(d\varphi)_{([e],y)} : T_{([e],y)}(G/K \times Y) \to T_y X,$$

can be regarded as

$$(d\varphi)_{([e],y)} : T_{[e]}(G/K) \oplus T_y Y \to T_y O_y \oplus T_y Y.$$

Denote $(d\varphi)_{([e],y)}|_{T_{[e]}(G/K)} : T_{[e]}(G/K) \to T_y O_y$ by $\tilde{\Psi}_y$. It is then obvious that $\Psi_y = \tilde{\Psi}_y \circ (d\pi)_e|_{\mathfrak{l}}$, as one can easily prove it in its matrix form

$$(d\varphi)_{([e],y)} = \begin{pmatrix} \tilde{\Psi}_y & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

Since $d\mu$ is the associated Riemannian measure on G/K, the product measure $d\mu\, dy$ is the associated Riemannian measure on the product Riemannian manifold $G/K \times Y'$. By Proposition 2.1 and the orthogonality condition (d),

$$\begin{aligned} J(y) = \left|\det(d\varphi)_{([e],y)}\right| &= \left|\begin{pmatrix} \tilde{\Psi}_y & 0 \\ 0 & \mathrm{id} \end{pmatrix}\right| \\ &= \left|\det \tilde{\Psi}_y\right| = \left|\det\big(\Psi_y \circ ((d\pi)_e|_{\mathfrak{l}})^{-1}\big)\right| = C\left|\det \Psi_y\right|, \end{aligned}$$

where $C = \left|\det\big((d\pi)_e|_{\mathfrak{l}}\big)\right|^{-1}$. □

Remark. Although formula (2–10) only holds on Y', since Y' is dense in Y and $J \in C^\infty(Y)$, we can get $J(y)$ for all $y \in Y$ by smooth continuation.

3. Integration over G-spaces

Occasionally we will be interested in some kinds of integration formulae. In this section we give some preliminaries on integration. The reader who is more interested in the eigenvalue distributions of the generalized random matrix ensembles may skip to Section 4 directly.

The next proposition generalizes the change-of-variables formula for multiple integration.

Proposition 3.1. *Let $\varphi : M \to N$ be a smooth map between two n-dimensional smooth manifolds M and N, and dy a smooth measure on N. If φ is a local diffeomorphism that is a d-sheeted covering map, then, for any $f \in C^\infty(N)$ with $f \geq 0$ or $f \in L^1(N, dy)$, we have*

$$\int_N f(y)\, dy = \frac{1}{d} \int_M f(\varphi(x))\, \varphi^*(dy). \tag{3–1}$$

Proof. It is a standard argument using a partition of unity. □

Remark. Formula (3–1) resembles a formula relating the degree of a map with the integration of volume forms on manifolds. When M, N are compact and oriented, then, under the conditions of Proposition 3.1 and up to a sign, formula (3–1) says nothing if not this. In general, however, the integration of differential forms is not suitable for us. What we need is a change-of-variables formula that ignores the negative sign.

As in the previous section, take a G-space X, where X is an n-dimensional smooth manifold and G is a Lie group. We then have a reduced map $\varphi : G/K \times Y \to X$. Suppose dx, dy, and $d\mu$ are smooth measures on X, Y, and G/K, respectively, with dx and $d\mu$ being G-invariant. Our goal is to convert the integration over X to integration over Y.

Proposition 3.2. *If conditions (a), (b), and (c) hold, then $\varphi : G/K \times Y' \to X'$ is a local diffeomorphism.*

Proof. Let e be the unit element in G. At each $([e], y) \in G/K \times Y'$ we have $d\varphi_{([e],y)}(0, v) = v$ for all $v \in T_yY'$, and so $T_yY' \subset \operatorname{Im}\big(d\varphi_{([e],y)}\big)$. Furthermore, $\varphi|_{G/K\times\{y\}} : G/K \times \{y\} \to O_y \cong G/G_y$ is a local diffeomorphism, so $T_yO_y \subset \operatorname{Im}\big(d\varphi_{([e],y)}\big)$. Thus, $d\varphi_{([e],y)}$ is surjective. But $\dim(G/K \times Y') = \dim X'$, so $d\varphi_{([e],y)}$ is in fact an isomorphism.

For general $([g], y) \in G/K \times Y'$, notice that $\varphi \circ l_g = \sigma_g \circ \varphi$, where $l_g([h], y) = ([gh], y)$, so $d\varphi_{([g],y)} \circ (dl_g)_{([e],y)} = (d\sigma_g)_{([e],y)} \circ d\varphi_{([e],y)}$, and $d\varphi_{([e],y)}$ being an isomorphism implies that $d\varphi_{([g],y)}$ is one as well. Thus, φ is everywhere regular and hence is a local diffeomorphism. □

To make Proposition 3.1 useful, we also require the following condition:

(e) The map $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map, with $d < +\infty$ (*covering condition*).

Theorem 3.3. *If conditions (a), (b), (c), and (e) hold, then*

$$\int_X f(x)\,dx = \frac{1}{d}\int_Y \biggl(\int_{G/K} f(\sigma_g(y))\,d\mu([g]) \biggr) J(y)\,dy \tag{3–2}$$

for all $f \in C^\infty(X)$ with $f \geq 0$ or $f \in L^1(X, dx)$, and where $J \in C^\infty(Y)$ is determined by formula (2–6).

Proof. By Proposition 3.2, $\varphi : G/K \times Y' \to X'$ is a local diffeomorphism. By the covering condition (e), φ is a d-sheeted covering map. So, by Proposition 3.1, for $f \in C^\infty(X)$ with $f \geq 0$ or $f \in L^1(X, dx)$, we have

$$\begin{aligned}
\int_X f(x)\,dx = \int_{X'} f(x)\,dx &= \frac{1}{d}\int_{G/K\times Y'} f\bigl(\varphi([g], y)\bigr)\,\varphi^*(dx)\\
&= \frac{1}{d}\int_{G/K\times Y'} f\bigl(\sigma_g(y)\bigr)\,J(y)\,d\mu([g])\,dy\\
&= \frac{1}{d}\int_{Y'}\biggl(\int_{G/K} f\bigl(\sigma_g(y)\bigr)\,d\mu([g])\biggr)\,J(y)\,dy\\
&= \frac{1}{d}\int_{Y}\biggl(\int_{G/K} f\bigl(\sigma_g(y)\bigr)\,d\mu([g])\biggr)J(y)\,dy
\end{aligned}$$

□

Corollary 3.4. *Under the same conditions as in the previous theorem, if furthermore $f(\sigma_g(x)) = f(x)$ for all $g \in G$ and $x \in X$, then*

$$\int_X f(x)\,dx = \frac{\mu(G/K)}{d}\int_Y f(y)\,J(y)\,dy. \tag{3–3}$$

□

To make this conclusion more useful, we give some criteria on when the map $\varphi : G/K \times Y' \to X'$ is a covering map.

Proposition 3.5. *Let M, N be smooth n-dimensional manifolds. An everywhere-regular smooth map $\varphi : M \to N$ is a d-sheeted covering map if and only if $\varphi^{-1}(y)$ has d points for each $y \in N$.*

Proof. The "only if" part is obvious; we prove the "if" part. For $y \in N$, let $\varphi^{-1}(y) = \{x_1, \dots, x_d\}$. Since φ is everywhere regular, there exist open neighborhoods U_i of x_i, $i = 1, \dots, d$, such that $U_i \cap U_j = \varnothing$ for $i \neq j$, and each $\varphi_i = \varphi|_{U_i} : U_i \to \varphi(U_i)$ is a diffeomorphism. Let $V = \bigcup_{i=1}^d \varphi(U_i)$ and $V_i = \varphi_i^{-1}(V)$. Then $\varphi|_{V_i}$ is also a diffeomorphism onto V. We conclude that $\varphi^{-1}(V) = \bigcup_{i=1}^d V_i$. In fact, if for all $z \in \varphi^{-1}(V)$ we set $z_i = \varphi_i^{-1}(\varphi(z))$, then $z_i \in \varphi^{-1}(\varphi(z))$ and $z_i \neq z_j$ for $i \neq j$. But, since $z \in \varphi^{-1}(\varphi(z))$ and $\bigl|\varphi^{-1}(\varphi(z))\bigr| = d$, this forces $z = z_{i_0}$ for some i_0. Hence $z \in \bigcup_{i=1}^d V_i$. Therefore, $\varphi^{-1}(V) = \bigcup_{i=1}^d V_i$ and the lemma is proved. □

Corollary 3.6. *Suppose conditions (a), (b), and (c) hold. If furthermore there exists $d \in \mathbb{N}$ such that, for all $y \in Y'$,*

(1) *the isotropy subgroup G_y coincides with K,*

(2) $|O_y \cap Y'| = d$,

then $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map.

Proof. By Proposition 3.2, φ is a local diffeomorphism. So, by Proposition 3.5, we need only show that $\varphi^{-1}(x)$ has d points for each $x \in X'$.

For $x \in Y'$, suppose that $O_x \cap Y' = \{y_1, \dots, y_d\}$. Then there exists $g_i \in G$ such that $\sigma_{g_i}(y_i) = x$ for each $i \in \{1, \dots, d\}$. It follows that $([g_i], y_i) \in \varphi^{-1}(x)$. On the other hand, if $([g], y) \in \varphi^{-1}(x)$, then $y = y_{i_0}$ for some $i_0 \in \{1, \dots, d\}$. We have $\sigma_{gg_{i_0}^{-1}}(x) = \sigma_g(y_{i_0}) = x$, that is, $gg_{i_0}^{-1} \in G_x = K$, and so $[g] = [g_{i_0}]$ and $([g], y) = ([g_{i_0}], y_{i_0})$. Thus, $\varphi^{-1}(x) = \big\{([g_1], y_1), \dots, ([g_d], y_d)\big\}$.

In general, for $x \in X'$, suppose that $\sigma_h(x) \in Y'$ for some $h \in G$. Then the relation $\varphi^{-1}(\sigma_h(x)) = l_h(\varphi^{-1}(x))$ reduces the general case to the previous one. □

Both Proposition 3.5 and Corollary 3.6 will be used in a forthcoming article devoted to concrete examples [An et al. 2005].

Remark. The converse of Corollary 3.6 is not true. That is, the isotropy subgroups G_y associated to $y \in Y'$ may change "suddenly", even if Y' is connected. For example, the group $SO(n)$ acts on $\mathbb{RP}^n$ smoothly if we regard $\mathbb{RP}^n$ as a quotient space obtained by gluing opposite points on the boundary of the closed unit ball B^n. If X_z is the image of $\{0\}$ and Y is the image of the segment $\{(x, 0, \dots, 0) \mid |x| \le 1\}$, then the conditions (a), (b), (c), and (e) hold. The isotropy subgroup associated with the image of a point in Y' that is an interior point of B^n is $\operatorname{diag}(1, SO(n-1))$, but, for the image of the point $(1, 0, \dots, 0)$, its isotropy subgroup is $\operatorname{diag}(\pm 1, O^{\pm}(n-1))$; here, $O^{\pm}(n-1) = \{g \in O(n-1) \mid \det g = \pm 1\}$. Other examples exhibiting similar phenomena will appear in [An et al. 2005], where we consider the group ensembles associated with complex semisimple Lie groups. When such sudden variation of the isotropy subgroups happens, it is in general an open problem whether we can make them be the same by enlarging the set X_z.

4. Generalized random matrix ensembles

We are ready to present the generalized random matrix ensembles.

Let G be a Lie group acting on an n-dimensional smooth manifold X by $\sigma : G \times X \to X$. For convenience, suppose X is a Riemannian manifold. Assume the induced Riemannian measure dx is G-invariant (note that we do not require the Riemannian structure on X to be G-invariant). Let Y be a closed submanifold of X, endowed the induced Riemannian measure dy, and let

$$K = \big\{g \in G \;\big|\; \sigma_g(y) = y, \text{ for all } y \in Y\big\}.$$

As in Section 2, we take the map $\varphi : G/K \times Y \to X$ with $\varphi\big([g], y\big) = \sigma_g(y)$. Let $X_z \subset X$ and $Y_z \subset Y$ be closed zero-measure subsets of X and Y, respectively. Set $X' = X \setminus X_z$ and $Y' = Y \setminus Y_z$. We assume that the conditions (a), (b), (c), and (d) of Section 2 hold. For the reader's convenience, we list them below.

(a) $X' = \bigcup_{y \in Y'} O_y$ (*invariance condition*).

(b) $T_y X = T_y O_y \oplus T_y Y$ for all $y \in Y'$ (*transversality condition*).

(c) $\dim G_y = \dim K$ for all $y \in Y'$ (*dimension condition*).

(d) $T_y Y \perp T_y O_y$ for all $y \in Y'$ (*orthogonality condition*).

Suppose $d\mu$ is a G-invariant smooth measure on G/K, and $p(x)$ is a G-invariant smooth function on X. Then, by Corollary 2.3, there is a quasi-smooth measure $d\nu$ on Y such that

$$\varphi^*(p(x)\,dx) = d\mu\,d\nu. \tag{4–1}$$

Definition. Let the conditions and notation be as above. The system

$$(G, \sigma, X, p(x)\,dx, Y, dy)$$

is called a *generalized random matrix ensemble*. The manifolds X and Y are called the *integration manifold* and the *eigenvalue manifold*, respectively. The measure $d\nu$ on Y determined by (4–1) is called the *generalized eigenvalue distribution*.

Recall that in Section 2 we have defined the map $\Psi_y : \mathfrak{l} \to T_y O_y$ by

$$\Psi_y(\xi) = \frac{d}{dt}\Big|_{t=0} \sigma_{\exp t\xi}(y) \qquad \text{for all } \xi \in \mathfrak{l},$$

where $\mathfrak{l}$ is a linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$. Thanks to the preliminaries in Section 2, we can compute the generalized eigenvalue distribution directly with the next theorem, which follows directly from Proposition 2.2, Corollary 2.3, Proposition 2.4, and Theorem 2.5.

Theorem 4.1. *Let $(G, \sigma, X, p(x)\,dx, Y, dy)$ be a generalized random matrix ensemble. The generalized eigenvalue distribution $d\nu$ is given by*

$$d\nu(y) \;=\; \mathcal{P}(y)\,dy \;=\; p(y)\,J(y)\,dy, \tag{4–2}$$

where

$$J(y) = C\,\big|\det \Psi_y\big|, \tag{4–3}$$

with $C = \big|\det\big((d\pi)_e|_{\mathfrak{l}}\big)\big|^{-1}$. □

The function $\mathcal{P}(y) = p(y)\, J(y)$ in (4–2) is the *generalized joint-density function*.

One of the fundamental problems in random matrix theory is to compute the eigenvalue distribution $d\nu$. In our generalized scheme, it is given by formulae (4–2) and (4–3). Note that the power of (4–3) is reflected in the fact that it provides a direct and unified method to compute the eigenvalue distributions of various kinds of random matrix ensembles. In [An et al. 2005] we show how all classical ensembles are included in our generalized scheme, and how the corresponding eigenvalue distributions can be derived from (4–2) and (4–3). We will also present various kinds of generalized ensembles that were not considered before, and compute their eigenvalue distributions explicitly.

Now, we consider the integration formula associated with the generalized random matrix ensemble. As in Section 3, we assume the following condition holds:

(e) The map $\varphi : G/K \times Y' \to X'$ is a d-sheeted covering map, with $d < +\infty$ (*covering condition*).

Theorem 4.2. *Let $(G, \sigma, X, p(x)\, dx, Y, dy)$ be a generalized random matrix ensemble. If the covering condition (e) holds, then we have the integration formula*

$$\int_X f(x)\, p(x)\, dx = \frac{1}{d} \int_Y \left(\int_{G/K} f\big(\sigma_g(y)\big)\, d\mu([g]) \right) d\nu(y) \tag{4–4}$$

for all $f \in C^\infty(X)$ with $f \geq 0$ or $f \in L^1(X, p(x)\, dx)$. If moreover $f(\sigma_g(x)) = f(x)$ for all $g \in G$ and $x \in X$, then

$$\int_X f(x)\, p(x)\, dx = \frac{\mu(G/K)}{d} \int_Y f(y)\, d\nu(y). \tag{4–5}$$

Proof. It is obvious from Theorem 3.3 and Corollary 3.4. □

If the measure $p(x)\, dx$ in (4–5) is a probability measure and we let $f = 1$, we get $\big(\mu(G/K)/d\big) \int_Y d\nu(y) = 1$. So, if G/K is compact, we can normalize the measure $d\mu$ such that $\mu(G/K) = d$, and then the generalized eigenvalue distribution $d\nu$ is a probability measure.

Remark. The condition $f \in C^\infty(X)$ in Theorem 4.2 is superfluous. It is sufficient to assume that f is measurable. The same is true for Proposition 3.1 and Theorem 3.3.

5. Special cases

In this section we discuss several classes of generalized random matrix ensembles: linear ensembles, nonlinear noncompact ensembles, compact ensembles, group ensembles, algebra ensembles, pseudo-group ensembles, and pseudo-algebra ensemble. These account for all kinds of classical random matrix ensembles and some new examples of generalized ensembles.

Linear ensemble and the nonlinear noncompact ensembles. Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$, in the sense of [Knapp 2002, Section 7.2]. The group G admits a global Cartan involution Θ, inducing a Cartan involution θ of $\mathfrak{g}$. Let the corresponding Cartan decomposition of $\mathfrak{g}$ be $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and set

$$K = \{g \in G \mid \Theta(g) = g\} \qquad \text{and} \qquad P = \exp(\mathfrak{p}).$$

K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, while P is a closed submanifold of G satisfying $T_e P = \mathfrak{p}$. The spaces $\mathfrak{p}$ and P are invariant under the adjoint action $\mathrm{Ad}|_K$ and the conjugate action σ of K, respectively. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let A be the connected subgroup of G with Lie algebra $\mathfrak{a}$. Set

$$M = \left\{k \in K \;\middle|\; (\mathrm{Ad}|_K)_k(\eta) = \eta, \text{ for all } \eta \in \mathfrak{a}\right\} = \left\{k \in K \;\middle|\; \sigma_k(a) = a, \text{ for all } a \in A\right\}.$$

It can be shown that there are Riemannian structures on $\mathfrak{p}$ and P inducing K-invariant Riemannian measures dX on $\mathfrak{p}$ and dx on P. They also induce Riemannian measures dY on $\mathfrak{a}$ and da on A. Further, there is a K-invariant smooth measure $d\mu$ on K/M. If $p_1(\xi)$ and $p_2(x)$ are K-invariant positive smooth functions on $\mathfrak{p}$ and P, then it can be proved that the systems

$$(K, \mathrm{Ad}|_K, \mathfrak{p}, p_1(\xi)\, dX(\xi), \mathfrak{a}, dY) \qquad \text{and} \qquad (K, \sigma, P, p_2(x)\, dx, A, da)$$

are generalized random matrix ensembles, which we call *linear ensemble* and *nonlinear noncompact ensemble*, respectively. It can be shown that the Gaussian ensemble and the chiral ensemble are particular examples of linear ensembles, while the new transfer matrix ensembles are particular examples of nonlinear noncompact ensemble.

Compact ensembles. Let G be a connected compact Lie group G with Lie algebra $\mathfrak{g}$. Suppose Θ is a global involution of G with induced involution $\theta = d\Theta$ on $\mathfrak{g}$. Let $K = \{g \in G \mid \Theta(g) = g\}$. Let $\mathfrak{p}$ be the eigenspace of θ of eigenvalue -1, and let $P = \exp(\mathfrak{p})$. Then P is invariant under the conjugate action σ of K. It was proved in [An and Wang 2006] that P is a closed submanifold of G satisfying $T_e P = \mathfrak{p}$, and that it is just the identity component of the set $\{g \in G \mid \Theta(g) = g^{-1}\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let A be the torus with Lie algebra $\mathfrak{a}$. There is a Riemannian structure on P that induces a K-invariant Riemannian measure dx on P and a Riemannian measure da on A. Let

$$M = \left\{k \in K \;\middle|\; \sigma_k(a) = a, \text{ for all } a \in A\right\}.$$

There is a K-invariant smooth measure $d\mu$ on K/M. Take $p(x)$ a K-invariant positive smooth function on P. It can be proved that the system

$$(K, \sigma, P, p(x)\, dx, A, da)$$

is a generalized random matrix ensemble, which we call *compact ensemble*. The circular ensemble and the Jacobi ensembles are particular examples of compact ensembles.

Group and algebra ensembles. Let G be a unimodular Lie group G with Lie algebra $\mathfrak{g}$. There are Riemannian structures on G and $\mathfrak{g}$ inducing a σ-invariant Riemannian measure dg on G and an Ad-invariant Riemannian measure dX on $\mathfrak{g}$, where σ denotes the conjugate action of G on itself. Let $p_1(g)$ and $p_2(\xi)$ be two functions on G and $\mathfrak{g}$, respectively, that are invariant under the corresponding actions of G. If there exists a closed submanifold Y of G such that

$$(G, \sigma, G, p(g)\, dg, Y, dy)$$

is a generalized random matrix ensemble, where dy is the induced Riemannian measure on Y, then we call it a *group ensemble*. And, if there exists a closed submanifold $\mathfrak{y}$ of $\mathfrak{g}$ such that

$$(G, \mathrm{Ad}, \mathfrak{g}, p(\xi)\, dX(\xi), \mathfrak{y}, dY)$$

is a generalized random matrix ensemble, where dY is the induced Riemannian measure on $\mathfrak{y}$, then we call it an *algebra ensemble*.

Among all unimodular Lie groups, connected compact Lie groups and connected complex semisimple Lie groups are of particular interest. For a connected compact Lie group G, we can let the submanifold Y of G be a maximal torus T of G, and let the submanifold $\mathfrak{y}$ of $\mathfrak{g}$ be the Lie algebra of T. For a connected complex semisimple Lie group G, we can let the submanifold $\mathfrak{y}$ of $\mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, and let the submanifold Y of G be the corresponding Cartan subgroup of G. For these cases, it can be proved that the systems $(G, \sigma, G, p(g)\, dg, Y, dy)$ and $(G, \mathrm{Ad}, \mathfrak{g}, p(\xi)\, dX(\xi), \mathfrak{y}, dY)$ are generalized random matrix ensembles.

Pseudogroup and pseudoalgebra ensembles. These are related to real reductive groups. Let G be a real reductive group with lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$, and $\mathfrak{h}_1, \dots, \mathfrak{h}_m$ a maximal set of mutually nonconjugate θ-stable Cartan subalgebras of $\mathfrak{g}$, with corresponding Cartan subgroups $H_1, \dots, H_m$ of G. Denote the sets of all regular elements in G and $\mathfrak{g}$ by G_r and $\mathfrak{g}_r$, respectively. Let $H_j' = H_j \cap G_r$ and $\mathfrak{h}_j' = \mathfrak{h}_j \cap \mathfrak{g}_r$. It is known that

$$G_r = \bigsqcup_{j=1}^{m} \bigcup_{g\in G} \sigma_g(H_j') \qquad \text{and} \qquad \mathfrak{g}_r = \bigsqcup_{j=1}^{m} \bigcup_{g\in G} \mathrm{Ad}_g(\mathfrak{h}_j')$$

(see [Knapp 2002, Theorem 7.108] and [Warner 1972, Proposition 1.3.4.1], respectively). Here, "$\sqcup$" means disjoint union. Each $\bigcup_{g\in G} \sigma_g(H_j')$ is an open set in

G, and each $\bigcup_{g\in G}\mathrm{Ad}_g(\mathfrak{h}_j')$ is an open set in $\mathfrak{g}$. Let

$$G_j=\overline{\bigcup_{g\in G}\sigma_g(H_j')}\qquad\text{and}\qquad \mathfrak{g}_j=\overline{\bigcup_{g\in G}\mathrm{Ad}_g(\mathfrak{h}_j')}.$$

It can be shown that some suitable Riemannian structures on G and $\mathfrak{g}$ induce, for each j, a σ-invariant measure dg_j on G_j, and an Ad-invariant measure dX_j on $\mathfrak{g}_j$, and that they also induce Riemannian measures dh_j on H_j and dY_j on $\mathfrak{h}_j$. It is known that

$$Z(H_j)=\{g\in G\mid \sigma_g(h)=h,\ \text{for all } h\in H_j\},$$
$$H_j=\{g\in G\mid \mathrm{Ad}_g(\xi)=\xi,\ \text{for all } \xi\in\mathfrak{h}_j\}.$$

Let $d\mu_j'$ and $d\mu_j$ be G-invariant measures on $G/Z(H_j)$ and G/H_j, respectively. In general, the spaces G_j and $\mathfrak{g}_j$ may have singularities, but this doesn't matter, since these spaces are closures of open submanifolds in G and $\mathfrak{g}$, whose boundaries have measure zero. If we ignore this ambiguity, it can be proved that

$$(G,\sigma,G_j,dg_j,H_j,dh_j)\qquad\text{and}\qquad (G,\mathrm{Ad},\mathfrak{g}_j,dX_j,\mathfrak{h}_j,dY_j)$$

are generalized random matrix ensembles, which we call *pseudogroup ensemble* and *pseudoalgebra ensemble*, respectively.

The classes introduced above do not exhaust all generalized ensembles. But they include all kinds of classical random matrix ensembles and some new examples of generalized ensembles, which will be studied in [An et al. 2005].

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References

[Altland and Zirnbauer 1997] A. Altland and M. R. Zirnbauer, "Novel symmetry classes in mesoscopic normal conducting-superconducting hybrid structures", *Phys. Rev. B* **55** (1997), 1142–1161.

[An and Wang 2006] J. An and Z. Wang, "On the realization of Riemannian symmetric spaces in Lie groups", *Topology and its Applications* **153**:7 (2006), 1008–1015. MR 2203015 Zbl 1086.53072 math.GT/0502028

[An et al. 2005] J. An, Z. Wang, and K. Yan, "A generalization of random matrix ensemble II: concrete examples and integration formulae", preprint, 2005. math-ph/0502021

[Caselle 1994] M. Caselle, "On the distribution of transmission eigenvalues in disordered wires", preprint, 1994. cond-mat/9410097

[Caselle 1996] M. Caselle, "A new classification scheme for random matrix theories", preprint, 1996. cond-mat/9610017

[Caselle and Magnea 2004] M. Caselle and U. Magnea, "Random matrix theory and symmetric spaces", *Phys. Rep.* **394**:2-3 (2004), 41–156. MR 2005d:82049

[Dueñez 2004] E. Dueñez, "Random matrix ensembles associated to compact symmetric spaces", *Comm. Math. Phys.* **244**:1 (2004), 29–61. MR 2005g:15039 Zbl 1041.15017

[Dyson 1970] F. J. Dyson, "Correlations between eigenvalues of a random matrix", *Comm. Math. Phys.* **19** (1970), 235–250. MR 43 #4398 Zbl 0221.62019

[Helgason 2000] S. Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs **83**, American Mathematical Society, Providence, RI, 2000. MR 2001h:22001

[Ivanov 2001] D. A. Ivanov, "Random-matrix ensembles in p-wave vortices", preprint, 2001. cond-mat/0103089

[Knapp 2002] A. W. Knapp, *Lie groups beyond an introduction*, Progress in Mathematics **140**, Birkhäuser, Boston, 2002. MR 2003c:22001 Zbl 1075.22501

[Mehta 1991] M. L. Mehta, *Random matrices*, Academic Press, Boston, 1991. MR 92f:82002 Zbl 0780.60014

[Titov et al. 2001] M. Titov, P. W. Brouwer, A. Furusaki, and C. Mudry, "Fokker–Planck equations and density of states in disordered quantum wires", *Phys. Rev. B* **63** (2001), 235–318.

[Warner 1972] G. Warner, *Harmonic analysis on semi-simple Lie groups*, vol. I, Springer, New York, 1972. MR 58 #16979 Zbl 0265.22020

[Zirnbauer 1996] M. R. Zirnbauer, "Riemannian symmetric superspaces and their origin in random-matrix theory", *J. Math. Phys.* **37**:10 (1996), 4986–5018. MR 97m:58012 Zbl 0871.58005

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