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CURVATURE OF SPECIAL ALMOST HERMITIAN MANIFOLDS

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We study the curvature of almost Hermitian manifolds and their special analogues via intrinsic torsion and representation theory. By deriving different formulae for the skew-symmetric part of the $*$-Ricci curvature, we find that some of these contributions are dependent on the approach used and, for the almost Hermitian case, we obtain tables that differ from those of Falcitelli, Farinola, and Salamon. We show how the exterior algebra may be used to explain some of these variations.

1. Introduction

Tricerri and Vanhecke [1981] gave a complete decomposition of the Riemannian curvature tensor R of an almost Hermitian manifold

$$\big(M, I, \langle\cdot,\cdot\rangle\big)$$

into irreducible $\mathrm{U}(n)$-components. These divide naturally into two groups, one forming the space $\mathcal{K}=\mathcal{K}(\mathfrak{u}(n))$ of algebraic curvature tensors for a Kähler manifold, and the other being its orthogonal complement $\mathcal{K}^{\perp}$.

Falcitelli et al. [1994] showed that the components of R in $\mathcal{K}^{\perp}$ are linearly determined by the covariant derivative $\nabla\xi$, where ∇ is the Levi-Civita connection and ξ is the intrinsic torsion of the $\mathrm{U}(n)$-structure on M. Gray and Hervella [1980] showed that, in general dimensions, ξ may be split into four components $\xi_1,\dots,\xi_4$ under the action of $\mathrm{U}(n)$. By using the minimal $\mathrm{U}(n)$-connection $\widetilde{\nabla}=\nabla+\xi$ of M, Falcitelli et al. display some tables showing whether the tensors $\widetilde{\nabla}\xi_i$ and $\xi_i\odot\xi_j$ contribute to the components of R in $\mathcal{K}^{\perp}$. This provides a unified approach to many of the curvature results obtained in [Gray 1976a].

The present paper is motivated by the interest in extending the above-mentioned results to special almost Hermitian manifolds. These are defined as almost Hermitian manifolds $\big(M, I, \langle\cdot,\cdot\rangle\big)$ equipped with a complex volume form

$$\Psi=\psi_{+}+i\psi_{-}.$$

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Equivalently, they are manifolds with structure group $SU(n)$. A detailed study of the intrinsic torsion $\eta+\xi$ of such manifolds was made in [Martín Cabrera 2005], extending results of Chiossi and Salamon [2002]; here, ξ is the intrinsic $U(n)$-torsion, as above, and η is essentially a 1-form. There is much current interest in $SU(n)$-structures, partly as generalisations of Calabi–Yau manifolds [Grantcharov et al. 2003; Banos 2002] and partly because of the role played by torsion connections with $SU(n)$ holonomy in string theory [Papadopoulos 1999; Gutowski et al. 2003].

For $SU(n)$ structures, the algebraic curvature tensors lie in $\mathcal{K}(\mathfrak{su}(n))$ and are automatically Ricci-flat. Therefore, one may compute the Ricci curvature Ric, and indeed the $*$-Ricci curvature Ric^*, in terms of the intrinsic $SU(n)$-torsion $\eta+\xi$. This enables us to find information about those $SU(n)$-components of the Riemannian curvature R that are determined by the tensors Ric and Ric^*. Some of these components are contained in $\mathcal{K}^\perp$, and others are contained in $\mathcal{K}$. This will allow us, on the one hand, to get more concrete information about some components of R contained in $\mathcal{K}^\perp$ and, on the other hand, to enlarge the tables of Falcitelli et al. with columns related to some components contained in $\mathcal{K}$.

In working out these contributions, we arrived at various alternative formulae for certain curvature components purely in terms of the intrinsic $U(n)$-torsion ξ. This leads to some table entries that are different from those obtained by Falcitelli et al. To try to account for this, we consider the identity $d^2=0$ in the exterior algebra. Applying this to the Kähler 2-form ω and considering a particular component leads indeed to a nontrivial relation between the tensors contributing to the curvature. One may view the relation $d^2\omega=0$ as one way of taking into account some of the information that the Levi-Civita connection connection $\nabla=\widetilde{\nabla}-\xi$ is torsion-free.

The paper is organized as follows. In Section 2 we present some preliminary material: definitions, results, notation, etc. Then, in Section 3 we derive some formulae relating the curvature and the intrinsic torsion. As an immediate application, we give an alternative proof of the result of Gray [1976b] that any nearly Kähler manifold of dimension 6 that is not Kähler is an Einstein manifold. We then proceed to computing the contributions of different components of the intrinsic torsion and its covariant derivative to the Ricci, $*$-Ricci and Riemannian curvatures. Because of representation theory, this behaves differently in dimensions 4 and 6 than in higher dimensions: in dimension 6, ξ splits into more $SU(3)$-components; in dimension 4, the space of curvature tensors is decomposed more finely under the action of $SU(2)$. This motivates us to display results and tables in two separate sections: Section 4 for high dimensions, $2n \geqslant 8$, and Section 5 for dimensions 6 and 4. Finally, in Section 6 we discuss identities derived from the exterior algebra.

Note. We will often use decompositions of tensor products without providing details, since such information can be readily obtained from available software.

2. Preliminaries

An *almost Hermitian manifold* is a $2n$-dimensional manifold M, $n > 0$, with a $\mathrm{U}(n)$-structure. This means that M is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$ and an orthogonal almost complex structure I. Each fibre $T_m M$ of the tangent bundle can be considered as a complex vector space by defining $ix = Ix$. We will write $T_m M_{\mathbb{C}}$ when we are regarding $T_m M$ as such a space.

We define a Hermitian scalar product $\langle \cdot, \cdot \rangle_{\mathbb{C}} = \langle \cdot, \cdot \rangle + i\omega(\cdot, \cdot)$, where ω is the Kähler form given by $\omega(x, y) = \langle x, Iy \rangle$. The real tangent bundle TM is identified with the cotangent bundle T^*M by the map $x \mapsto \langle \cdot, x \rangle = x$. Similarly, the conjugate complex vector space $\overline{T_m M_{\mathbb{C}}}$ is identified with the dual complex space $T_m^* M_{\mathbb{C}}$ by the map $x \mapsto \langle \cdot, x \rangle_{\mathbb{C}} = x_{\mathbb{C}}$. It follows immediately that $x_{\mathbb{C}} = x + iIx$.

If we consider the spaces $\bigwedge^p T_m^* M_{\mathbb{C}}$ of skew-symmetric complex forms, one can check that $x_{\mathbb{C}} \wedge y_{\mathbb{C}} = (x + iIx) \wedge (y + iIy)$. There are natural extensions of the scalar products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ to $\bigwedge^p T_m^* M$ and $\bigwedge^p T_m^* M_{\mathbb{C}}$, defined respectively by

$$\langle a, b \rangle = \frac{1}{p!} \sum_{i_1, \dots, i_p = 1}^{2n} a(e_{i_1}, \dots, e_{i_p})\, b(e_{i_1}, \dots, e_{i_p}),$$

$$\langle a_{\mathbb{C}}, b_{\mathbb{C}} \rangle_{\mathbb{C}} = \frac{1}{p!} \sum_{i_1, \dots, i_p = 1}^{n} a_{\mathbb{C}}(u_{i_1}, \dots, u_{i_p})\, \overline{b_{\mathbb{C}}(u_{i_1}, \dots, u_{i_p})},$$

where $e_1, \dots, e_{2n}$ is an orthonormal basis for real vectors, and $u_1, \dots, u_n$ is a unitary basis for complex vectors.

The following conventions will be used in this paper. If b is a $(0, s)$-tensor, we write

$$I_{(i)} b(X_1, \dots, X_i, \dots, X_s) = -b(X_1, \dots, IX_i, \dots, X_s),$$
$$Ib(X_1, \dots, X_s) = (-1)^s b(IX_1, \dots, IX_s).$$

Tricerri and Vanhecke [1981] gave a complete decomposition of the Riemannian curvature tensor R of an almost Hermitian manifold $\big(M, I, \langle \cdot, \cdot \rangle\big)$ into irreducible $\mathrm{U}(n)$-components. As indicated above, some of these components, constituting a $\mathrm{U}(n)$-space denoted by $\mathcal{K} = \mathcal{K}(\mathfrak{u}(n))$, are the only components that can occur when M is a Kähler manifold. In this text we will follow the notation used in [Falcitelli et al. 1994] for such components. Likewise, we will adopt the formalism used in [Salamon 1989] and [Falcitelli et al. 1994] for irreducible $\mathrm{U}(n)$-modules. Thus, for $n \geqslant 2$,

$$\mathcal{K} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2,$$

where $\mathcal{C}_3 \cong [\sigma_0^{2,2}]$, $\mathcal{K}_1 \cong \mathbb{R}$, $\mathcal{K}_2 \cong [\lambda_0^{1,1}]$, and $+$ denotes direct sum. We recall that $\lambda_0^{p,q}$ is a complex irreducible $\mathrm{U}(n)$-module coming from the (p, q)-part of the complex exterior algebra, and that its corresponding dominant weight in standard coordinates is given by $(1, \dots, 1, 0, \dots, 0, -1, \dots, -1)$, where 1 and -1 are repeated p and q times, respectively. By analogy with the exterior algebra, there are also irreducible $\mathrm{U}(n)$-modules $\sigma_0^{p,q}$, with dominant weights $(p, 0, \dots, 0, -q)$ coming from the symmetric algebra. The notation $[\![V]\!]$ stands for the real vector space underlying a complex vector space V, and $[W]$ denotes a real vector space that admits W as its complexification.

Moreover, let Ric and Ric^* be the Ricci and $*$-Ricci curvatures, defined by

$$\mathrm{Ric}(X, Y) = \langle R_{X,e_i} Y,\ e_i\rangle, \qquad \mathrm{Ric}^*(X, Y) = \langle R_{X,e_i} IY,\ Ie_i\rangle,$$

where $R_{X,Y} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, and the summation convention is used.

The components of the curvature R in $\mathcal{K}_1$ and $\mathcal{K}_2$ are determined by, respectively, the trace and the trace-free components of $\mathrm{Ric}_H + 3\,\mathrm{Ric}^*_H$ (see [Tricerri and Vanhecke 1981]), where b_H indicates the Hermitian part of a bilinear form b, that is, the part satisfying $b_H(IX, IY) = b_H(X, Y)$. Note that Ric^*_H coincides with the symmetric part of Ric^*.

The remaining components of R, not included in $\mathcal{K}$, are contained in a $\mathrm{U}(n)$-space denoted by $\mathcal{K}^\perp$. For $n \geqslant 4$, one has [Falcitelli et al. 1994]:

$$\mathcal{K}^\perp = \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,$$

where $\mathcal{K}_{-1} \cong \mathbb{R}$, $\mathcal{K}_{-2} \cong [\lambda_0^{1,1}]$, $\mathcal{C}_4 \cong [\lambda_0^{2,2}]$, $\mathcal{C}_5 \cong [\![U]\!]$, $\mathcal{C}_6 \cong [\![\lambda^{2,0}]\!]$, $\mathcal{C}_7 \cong [\![V]\!]$, and $\mathcal{C}_8 \cong [\![\sigma^{2,0}]\!]$. The irreducible $\mathrm{U}(n)$-modules U and V have dominant weights $(2, 2, 0, \dots, 0)$ and $(2, 1, 0, \dots, 0, -1)$. For $n = 3$, the decomposition of $\mathcal{K}^\perp$ is formed by the same summands but omitting $\mathcal{C}_4$. Finally, when $n = 2$ we have to omit $\mathcal{K}_{-2}$, $\mathcal{C}_4$, and $\mathcal{C}_7$.

We are dealing with G-structures where G is a subgroup of the linear group $\mathrm{GL}(m, \mathbb{R})$. If M possesses a G-structure, then there always exists a G-connection defined on M. Moreover, if $\big(M^m, \langle \cdot\,, \cdot \rangle\big)$ is an orientable m-dimensional Riemannian manifold and G a closed and connected subgroup of $\mathrm{SO}(m)$, then there exists a unique metric G-connection $\widetilde{\nabla}$ such that $\xi_x = \widetilde{\nabla}_x - \nabla_x$ takes its values in $\mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ denotes the orthogonal complement in $\mathfrak{so}(m)$ of the Lie algebra $\mathfrak{g}$ of G, and ∇ is the Levi-Civita connection [Salamon 1989; Cleyton and Swann 2004]. The tensor ξ is the *intrinsic torsion* of the G-structure, and $\widetilde{\nabla}$ is called the *minimal G-connection*.

For $\mathrm{U}(n)$-structures, the minimal $\mathrm{U}(n)$-connection is given by $\widetilde{\nabla} = \nabla + \xi$, with

$$\xi_X Y = -\tfrac{1}{2} I(\nabla_X I)Y, \tag{2–1}$$

see [Falcitelli et al. 1994]. Since $\mathrm{U}(n)$ stabilizes the Kähler form ω, it follows that $\widetilde{\nabla}\omega = 0$. Moreover, $\xi_X(IY) + I(\xi_X Y) = 0$ implies $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(n)^\perp$. Thus, one can identify the $\mathrm{U}(n)$-components of ξ with those of $\nabla\omega$:

(1) if $n = 1$, $\xi \in T^*M \otimes \mathfrak{u}(1)^\perp = \{0\}$;

(2) if $n = 2$, $\xi \in T^*M \otimes \mathfrak{u}(2)^\perp = \mathcal{W}_2 + \mathcal{W}_4$;

(3) if $n \geqslant 3$, $\xi \in T^*M \otimes \mathfrak{u}(n)^\perp = \mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3 + \mathcal{W}_4$.

The summands $\mathcal{W}_i$ are the irreducible $U(n)$-modules given by Gray and Hervella [1980], so $\mathcal{W}_1 \cong [\![\lambda^{3,0}]\!]$, $\mathcal{W}_2 \cong [\![A]\!]$, $\mathcal{W}_3 \cong [\![\lambda_0^{2,1}]\!]$, and $\mathcal{W}_4 \cong [\![\lambda^{1,0}]\!]$, where $A \subset \lambda^{1,0} \otimes \lambda^{2,0}$ is the irreducible $U(n)$-module with dominant weight $(2, 1, 0, \dots, 0)$. In the following, ξ_i will denote the component in $\mathcal{W}_i$ of the torsion tensor ξ.

Falcitelli et al. [1994] proved that the components of R in $\mathcal{K}^\perp$ are linearly determined by the covariant derivative $\nabla\xi$ with respect to the Levi-Civita connection ∇. To prove this result, they consider the space $\mathcal{R} = \mathcal{K} + \mathcal{K}^\perp$ of curvature tensors (we recall that $\mathcal{R}$ is the kernel of the mapping $\odot^2\big(\bigwedge^2 T_m^*M\big) \to \bigwedge^4 T_m^*M$ defined by wedging 2-forms together). Then, they deduce that the orthogonal projection $\pi^\perp = (\pi_2 \circ \pi_1)\,|_{\mathcal{R}} : \mathcal{R} \to \mathcal{K}^\perp$ can be expressed as the restriction to $\mathcal{R}$ of the composition map $\pi_2 \circ \pi_1$, where $\pi_1 : \bigwedge^2 T_m^*M \otimes \bigwedge^2 T_m^*M \to \bigwedge^2 T_m^*M \otimes \mathfrak{u}(n)^\perp$ is the orthogonal projection, and $\pi_2 : \bigwedge^2 T_m^*M \otimes \mathfrak{u}(n)^\perp \to \mathcal{K}^\perp$ is a certain $U(n)$-equivariant homomorphism. Since we have the identity [Falcitelli et al. 1994]

$$\begin{aligned}\pi_1(R)(X, Y, Z, W)&\\ &= \langle(\nabla_X I\xi)_Y\, IZ,\ W\rangle - \langle(\nabla_Y I\xi)_X\, IZ,\ W\rangle\\ &= \langle(\nabla_X\xi)_Y\, Z,\ W\rangle - \langle(\nabla_Y\xi)_X\, Z,\ W\rangle + 2\langle\xi_X\xi_Y Z,\ W\rangle - 2\langle\xi_Y\xi_X Z,\ W\rangle,\end{aligned}$$

with the third and fourth summands in $\bigwedge^2 T_m^*M \otimes \mathfrak{u}(n)$, and since π_2 is $U(n)$-equivariant, it follows that the components of $\pi^\perp(R)$ in $\mathcal{K}^\perp$ are linear functions of the components of $\nabla\xi$. Now, taking the $U(n)$-connection $\widetilde{\nabla} = \nabla + \xi$ into account, one obtains

$$\begin{aligned}(2\text{–}2)\quad \pi_1(R)(X, Y, Z, W)&\\ &= \langle(\widetilde{\nabla}_X\xi)_Y Z, W\rangle - \langle(\widetilde{\nabla}_Y\xi)_X Z, W\rangle + \langle\xi_{\xi_X Y} Z, W\rangle - \langle\xi_{\xi_Y X} Z, W\rangle.\end{aligned}$$

From this equation and by considering the image $\pi_2 \circ \pi_1(R)$, Falcitelli et al. give some tables that show whether the tensors $\widetilde{\nabla}\xi_i$ and $\xi_i \odot \xi_j$ contribute to the components of R in $\mathcal{K}^\perp$.

Here, we also consider manifolds equipped with an $SU(n)$-structure. Such manifolds are called *special almost Hermitian manifolds*. These are almost Hermitian manifolds $\big(M, I, \langle\cdot,\cdot\rangle\big)$ equipped with a complex volume form $\Psi = \psi_+ + i\psi_-$ such that $\langle\Psi, \Psi\rangle_{\mathbb{C}} = 1$. Note that $I_{(i)}\psi_+ = \psi_-$. See [Martín Cabrera 2005] for details and more exhaustive information, or [Bryant 1999; Joyce 2000; Hitchin 1997].

For a special almost Hermitian $2n$-manifold M, we have the intrinsic torsion $\eta + \xi \in T^*M \otimes \mathbb{R}\omega + T^*M \otimes \mathfrak{u}(n)^\perp = T^*M \otimes \mathfrak{su}(n)^\perp$ and the minimal $\mathrm{SU}(n)$-connection $\overline{\nabla} = \nabla + \eta + \xi$. Since $\overline{\nabla}$ is metric and $\eta \in T^*M \otimes \mathbb{R}\omega$, we have $\langle Y, \eta_X Z\rangle = \hat{\eta}(X)\omega(Y, Z)$, where $\hat{\eta}$ is a 1-form. Hence,

$$\eta_X Y = \hat{\eta}(X) I Y.$$

In [Martín Cabrera 2005] it is shown that the 1-form $\hat{\eta}$ is given by

$$-I\hat{\eta} = \frac{1}{2^{n-1}n} *(*\, d\psi_+ \wedge \psi_+ + *\, d\psi_- \wedge \psi_-) - \frac{1}{2n} I\, d^*\omega,$$

where $*$ is the Hodge star operator and d^* the coderivative. This formula simplifies for $n \geqslant 3$, since then $*\, d\psi_+ \wedge \psi_+ = *\, d\psi_- \wedge \psi_-$, and one sees that $nI\hat{\eta} - \frac{1}{2} I\, d^*\omega$ is essentially the coefficient of Ψ in the $(n, 1)$-part of $d\Psi$. The other part of the intrinsic torsion $\xi \in T^*M \otimes \mathfrak{u}(n)^\perp$ is still given by equation (2–1).

The tensors ω, ψ_+, and ψ_- are stabilised by the $\mathrm{SU}(n)$-action, and we have $\overline{\nabla}\omega = 0$, $\overline{\nabla}\psi_+ = 0$, and $\overline{\nabla}\psi_- = 0$. Moreover, one can check that $\eta\omega = 0$ and obtain $\nabla\omega = -\xi\omega \in T^*M \otimes \mathfrak{u}(n)^\perp$. In general, the above-mentioned $\mathrm{U}(n)$-spaces $\mathcal{W}_i$ are also irreducible as $\mathrm{SU}(n)$-spaces. The only exceptions are $\mathcal{W}_1$ and $\mathcal{W}_2$ when $n = 3$. In fact, for that case, we have the following decompositions into irreducible $\mathrm{SU}(3)$-components:

$$\mathcal{W}_i = \mathcal{W}_i^+ + \mathcal{W}_i^-, \qquad i = 1, 2,$$

where the spaces $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$ consist of those tensors $a \in \mathcal{W}_i \subseteq T^*M \otimes \bigwedge^2 T^*M$ such that the bilinear form $r(a)$, defined by $2r(a)(x, y) = \langle x \lrcorner \psi_+,\ y \lrcorner a\rangle$, is, respectively, symmetric or skew-symmetric, see [Martín Cabrera 2005; Chiossi and Salamon 2002]. The components of the tensor ξ in $\mathcal{W}_i^+$ and $\mathcal{W}_i^-$, $i = 1, 2$, will be denoted by ξ_i^+ and ξ_i^-. Writing $\eta \in \mathcal{W}_5 \cong T^*M$, the intrinsic $\mathrm{SU}(n)$-torsion $\xi + \eta$ is contained in $\big(T^*M \otimes \mathfrak{u}(n)^\perp\big) + \mathcal{W}_5$. The space $\mathcal{W}_5$ is always $\mathrm{SU}(n)$-irreducible.

From the equations $\overline{\nabla}\psi_+ = 0$ and $\overline{\nabla}\psi_- = 0$, we have $\nabla\psi_+ = -\xi\psi_+ - \eta\psi_+$ and $\nabla\psi_- = -\xi\psi_- - \eta\psi_-$. Moreover, for $n \geqslant 2$, it is shown in [Martín Cabrera 2005] that

$$\begin{gathered} \xi_X\psi_+,\ \xi_X\psi_- \in [\![\lambda^{n-2,0}]\!] \wedge \omega, \\ \eta_X\psi_+ = n\,\hat{\eta}(X)\psi_- \qquad \text{and} \qquad \eta_X\psi_- = -n\,\hat{\eta}(X)\psi_+. \end{gathered} \tag{2–3}$$

When considering curvature, note that the module $\mathcal{C}_3 = \mathcal{K}(\mathfrak{su}(n))$ in $\mathcal{K}$ consists of the algebraic curvature tensors for a metric with holonomy algebra $\mathfrak{su}(n)$.

3. Some curvature formulae

For special almost Hermitian $2n$-manifolds, results and tables given in [Falcitelli et al. 1994] are still valid with respect to the tensors $\widetilde{\nabla}\xi_i$ and $\xi_i \odot \xi_j$. Here, $\widetilde{\nabla} = \overline{\nabla} - \eta$ is the minimal $\mathrm{U}(n)$-connection, with $\overline{\nabla}$ denoting the minimal $\mathrm{SU}(n)$-connection.

For $\mathrm{SU}(n)$-structures, the additional information coming from η will allow us to compute the components of R in $\mathcal{K}_1$ and $\mathcal{K}_2$ in terms of the intrinsic torsion $\eta+\xi$. To achieve this, we compute the difference between the Ricci and the $*$-Ricci curvatures. In the first instance, we only need the almost Hermitian structure.

Lemma 3.1. *If M is an almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal $\mathrm{U}(n)$-connection $\widetilde{\nabla} = \nabla + \xi$, then*

$$
\begin{aligned}
\operatorname{Ric}^*&(X, Y) - \operatorname{Ric}(X, Y)\\
&= 2\langle (\nabla_{e_i} I\xi)_X IY,\ e_i\rangle - 2\langle (\nabla_X I\xi)_{e_i} IY,\ e_i\rangle,\\
&= 2\langle (\widetilde{\nabla}_{e_i}\xi)_X Y,\ e_i\rangle - 2\langle (\widetilde{\nabla}_X\xi)_{e_i} Y,\ e_i\rangle + 2\langle \xi_{\xi_{e_i}X} Y,\ e_i\rangle - 2\langle \xi_{\xi_X e_i} Y,\ e_i\rangle.
\end{aligned}
$$

Proof. It is straightforward to check that

$$\operatorname{Ric}^*(X, Y) - \operatorname{Ric}(X, Y) = -(R_{X,e_i}\omega)(IY, e_i). \tag{3–1}$$

However, the so-called Ricci formula [Besse 1987, p. 26] implies that

$$-(R_{X,e_i}\omega)(IY, e_i) = \tilde{\mathbf{a}}(\nabla^2\omega)_{X,e_i}(IY, e_i), \tag{3–2}$$

where $\tilde{\mathbf{a}} : T^*M \otimes T^*M \otimes \bigwedge^2 T^*M \to \bigwedge^2 T^*M \otimes \bigwedge^2 T^*M$ is the alternation map.

The required identities follow from equations (3–1) and (3–2), by taking into account that $\widetilde{\nabla}\omega = 0$. □

The components of R in $\mathcal{K}_{-1}$ and $\mathcal{K}_{-2}$ are determined by the trace and trace-free parts of $\operatorname{Ric}^*_H - \operatorname{Ric}_H$. Similarly, the $\mathcal{C}_6$-component of R is determined by the skew-symmetric (or anti-Hermitian) part $\operatorname{Ric}^*_{AH}$ of Ric^*. Moreover, the anti-Hermitian part Ric_{AH} of the Ricci curvature, which satisfies $\operatorname{Ric}_{AH}(IX, IY) = -\operatorname{Ric}_{AH}(X, Y)$, determines the component of R in $\mathcal{C}_8$. These assertions motivate the expressions contained in the next lemma.

Lemma 3.2. *If M is an almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal $\mathrm{U}(n)$-connection $\widetilde{\nabla} = \nabla + \xi$, then*

$$
\begin{aligned}
(\operatorname{Ric}^*_H - \operatorname{Ric}_H)(X, Y) &= \langle (\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle - \langle (\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle\\
&\quad + \langle (\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle - \langle (\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle + \langle \xi_{\xi_{e_i}X} Y, e_i\rangle\\
&\quad - \langle \xi_{\xi_X e_i} Y, e_i\rangle + \langle \xi_{\xi_{e_i}IX} IY, e_i\rangle - \langle \xi_{\xi_{IX} e_i} IY, e_i\rangle,
\end{aligned} \tag{3–3}
$$

$$
\begin{aligned}
2\operatorname{Ric}^*_{AH}(X, Y) &= \langle (\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle - \langle (\widetilde{\nabla}_{e_i}\xi)_Y X, e_i\rangle - \langle (\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle\\
&\quad + \langle (\widetilde{\nabla}_{e_i}\xi)_{IY} IX, e_i\rangle - \langle (\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle + \langle (\widetilde{\nabla}_Y\xi)_{e_i} X, e_i\rangle\\
&\quad + \langle (\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle - \langle (\widetilde{\nabla}_{IY}\xi)_{e_i} IX, e_i\rangle + \langle \xi_{\xi_X e_i} Y, e_i\rangle\\
&\quad - \langle \xi_{\xi_Y e_i} X, e_i\rangle - \langle \xi_{\xi_{IX} e_i} IY, e_i\rangle + \langle \xi_{\xi_{IY} e_i} IX, e_i\rangle,
\end{aligned} \tag{3–4}
$$

$$\begin{aligned}
2\,\mathrm{Ric}_{AH}(X,Y) = {} & -\langle(\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle + \langle(\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle - \langle(\widetilde{\nabla}_{e_i}\xi)_Y X, e_i\rangle \\
& + \langle(\widetilde{\nabla}_Y\xi)_{e_i} X, e_i\rangle + \langle(\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle - \langle(\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle \\
& + \langle(\widetilde{\nabla}_{e_i}\xi)_{IY} IX, e_i\rangle - \langle(\widetilde{\nabla}_{IY}\xi)_{e_i} IX, e_i\rangle - \langle\xi_{\xi_{e_i}X} Y, e_i\rangle \\
& + \langle\xi_{\xi_X e_i} Y, e_i\rangle - \langle\xi_{\xi_{e_i}Y} X, e_i\rangle + \langle\xi_{\xi_Y e_i} X, e_i\rangle + \langle\xi_{\xi_{e_i}IX} IY, e_i\rangle \\
& - \langle\xi_{\xi_{IX}e_i} IY, e_i\rangle + \langle\xi_{\xi_{e_i}IY} IX, e_i\rangle - \langle\xi_{\xi_{IY}e_i} IX, e_i\rangle.
\end{aligned} \tag{3–5}$$

Proof. This follows from Lemma 3.1 together with $\langle\xi_{\xi_{e_i}X}Y, e_i\rangle = \langle\xi_{\xi_{e_i}Y}X, e_i\rangle$. $\square$

Up to this point, we have not said anything particular to $\mathrm{SU}(n)$-structures. We now give a first result that uses the complex volume form Ψ.

Lemma 3.3. *If M is a special almost Hermitian $2n$-manifold, $n \geqslant 2$, with complex volume form $\Psi = \psi_+ + i\psi_-$ and minimal $\mathrm{SU}(n)$-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$, then*

$$\mathrm{Ric}^*(X,Y) = -n\,d\hat{\eta}(X, IY) - \langle\xi_X e_i,\ \xi_{IY} I e_i\rangle, \tag{3–6}$$

$$\begin{aligned}
\mathrm{Ric}(X,Y) = {} & -n\,d\hat{\eta}(X, IY) - \langle\xi_X e_i,\ \xi_{IY} I e_i\rangle - 2\langle(\widetilde{\nabla}_{e_i}\xi)_X Y,\ e_i\rangle \\
& + 2\langle(\widetilde{\nabla}_X\xi)_{e_i} Y,\ e_i\rangle - 2\langle\xi_{\xi_{e_i}X} Y,\ e_i\rangle + 2\langle\xi_{\xi_X e_i} Y,\ e_i\rangle.
\end{aligned} \tag{3–7}$$

Proof. Start by noticing that $\langle R_{X,Y}\psi_+,\ \psi_-\rangle = -2^{n-2}\langle R_{X,Y} I e_i,\ e_i\rangle$. By the first Bianchi identity, we have

$$\langle R_{X,Y}\psi_+, \psi_-\rangle = -2^{n-1}\,\mathrm{Ric}^*(X, IY). \tag{3–8}$$

On the other hand, using the Ricci formula $-R_{X,Y}\psi_+ = \tilde{\mathbf{a}}(\nabla^2\psi_+)_{(X,Y)}$ and taking $\overline{\nabla} = \nabla + \eta + \xi$ into account, we obtain

$$\begin{aligned}
-R_{X,Y}\psi_+ = {} & n\,d\hat{\eta}(X,Y)\psi_- + n\,\hat{\eta}(X)(\xi_Y\psi_-) - n\,\hat{\eta}(Y)(\xi_X\psi_-) \\
& + Y \lrcorner \bigl(\nabla_X(\xi\psi_+)\bigr) - X \lrcorner \bigl(\nabla_Y(\xi\psi_+)\bigr).
\end{aligned}$$

Using the inclusions of (2–3), we have $\langle\xi_X\psi_+, \psi_-\rangle = 0$, $\langle\xi_X\psi_-, \psi_-\rangle = 0$, and $\langle Y \lrcorner \bigl(\nabla_X(\xi\psi_+)\bigr), \psi_-\rangle = -\langle\xi_X(\xi_Y\psi_+), \psi_-\rangle$. This gives

$$\langle R_{X,Y}\psi_+, \psi_-\rangle = -n\,2^{n-1} d\hat{\eta}(X,Y) - 2^{n-1}\langle\xi_X e_i, \xi_Y I e_i\rangle. \tag{3–9}$$

Using equations (3–8), (3–9), and Lemma 3.1, we obtain the required identities for Ric^* and Ric. $\square$

Theorem 3.4. *If M is a special almost Hermitian $2n$-manifold, $n \geqslant 2$, that is Kähler, then $\mathrm{Ric}^* = \mathrm{Ric}$, and*

(1) *if $d\hat{\eta} = \lambda\omega$, for some $\lambda \in \mathbb{R} \setminus \{0\}$, then the manifold is Einstein; or*

(2) *if the 1-form $\hat{\eta}$ is closed, then the manifold is Ricci-flat.*

Proof. This is an immediate consequence of the previous lemma. □

Gray proved that any nearly Kähler (type $\mathcal{W}_1$) connected 6-manifold that is not Kähler is Einstein. Here we give an alternative proof.

Theorem 3.5 [Gray 1976b]. *If M is a special almost Hermitian connected 6-manifold of type $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5$ that is not of type $\mathcal{W}_5$, then M is an Einstein manifold such that* $\mathrm{Ric} = 5\,\mathrm{Ric}^* = 5\alpha\langle\cdot\,,\cdot\rangle$, *where* $\alpha = (w_1^+)^2 + (w_1^-)^2$ *with* $\nabla\omega = w_1^+\psi_+ + w_1^-\psi_-$.

Proof. We already know that $\alpha = (w_1^+)^2 + (w_1^-)^2$ is a positive constant and the 1-form $\hat{\eta}$ is closed (see [Martín Cabrera 2005, Theorem 3.7]). Since $\nabla\omega = -\xi\omega$ and $\nabla\omega = w_1^+\psi_+ + w_1^-\psi_-$, we have

$$2\langle Y,\ \xi_X Z\rangle = w_1^-\psi_+(X, Y, Z) - w_1^+\psi_-(X, Y, Z).$$

Therefore, using

$$\langle X\lrcorner\psi_+,\ Y\lrcorner\psi_+\rangle = \langle X\lrcorner\psi_-,\ Y\lrcorner\psi_-\rangle = 2\langle X, Y\rangle,$$
$$\langle X\lrcorner\psi_+,\ Y\lrcorner\psi_-\rangle = -2\omega(X, Y),$$

we get

$$\langle\xi_X e_i,\ \xi_Y e_i\rangle = \langle e_j,\ \xi_X e_i\rangle\langle e_j,\ \xi_Y e_i\rangle = \alpha\langle X, Y\rangle. \tag{3–10}$$

Moreover, since $\xi \in \mathcal{W}_1^+ + \mathcal{W}_1^-$ and $\widetilde{\nabla}$ is a $\mathrm{U}(3)$-connection, the $(0,3)$-tensors $\langle\cdot\,,\ \xi.\cdot\rangle$ and $\langle\cdot\,,\ (\widetilde{\nabla}_X\xi).\cdot\rangle$ are skew symmetric [Gray and Hervella 1980]. Thus, from (3–7), we get

$$\mathrm{Ric}(X, Y) = 5\langle\xi_X e_i,\ \xi_Y e_i\rangle = 5\alpha\langle X, Y\rangle.$$

We recall that $\langle Y,\ \xi_{IX} IZ\rangle = -\langle Y,\ \xi_X Z\rangle$ for $\xi \in \mathcal{W}_1$, and note that the contractions $\langle(\widetilde{\nabla}_X\xi)_{e_i} Y,\ e_i\rangle$ and $\langle(\widetilde{\nabla}_{e_i}\xi)_X Y,\ e_i\rangle$ both vanish. In fact, the last term is a skew-symmetric 2-form, and the remaining summands in the expression for Ric are symmetric. □

Remark 3.6. Theorem 3.5 can be extended to connected almost Hermitian 6-manifolds which are nearly Kähler but not Kähler. In fact, one can define a complex volume form on an open neighbourhood U of a point where $\nabla\omega \neq 0$, by using the $(3,0)$-component of this tensor. Then, U is a special almost Hermitian 6-manifold of type $\mathcal{W}_1^+ + \mathcal{W}_1^- + \mathcal{W}_5$. Therefore, $\mathrm{Ric} = 5\,\mathrm{Ric}^* = 5\alpha\langle\cdot\,,\cdot\rangle$ on U. Since the manifold is connected, it follows that $\mathrm{Ric} = 5\alpha\langle\cdot\,,\cdot\rangle$ everywhere.

The expressions (3–6) and (3–7) for Ric^* and Ric allow us to compute $3\,\mathrm{Ric}^*_H + \mathrm{Ric}_H$ and study the contributions of the intrinsic torsion of the $\mathrm{SU}(n)$-structure to the components of R in $\mathcal{K}_1$ and $\mathcal{K}_2$.

Lemma 3.7. *If M is a special almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal* $\mathrm{SU}(n)$*-connection* $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$, *then*

$$
\begin{aligned}
(3\text{–}11)\quad &(3\,\mathrm{Ric}^*_H + \mathrm{Ric}_H)(X, Y)\\
&= -2n\, d\hat{\eta}(X, IY) + 2n\, d\hat{\eta}(IX, Y) - \langle(\widetilde{\nabla}_{e_i}\xi)_X Y, e_i\rangle\\
&\quad + \langle(\widetilde{\nabla}_X\xi)_{e_i} Y, e_i\rangle - \langle(\widetilde{\nabla}_{e_i}\xi)_{IX} IY, e_i\rangle + \langle(\widetilde{\nabla}_{IX}\xi)_{e_i} IY, e_i\rangle\\
&\quad - \langle\xi_{\xi_{e_i}X} Y, e_i\rangle + \langle\xi_{\xi_X e_i} Y, e_i\rangle - \langle\xi_{\xi_{e_i}IX} IY, e_i\rangle\\
&\quad + \langle\xi_{\xi_{IX}e_i} IY, e_i\rangle - 2\langle\xi_X e_i, \xi_{IY} Ie_i\rangle - 2\langle\xi_Y e_i, \xi_{IX} Ie_i\rangle.
\end{aligned}
$$

Finally, we record an alternative to equation (3–4):

Lemma 3.8. *If M is an almost Hermitian $2n$-manifold, $n \geqslant 2$, with minimal* $\mathrm{U}(n)$*-connection* $\widetilde{\nabla} = \nabla + \xi$, *then*

$$
\begin{aligned}
(3\text{–}12)\qquad \mathrm{Ric}^*_{AH}(X, Y) &= \langle(\widetilde{\nabla}_{e_i}\xi)_{Ie_i} IX, Y\rangle - \langle\xi_{I\xi_{e_i}e_i} IX, Y\rangle\\
&= -\langle(\nabla_{e_i} I\xi)_{Ie_i} X, Y\rangle.
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
-2\,&\mathrm{Ric}^*(X, IY) - 2\,\mathrm{Ric}^*(IX, Y)\\
&= \langle R_{e_i, Ie_i} X, Y\rangle - \langle R_{e_i, Ie_i} IX, IY\rangle\\
&= 4\langle\widetilde{\nabla}_{e_i}\xi_{Ie_i} X, Y\rangle - 4\langle\xi_{Ie_i}\widetilde{\nabla}_{e_i} X, Y\rangle - 4\langle\xi_{\widetilde{\nabla}_{e_i} Ie_i} X, Y\rangle + 4\langle\xi_{\xi_{e_i} Ie_i} X, Y\rangle,
\end{aligned}
$$

from which the lemma follows. ◻

4. High dimensions

In this section, we consider special almost Hermitian manifolds of dimension higher than or equal to 8. For such manifolds, the decomposition into $\mathrm{SU}(n)$-irreducible modules of the space of curvature tensors $\mathcal{R}$ is the same as that coming from the action of $\mathrm{U}(n)$. Thus,

$$
\mathcal{R} = \mathcal{K} + \mathcal{K}^\perp = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,
$$

where all $\mathcal{K}_i$ and $\mathcal{C}_j$ are also $\mathrm{SU}(n)$-irreducible spaces. Our aim here is to see whether different components of the intrinsic torsion of the $\mathrm{SU}(n)$-structure contribute to the components of the curvature.

We start by studying such contributions to the $\mathrm{SU}(n)$-components of the Ricci and $*$-Ricci curvatures. For $n \geqslant 3$, the spaces $\mathcal{R}ic$ and $\mathcal{R}ic^*$ of such tensors admit the following decompositions into $\mathrm{SU}(n)$-irreducible modules

$$
\mathcal{R}ic = \mathbb{R}\langle\,\cdot\,,\cdot\,\rangle + [\lambda_0^{1,1}] + [\![\sigma^{2,0}]\!], \qquad \mathcal{R}ic^* = \mathbb{R}\langle\,\cdot\,,\cdot\,\rangle + [\lambda_0^{1,1}] + [\![\lambda^{2,0}]\!].
$$

	Ric^* (3–6)			Ric (3–7)		
$2n \geqslant 8$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\lambda^{2,0}]\!]$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\sigma^{2,0}]\!]$
$d\hat{\eta}$	✓	✓	✓	✓	✓	
$\overline{\nabla}\xi_1,\ \eta\xi_1$						
$\overline{\nabla}\xi_2,\ \eta\xi_2$						✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$					✓	
$\overline{\nabla}\xi_4,\ \eta\xi_4$				✓	✓	✓
$\xi_1 \otimes \xi_1$	✓	✓		✓	✓	
$\xi_2 \otimes \xi_2$	✓	✓		✓	✓	
$\xi_3 \otimes \xi_3$	✓	✓		✓	✓	✓
$\xi_4 \otimes \xi_4$	✓	✓		✓	✓	✓
$\xi_1 \odot \xi_2$		✓			✓	
$\xi_1 \odot \xi_3$			✓			✓
$\xi_1 \odot \xi_4$			✓			
$\xi_2 \odot \xi_3$			✓			✓
$\xi_2 \odot \xi_4$			✓			✓
$\xi_3 \odot \xi_4$		✓			✓	

Table 1. Ricci curvatures, $2n \geqslant 8$.

Taking into account the symmetry properties and types of the Gray–Hervella components ξ_i of ξ, we obtain:

Theorem 4.1. *Let M be a special almost Hermitian $2n$-manifold, $2n \geqslant 8$, with minimal $\mathrm{SU}(n)$-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$. The tensors $d\hat{\eta}$, $\overline{\nabla}\xi$, and $\xi_i \odot \xi_j$ contribute to the components of the $*$-Ricci curvature Ric^* via equation (3–6) and to the Ricci curvature Ric via equation (3–7) if and only if there is a tick in the corresponding place in Table 1.*

Using in addition that $\langle \xi_{\xi_X e_i} Y,\ e_i \rangle = -\langle \xi_X e_i,\ \xi_{e_i} Y \rangle$, we get part (1) of the next theorem. Part (2) is proved in [Falcitelli et al. 1994].

Theorem 4.2. *If M is a special almost Hermitian $2n$-manifold, $2n \geqslant 8$, with minimal $\mathrm{SU}(n)$-connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$, then*

(1) *Using equations (3–3), (3–4), (3–5), and (3–11), each of the tensors $\overline{\nabla}\xi_i$, $\eta\xi_i$, and $\xi_i \odot \xi_j$ contributes to the components of R in $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_6$, and $\mathcal{C}_8$ if and only if there is a tick in the corresponding place in Table 2.*

	(3–11)		(3–3)		(3–4)	(3–12)	(3–5)	[Falcitelli et al.]		
$2n \geqslant 8$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_{-1}$	$\mathcal{K}_{-2}$	$\mathcal{C}_6$	$\mathcal{C}_6$	$\mathcal{C}_8$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_7$
$d\hat{\eta}$	✓	✓								
$\overline{\nabla}\xi_1,\ \eta\xi_1$					✓	✓				
$\overline{\nabla}\xi_2,\ \eta\xi_2$					✓	✓	✓		✓	✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$		✓		✓		✓		✓		✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$	✓	✓	✓	✓	✓	✓	✓			
$\xi_1 \otimes \xi_1$	✓	✓	✓	✓				✓*		
$\xi_2 \otimes \xi_2$	✓	✓	✓	✓				✓		
$\xi_3 \otimes \xi_3$	✓	✓					✓			✓
$\xi_4 \otimes \xi_4$	✓	✓					✓			
$\xi_1 \odot \xi_2$		✓		✓				✓		
$\xi_1 \odot \xi_3$					✓		✓		✓	✓
$\xi_1 \odot \xi_4$					✓	✓				
$\xi_2 \odot \xi_3$					✓		✓		✓	✓
$\xi_2 \odot \xi_4$					✓	✓	✓		✓	✓
$\xi_3 \odot \xi_4$		✓				✓				✓

*absent when $2n = 8$

Table 2. Curvature complementary to $\mathcal{C}_3 = \mathcal{K}(\mathfrak{su}(n))$, $2n \geqslant 8$.

(2) *Taking the image* $\pi_2 \circ \pi_1(R)$ *into account, where* $\pi_1(R)$ *is given by* (2–2), *each of the tensors* $\overline{\nabla}\xi_i,\ \eta\xi_i$, *and* $\xi_i \odot \xi_j$ *contributes to the components of* R *in* $\mathcal{C}_4$, $\mathcal{C}_5$, *and* $\mathcal{C}_7$ *if and only if there is a tick in the corresponding place in Table 2.*

For part (1), we emphasize that the columns for $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_6$, and $\mathcal{C}_8$ are obtained by a different method than that in [Falcitelli et al. 1994], and that for $\mathcal{C}_6$ this even leads to a different result. In particular, we claim that the tensors $\widetilde{\nabla}\xi_3$ and $\xi_3 \odot \xi_4$ do not contribute to the $\mathcal{C}_6$-component of R, but that $\widetilde{\nabla}\xi_1$ and $\eta\xi_1$ do. Thus, the contributions of the different tensors to the distinct components of R depend on the choice of the current expression that we use; different expressions may lead to different behaviour in the contributions. For the $\mathcal{C}_6$-component of R, we get a third formula from equation (3–12), which we also list in Table 2. A partial explanation for these different results will be given in Section 6. Note that the entries for $\mathcal{C}_6$ in Table 2 only involve the intrinsic $\mathrm{U}(n)$-torsion. The $[\![\lambda^{2,0}]\!]$-column of Table 1 provides yet another description of the $\mathcal{C}_6$-component using the $\mathrm{SU}(n)$-structure.

5. Low dimensions

We consider in turn special almost Hermitian manifolds of dimension 6 and 4.

Six dimensions. The decomposition of the space of curvature tensors $\mathcal{R}$ into irreducible SU(3)-modules has the same subspaces as for U(3). Thus,

$$\mathcal{R} = \mathcal{K} + \mathcal{K}^{\perp} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_{-1} + \mathcal{K}_{-2} + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8,$$

with $\mathcal{K}_i$ and $\mathcal{C}_j$ all SU(3)-irreducible. As we noted above, the summand $\mathcal{C}_4$ is absent in this dimension. On the other hand, the U(3)-intrinsic torsion splits under SU(3) as $\xi = \xi_1^+ + \xi_1^- + \xi_2^+ + \xi_2^- + \xi_3 + \xi_4$, where $\xi_i = \xi_i^+ + \xi_i^-$, $i = 1, 2$. This was briefly described in Section 2, and more detailed information is contained in [Chiossi and Salamon 2002] and [Martín Cabrera 2005].

The next result concerns the contributions of the components of ξ to the components of the Ricci and the $*$-Ricci curvatures, and then to the curvature components complementary to $\mathcal{C}_3$.

Theorem 5.1. *Let M be a special almost Hermitian 6-manifold with* SU(3)-*connection $\overline{\nabla} = \nabla + \eta + \xi = \widetilde{\nabla} + \eta$. The tensors $d\hat{\eta}$, $\overline{\nabla}\zeta$, $\eta\zeta$, and $\zeta \odot \vartheta$, for $\zeta, \vartheta = \xi_1^+, \xi_1^-, \xi_2^+, \xi_2^-, \xi_3, \xi_4$, contribute to the components of* Ric^* *and* Ric *if and only if there is a tick in the corresponding place in Table 3.*

The corresponding contributions to the curvature components $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_{-1}$, $\mathcal{K}_{-2}$, $\mathcal{C}_6$, and $\mathcal{C}_8$, via equations (3–3), (3–4), (3–5), *and* (3–11), *and to the components $\mathcal{C}_5$ and $\mathcal{C}_7$ via $\pi_2 \circ \pi_1(R)$ are given in Table 4* (cf. [Falcitelli et al. 1994]).

Four dimensions. The U(2)-decomposition of the space of curvature tensors $\mathcal{R}$ is given by

$$\mathcal{R} = \mathcal{K} + \mathcal{K}^{\perp} = \mathcal{C}_3 + \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_{-1} + \mathcal{C}_5 + \mathcal{C}_6 + \mathcal{C}_8.$$

When we consider the SU(2)-action, only the modules $\mathcal{C}_3$, $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_{-1}$ remain irreducible. To describe the decompositions of $\mathcal{C}_5$ and $\mathcal{C}_6$ into SU(2)-irreducible modules, we will make use of tensors defined by

$$\chi(a, b) = 6\, a \odot b - a \wedge b,$$

for all $a, b \in \bigwedge^2 T^*M$, where $\odot$ denotes the symmetric product given by $2\, a \odot b = a \otimes b + b \otimes a$. The relevant decompositions are now given by

(1) $\mathcal{C}_5 = \mathcal{C}_5^{++} + \mathcal{C}_5^{--} + \mathcal{C}_5^{+-}$, where $\mathcal{C}_5^{++} = \mathbb{R}\chi(\psi_+, \psi_+)$, $\mathcal{C}_5^{--} = \mathbb{R}\chi(\psi_-, \psi_-)$, and $\mathcal{C}_5^{+-} = \mathbb{R}\chi(\psi_+, \psi_-)$,

(2) $\mathcal{C}_6 = \mathcal{C}_6^{+} + \mathcal{C}_6^{-}$, where $\mathcal{C}_6^{+} = \mathbb{R}\chi(\psi_+, \omega)$ and $\mathcal{C}_6^{-} = \mathbb{R}\chi(\psi_-, \omega)$.

For the intrinsic torsion, the U(2)-decomposition of ξ is given by

$$\xi = \xi_2 + \xi_4 \in \mathcal{W} = \mathcal{W}_2 + \mathcal{W}_4.$$

	Ric* (3–6)			Ric (3–7)		
$2n=6$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\lambda^{2,0}]\!]$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\sigma^{2,0}]\!]$
$d\hat{\eta}$	✓	✓	✓	✓	✓	
$\overline{\nabla}\xi_1^{\pm},\ \eta\xi_1^{\pm}$						
$\overline{\nabla}\xi_2^{\pm},\ \eta\xi_2^{\pm}$						✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$					✓	
$\overline{\nabla}\xi_4,\ \eta\xi_4$				✓	✓	✓
$\xi_1^{\pm}\otimes\xi_1^{\pm}$	✓			✓		
$\xi_2^{\pm}\otimes\xi_2^{\pm}$	✓	✓		✓	✓	
$\xi_3\otimes\xi_3$	✓	✓		✓	✓	✓
$\xi_4\otimes\xi_4$	✓	✓		✓	✓	✓
$\xi_1^{+}\odot\xi_1^{-}$						
$\xi_1^{\pm}\odot\xi_2^{\pm}$		✓			✓	
$\xi_1^{\pm}\odot\xi_2^{\mp}$						
$\xi_1^{\pm}\odot\xi_3$						✓
$\xi_1^{\pm}\odot\xi_4$			✓			
$\xi_2^{+}\odot\xi_2^{-}$		✓			✓	
$\xi_2^{\pm}\odot\xi_3$			✓			✓
$\xi_2^{\pm}\odot\xi_4$			✓			✓
$\xi_3\odot\xi_4$		✓			✓	

Table 3. Ricci curvatures, $2n=6$.

Under SU(2), we have $\mathcal{W}_2\cong\mathcal{W}_4\cong T^*M$, which, as we will see, gives rise to different choices of decompositions of ξ.

For an SU(2)-structure, we have $\nabla\omega\in\mathcal{W}=T^*M\otimes\psi_+ + T^*M\otimes\psi_-$. Consequently, $\nabla\omega=\xi_+\otimes\psi_+ + \xi_-\otimes\psi_-$, where ξ_+ and ξ_- are 1-forms. Moreover,

$$2\langle Y,\xi_X Z\rangle = -\xi_+(X)\psi_-(Y,Z)+\xi_-(X)\psi_+(Y,Z),$$

so $\xi=\xi_+ + \xi_-$, where

$$2\langle Y,(\xi_+)_X Z\rangle = -\xi_+(X)\psi_-(Y,Z),\qquad 2\langle Y,(\xi_-)_X Z\rangle = \xi_-(X)\psi_+(Y,Z).$$

The two decompositions of ξ are related as follows:

$$\xi\in\mathcal{W}_2 \text{ if and only if } \xi_+ = I\xi_-;\qquad \xi\in\mathcal{W}_4 \text{ if and only if } \xi_+ = -I\xi_-.$$

	(3–11)		(3–3)		(3–4)	(3–5)	[Falcitelli et al.]	
$2n=6$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_{-1}$	$\mathcal{K}_{-2}$	$\mathcal{C}_6$	$\mathcal{C}_8$	$\mathcal{C}_5$	$\mathcal{C}_7$
$d\hat{\eta}$	✓	✓						
$\overline{\nabla}\xi_1^{\pm},\ \eta\xi_1^{\pm}$					✓			
$\overline{\nabla}\xi_2^{\pm},\ \eta\xi_2^{\pm}$					✓	✓	✓	✓
$\overline{\nabla}\xi_3,\ \eta\xi_3$		✓		✓				✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$	✓	✓	✓	✓	✓	✓		
$\xi_1^{\pm}\otimes\xi_1^{\pm}$	✓		✓					
$\xi_2^{\pm}\otimes\xi_2^{\pm}$	✓	✓	✓	✓				
$\xi_3\otimes\xi_3$	✓	✓				✓		✓
$\xi_4\otimes\xi_4$	✓	✓				✓		
$\xi_1^{+}\odot\xi_1^{-}$								
$\xi_1^{\pm}\odot\xi_2^{\pm}$		✓		✓				
$\xi_1^{\pm}\odot\xi_2^{\mp}$								
$\xi_1^{\pm}\odot\xi_3$						✓	✓	
$\xi_1^{\pm}\odot\xi_4$					✓			
$\xi_2^{+}\odot\xi_2^{-}$		✓		✓				
$\xi_2^{\pm}\odot\xi_3$					✓	✓	✓	✓
$\xi_2^{\pm}\odot\xi_4$					✓	✓	✓	✓
$\xi_3\odot\xi_4$		✓						✓

Table 4. Curvature complementary to $\mathcal{C}_3=\mathcal{K}(\mathfrak{su}(n))$, $2n=6$.

The next theorem deals with the contributions of the components of the intrinsic torsion to the tensors Ric* and Ric. First, in dimension four, $\mathcal{R}ic^*$ decomposes under SU(2) as

$$\mathcal{R}ic^*=\mathbb{R}\langle\cdot,\cdot\rangle+[\lambda_0^{1,1}]+\mathbb{R}\psi_++\mathbb{R}\psi_-.$$

Theorem 5.2. *Let M be a special almost Hermitian 4-manifold with minimal* SU(2)*-connection $\overline{\nabla}=\nabla+\eta+\xi=\widetilde{\nabla}+\eta$. The curvature contributions corresponding to Theorems 4.1 and 4.2 via the decompositions $\xi=\xi_2+\xi_4$ and $\xi=\xi_++\xi_-$ are given in Tables 5 and 6.*

Proof. The absence of $\mathcal{K}_{-2}$ in the decomposition of $\mathcal{R}$ comes from the fact that

$$\left(\mathrm{Ric}^*_H-\mathrm{Ric}_H\right)(X,Y)=\beta\,\langle X,Y\rangle, \tag{5–1}$$

	Ric* (3–6)				Ric (3–7)		
$2n=4$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$\mathbb{R}\psi_+$	$\mathbb{R}\psi_-$	$\mathbb{R}$	$[\lambda_0^{1,1}]$	$[\![\sigma^{2,0}]\!]$
$d\hat{\eta}$	✓	✓	✓	✓	✓	✓	
$\overline{\nabla}\xi_2,\ \eta\xi_2$							✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$					✓		✓
$\xi_2 \otimes \xi_2$	✓	✓			✓	✓	
$\xi_4 \otimes \xi_4$	✓	✓			✓	✓	✓
$\xi_2 \odot \xi_4$						✓	✓
$\overline{\nabla}\xi_+,\ \eta\xi_+$					✓		✓
$\overline{\nabla}\xi_-,\ \eta\xi_-$					✓		✓
$\xi_+ \otimes \xi_+$					✓		✓
$\xi_- \otimes \xi_-$					✓		✓
$\xi_+ \odot \xi_-$	✓	✓			✓	✓	✓

Table 5. Ricci curvatures, $2n=4$.

where $\beta = \langle (\widetilde{\nabla}_{e_i}\xi)_{e_j} e_j,\ e_i\rangle + \langle \xi_{\xi_{e_i} e_j} e_j,\ e_i\rangle$. Therefore, by (3–4), we have

$$\text{(5–2)}\quad \big(3\,\mathrm{Ric}^*_H + \mathrm{Ric}_H\big)(X,Y) = -\beta\langle X,Y\rangle - 4d\hat{\eta}(X,IY) + 4d\hat{\eta}(IX,Y) \\ -2\langle \xi_X e_i,\ \xi_{IY} I e_i\rangle - 2\langle \xi_Y e_i,\ \xi_{IX} I e_i\rangle.$$

Using equations (5–1) and (5–2), the tables follow. □

Remark 5.3. We list some direct consequences of the results and tables presented here and in Section 4:

(1) if $\xi \in \mathcal{W}_3$, the components of R in $\mathcal{K}_{-1}$, $\mathcal{C}_5$, and $\mathcal{C}_6$ vanish;

(2) if $\xi \in \mathcal{W}_3 + \mathcal{W}_4$ and $d\hat{\eta}$ is Hermitian, the components of R in $\mathcal{C}_5$ and $\mathcal{C}_6$ vanish;

(3) if $\xi \in \mathcal{W}_1 + \mathcal{W}_2$ and $d\hat{\eta}$ is Hermitian, the component of R in $\mathcal{C}_6$ vanishes;

(4) if $n=2$ and $d\hat{\eta}$ is Hermitian, then the component of R in $\mathcal{C}_6$ vanishes.

There are more consequences of this sort, but they have already been pointed out in [Falcitelli et al. 1994].

Remark 5.4. For special almost Hermitian 2-manifolds, we have the following identity, obtained in [Martín Cabrera 2005]:

$$K(\psi_+,\psi_-) = d\hat{\eta}(\psi_+,\psi_-) = d\eta_+(\psi_+) + d\eta_-(\psi_-) - \eta_+^2 - \eta_-^2,$$

where K denotes the sectional curvature and $\hat{\eta} = \eta_+\psi_- - \eta_-\psi_+$.

$2n=4$	$\mathcal{K}_1$	$\mathcal{K}_2$	$\mathcal{K}_{-1}$	$\mathcal{C}_6^+$	$\mathcal{C}_6^-$	$\mathcal{C}_8$	$\mathcal{C}_5^{++}$	$\mathcal{C}_5^{--}$	$\mathcal{C}_5^{+-}$
$d\hat{\eta}$	✓	✓							
$\overline{\nabla}\xi_2,\ \eta\xi_2$				✓	✓	✓	✓	✓	✓
$\overline{\nabla}\xi_4,\ \eta\xi_4$	✓		✓	✓	✓	✓			
$\xi_2\otimes\xi_2$	✓	✓	✓						
$\xi_4\otimes\xi_4$	✓	✓				✓			
$\xi_2\odot\xi_4$				✓	✓	✓	✓	✓	✓
$\overline{\nabla}\xi_+,\ \eta\xi_+$	✓		✓		✓	✓	✓	✓	✓
$\overline{\nabla}\xi_-,\ \eta\xi_-$	✓		✓	✓		✓	✓	✓	✓
$\xi_+\otimes\xi_+$	✓		✓			✓		✓	✓
$\xi_-\otimes\xi_-$	✓		✓			✓	✓		✓
$\xi_+\odot\xi_-$	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 6. Curvature complementary to $\mathcal{C}_3=\mathcal{K}(\mathfrak{su}(n))$, $2n=4$.

6. Identities from exterior algebra

As remarked in Section 4, one may see different contributions to the module $\mathcal{C}_6\cong[\![\lambda^{2,0}]\!]$ by using different computations of the curvature. This is because of nontrivial identities that relate the components of $\widetilde{\nabla}\xi_i$ and $\xi_j\odot\xi_k$. Such an identity for the $[\![\lambda^{2,0}]\!]$-components may be obtained by comparing equations (3–4) and (3–12). However, we claim that this information may also be obtained from the exterior algebra of a $\mathrm{U}(n)$-manifold.

Consider the Kähler 2-form ω. Being a differential form, it satisfies $d^2\omega=0$. However, since the Levi-Civita connection ∇ is torsion-free, we may compute $d^2\omega$ using ∇. Writing $\nabla=\widetilde{\nabla}-\xi$ and using that $\widetilde{\nabla}\omega=0$, we first have

$$\tfrac{1}{2}d\omega(Y,Z,W)=\langle\xi_Y Z, IW\rangle+\langle\xi_W Y, IZ\rangle+\langle\xi_Z W, IY\rangle.$$

Now, $d^2\omega=\mathbf{a}(\widetilde{\nabla}d\omega)-\mathbf{a}(\xi\,d\omega)$, where $\mathbf{a}: T^*M\otimes\bigwedge^3T^*M\to\bigwedge^4T^*M$ is the alternation map. One computes that these two terms are the expressions obtained by summing, respectively, $\varepsilon\langle(\widetilde{\nabla}_X\xi)_Y Z, IW\rangle$ and $\varepsilon\langle\xi_{\xi_X Y}Z, IW\rangle$ over all permutations of (X,Y,Z,W), where ε is the sign of the permutation.

We have

$$\bigwedge^4T^*M=[\![\lambda^{4,0}]\!]+[\![\lambda^{3,1}]\!]+[\![\lambda^{2,0}]\!]\omega+[\lambda_0^{2,2}]+[\lambda_0^{1,1}]\omega+\mathbb{R}\omega^2,$$

so, in order to compute the $[\![\lambda^{2,0}]\!]$-component of $d^2\omega$, we contract with ω on the first two arguments and then take the projection to $[\![\lambda^{2,0}]\!]$, which is the (-1)-eigenspace of I acting on 2-forms. Using the symmetries of the components of ξ, one obtains that the $[\![\lambda^{2,0}]\!]$-component of $d^2\omega$ is

$$\begin{aligned}(6\text{–}1)\quad 0 = {} & 3\langle(\widetilde{\nabla}_{e_i}\xi_1)_{e_i}X,\, Y\rangle - \langle(\widetilde{\nabla}_{e_i}\xi_3)_{e_i}X,\, Y\rangle + (n-2)\langle(\widetilde{\nabla}_{e_i}\xi_4)_{e_i}X,\, Y\rangle \\ & + \langle(\xi_3)_X e_i,\, (\xi_1)_{e_i}Y\rangle - \langle(\xi_3)_Y e_i,\, (\xi_1)_{e_i}X\rangle \\ & + \langle(\xi_3)_X e_i,\, (\xi_2)_{e_i}Y\rangle - \langle(\xi_3)_Y e_i,\, (\xi_2)_{e_i}X\rangle \\ & - \tfrac{n-5}{n-1}\langle(\xi_1)_{\xi_{4e_i}e_i}X,\, Y\rangle - \tfrac{n-2}{n-1}\langle(\xi_2)_{\xi_{4e_i}e_i}X,\, Y\rangle + \langle(\xi_3)_{\xi_{4e_i}e_i}X,\, Y\rangle.\end{aligned}$$

We conclude that, in general dimensions, there is a nontrivial linear relation between the $[\![\lambda^{2,0}]\!]$-components of $\widetilde{\nabla}\xi_1$, $\widetilde{\nabla}\xi_3$, $\widetilde{\nabla}\xi_4$, $\xi_1\odot\xi_3$, $\xi_1\odot\xi_4$, $\xi_2\odot\xi_3$, $\xi_2\odot\xi_4$, and $\xi_3\odot\xi_4$. By 'nontrivial' we mean that no coefficient is zero, so this relation may be used to write any of the terms as a linear combination of the others. Interestingly, when $2n = 10$ this relation does not involve $\xi_1 \odot \xi_4$.

This is sufficient to explain the difference between the ticks in the $\mathscr{C}_6$ column in [Falcitelli et al. 1994] and those we obtained from equation (3–4). An extra coincidence in the coefficients explains the differences between our results from (3–4) and (3–12).

One may try to apply the above approach to the other modules that $\bigwedge^4 T^*M$ has in common with the space of curvature tensors, namely $[\lambda_0^{2,2}]$, $[\lambda_0^{1,1}]\omega$, and $\mathbb{R}\omega^2$. However, this is not so rewarding, because of the higher multiplicities that these modules have in the relevant decompositions. Indeed, $\mathscr{C}_6 \cong [\![\lambda^{2,0}]\!]$ is distinguished by occurring only with multiplicity one or zero in the modules for $\widetilde{\nabla}\xi_i$ and $\xi_i \otimes \xi_j$.

In [Falcitelli et al. 1994] it is pointed out that, if $\xi \in \mathscr{W}_4$, the components of R in $\mathscr{C}_4$, $\mathscr{C}_5$, $\mathscr{C}_6$, and $\mathscr{C}_7$ vanish. We indicate how equation (6–1) gives an alternative proof of this result, for $n > 2$. By using Tables 2 and 4, the vanishing of the components in $\mathscr{C}_4$, $\mathscr{C}_5$, and $\mathscr{C}_7$ is in fact immediate. On the other hand, equations (3–4) and (6–1) give the vanishing of the component in $\mathscr{C}_6$.

Finally, a comparison of Tables 1 and 2 reveals another relation on special almost Hermitian manifolds: the $[\![\lambda^{2,0}]\!]$-part of $d\hat{\eta}$ carries all the information from the corresponding components of $\widetilde{\nabla}\xi_i$ modulo the $[\![\lambda^{2,0}]\!]$-parts of $\xi_1 \odot \xi_3$, $\xi_1 \odot \xi_4$, $\xi_2 \odot \xi_3$, and $\xi_2 \odot \xi_4$. This relation is obtainable by considering the $(n, 2)$-part of the equation $d^2\Psi = 0$, where Ψ is the complex volume, see [Martín Cabrera 2005].

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