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## THE CONVOLUTION SUM $\sum_{m < n/8} \sigma(m) \sigma(n-8m)$

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**The convolution sum  $\sum_{m < n/8} \sigma(m) \sigma(n-8m)$  is evaluated for all  $n \in \mathbb{N}$ . This evaluation is used to determine the number of representations of  $n$  by the quadratic form  $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 2x_7^2 + 2x_8^2$ .**

### 1. Introduction

Let  $\mathbb{N}$  denote the set of natural numbers. For  $n \in \mathbb{N}$  and  $k \in \mathbb{N}$ , we set

$$\sigma_k(n) = \sum_{d|n} d^k,$$

where  $d$  runs through the positive integers dividing  $n$ . If  $n \notin \mathbb{N}$ , we set  $\sigma_k(n) = 0$ . We write  $\sigma(n)$  for  $\sigma_1(n)$ . We define the convolution sum  $W_k(n)$  by

$$(1) \quad W_k(n) := \sum_{m < n/k} \sigma(m) \sigma(n-km),$$

where  $m$  runs through the positive integers  $< n/k$ . The sum  $W_k(n)$  has been evaluated for  $k = 1, 2, 3, 4, 9$  for all  $n \in \mathbb{N}$ , and for  $k = 5$  for  $n \equiv 8 \pmod{16}$ ,  $n \not\equiv 0 \pmod{5}$ ; see [Besge 1862; Huard et al. 2002] for  $k = 1$ , [Huard et al. 2002; Melfi 1998a; 1998b] for  $k = 2$ , [Huard et al. 2002; Melfi 1998a; 1998b; Williams 2004] for  $k = 3$ , [Huard et al. 2002; Melfi 1998a; 1998b] for  $k = 4$ , [Melfi 1998a; 1998b; Williams 2004; 2005] for  $k = 9$ , and [Melfi 1998a; 1998b] for  $k = 5$ . In this paper, we evaluate  $W_k(n)$  for  $k = 8$  and all  $n \in \mathbb{N}$ .

Let  $\mathbb{Z}, \mathbb{R}, \mathbb{C}$  denote the sets of integers, real numbers, and complex numbers, respectively. For  $q \in \mathbb{C}$  with  $|q| < 1$ , we set [Ramanujan 1916, eqn. (92), p. 151]

$$(2) \quad \Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

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For  $n \in \mathbb{N}$ , we define  $k(n) \in \mathbb{Z}$  by

$$(3) \quad (\Delta(q^2) \Delta(q^4))^{1/6} = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 := \sum_{n=1}^{\infty} k(n) q^n.$$

Clearly,  $k(n) = 0$  for  $n \equiv 0 \pmod{2}$ . By Euler’s identity [Knopp 1970, Corollary 5, p. 37], we have

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m+1)/2} = 1 - q - q^2 + q^5 + q^7 - q^{12} - \dots .$$

Using MAPLE, we find

$$q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = q - 4q^3 - 2q^5 + 24q^7 - 11q^9 - \dots ,$$

so that the first few values of  $k(2n-1)$ , for  $n \in \mathbb{N}$ , are

$$k(1) = 1, \quad k(3) = -4, \quad k(5) = -2, \quad k(7) = 24, \quad k(9) = -11.$$

The evaluation of  $W_8(n)$  is given in Theorem 1 and proved in Section 2.

**Theorem 1.** For  $n \in \mathbb{N}$ ,

$$\begin{aligned} \sum_{m < n/8} \sigma(m) \sigma(n-8m) &= \frac{1}{192} \sigma_3(n) + \frac{1}{64} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{16} \sigma_3\left(\frac{n}{4}\right) + \frac{1}{3} \sigma_3\left(\frac{n}{8}\right) \\ &\quad + \left(\frac{1}{24} - \frac{n}{32}\right) \sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma\left(\frac{n}{8}\right) - \frac{1}{64} k(n). \end{aligned}$$

As an application of Theorem 1, we evaluate in Section 3 the number

$$N(n) = \text{card}\{(x_1, x_2, \dots, x_8) \in \mathbb{Z}^8 \mid n = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 2x_7^2 + 2x_8^2\}.$$

We prove:

**Theorem 2.** For  $n \in \mathbb{N}$ ,

$$N(n) = 4\sigma_3(n) - 4\sigma_3\left(\frac{n}{2}\right) - 16\sigma_3\left(\frac{n}{4}\right) + 256\sigma_3\left(\frac{n}{8}\right) + 4k(n).$$

**2. Proof of Theorem 1**

Let  $q \in \mathbb{C}$  be such that  $|q| < 1$ . The Eisenstein series  $L(q)$ ,  $M(q)$ ,  $N(q)$  are

$$(4) \quad L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n,$$

$$(5) \quad M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

$$(6) \quad N(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n,$$

see for example [Berndt 1989, p. 318; Ramanujan 1916, eqn. (25), p. 140]. It was shown in [Ramanujan 1916, eqn. (44)] that

$$(7) \quad \Delta(q) = \frac{1}{1728} (M(q)^3 - N(q)^2).$$

First, we determine the generating function of  $W_8(n)$  for  $n \in \mathbb{N}$ . By (4), we have

$$\begin{aligned} (1 - L(q))(1 - L(q^8)) &= \left(24 \sum_{l=1}^{\infty} \sigma(l)q^l\right) \left(24 \sum_{m=1}^{\infty} \sigma(m)q^{8m}\right) = 576 \sum_{l,m=1}^{\infty} \sigma(l)\sigma(m)q^{l+8m} \\ &= 576 \sum_{n=1}^{\infty} q^n \sum_{\substack{l,m=1 \\ l+8m=n}}^{\infty} \sigma(l)\sigma(m) = 576 \sum_{n=1}^{\infty} q^n \sum_{1 \leq m < n/8} \sigma(m)\sigma(n-8m), \end{aligned}$$

that is,

$$(1 - L(q))(1 - L(q^8)) = 576 \sum_{n=1}^{\infty} W_8(n)q^n,$$

so that

$$(8) \quad \sum_{n=1}^{\infty} W_8(n)q^n = \frac{1}{576} - \frac{1}{576}L(q) - \frac{1}{576}L(q^8) + \frac{1}{576}L(q)L(q^8).$$

Next, we use (4) and (8) to determine the power-series expansion of  $L(q)L(q^8)$  in powers of  $q$ . Letting  $q \mapsto q^8$  in (4) yields

$$(9) \quad L(q^8) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{8n} = 1 - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{8}\right)q^n.$$

From (4), (8), and (9) we deduce

$$(10) \quad L(q)L(q^8) = 1 + \sum_{n=1}^{\infty} \left(576 W_8(n) - 24\sigma(n) - 24\sigma\left(\frac{n}{8}\right)\right)q^n.$$

Our next objective is to determine the power series of  $(L(q) - 8L(q^8))^2$  in powers of  $q$ . The result,

$$(11) \quad L(q)^2 = 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n,$$

is classical, see for example [Glaisher 1885a; 1885b]. Letting  $q \mapsto q^8$  in (11), we obtain

$$(12) \quad L(q^8)^2 = 1 + \sum_{n=1}^{\infty} \left( 240\sigma_3\left(\frac{n}{8}\right) - 36n\sigma\left(\frac{n}{8}\right) \right) q^n.$$

Hence, by (10), (11), and (12), we obtain

$$\begin{aligned} (L(q) - 8L(q^8))^2 &= L(q)^2 + 64L(q^8)^2 - 16L(q)L(q^8) \\ &= 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n \\ &\quad + 64 \left( 1 + \sum_{n=1}^{\infty} \left( 240\sigma_3\left(\frac{n}{8}\right) - 36n\sigma\left(\frac{n}{8}\right) \right) q^n \right) \\ &\quad - 16 \left( 1 + \sum_{n=1}^{\infty} \left( 576W_8(n) - 24\sigma(n) - 24\sigma\left(\frac{n}{8}\right) \right) q^n \right), \end{aligned}$$

that is,

$$(13) \quad \begin{aligned} (L(q) - 8L(q^8))^2 &= 49 + \sum_{n=1}^{\infty} \left( 240\sigma_3(n) + 15360\sigma_3\left(\frac{n}{8}\right) + (384 - 288n)\sigma(n) \right. \\ &\quad \left. + (384 - 2304n)\sigma\left(\frac{n}{8}\right) - 9216W_8(n) \right) q^n. \end{aligned}$$

Next, we use Ramanujan's evaluation of  $L(q)$  [Berndt 1991, p. 129] and the principle of duplication [Berndt 1991, p. 125] to determine the power-series expansion of  $(L(q) - 8L(q^8))^2$  in another way. For  $z \in \mathbb{C}$  with  $|z| < 1$ , we set

$$(14) \quad w(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{z^n}{n!},$$

where  ${}_2F_1$  is the Gaussian hypergeometric function and  $(a)_n$  is the Pochhammer symbol, see for example [Copson 1935, p. 247; Rainville 1971, p. 45]. Clearly,  $w(0) = 1$ . The infinite series (14) diverges at  $z = 1$  [Copson 1935, p. 249], so that

$w(1) = +\infty$ . For  $x \in \mathbb{R}$  with  $0 \leq x < 1$ , we have

$$w(x) = 1 + \sum_{n=1}^{\infty} \frac{(2n!)^2}{(n!)^4 2^{4n}} x^n \geq 1$$

so that

$$(15) \quad w(x) \neq 0, \quad 0 \leq x < 1.$$

The derivative with respect to  $x$  of the function

$$(16) \quad y(x) := \pi \frac{w(1-x)}{w(x)}, \quad 0 < x < 1,$$

is

$$(17) \quad y'(x) = \frac{-1}{x(1-x)w(x)^2}, \quad 0 < x < 1,$$

see [Berndt 1989, p. 87]. Thus, by (15) and (17), we have

$$(18) \quad y'(x) < 0, \quad 0 < x < 1.$$

Hence, as  $x$  increases from 0 to 1,  $y(x)$  strictly decreases

$$\text{from } y(0) = \pi \frac{w(1)}{w(0)} = +\infty \quad \text{to} \quad y(1) = \pi \frac{w(0)}{w(1)} = 0.$$

Now, restrict  $q$  so that  $q \in \mathbb{R}$  and  $0 < q < 1$ . Thus,  $0 < -\log q < +\infty$ . Hence, there is a unique value of  $x$  between 0 and 1 such that

$$(19) \quad y(x) = -\log q.$$

Ramanujan gave in his notebooks [Ramanujan 1957] the following formulae for  $L(q)$ ,  $M(q)$  and  $N(q)$ , which are proved in [Berndt 1991, pp. 126–129]:

$$(20) \quad L(q) = (1-5x)w^2 + 12x(1-x)w \frac{dw}{dx},$$

$$(21) \quad M(q) = (1+14x+x^2)w^4,$$

$$(22) \quad N(q) = (1+x)(1-34x+x^2)w^6.$$

From (7), (21), and (22), we obtain

$$(23) \quad \Delta(q) = \frac{x(1-x)^4 w^{12}}{2^4}.$$

Applying the principle of duplication [Berndt 1991, p. 125]

$$q \mapsto q^2, \quad x \mapsto \left( \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right)^2, \quad w \mapsto \left( \frac{1+\sqrt{1-x}}{2} \right) w$$

to (20), (21), and (23), we obtain

$$(24) \quad L(q^2) = (1 - 2x)w^2 + 6x(1 - x)w \frac{dw}{dx},$$

$$(25) \quad M(q^2) = (1 - x + x^2)w^4,$$

$$(26) \quad \Delta(q^2) = \frac{x^2(1 - x)^2w^{12}}{2^8}.$$

Formulae (24) and (25) are given in [Berndt 1991, pp. 122, 126]. Applying the principle of duplication to (24), (25), and (26), we obtain

$$(27) \quad L(q^4) = \left(1 - \frac{5}{4}x\right)w^2 + 3x(1 - x)w \frac{dw}{dx},$$

$$(28) \quad M(q^4) = \left(1 - x + \frac{1}{16}x^2\right)w^4,$$

$$(29) \quad \Delta(q^4) = \frac{x^4(1 - x)w^{12}}{2^{16}}.$$

Formulae (27) and (28) can be deduced from [Berndt 1991, pp. 122, 127]. From (3), (26), and (29), we obtain

$$(30) \quad x\sqrt{1-x} w^4 = 16 \sum_{n=1}^{\infty} k(n)q^n.$$

Applying the duplication principle to (27) and (28), we have

$$(31) \quad L(q^8) = \left(\frac{5}{8} - \frac{11}{16}x + \frac{3}{8}\sqrt{1-x}\right)w^2 + \frac{3}{2}x(1-x)w \frac{dw}{dx},$$

$$(32) \quad M(q^8) = \left(\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x}\right)w^4,$$

see [Cheng and Williams 2004, p. 564]. From (20) and (31), we deduce

$$(33) \quad L(q) - 8L(q^8) = \left(-4 + \frac{1}{2}x - 3\sqrt{1-x}\right)w^2.$$

Squaring both sides of (33), we have

$$(34) \quad (L(q) - 8L(q^8))^2 = \left(25 - 13x + \frac{1}{4}x^2 + 24\sqrt{1-x} - 3x\sqrt{1-x}\right)w^4.$$

From (21), (25), and (28), we obtain

$$(35) \quad w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4),$$

$$(36) \quad xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2),$$

$$(37) \quad x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4).$$



Using (30), (35), (36), and (37) in (32), we obtain

$$M(q^8) = -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{4}\sum_{n=1}^{\infty}k(n)q^n.$$

Thus,

$$(38) \quad \sqrt{1-x}w^4 = \frac{1}{15}M(q^2) - \frac{6}{5}M(q^4) + \frac{32}{15}M(q^8) + 8\sum_{n=1}^{\infty}k(n)q^n.$$

Now, using (30), (35), (36), (37), and (38) in (34), we deduce

$$(39) \quad (L(q) - 8L(q^8))^2 = \frac{4}{5}M(q) - \frac{3}{5}M(q^2) - \frac{12}{5}M(q^4) + \frac{256}{5}M(q^8) + 144\sum_{n=1}^{\infty}k(n)q^n.$$

Appealing to (5) and (39), we obtain

$$(40) \quad (L(q) - 8L(q^8))^2 = 49 + \sum_{n=1}^{\infty} \left( 192\sigma_3(n) - 144\sigma_3\left(\frac{n}{2}\right) - 576\sigma_3\left(\frac{n}{4}\right) + 12288\sigma_3\left(\frac{n}{8}\right) + 144k(n) \right) q^n.$$

Equating coefficients of  $q^n$  in (13) and (40), we deduce that

$$\begin{aligned} 240\sigma_3(n) + 15360\sigma_3\left(\frac{n}{8}\right) + (384 - 288n)\sigma(n) + (384 - 2304n)\sigma\left(\frac{n}{8}\right) - 9216W_8(n) \\ = 192\sigma_3(n) - 144\sigma_3\left(\frac{n}{2}\right) - 576\sigma_3\left(\frac{n}{4}\right) + 12288\sigma_3\left(\frac{n}{8}\right) + 144k(n), \end{aligned}$$

from which the asserted formula for  $W_8(n)$  follows. □

### 3. The number of representations of $n$ by the quadratic form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 2x_7^2 + 2x_8^2$$

*Proof of Theorem 2.* Let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For  $l \in \mathbb{N}_0$ , we set

$$r_4(l) = \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = l\},$$

so that  $r_4(0) = 1$ . The number  $N(n)$  of representations of  $n$  by the quadratic form  $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 2x_7^2 + 2x_8^2$  is

$$\begin{aligned}
 N(n) &= \text{card}\{(x_1, \dots, x_8) \in \mathbb{Z}^8 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_5^2 + 2x_6^2 + 2x_7^2 + 2x_8^2 = n\} \\
 &= \sum_{\substack{l, m \in \mathbb{N}_0 \\ l+2m=n}} \left( \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = l}} 1 \right) \left( \sum_{\substack{(x_5, x_6, x_7, x_8) \in \mathbb{Z}^4 \\ x_5^2 + x_6^2 + x_7^2 + x_8^2 = m}} 1 \right) = \sum_{\substack{l, m \in \mathbb{N}_0 \\ l+2m=n}} r_4(l) r_4(m) \\
 &= r_4(0) r_4\left(\frac{n}{2}\right) + r_4(n) r_4(0) + \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} r_4(l) r_4(m) \\
 &= r_4(n) + r_4\left(\frac{n}{2}\right) + \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} r_4(l) r_4(m).
 \end{aligned}$$

It is a classical result of Jacobi — see for example [Spearman and Williams 2000] — that

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4 \nmid d}} d = 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right), \quad n \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
 N(n) &= 8\sigma(n) - 32\sigma\left(\frac{n}{4}\right) + 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{8}\right) \\
 &\quad + \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} \left(8\sigma(l) - 32\sigma\left(\frac{l}{4}\right)\right) \left(8\sigma(m) - 32\sigma\left(\frac{m}{4}\right)\right).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 N(n) &- 8\sigma(n) - 8\sigma\left(\frac{n}{2}\right) + 32\sigma\left(\frac{n}{4}\right) + 32\sigma\left(\frac{n}{8}\right) \\
 &= 64 \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} \sigma(l)\sigma(m) - 256 \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} \sigma\left(\frac{l}{4}\right)\sigma(m) \\
 &\quad - 256 \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} \sigma(l)\sigma\left(\frac{m}{4}\right) + 1024 \sum_{\substack{l, m \in \mathbb{N} \\ l+2m=n}} \sigma\left(\frac{l}{4}\right)\sigma\left(\frac{m}{4}\right) \\
 &= 64 \sum_{m < n/2} \sigma(m)\sigma(n-2m) - 256 \sum_{\substack{l, m \in \mathbb{N} \\ 4l+2m=n}} \sigma(l)\sigma(m) \\
 &\quad - 256 \sum_{\substack{l, m \in \mathbb{N} \\ l+8m=n}} \sigma(l)\sigma(m) + 1024 \sum_{\substack{l, m \in \mathbb{N} \\ 4l+8m=n}} \sigma(l)\sigma(m) \\
 &= 64 W_2(n) - 256 \sum_{l < n/4} \sigma(l)\sigma\left(\frac{n}{2} - 2l\right) \\
 &\quad - 256 \sum_{m < n/8} \sigma(m)\sigma(n-8m) + 1024 \sum_{m < n/8} \sigma(m)\sigma\left(\frac{n}{4} - 2m\right),
 \end{aligned}$$

that is,

$$(41) \quad N(n) = 8\sigma(n) + 8\sigma\left(\frac{n}{2}\right) - 32\sigma\left(\frac{n}{4}\right) - 32\sigma\left(\frac{n}{8}\right) \\ + 64W_2(n) - 256W_2\left(\frac{n}{2}\right) - 256W_8(n) + 1024W_2\left(\frac{n}{4}\right).$$

From [Huard et al. 2002, Theorem 2], we have

$$(42) \quad W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{n}{8}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{2}\right).$$

Appealing to (41), (42), and Theorem 1, we obtain

$$N(n) = 4\sigma_3(n) - 4\sigma_3\left(\frac{n}{2}\right) - 16\sigma_3\left(\frac{n}{4}\right) + 256\sigma_3\left(\frac{n}{8}\right) + 4k(n),$$

as claimed. □

The values of  $N(n)$ ,  $\sigma_3(n)$  and  $k(n)$  for  $n = 1, 2, \dots, 10$  are as follows:

$n$	1	2	3	4	5	6	7	8	9	10
$N(n)$	8	32	96	240	496	896	1472	2160	2984	4032
$\sigma_3(n)$	1	9	28	73	126	252	344	585	757	1134
$k(n)$	1	0	-4	0	-2	0	24	0	-11	0

### Note added in proof

Since this paper was written, the sums  $W_k(n)$  have been evaluated for  $k = 5, 6, 7, 12, 16, 18$  and  $24$ . See M. Lemire and K. S. Williams, *Bull. Austral. Math. Soc.* **73** (2006), 107–115; A. Alaca, S. Alaca and K. S. Williams, *Adv. Th. App. Math.* **1** (2006), 27–48, *Int. Math. Forum* **2** (2007), 45–68, *Math. J. Okayama Univ.* (in press), *Canad. Math. Bull.* (to appear); and S. Alaca and K. S. Williams, *J. Number Th.* (in press).

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