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A COMPARISON BETWEEN RIEFFEL'S AND KONTSEVICH'S DEFORMATION QUANTIZATIONS FOR LINEAR POISSON TENSORS

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We consider Rieffel's deformation quantization and Kontsevich's star product on the duals of Lie algebras equipped with linear Poisson brackets. We give the equivalence operator between these star products and compare properties of these two main examples of deformation quantization. We show that Rieffel's deformation can be obtained by considering oriented graphs $\vec{\Gamma}$. We also prove that Kontsevich's star product provides a deformation quantization by partial embeddings on the space $C_c^\infty(\mathfrak{g})$ of C^∞ compactly supported functions on a general Lie algebra $\mathfrak{g}$ and a strict deformation quantization on the space $\mathcal{S}(\mathfrak{g}^*)$ of Schwartz functions on the dual $\mathfrak{g}^*$ of a nilpotent Lie algebra $\mathfrak{g}$. Finally, we give the explicit formulae of these two star products and we deduce the weights of graphs occurring in these expressions.

1. Introduction

The study of star products, that is, associative deformations of pointwise product of functions, was introduced by Vey [1975] and by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [Bayen et al. 1978a; 1978b] as a tool for the quantization of a symplectic or Poisson manifold [Lichnerowicz 1983]. The problem of existence of star products has been solved in stages. First, mathematicians proved the existence of star products on symplectic manifolds [De Wilde and Lecomte 1983], then on general Poisson manifolds [Rieffel 1989; 1990; Kontsevich 1997].

Star products are used as a new approach for the quantization of classical mechanical systems. A classical mechanical system is given by its phase space, which is a C^∞-manifold M equipped with a Poisson bracket $\{\ ,\ \}$. To quantize this system, one selects a suitable algebra A of C^∞ functions on M equipped with the usual product. One then deforms this product in the direction of the Poisson bracket. That is, if $\hbar$ denotes the deformation parameter ("Planck's constant"), taking real values in some interval about 0, then one tries to define a family $\star_\hbar$ of associative

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products in such a way that for $f, g \in A$ one has

$$f \star_\hbar g \to fg$$

and

$$(f \star_\hbar g - g \star_\hbar f)/i\hbar \to \{f, g\} \quad \text{as } \hbar \to 0.$$

This is the property characterizing how a deformation quantization is related to a given Poisson bracket.

Many examples of star products have appeared in the literature [Gutt 1983; Fedosov 1994], but almost all deal only with formal deformations in which $f \star_\hbar g$ is not a function, but rather a formal power series in $\hbar$ with functions as coefficients. Some 15 years ago Rieffel introduced the notions of strict deformation quantization [1989] and of deformation quantization by partial embeddings [1990], a framework in which the convergence question could be handled.

More recently, Kontsevich [1997] solved the problem of existence of star products on any finite-dimensional Poisson manifold. He built a star product on $\mathbb{R}^d$ equipped with any Poisson structure α. This star product was defined by considering oriented graphs $\vec{\Gamma}$, differential operators $B_{\vec{\Gamma}}(\alpha)$ and weights $w_{\vec{\Gamma}}$ associated to these graphs.

Here we compare these two main examples of deformation quantization for linear Poisson tensors. We prove that Rieffel's deformation can be obtained by Kontsevich's method and is equivalent to Kontsevich's star product on the dual of Lie algebras. We also show that Kontsevich's star product provides a deformation quantization by partial embeddings on the space $C_c^\infty(\mathfrak{g})$ of C^∞ compactly supported functions on a general Lie algebra $\mathfrak{g}$ and a strict deformation quantization on the space $\mathcal{S}(\mathfrak{g}^*)$ of Schwartz functions on the dual $\mathfrak{g}^*$ of a nilpotent Lie algebra $\mathfrak{g}$. We study different properties of these two star products, give their explicit expressions and derive the rationality of the weights of all graphs occurring in these expressions.

In Section 2, we generalize Kontsevich's construction of a star product $\star_\alpha^K$ on $\mathbb{R}^d$, defining *K-star products* on $\mathbb{R}^d$ by considering oriented graphs $\vec{\Gamma}$, differential operators $B_{\vec{\Gamma}}(\alpha)$ and (variable) weights $a_{\vec{\Gamma}}$ associated to these graphs.

In Section 3, we recall Rieffel's construction for duals of Lie algebras and prove that the deformation in his construction can be obtained by considering oriented graphs as Kontsevich did [1997]. We also give the equivalence operator between Rieffel's and Kontsevich's star products. Section 4 compares certain properties of these two main examples of deformation quantization. In Section 5, we prove that for Lie algebras equipped with linear Poisson brackets, the Kontsevich star product provides a convergent deformation quantization. Finally, in Section 6, we give the explicit expressions of the Rieffel and Kontsevich star products. We deduce the forms of graphs occurring in these expressions and the corresponding weights.

2. Generalization of the Kontsevich star product on the duals of Lie algebras

Kontsevich [1997] solved the problem of the existence of star products on any finite-dimensional Poisson manifold. He built a star product $\star^K_\alpha$ on $\mathbb{R}^d$ equipped with any Poisson bracket α. This star product is defined using oriented graphs $\vec{\Gamma}$. To each graph $\vec{\Gamma}$, Kontsevich associates, via a geometric construction, a bidifferential operator $C_{\vec{\Gamma}}(\alpha)$ and a weight $w_{\vec{\Gamma}}$. In [Arnal et al. 1999; Ben Amar 2003], we generalized Kontsevich's star product slightly, as follows.

To associate to each graph Γ an m-differential operator $C_\Gamma(\alpha)$, we consider certain oriented admissible graphs $\vec{\Gamma}$. Let $A = \{p_1, \dots, p_n\}$ and $B = \{q_1, \dots, q_m\}$ be finite sets. The points of A and B are the vertices of Γ. We place the vertices in A on the upper half-plane

$$H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\},$$

and those in B on the real axis, as in [Kontsevich 1997]. The edges are arrows $\overrightarrow{px}$ — which we write simply px — from a vertex p in A to a vertex x in $A \cup B$. (Kontsevich demands that $x \neq p$, but here we allow edges from a vertex to itself.) Exactly two arrows originate at each vertex p in A. Now we define an orientation of Γ by choosing total orderings $\leq$ on A and on the set of edges of Γ which are compatible in the sense that $p \leq q$ implies $px \leq qy$ for all $p, q \in A$ and all $x, y \in A \cup B$. For $x \in A \cup B$, we define the differential operator

$$\partial_{\operatorname{End} x} := \partial_{l_1 \dots l_s},$$

where $e_{l_1} < \cdots < e_{l_s}$ are the arrows ending at the point x, ordered according to $\leq$.

Denote the edges starting at $p_i \in A$ by $e_{l_{i,1}} < e_{l_{i,2}}$. For any Poisson structure α, the m-differential operator $C_{\vec{\Gamma}}(\alpha)$ is by definition

$$C_{\vec{\Gamma}}(\alpha)(f_1, \dots, f_m) = \sum_{1 \leq l_{1,1}, l_{1,2}, \dots, l_{n,2} \leq d} \prod_{i=1}^{n} \partial_{\operatorname{End} p_i} \alpha^{l_{i,1} l_{i,2}} \prod_{k=1}^{m} \partial_{\operatorname{End} q_k} f_k,$$

where $f_1, \dots, f_m$ are C^∞ functions on $\mathbb{R}^d$.

Consider the chosen total ordering $p_1 < p_2 < \cdots < p_n$ on A. For any graph Γ, define a particular orientation $\operatorname{lex} \Gamma$ by ordering the set of edges lexicographically:

$$\begin{aligned} p_i p_j \leq p_{i'} p_{j'} &\iff i < i' \text{ or } i = i' \text{ and } j \leq j', \\ p_i p_j \leq p_{i'} q_k &\iff i \leq i', \\ p_i q_k \leq p_{i'} q_{k'} &\iff i < i' \text{ or } i = i' \text{ and } k \leq k'. \end{aligned}$$

Finally, starting with $\operatorname{lex} \Gamma$, we symmetrize Γ by the action of the group S_n of all permutations of vertices $p_1, \dots, p_n$. Let Γ^σ be the graph obtained from Γ by relabeling $p_{\sigma^{-1}(i)}$ the vertices p_i. Each edge $p_i p_j$ or $p_i q_k$ of $\operatorname{lex}\Gamma$ becomes an edge $p_{\sigma^{-1}(i)} p_{\sigma^{-1}(j)}$ or $p_{\sigma^{-1}(i)} q_k$ of $\operatorname{lex}\Gamma^\sigma$.

We now set

$$C_\Gamma(\alpha) = \sum_{\sigma \in S_n} \frac{1}{n!} \varepsilon(\tilde{\sigma}) C_{\operatorname{lex}\Gamma^\sigma}(\alpha).$$

For your question, we are ranging over ALL graphs on $n+m$ points satisfying the constraints specified in the second paragraph of section 2.

For fixed n and m, let $V_{n,m}$ be the space of all lex-ordered graphs Γ on $n+m$ points p_i, q_i (ranging over the upper-half plane and the real axis, respectively) that satisfy the constraints specified in the second paragraph of this section. If $\gamma = \sum a_i \Gamma_i$ is a linear combination of graphs Γ_i in $V_{n,m}$, then C_γ will be by definition

$$C_\gamma = \sum a_i C_{\Gamma_i}.$$

We now restrict ourselves to linear Poisson structures on $\mathbb{R}^d$. Let ∂_i be the differentiation in the i-th coordinate; then a linear Poisson tensor α is given by

$$\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j = \sum_{i,j,k} C_{ij}^k x_k \partial_i \wedge \partial_j,$$

where the C_{ij}^k are the structure constants of a Lie algebra $\mathfrak{g}$. Hence, if $(E_i)_{i=1,\dots,d}$ is a basis of $\mathfrak{g}$ dual to the canonical basis of $\mathbb{R}^d$, we have

$$[E_i, E_j] = \sum_{i,j,k} C_{ij}^k E_k.$$

We see that the second derivative of each α^{ij} vanishes. Thus we consider only graphs Γ where at most one edge ends at p_i, for every i.

To define star products based on such graphs, we fix $m = 2$; the functions f and g are placed respectively on the two points, q_1 and q_2, of B.

Definition 2.1 (K-star product on $\mathbb{R}^d$). Let $(\gamma_n)_{n\geq 1}$ be an element of $\prod_{n\geq 1} V_{n,2}$ and let α be a linear Poisson structure on $\mathbb{R}^d$. For all f, g in $C^\infty(\mathbb{R}^d)$, the formal series

$$f \star_\alpha g = fg + \sum_{n=1}^{\infty} C_{\gamma_n}(\alpha)(f, g)$$

is called a *K-star product* if for all h in $C^\infty(\mathbb{R}^d)$,

$$\sum_{r+s=n} C_{\gamma_r}(\alpha)\big(C_{\gamma_s}(\alpha)(f, g), h\big) = \sum_{r+s=n} C_{\gamma_r}(\alpha)\big(f, C_{\gamma_s}(\alpha)(g, h)\big)$$

and

$$C_{\gamma_1}(\alpha)(f, g) - C_{\gamma_1}(\alpha)(g, f) = 2\langle \alpha, df \wedge dg \rangle.$$

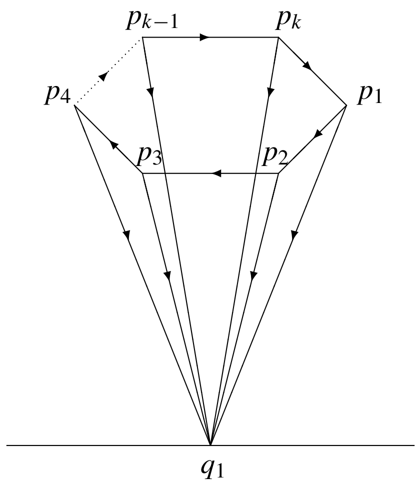

Figure 1. The wheel graph W_k.

Definition 2.2 (Wheel graph). A *wheel graph* of size k is a graph W_k with k vertices $p_1, \dots, p_k \in A$, one vertex $q_1 \in B$, and edge set $\{p_i p_{i+1}, p_i q_1 : 1 \le i \le k\}$ (where $p_{k+1} := p_1$). See Figure 1.

In [Ben Amar 2003] we showed that every K-star product for a linear Poisson structure α is given by a universal integral formula on any Lie algebra $\mathfrak{g}$:

$$(f \star_\alpha g)(\xi) = \int_{\mathfrak{g}^2} \hat{f}(X)\hat{g}(Y) \frac{F(X)F(Y)}{F(X \times_\alpha Y)} e^{2i\pi\langle \xi, X \times_\alpha Y\rangle} \, dX \, dY$$

for all f, g in the space $\mathcal{P}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$ (or f, g smooth functions such that $\hat{f}$ and $\hat{g}$ are compactly supported with a sufficiently small support). Here $X \times_\alpha Y$ is given by the Backer–Campbell–Hausdorff formula, and

$$F(X) = 1 + \sum_{n=1}^{\infty} \sum_{|k|=n} a_{k_1 \dots k_p} \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_1} \dots \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_p}.$$

These star products are all equivalent to the star product $\star_\alpha^K$ built by Kontsevich.

By results of Kontsevich [1997] and Shoikhet [2001], $\star_\alpha^K$ can be written as

$$(f \star_\alpha^K g)(\xi) = \int_{\mathfrak{g}^2} \hat{f}(X)\hat{g}(Y) \frac{J(X)J(Y)}{J(X \times_\alpha Y)} e^{2i\pi\langle \xi, X \times_\alpha Y\rangle} \, dX \, dY$$

for all f, g in $\mathcal{P}(\mathfrak{g}^*)$ (or f, g smooth functions such that $\hat{f}$ and $\hat{g}$ are compactly supported with a sufficiently small support), where

$$J(X) = \det\left(\frac{\operatorname{sh} \operatorname{ad}(X/2)}{\operatorname{ad}(X/2)}\right)^{1/2};$$

see [Arnal and Ben Amar 2001], for instance.

An *equivalence operator* T between $\star_\alpha^K$ and $\star_\alpha$, that is, one satisfying

$$f \star_\alpha g = T^{-1}(Tf \star_\alpha^K Tg),$$

is given by $(Tf)(\xi) = \int_{\mathfrak{g}} \hat{f}(X)H(X)\, e^{2i\pi\langle\xi,X\rangle}\, dX$ if

$$F(X) = J(X)H(X).$$

T can be expressed as a formal series of differential operators, each being a linear combination of products of operators T_k. Indeed, for each positive integer k,

$$(T_k f)(\xi) = (2i\pi)^k \int_{\mathfrak{g}} \hat{f}(X)\, \mathrm{tr}(\mathrm{ad}\, X)^k\, e^{2i\pi\langle\xi,X\rangle}\, dX.$$

Thus T_k has the expression

$$(T_k f) = \sum_{i_1\dots i_k} \sum_{j_1\dots j_k} C_{i_1 j_1}^{j_2} C_{i_2 j_2}^{j_3} \dots C_{i_{k-1} j_{k-1}}^{j_k} C_{i_k j_k}^{j_1} \partial_{i_1\dots i_k} f.$$

Thus we see that T_k is the operator associated to the wheel graph W_k. We can write

$$T = \mathrm{Id} + \sum_{n=1}^{\infty} \sum_{|k|=n} b_{k_1\dots k_p} T_{k_1} \circ \dots \circ T_{k_p}$$

if

$$H(X) = 1 + \sum_{n=1}^{\infty} \sum_{|k|=n} b_{k_1\dots k_p}\, \mathrm{tr}(2i\pi\, \mathrm{ad}\, X)^{k_1} \dots \mathrm{tr}(2i\pi\, \mathrm{ad}\, X)^{k_p}.$$

3. Rieffel's deformation quantization

Rieffel [1990] constructed a deformation quantization on a general Lie algebra equipped with the linear Poisson bracket as follows.

Let exp be the exponential map from $\mathfrak{g}$ to its simply connected Lie group G and let U be an open neighborhood of $0 \in \mathfrak{g}$ on which exp is a diffeomorphism into G and such that $U = -U$. We identify U with its image in G. Let C be an open convex neighborhood of $0 \in \mathfrak{g}$ such that as $C^3 \subseteq U$ in G and such that $C = -C$ ($= C^{-1}$ in G). Now let $\mathfrak{g}_\hbar$ be the Lie algebra $\mathfrak{g}$ as an additive group but equipped with the Lie bracket $[\ ,\ ]_\hbar$ defined by

$$[\ ,\ ]_\hbar = \hbar\, [\ ,\ ].$$

Let $G_\hbar$ denote the simply connected Lie group of $\mathfrak{g}_\hbar$. For $\hbar = 0$, we set $G_0 = \mathfrak{g}$. For $\hbar$ nonzero, $G_\hbar$ is isomorphic to G and map exp is a diffeomorphism on $U_\hbar = \hbar^{-1}U$. Let $\times_\hbar$ be the partially defined group product on $U_\hbar$ coming from $G_\hbar$. Then

$$X \times_\hbar Y = \hbar^{-1}((\hbar X).(\hbar Y)),$$

where $(\hbar X).(\hbar Y)$ is the group product in U obtained from G. Let $C_\hbar = \hbar^{-1}C$. Then one has $C_\hbar^3 \subset U_\hbar$ for the product $\times_\hbar$.

Rieffel considered the space $C_c^\infty(\mathfrak{g})$ of C^∞ functions compactly supported on $\mathfrak{g}$. For $\hbar$ sufficiently small, the supports of two elements φ and ψ in $C_c^\infty(\mathfrak{g})$ are in $C_\hbar$. He then defined the convolution $\varphi *_\hbar \psi$ for the left Haar measure $\omega_\hbar(X)\,dX$ by

$$(\varphi *_\hbar \psi)(X) = \int_{\mathfrak{g}} \varphi(Y)\psi(Y^{-1} \times_\hbar X)\omega_\hbar(Y)\,dY,$$

where $\omega(Y) = \det(d(\exp Y))$ and $\omega_\hbar(Y) = \omega(\hbar Y)$.

The product constructed by Rieffel can be written on $\mathfrak{g}^*$ and it is a K-star product as defined in [Ben Amar 2003]. In the following, we suppose given two functions f and g on $\mathfrak{g}^*$ whose Fourier transforms $\hat{f} = \varphi$ and $\hat{g} = \psi$ are smooth and compactly supported, and we consider only the $\hbar's$ sufficiently small such that the supports of $\hat{f}$ and $\hat{g}$ are both in $C_\hbar$. Then we can write the Rieffel star product $f \star_\hbar^R g$ for all $\xi \in \mathfrak{g}^*$ as

$$(f \star_\hbar^R g)(\xi) = \int_{\mathfrak{g}\times\mathfrak{g}} \varphi(Y)\psi(Y^{-1} \times_\hbar X)\omega_\hbar(Y)\, e^{2i\pi\langle X,\,\xi\rangle}\,dY\,dX.$$

Let $Z = Y^{-1} \times_\hbar X$ then $\omega_\hbar(Z)\,dZ = \omega_\hbar(X)\,dX$. We obtain

$$\begin{aligned}(f \star_\hbar^R g)(\xi) &= \int_{\mathfrak{g}^2} \varphi(Y)\psi(Z)\frac{\omega_\hbar(Y)\omega_\hbar(Z)}{\omega_\hbar(Y \times_\hbar Z)}\, e^{2i\pi\langle Y\times_\hbar Z,\,\xi\rangle}\,dY\,dZ \\ &= \int_{\mathfrak{g}^2} \hat{f}(Y)\hat{g}(Z)\frac{\omega_\hbar(Y)\omega_\hbar(Z)}{\omega_\hbar(Y \times_\hbar Z)}\, e^{2i\pi\langle Y\times_\hbar Z,\,\xi\rangle}\,dY\,dZ.\end{aligned}$$

By the integral formula above, we see that $\star_\hbar^R$ is a K-star product associated to the function $\omega_\hbar$. We conclude that Rieffel's deformation quantization can be obtained by considering oriented graphs, differential operators and weights associated to different graphs, as in [Kontsevich 1997].

The Rieffel star product $\star_\hbar^R$ is also associated to the function $J_\hbar^2$. In fact, the differential $d\exp$ of the exponential map is given by (see [Varadarajan 1974])

$$d(\exp X) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}(\operatorname{ad} X)^n = \frac{1-e^{-\operatorname{ad} X}}{\operatorname{ad} X} = e^{-\frac{1}{2}\operatorname{ad} X}\,\frac{\operatorname{sh}\operatorname{ad}(X/2)}{\operatorname{ad}(X/2)}.$$

It follows that

$$\omega_\hbar(X) = \det(d(\exp \hbar X)) = \det\left(e^{-\frac{1}{2}\operatorname{ad}\hbar X}\,\frac{\operatorname{sh}\operatorname{ad}(\hbar X/2)}{\operatorname{ad}(\hbar X/2)}\right) = e^{-\frac{1}{2}\operatorname{tr}(\operatorname{ad}\hbar X)}\,J_\hbar^2(X).$$

The wheel operator T_1 associated to the wheel graph W_1 (Definition 2.2) is a universal derivation for K-star products [Ben Amar 2003, p. 340]. Then $\exp\bigl(-\frac{\hbar}{4i\pi}T_1\bigr)$ is an automorphism, so the term $e^{-\frac{1}{2}}\operatorname{tr}(\operatorname{ad}\hbar X)$ plays no role and the K-star product associated to $\omega_\hbar$ coincides with the K-star product defined by $J_\hbar^2$.

We now show that $\star_\hbar^R$ is associated to $J_\hbar^2$ by direct computation. We start with:

Lemma 3.1. *Let X and Y be two elements in the Lie algebra $\mathfrak{g}$. Then one has*

$$e^{-\frac{1}{2}\operatorname{tr}(\operatorname{ad}(X\times_\hbar Y))}=e^{-\frac{1}{2}\operatorname{tr}(\operatorname{ad}(X+Y))}=e^{-\frac{1}{2}\operatorname{tr}(\operatorname{ad}X)}e^{-\frac{1}{2}\operatorname{tr}(\operatorname{ad}Y)}.$$

Proof. The Campbell–Hausdorff gives

$$\begin{aligned}X\times_\hbar Y&=\sum_{n=1}^{\infty}\frac{\hbar^{n-1}}{n}\sum_{m\geqslant 1}\frac{(-1)^{m-1}}{m}\sum_{(p,q)\in\mathcal{P}_m^n}\frac{1}{p!q!}\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m})\\&=X+Y+\sum_{n=2}^{\infty}\frac{\hbar^{n-1}}{n}\sum_{m\geqslant 1}\frac{(-1)^{m-1}}{m}\sum_{(p,q)\in\mathcal{P}_m^n}\frac{1}{p!q!}\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m}),\end{aligned}$$

where

$\mathcal{P}_m^n=\big\{(p_1,\dots,p_m,q_1,\dots,q_m): p_j+q_j\geqslant 1 \text{ for all } j \text{ such that } \sum p_j+q_j=n\big\}$,

$X^{p_j}=X,X,\dots,X$ (p_j times), $Y^{q_j}=Y,Y,\dots,Y$ (q_j times), and

$$\Psi(Z_1,\dots,Z_n)=[Z_1,[Z_2,[\dots[Z_{n-1},Z_n]\dots]]]$$

(see [Varadarajan 1974, 2.15]). But if $n>1$,

$$\operatorname{tr}(\operatorname{ad}(\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m})))=0\quad\text{for all } (p,q)\in\mathcal{P}_m^n.$$

Indeed, for all X, Y in $\mathfrak{g}$ one has $\operatorname{tr}(\operatorname{ad}[X,Y])=\operatorname{tr}(\operatorname{ad}X\operatorname{ad}Y-\operatorname{ad}Y\operatorname{ad}X)=0$. Now, if $p_1>0$ one has

$$\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m})=[X,Z],$$

where $Z=\Psi(X^{p_1-1},Y^{q_1},\dots,X^{p_m},Y^{q_m})$. Then

$$\operatorname{tr}(\operatorname{ad}(\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m})))=\operatorname{tr}(\operatorname{ad}[X,Z])=0$$

If $p_1=0$,

$$\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m})=[Y,Z]$$

where $Z=\Psi(Y^{q_1-1},\dots,X^{p_m},Y^{q_m})$. Then

$$\operatorname{tr}(\operatorname{ad}\Psi(X^{p_1},Y^{q_1},\dots,X^{p_m},Y^{q_m}))=\operatorname{tr}(\operatorname{ad}[Y,Z])=0.$$

The equality in the lemma follows. □

Proposition 3.2 (Integral formula). *Let f and g be functions on $\mathfrak{g}^*$ such that $\hat{f}$ and $\hat{g}$ are smooth compactly supported. Then $\star_\hbar^R$ can be written by the following integral formula*

$$(f\star_\hbar^R g)(\xi)=\int_{\mathfrak{g}^2}\hat{f}(X)\hat{g}(Y)\frac{J_\hbar^2(X)J_\hbar^2(Y)}{J_\hbar^2(X\times_\hbar Y)}\,e^{2i\pi\langle X\times_\hbar Y,\,\xi\rangle}\,dX\,dY$$

where

$$J_\hbar^2(X) = J^2(\hbar X) = \det\left(\frac{\operatorname{sh}\operatorname{ad}(\hbar X/2))}{\operatorname{ad}(\hbar X/2)}\right).$$

Proof. Using Lemma 3.1 we get

$$\frac{\omega_\hbar(X)\omega_\hbar(Y)}{\omega_\hbar(X\times_\hbar Y)} = \frac{e^{-\frac{1}{2}\hbar\operatorname{tr}(\operatorname{ad}X)}e^{-\frac{1}{2}\hbar\operatorname{tr}(\operatorname{ad}Y)}}{e^{-\frac{1}{2}\hbar\operatorname{tr}(\operatorname{ad}(X\times_\hbar Y))}} \times \frac{J_\hbar^2(X)J_\hbar^2(Y)}{J_\hbar^2(X\times_\hbar Y)} = \frac{J_\hbar^2(X)J_\hbar^2(Y)}{J_\hbar^2(X\times_\hbar Y)}. \qquad \square$$

The Rieffel star product $\star_\hbar^R$ is equivalent to Kontsevich star product $\star_\hbar^K$ through an intertwining operator T defined by

$$(Tf)(\xi) = \int_{\mathfrak{g}} \hat{f}(X) J_\hbar(X)\, e^{2i\pi\langle X,\,\xi\rangle}\, dX.$$

The star product $\star_\hbar^R$ is also equivalent to Gutt star product [1983]. This product, denoted by $\star_\hbar^G$, is associated to the complete symmetrization mapping between the space $\mathcal{P}(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$ and the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$

$$(f \star_\hbar^G g)(\xi) = \int_{\mathfrak{g}^2} \hat{f}(X)\hat{g}(Y)\, e^{2i\pi\langle X\times_\hbar Y,\,\xi\rangle}\, dX\, dY.$$

The equivalence between $\star_\hbar^R$ and $\star_\hbar^G$ is through an intertwining operator T':

$$T'(f \star_\hbar^R g) = T'(f) \star_\hbar^G T'(g).$$

The operator T' is defined by

$$(T'f)(\xi) = \int_{\mathfrak{g}} \hat{f}(X) J_\hbar^2(X)\, e^{2i\pi\langle X,\,\xi\rangle}\, dX.$$

The operators T and T' are composed of wheel operators, as discussed in the previous section, and we see immediately that $T' = T \circ T$.

Finally, we remark that $\star_\hbar^R$ is equivalent to $\star_\hbar^G$ also through the intertwining operator T'' defined by

$$(T''f)(\xi) = \int_{\mathfrak{g}} \hat{f}(X)\omega_\hbar(X)\, e^{2i\pi\langle X,\,\xi\rangle}\, dX.$$

Indeed, we have

$$\omega_\hbar(X) = e^{-\frac{1}{2}\operatorname{tr}(\operatorname{ad}\hbar X)} J_\hbar^2(X).$$

We obtain

$$T'' = e^{-\frac{\hbar}{4i\pi}T_1} \circ T',$$

the operator composed with T' being an automorphism. The equivalence between $\star_\hbar^R$ and $\star_\hbar^G$ is through T' or T''.

4. Comparison of properties of the Rieffel and Kontsevich star products

Definition 4.1 (Symmetric star products). A star product on a Poisson manifold M is said to be *symmetric* if C_r is symmetric for r even and skew-symmetric for r odd; that is, if

$$C_r(f, g) = (-1)^r C_r(g, f) \quad \text{for all } f, g \in C^\infty(M).$$

In [Ben Amar 2003] we have shown that a K-star product $\star_\hbar$ is symmetric if and only if $\star_\hbar$ is associated to a function F such that

$$F(X) = e^{2i\pi a_1 \operatorname{tr}(\operatorname{ad} X)} G(X)$$

where G is an even function, or again

$$F(X) = e^{2i\pi a_1 \operatorname{tr}(\operatorname{ad} X)} \left(1 + \sum_{n=1}^{\infty} \sum_{\substack{k_1 \dots k_p \\ |k|=2n}} b_{k_1 \dots k_p} \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_1} \dots \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_p} \right).$$

In this way, we can associate $\star_\hbar$ to the even function G itself.

Although the function ω is not even, we can deduce that the Rieffel star product $\star_\hbar^R$ is symmetric since it is also associated to the even function J^2 (see Section 3).

Symmetry is a property related to realness for star products. This notion was considered by Lichnerowicz.

Definition 4.2 (Real star products). A K-star product $\star_\hbar$ is said to be *real* if

$$\overline{f \star_\hbar g} = \overline{g} \star_\hbar \overline{f}.$$

for any smooth functions f and g whose Fourier transforms $\hat{f}$, $\hat{g}$ have sufficiently small support.

In [Ben Amar 2003] we showed that a symmetric K-star product associated to a function F, which we can suppose to be even since the term $e^{a_1 \operatorname{tr}(\operatorname{ad} X)}$ does not play any role, is real if and only if F is a real function. We can deduce the following:

Proposition 4.3 (Symmetry and reality). *The Rieffel star product $\star_\hbar^R$ and the Kontsevich star product $\star_\hbar^K$ are symmetric and real.*

Note that the Rieffel star product is defined by the entire function ω, while the Kontsevich star product is associated to the function J, which is not entire. But the function J is holomorphic on a neighborhood of zero and this property is needed in showing certain results. Then we will define analytic star products to characterize a certain class of K-star products given by functions holomorphic near zero.

Definition 4.4 (Analytic star products). A K-star product defined by a function $F(X) = J(X)e^{h(X)}$ is said to be *analytic* if the series

$$h(x) = \sum_{k_1 \dots k_p} a_{k_1 \dots k_p} \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_1} \dots \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_p}$$

has a strictly positive convergence radius: there exists $r > 0$ such that

$$\sum_{k_1 \dots k_p} |a_{k_1 \dots k_p}| \, r^{k_1 + \dots + k_p} < \infty.$$

A K-star product $\star_\hbar$ is analytic if and only if it is associated to a function F holomorphic on a neighborhood of zero [Ben Amar 2003].

Definition 4.5 (Closed star products). A star product on $\mathfrak{g}^*$ is said to be *closed* if

$$\int_{\mathfrak{g}^*} (f \star_\hbar g)(\xi) \, d\xi = \int_{\mathfrak{g}^*} f(\xi) g(\xi) \, d\xi$$

for all smooth compactly supported f, g.

In [Ben Amar 2003], we have proved that if $\mathfrak{g}$ is not a unimodular Lie algebra then an analytic K-star product $\star_\hbar$ on $\mathfrak{g}^*$ is not closed. An analytic K-star product will be called closed if and only if it is closed for any unimodular Lie algebra $\mathfrak{g}$.

We now recall the definitions of relative star products and strict Kontsevich star products.

Definition 4.6 (Relative K-star products). A K-star product $\star_\hbar$ is said to be *relative* (to the algebra of invariant polynomial functions) if

$$f \star_\hbar g = f.g$$

for every linear Poisson bracket and all invariant polynomial functions f, g.

Definition 4.7 (Strict Kontsevich star products [Arnal and Ben Amar 2001]). A K-star product associated to a function F is said to be *strict* if there exists a function f holomorphic on a neighborhood of 0 such that

$$f(0) = 1 \quad \text{and} \quad F(X) = \det f(\operatorname{ad} X).$$

We can deduce from the results of [Arnal and Ben Amar 2001] that a K-star product $\star_\hbar$ is a strict relative star product if and only if

$$F(X) = J(X) \exp\left(\sum_{s=0}^{\infty} a_{2s+1} \operatorname{tr}(2i\pi \operatorname{ad} X)^{2s+1}\right).$$

Proposition 4.8 (Relativity). *The Rieffel star product $\star_\hbar^R$ is an analytic strict K-star product but $\star_\hbar^R$ is not relative to the algebra of polynomial functions.*

We have shown that if an analytic K-star product $\star_\hbar$ is closed then it is also relative and that any strict and relative K-star product is closed [Ben Amar 2003]. Thus, we can conclude that $\star_\hbar^R$ is not closed.

We summarize these results as follows.

Proposition 4.9 (Comparison). *The Rieffel star product $\star_\hbar^R$ and the Kontsevich star product $\star_\hbar^K$ are analytic, strict, symmetric and real K-star products.*

$\star_\hbar^R$ is defined by the entire function ω but it is not closed and not relative.

$\star_\hbar^K$ is closed and relative but it is associated to the nonentire function J.

5. Convergence notions

Rieffel [1989; 1990] introduced the notion of deformation quantization by partial embeddings and the notion of strict deformation quantization, a framework in which the convergence question could be handled. He showed that the deformation quantization constructed in [Rieffel 1990] provides a deformation quantization by partial embeddings on the space $C_c^\infty(\mathfrak{g})$ of C^∞ compactly supported functions on a general Lie algebra $\mathfrak{g}$.

Here we will choose two neighborhoods U and C of $0 \in \mathfrak{g}$ as in Section 3, but C must be chosen in such a way that the function J is holomorphic on C. This is possible since J is holomorphic near zero. In fact it is sufficient that C be contained in the ball of radius 2π. We remark that $J_\hbar$ is holomorphic on $C_\hbar$, since J is holomorphic on C.

Thus, for all $\hbar$ in $\mathbb{R}$, we can write the integral formula of the Kontsevich star product $\star_\hbar^K$ as follows. For all f, g polynomial functions or such that $\hat{f}, \hat{g}$ are smooth functions and compactly supported, we put

$$(f \star_\hbar^K g)(\xi) = \int_{\mathfrak{g}^2} \hat{f}(X)\hat{g}(Y)\frac{J_\hbar(X)J_\hbar(Y)}{J_\hbar(X \times_\hbar Y)} e^{2i\pi\langle X\times_\hbar Y,\xi\rangle}\, dX\, dY.$$

In the following, we will consider for each $\hbar$ the C^*-norm from the reduced group C^*-algebra $C_r^*(G_\hbar)$ (see [Pedersen 1979] for the definition) and we remark that $C_c^\infty(C_\hbar)$ is embedded in $C_c^\infty(G_\hbar)$.

We next recall the definition of a deformation quantization by partial embeddings, in which the notion of convergence is precise. In this definition A, $P_\hbar$ and $Q_\hbar$ play the roles of $C_c^\infty(\mathfrak{g})$, $C_c^\infty(U_\hbar)$ and $C_c^\infty(C_\hbar)$, respectively.

Definition 5.1 (Deformation quantization by partial embeddings [Rieffel 1990]). Let A be a commutative algebra equipped with a C^*-norm and a Poisson bracket $\{\ ,\ \}$. A *deformation quantization of A by partial embeddings* in the direction of $\{\ ,\ \}$ is an interval I of real numbers with 0 as center, together with, for each $\hbar \in I$, a C^*-algebra $A_\hbar$ and linear subspaces $P_\hbar \supseteq Q_\hbar$ of A, together with a linear embedding $\ell_\hbar$ of $P_\hbar$ into $A_\hbar$ satisfying the following conditions:

(1) A_0 is the C^*-completion of A, with ℓ_0 the inclusion map, and $P_0 = Q_0 = A$.
(2) $P_{-\hbar} = P_\hbar$ and $Q_{-\hbar} = Q_\hbar$.
(3) If $|\hbar| \leq |\hbar_0|$ then $P_\hbar \supseteq P_{\hbar_0}$ and $Q_\hbar \supseteq Q_{\hbar_0}$.
(4) For every $a \in A$ there is an $\hbar \neq 0$ such that $a \in Q_\hbar$.
(5) If $a, b \in Q_\hbar$, there is a (unique) element $a \star_\hbar b$ in $P_\hbar$ such that

$$\ell_\hbar(a)\ell_\hbar(b) = \ell_\hbar(a \star_\hbar b).$$

(6) For every $a, b \in A$, the value of $\|(a \star_\hbar b - b \star_\hbar a)/i\hbar - \{a, b\}\|_\hbar$ converges to zero as $\hbar$ goes to zero, where $\| \ \|_\hbar$ is the norm of $A_\hbar$ pulled back to $P_\hbar$ by $\ell_\hbar$.
(7) For any $a \in A$, the function $\hbar \mapsto \|a\|_\hbar$ is lower semicontinuous for the values of $\hbar$ for which $a \in Q_\hbar$.

To show that Kontsevich's star product provides a deformation quantization by partial embeddings as defined above, we first recall this result:

Proposition 5.2 (Fourier transform of the Poisson bracket [Rieffel 1990]). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{g}^*$ its dual. The Fourier transform of the Poisson bracket on the space $\mathcal{S}(\mathfrak{g}^*)$ of Schwartz functions on $\mathfrak{g}^*$ is given for $\varphi, \psi \in \mathcal{S}(\mathfrak{g})$ by*

$$\{\varphi, \psi\}(X) = -2i\pi \int_{\mathfrak{g}} \varphi(Y)\big(\langle [Y, X], d\psi(X-Y)\rangle + \psi(X-Y)\operatorname{tr}(\operatorname{ad} Y)\big)\, dY.$$

Theorem 5.3 (Convergence of the Kontsevich star product). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $\{\ ,\ \}$ be the Poisson bracket on $A = C_c^\infty(\mathfrak{g})$ (the set of C^∞ functions compactly supported on $\mathfrak{g}$) obtained by Fourier transform of the linear Poisson bracket on $\mathfrak{g}^*$.*

Using the previous notations, let $P_\hbar = C_c^\infty(U_\hbar)$ and $Q_\hbar = C_c^\infty(C_\hbar)$, with their embeddings in $C_r^*(G_\hbar)$. This structure together with the Kontsevich star product $\star_\hbar^K$ provides a deformation quantization of A by partial embeddings in the direction of $-(2\pi)^{-1}\{\ ,\ \}$.

Proof. Properties (1)–(5) are easily derived from our hypotheses. Let φ and ψ be compactly supported C^∞ functions on $\mathfrak{g}$. We consider values of $\hbar$ sufficiently small so that the supports of φ and ψ are both in $C_\hbar$. Note that $\omega_\hbar$ and $J_\hbar$ don't vanish on $C_\hbar$ since $C_\hbar$ is contained in the ball of radius $2\pi\hbar^{-1}$.

On $C_\hbar$, $\star_\hbar^R$ is equivalent to $\star_\hbar^K$ either through the operator T given in Section 3 or through the operator T''' defined by

$$(T''' f)(\xi) = \int_{\mathfrak{g}} \hat{f}(X) \frac{\omega_\hbar(X)}{J_\hbar(X)}\, e^{2i\pi\langle X, \xi\rangle}\, dX.$$

Using this equivalence operator, we can write

$$(\varphi \star_\hbar^K \psi)\frac{J_\hbar}{\omega_\hbar} = \Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big) \star_\hbar^R \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big) = \Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big) *_\hbar \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big),$$

where $*_\hbar$ is the convolution of functions in $C_c^\infty(C_\hbar)$ for the left Haar measure $\omega_\hbar(X)\,dX$. Thus, for all Z in $\mathfrak{g}$ one has

$$(\varphi \star_\hbar^K \psi)(Z) = \frac{\omega_\hbar}{J_\hbar}(Z) \int_{\mathfrak{g}} \Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(X) \, \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X^{-1} \times_\hbar Z)\, \omega_\hbar(X)\, dX.$$

We remark, following [Rieffel 1990], that if $\varphi(X) \neq 0$ and $\psi(X^{-1} \times_\hbar Z) \neq 0$ then $X \in C_\hbar$ and $X^{-1} \times_\hbar Z \in C_\hbar$. Then the integral above is well defined since $C_\hbar^3 \subseteq U_\hbar$. Thus if we consider that the integral vanishes when $X^{-1} \times_\hbar Z$ is not defined, we can see that $\varphi \star_\hbar^K \psi$ lies in $C_c^\infty(\mathfrak{g})$.

By a change of variables we obtain

$$\begin{aligned}\Big((\varphi \star_\hbar^K \psi - \psi \star_\hbar^K \varphi)\Big(\frac{J_\hbar}{\omega_\hbar}\Big)(Z)\Big)/\hbar \\ = \frac{1}{\hbar}\int_{\mathfrak{g}} \Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X \times_\hbar Z)\, \Delta_\hbar(X)^{-1} \omega_\hbar(X)\, dX \\ - \frac{1}{\hbar}\int_{\mathfrak{g}} \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(Z \times_\hbar X)\Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\, \omega_\hbar(X)\, dX,\end{aligned}$$

where $\Delta_\hbar(X)$ is the modular function for $G_\hbar$ defined by $\Delta_\hbar(X) = e^{-\hbar\,\mathrm{tr}(\mathrm{ad}\,X)}$.

Let

$$A_\hbar(Z, X) = \frac{1}{\hbar}\Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\Big(\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X \times_\hbar Z) - \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X + Z)\Big),$$

$$B_\hbar(Z, X) = \frac{1}{\hbar}\Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\Big(\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X + Z)\Big(\Delta_\hbar^{-1}(X) - 1\Big)\Big),$$

$$D_\hbar(Z, X) = \frac{1}{\hbar}\Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\Big(\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(Z \times_\hbar X\Big) - \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(Z + X)\Big).$$

Then a simple computation leads to

$$\begin{aligned}((\varphi \star_\hbar^K \psi - \psi \star_\hbar^K \varphi)(Z))/\hbar \\ = \Big(\frac{\omega_\hbar}{J_\hbar}\Big)(Z) \int_{\mathfrak{g}} (A_\hbar(Z, X)\Delta_\hbar^{-1}(X) + (B_\hbar - D_\hbar)(Z, X))\omega_\hbar(X)\, dX.\end{aligned}$$

One has

$$X \times_\hbar Z = X + Z + \tfrac{1}{2}\hbar\, [X, Z] + \hbar^2 R(\hbar, Z, X)$$

where R is a continuous function. Set

$$M(\hbar, Z, X) = \tfrac{1}{2}[X, Z] + \hbar\, R(\hbar, Z, X)$$

If we write the Taylor formula for the function $\psi \dfrac{J_\hbar}{\omega_\hbar}$ about $X+Z$, we obtain

$$\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X \times_\hbar Z) = \Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X+Z) + \hbar \Big\langle M(\hbar, Z, X),\ d\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X+Z)\Big\rangle$$
$$+ \Big(\frac{\hbar^2}{2}\Big)\, d^2\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)\big(X+Z+\tau\hbar\, M(\hbar, Z, X)\big)\,\big(M(\hbar, Z, X),\ M(\hbar, Z, X)\big),$$

where $\tau \in [0, 1]$ depends on $\hbar$, Z and X. Then

$$A_\hbar(Z, X) - \Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\Big\langle \tfrac{1}{2}[X, Z],\ d\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X+Z)\Big\rangle$$
$$= \hbar\,\Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\Big\langle R(\hbar, Z, X),\ d\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)(X+Z)\Big\rangle$$
$$+\frac{\hbar}{2}\Big(\varphi \frac{J_\hbar}{\omega_\hbar}\Big)(-X)\, d^2\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big)\big(X+Z+\tau\hbar\, M(\hbar,Z,X)\big)\,\big(M(\hbar,Z,X),\ M(\hbar,Z,X)\big).$$

Since $\omega_\hbar$ and $J_\hbar$ are holomorphic functions on $C_\hbar$, we see that $J_\hbar/\omega_\hbar$ converges uniformly on compact subsets to 1 and $\psi(J_\hbar/\omega_\hbar)$ converges uniformly on compact subsets to ψ. Now, the differential of $\psi(J_\hbar/\omega_\hbar)$ is given by

$$d\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big) = d(\psi)\frac{J_\hbar}{\omega_\hbar} + \psi\, d\Big(\frac{J_\hbar}{\omega_\hbar}\Big),$$

where

$$d\Big(\frac{J_\hbar}{\omega_\hbar}\Big) = \frac{d(J_\hbar)\omega_\hbar - J_\hbar d(\omega_\hbar)}{{\omega_\hbar}^2}.$$

It is not hard to show that $d(J_\hbar)$ and $d(\omega_\hbar)$ converge uniformly on compact subsets to 0. Then $d(J_\hbar/\omega_\hbar)$ also converges uniformly on compact subsets to 0. Thus

$$d\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big) \longrightarrow d\psi \quad \text{uniformly on compact subsets.}$$

In the same way we can show that

$$d^2\Big(\psi \frac{J_\hbar}{\omega_\hbar}\Big) \longrightarrow d^2\psi \quad \text{uniformly on compact subsets.}$$

The functions $d^2\psi$, R and M are continuous, hence bounded on compact sets. It follows that $A_\hbar(Z, X)$ converges uniformly on compact subsets of $C_\hbar^2 \times C_\hbar$ to

$$\varphi(-X)\big\langle \tfrac{1}{2}[X, Z],\ d\psi(X+Z)\big\rangle$$

as $\hbar$ goes to zero.

Similarly, since $\Delta_\hbar$ and $\omega_\hbar$ are entire functions on $\mathfrak{g}$, they converge to 1 uniformly on compact subsets, and we deduce that

$$A_\hbar(Z, X)\Delta_\hbar(X)^{-1}\omega_\hbar(X) \longrightarrow \varphi(-X)\big\langle \tfrac{1}{2}[X, Z],\ d\psi(X+Z)\big\rangle$$

uniformly on compact subsets. We can also show that

$$D_\hbar(Z, X)\omega_\hbar(X) \longrightarrow \varphi(-X)\left\langle \tfrac{1}{2}[Z, X],\, d\psi(Z+X)\right\rangle$$

uniformly on compact subsets to Finally, it is clear that

$$B_\hbar(Z, X) \longrightarrow \varphi(-X)\,\psi(X+Z)\ \operatorname{tr}(\operatorname{ad} X)$$

uniformly on compact subsets. Then $B_\hbar(Z, X)\omega_\hbar(X)$ also converges to the same limit. Now, $\omega_\hbar/J_\hbar$ converges uniformly on compact sets to 1. Thus, we deduce that

$$\big((\varphi \star_\hbar^K \psi)(Z) - (\psi \star_\hbar^K \varphi)(Z)\big)/i\hbar$$

converges uniformly on compact subsets to

$$\frac{1}{i}\int_{\mathfrak{g}} \varphi(-X)\big(\langle [X, Z],\, d\psi(Z+X)\rangle + \psi(Z+X)\operatorname{tr}(\operatorname{ad} X)\big)\, dX$$

as $\hbar$ goes to zero. But if we compare this formula to the result in Proposition 5.2, we obtain that

$$\big((\varphi \star_\hbar^K \psi)(Z) - (\psi \star_\hbar^K \varphi)(Z)\big)/i\hbar$$

converges uniformly on compact sets to $-(2\pi)^{-1}\{\varphi, \psi\}(Z)$.

For φ and ψ fixed, if $\hbar$ varies in a compact interval it is clear that there exists a compact set containing the supports of all the $\varphi \star_\hbar^K \psi$. Hence

$$(\varphi \star_\hbar^K \psi - \psi \star_\hbar^K \varphi)/i\hbar$$

converges to $-(2\pi)^{-1}\{\varphi, \psi\}$ for the inductive limit topology, and so also for the L^1-norm for dX. Therefore

$$(\varphi \star_\hbar^K \psi - \psi \star_\hbar^K \varphi)\omega_\hbar/i\hbar$$

converges to $-(2\pi)^{-1}\{\varphi, \psi\}$ for the L^1-norm for dX.

Since the L^1 norm for $\omega_\hbar(X)\,dX$ dominates the norms $\|\ \ \|_\hbar$ of $C_r^*(G_\hbar)$, we conclude that

$$\big\|(\varphi \star_\hbar^K \psi - \psi \star_\hbar^K \varphi)/i\hbar + (2\pi)^{-1}\{\varphi, \psi\}\big\|_\hbar$$

converges to zero as $\hbar$ goes to zero. This proves property (6) in Definition 5.1.

The proof of property (7) is given in [Rieffel 1990]. In fact, the C^*-norms of the reduced group C^*-algebras $C_r^*(G_\hbar)$ coincide in our case with the C^*-norms considered by Rieffel. □

The same proof can be carried out for a general K-star product $\star_\hbar$ if it is associated to a function F holomorphic near zero, that is, if $\star_\hbar$ is an analytic K-star product.

Corollary 5.4 (Generalization for analytic K-star products). *Let $\star_\hbar$ be an analytic K-star product. Then $\star_\hbar$ provides a deformation quantization by partial embeddings on the space $C_c^\infty(\mathfrak{g})$ of C^∞ compactly supported functions on a general Lie algebra $\mathfrak{g}$.*

Rieffel [1990] showed that in the case of a nilpotent Lie algebra $\mathfrak{g}$, the star product $\star_\hbar^R$ provides a strict deformation quantization on the space $\mathcal{S}(\mathfrak{g}^*)$ of Schwartz functions on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$.

Definition 5.5 (Strict deformation quantization [Rieffel 1989]). Let M be a manifold equipped with a Poisson bracket $\{\ ,\ \}$ and let A be a subalgebra of $C_0(M)$ (the algebra of functions vanishing at infinity) that consists of smooth functions, contains $C_c^\infty(M)$ (the space of smooth functions of compact support) and is carried into itself by the Poisson bracket. By a *strict deformation quantization* of M in the direction of $\{\ ,\ \}$ we mean an open interval I of real numbers centered at 0, together with, for each $\hbar \in I$, an associative product $\star_\hbar$, an involution $^{*_\hbar}$ and a C^*-norm $\|\ \|_\hbar$ (for $\star_\hbar$ and $^{*_\hbar}$) on A, which for $\hbar = 0$ become the original pointwise product, complex conjugation involution, and supremum norm, and satisfying the following properties:

(1) For every $f \in A$, the function $\hbar \mapsto \|f\|_\hbar$ is continuous.

(2) For every $f, g \in A$,

$$\left\|(f \star_\hbar g - g \star_\hbar f)/i\hbar - \{f, g\}\right\|_\hbar$$

converges to 0 as $\hbar$ goes to 0.

In the nilpotent case, it was proved in [Ben Amar 1995] that the Rieffel star product $\star_\hbar^R$ coincides with the Gutt star product $\star_\hbar^G$. At the same time, Arnal [1998] showed by a direct computation of weights that in this case the Kontsevich star product $\star_\hbar^K$ coincides with $\star_\hbar^G$. Thus, in the case of a nilpotent Lie algebra $\mathfrak{g}$, we have $\star_\hbar^R = \star_\hbar^K = \star_\hbar^G$. In fact, if $\mathfrak{g}$ is nilpotent, $\operatorname{tr}(\operatorname{ad} X)^k = 0$ for all $X \in \mathfrak{g}$ and all $k \in \mathbb{N}^*$. Thus the function F associated to any K-star product

$$F(X) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{k_1 \dots k_p \\ |k|=n}} a_{k_1 \dots k_p} \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_1} \dots \operatorname{tr}(2i\pi \operatorname{ad} X)^{k_p}$$

equals 1 on $\mathfrak{g}$. It follows that all K-star products coincide in this case.

Finally, we can generalize the result proved in [Rieffel 1990] to all K-star products.

Theorem 5.6 (Generalization for K-star products). *Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra and let $\{\ ,\ \}$ be the linear Poisson bracket on the space $\mathcal{S}(\mathfrak{g}^*)$*

of Schwartz functions on $\mathfrak{g}^*$. *For any* $\hbar \in \mathbb{R}$ *and* $f \in \mathcal{S}(\mathfrak{g}^*)$ *we put*

$$f^*(X) = \overline{f}(X),$$

where we use the C^* *norms* $\| \ \|_\hbar$ *of the group* C^**-algebras* $C^*(G_\hbar)$. *This structure, together with any K-star product* $\star_\hbar$, *provides a strict deformation quantization of* $\mathcal{S}(\mathfrak{g}^*)$ *in the direction of* $(-2\pi)^{-1}\{\ ,\ \}$.

6. Explicit expressions of Rieffel's and Kontsevich's star products

Let $(E_i)_{i=1,\dots,d}$ be a basis of $\mathfrak{g}$ dual to the canonical basis of $\mathbb{R}^d$. We know from [Ben Amar 2003] that a symmetric graded star product $\star$ is entirely determined by the expression of $E_i \star f$. We will compute these expressions for Rieffel's and Kontsevich's star products as the Fourier transform of a distribution with $\{0\}$ support.

Theorem 6.1 (Explicit expressions for $\star_\hbar^R$ and $\star_\hbar^K$). *Let* $(E_i)_{i=1,\dots,d}$ *be a basis of the Lie algebra* $\mathfrak{g}$ *and let* C_{ij}^k *be the structure constants for* $\mathfrak{g}$. *For any* f *in the space* $\mathcal{P}(\mathfrak{g}^*)$ *of polynomial functions on* $\mathfrak{g}^*$ *or* f *with* $\hat{f}$ *smooth and compactly supported*, *we can write* $\star_\hbar^R$ *and* $\star_\hbar^K$ *as follows*:

$$E_i \star_\hbar f(\xi) = \sum_{k\geqslant 0} \frac{\hbar^k}{(2i\pi)^k}\frac{B_k}{k!}\Bigg(\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}} C_{i_1 i}^{j_1} C_{i_2 j_1}^{j_2} \dots C_{i_k j_{k-1}}^{j_k} \xi_{j_k} \partial_{i_1 \dots i_k} f + L \sum_{\substack{i_2,\dots,i_k\\ j_1,\dots,j_k}} C_{j_1 i}^{j_2} C_{i_k j_k}^{j_1} C_{i_{k-1} j_{k-1}}^{j_k} \dots C_{i_2 j_2}^{j_3} \partial_{i_2 \dots i_k} f \Bigg),$$

where B_k *is the k-th Bernoulli number and L equals 1 for* $\star_\hbar^R$ *and* $\frac{1}{2}$ *for* $\star_\hbar^K$.

Proof. Let Φ be a smooth function such that $\hat{\Phi}$ is smooth with sufficiently small support near 0. If $X = \sum_{j=1}^d x_j E_j$ and $Y = \sum_{j=1}^d y_j E_j$, we have

$$\begin{aligned} &\langle E_i \star_\hbar^R f(X \times_\hbar Y),\ \Phi(X \times_\hbar Y)\rangle \\ &\qquad = \big\langle \hat{f}(Y) J_\hbar^2(Y),\ \langle \hat{E}_i(X) J_\hbar^2(X),\ \hat{\Phi}(X \times_\hbar Y) J_\hbar^{-2}(X \times_\hbar Y)\rangle\big\rangle \\ &\qquad = \left\langle \hat{f}(Y),\ \frac{1}{2i\pi}\frac{\partial}{\partial x_i}\left(\hat{\Phi}(X \times_\hbar Y) \frac{J_\hbar^2(X) J_\hbar^2(Y)}{J_\hbar^2(X \times_\hbar Y)}\right)\bigg|_{X=0}\right\rangle. \end{aligned}$$

In this formula, we denote in the distribution sense

$$\langle f(X), \Phi(X)\rangle = \int_{\mathfrak{g}} f(X)\Phi(X)\, dX.$$

The functions $(X, Y) \mapsto X \times_\hbar Y$ and $X \mapsto J_\hbar^2(X)$ being holomorphic on $C_\hbar$, this expression holds since the support of $\hat{\Phi}$ is in $C_\hbar$ for $\hbar$ sufficiently small. Now recall from [Ben Amar 1995] that

$$\frac{\partial}{\partial x_i}(X \times_\hbar Y)|_{X=0} = \sum_{k\geqslant 0} \hbar^k \frac{B_k}{k!} (\operatorname{ad} Y)^k E_i .$$

Moreover,

$$\begin{aligned} J_\hbar^2(X) &= \prod_{k=1}^{\infty} \exp\biggl(\frac{B_{2k}\hbar^{2k}}{2k(2k)!} \operatorname{tr}(\operatorname{ad} X)^{2k} \biggr) \\ &= \prod_{k=1}^{\infty} \exp\biggl(\frac{B_{2k}\hbar^{2k}}{2k(2k)!} \sum_{i_1,\dots,i_{2k}} x_{i_1} \dots x_{i_{2k}} \sum_{j_1,\dots,j_{2k}} C_{i_1 j_1}^{j_{2k}} C_{i_2 j_2}^{j_1} \dots C_{i_{2k} j_{2k}}^{j_{2k-1}} \biggr). \end{aligned}$$

Therefore $\frac{\partial}{\partial x_i} J_\hbar^2(X)|_{X=0} = 0$ and

$$\begin{aligned} &\frac{\partial}{\partial x_i} J_\hbar^2(X \times_\hbar Y)|_{X=0} \\ &\quad = J_\hbar^2(Y) \sum_{k=2}^{\infty} \sum_{\ell=1}^{[k/2]} \hbar^k \frac{B_{2\ell}}{(2\ell)!} \frac{B_{k-2\ell}}{(k-2\ell)!} \sum_{i_2,\dots,i_k} y_{i_2} \dots y_{i_k} \\ &\qquad \times \sum_{j_1,\dots,j_k} C_{j_k j_1}^{j_2} C_{i_{2\ell} j_{2\ell}}^{j_1} C_{i_{2\ell-1} j_{2\ell-1}}^{j_{2\ell}} \dots C_{i_2 j_2}^{j_3} C_{i_{2\ell+1} i}^{j_{2\ell+1}} C_{i_{2\ell+2} j_{2\ell+1}}^{j_{2\ell+2}} \dots C_{i_k j_{k-1}}^{j_k} . \end{aligned}$$

But if $2\ell < k$ one has

$$\begin{aligned} &\sum_{\substack{i_2,\dots,i_k \\ j_1,\dots,j_k}} y_{i_2} \dots y_{i_k} C_{j_k j_1}^{j_2} C_{i_{2\ell} j_{2\ell}}^{j_1} C_{i_{2\ell-1} j_{2\ell-1}}^{j_{2\ell}} \dots C_{i_2 j_2}^{j_3} C_{i_{2\ell+1} i}^{j_{2\ell+1}} C_{i_{2\ell+2} j_{2\ell+1}}^{j_{2\ell+2}} \dots C_{i_k j_{k-1}}^{j_k} \\ &\quad = \operatorname{tr}\biggl(\sum_{i_2,\dots,i_k} y_{i_2} \dots y_{i_k} \operatorname{ad} E_{i_{2\ell}} \operatorname{ad} E_{i_{2\ell-1}} \dots \operatorname{ad} E_{i_2} \sum_{j_{2\ell+1},\dots,j_k} \bigl(C_{i_{2\ell+1} i}^{j_{2\ell+1}} C_{i_{2\ell+2} j_{2\ell+1}}^{j_{2\ell+2}} \dots \\ &\qquad\qquad C_{i_k j_{k-1}}^{j_k} \operatorname{ad} E_{j_k} \bigr) \biggr) \\ &\quad = \operatorname{tr}\biggl(\sum_{i_2,\dots,i_k} y_{i_2} \dots y_{i_k} \operatorname{ad} E_{i_{2\ell}} \operatorname{ad} E_{i_{2\ell-1}} \dots \operatorname{ad} E_{i_2} \\ &\qquad\qquad \operatorname{ad}[E_{i_k}, [E_{i_{k-1}}, [\dots [E_{i_{2\ell+2}}, [E_{i_{2\ell+1}}, E_i]] \dots]]] \biggr) \\ &\quad = \operatorname{tr}\bigl(\operatorname{ad} Y \operatorname{ad} Y \dots \operatorname{ad} Y \operatorname{ad}[Y, [Y, [\dots [Y, [Y, E_i]] \dots]]] \bigr) \\ &\quad = \operatorname{tr}\bigl(\operatorname{ad} Y \operatorname{ad} Y \dots \operatorname{ad} Y \operatorname{ad}[Y, Z] \bigr) = 0 \qquad (\text{setting } Z = [Y, [\dots [Y, [Y, E_i]] \dots]]). \end{aligned}$$

Thus

$$\begin{aligned} &\frac{\partial}{\partial x_i} J_\hbar^2(X \times_\hbar Y)|_{X=0} \\ &\qquad = J_\hbar^2(Y) \sum_{k=2}^{\infty} \hbar^k \frac{B_k}{k!} \sum_{i_2,\dots,i_k} y_{i_2} \dots y_{i_k} \sum_{j_1,\dots,j_k} C_{i j_1}^{j_2} C_{i_k j_k}^{j_1} C_{i_{k-1} j_{k-1}}^{j_k} \dots C_{i_2 j_2}^{j_3} . \end{aligned}$$

It follows that

$$\frac{\partial}{\partial x_i}\left(\frac{J_\hbar^2(X)J_\hbar^2(Y)}{J_\hbar^2(X\times_\hbar Y)}\right)\Bigg|_{X=0}$$
$$= -\sum_{k=2}^{\infty}\hbar^k\frac{B_k}{k!}\sum_{i_2,\dots,i_k} y_{i_2}\dots y_{i_k}\sum_{j_1,\dots,j_k} C_{ij_1}^{j_2}C_{i_kj_k}^{j_1}C_{i_{k-1}j_{k-1}}^{j_k}\dots C_{i_2j_2}^{j_3}.$$

Finally we get

$$E_i\star_\hbar^R f(\xi) = \sum_{k\geqslant 0}\frac{\hbar^k}{(2i\pi)^k}\frac{B_k}{k!}\Bigg(\sum_{\substack{i_1,\dots,i_k\\ j_1,\dots,j_k}} C_{i_1i}^{j_1}C_{i_2j_1}^{j_2}\dots C_{i_kj_{k-1}}^{j_k}E_{j_k}\partial_{i_1\dots i_k}f$$
$$+\sum_{\substack{i_2,\dots,i_k\\ j_1,\dots,j_k}} C_{j_1i}^{j_2}C_{i_kj_k}^{j_1}C_{i_{k-1}j_{k-1}}^{j_k}\dots C_{i_2j_2}^{j_3}\partial_{i_2\dots i_k}f\Bigg).$$

Similar computations yield the explicit expression of $\star_\hbar^K$. □

Using Theorem 6.1 we can find the forms of all graphs occurring in $\star_\hbar^R$ and $\star_\hbar^K$.

Corollary 6.2 (Graph shapes). *For each k, the only graphs occurring in the expression of $E_i\star_\hbar^R f$ and $E_i\star_\hbar^K f$ are one graph Γ_k without any wheels and one graph W_k' with exactly one wheel of size k. Γ_k has k vertices $p_1,\dots,p_k$ in A and* 2 *vertices q_1, q_2 in B, and its edges are p_1q_1, p_1q_2, and p_ip_{i-1}, p_iq_2 for $2\le i\le k$* (see Figure 2). *W_k' has the same vertices as Γ_k and its edges are p_1q_1, p_1p_k, and p_ip_{i-1}, p_iq_2 for $2\le i\le k$* (see Figure 3).

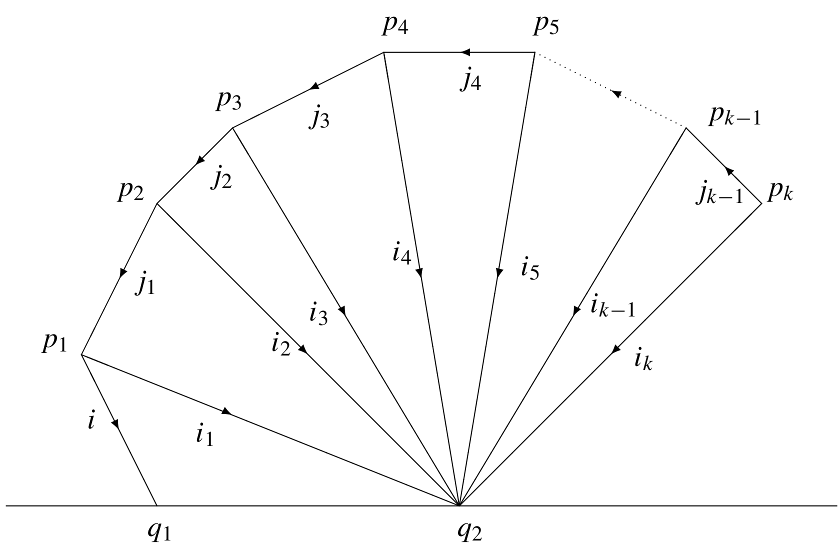

Figure 2. The graph Γ_k.

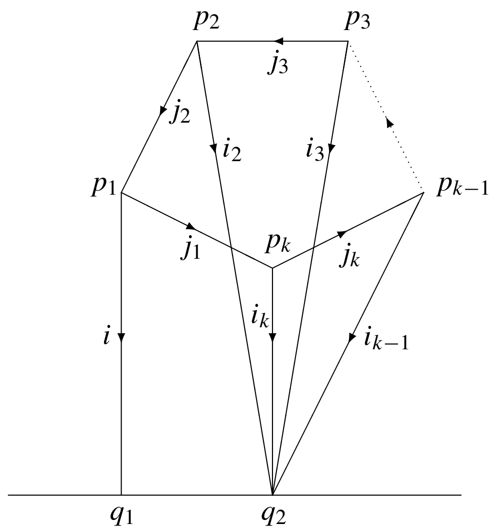

Figure 3. The graph W_k'.

We remark that for each k, there are $k!2^k$ graphs having the same weight. These graphs are obtained from Γ_k or W_k' by permutation of the vertices $p_1, \dots, p_k$ or of the edges coming from the vertices. We can obtain the weights of all these graphs from Theorem 6.1. These weights are rational.

Corollary 6.3 (Weight comparison). *Rieffel's deformation quantization, defined by considering convolution of functions, can be obtained by using the method of oriented graphs. The graphs and the corresponding differential operators are defined as in [Kontsevich 1997], but the weights of these graphs are different.*

In the case of Rieffel's star product the weight of the graph Γ_k coincides with the weight of the graph W_k'. This weight is given by

$$w^{(R)}_{\Gamma_k} = w^{(R)}_{W_k'} = \frac{B_k}{(k!)^2}.$$

For Kontsevich's star product the weight of the graph Γ_k is

$$w^{(K)}_{\Gamma_k} = \frac{B_k}{(k!)^2},$$

and the weight of the graph W_k' is

$$w^{(K)}_{W_k'} = \frac{1}{2}\frac{B_k}{(k!)^2}.$$

We cannot deduce directly from Theorem 6.1 the weights of the graphs $\Gamma_{k,2\ell}$ ($2 \le 2\ell < k$) having the same vertices as Γ_k and edges $p_1p_{2\ell}$, p_1p_k, $p_{2\ell+1}q_1$, $p_{2\ell+1}q_2$, and p_ip_{i-1}, p_iq_2 for $2 \le i \le 2l$ and $2l+2 \le i \le k$ (Figure 4).

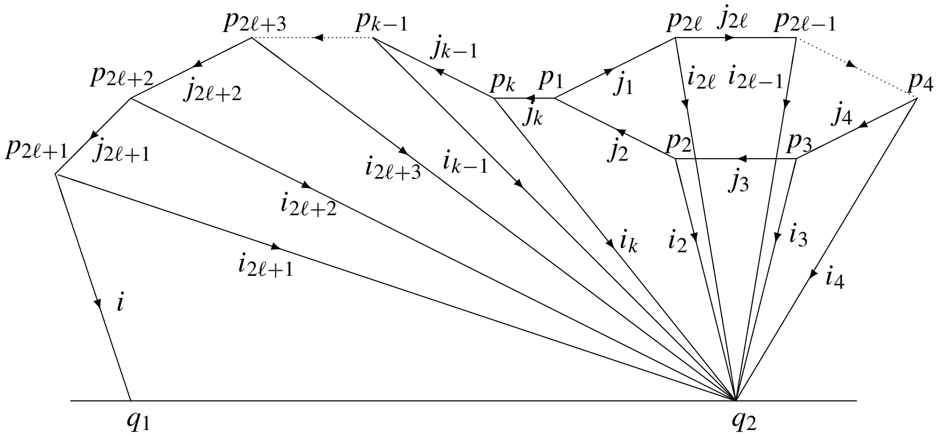

Figure 4. The graph $\Gamma_{k,2\ell}$.

These graphs do not occur in the expressions of $E_i \star_\hbar^R f$ and $E_i \star_\hbar^K f$. But we conjecture that the coefficients that appear respectively in $(\partial/\partial x_i) J_\hbar(X \times_\hbar Y)|_{X=0}$ and $(\partial/\partial x_i) J_\hbar^2(X \times_\hbar Y)|_{X=0}$ give these weights.

Conjecture 6.4 (Weights of graphs $\Gamma_{k,2\ell}$). *The weights of the graphs $\Gamma_{k,2\ell}$ are*

$$w^{(K)}_{\Gamma_{k,2\ell}} = \frac{1}{k!}\frac{B_{k-2\ell}}{(k-2\ell)!}\frac{B_{2\ell}}{2(2\ell)!} = \frac{1}{\binom{k}{2\ell}}\frac{B_{k-2\ell}}{((k-2\ell)!)^2}\frac{B_{2\ell}}{2((2\ell)!)^2} = \frac{1}{\binom{k}{2\ell}} w^{(K)}_{\Gamma_{k-2\ell}}.w^{(K)}_{W'_{2\ell}},$$

$$w^{(R)}_{\Gamma_{k,2\ell}} = \frac{1}{k!}\frac{B_{k-2\ell}}{(k-2\ell)!}\frac{B_{2\ell}}{(2\ell)!} = \frac{1}{\binom{k}{2\ell}}\frac{B_{k-2\ell}}{((k-2\ell)!)^2}\frac{B_{2\ell}}{((2\ell)!)^2} = \frac{1}{\binom{k}{2\ell}} w^{(R)}_{\Gamma_{k-2\ell}}.w^{(R)}_{W'_{2\ell}}.$$

From Theorem 6.1 we can also deduce that the weights (for $\star_\hbar^R$ and $\star_\hbar^K$) of any graph Γ having at least one parachute — that is, a wheel linked only with q_2, as in Figure 5 — vanish. (In [Ben Amar 2003] we prove this fact directly for $\star_\hbar^K$.)

Corollary 6.5 (Graphs having a parachute). *The weights (for $\star_\hbar^R$ and $\star_\hbar^K$) of any graph Γ having at least one parachute vanish.*

Corollary 6.6 (Rationality of weights). *If f and g are polynomial functions on $\mathfrak{g}^*$, all the weights of graphs occurring in the expression of $f \star_\hbar^R g$ and $f \star_\hbar^K g$ are rational numbers.*

Proof. If X is in $\mathfrak{g}$ and f in $\mathcal{P}(\mathfrak{g}^*)$, it is an easy consequence of Corollary 6.3 that the weights of all graphs which occur in the expression of $X \star_\hbar^R f$ are rational numbers.

The corollary follows by induction on the degree of f: If f is an homogeneous polynomial function of the form $f = Xf'$, there exists a polynomial function f'' such that

$$Xf' = X \star_\hbar^R f' + f'', \quad \text{with } \deg f'' \leq \deg f - 1.$$

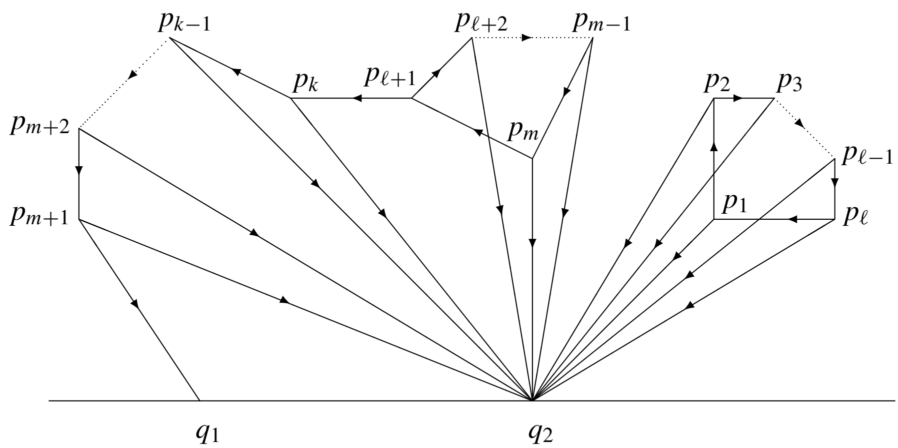

Figure 5. A graph Γ having a parachute ($w_\Gamma^{(R)} = w_\Gamma^{(K)} = 0$).

Then

$$f \star_\hbar^R g = (Xf') \star_\hbar^R g = X \star_\hbar^R (f' \star_\hbar^R g) + f'' \star_\hbar^R g.$$

But the weights of graphs in $X \star_\hbar^R (f' \star_\hbar^R g)$ and in $f'' \star_\hbar^R g$ are all rational numbers. □

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