Pacific Journal of Mathematics

q-CARTAN MATRICES AND COMBINATORIAL INVARIANTS OF DERIVED CATEGORIES FOR SKEWED-GENTLE ALGEBRAS

CHRISTINE BESSENRODT AND THORSTEN HOLM

Volume 229 No. 1 January 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 229, No. 1, 2007

q-CARTAN MATRICES AND COMBINATORIAL INVARIANTS OF DERIVED CATEGORIES FOR SKEWED-GENTLE ALGEBRAS

CHRISTINE BESSENRODT AND THORSTEN HOLM

Cartan matrices are of fundamental importance in representation theory. For algebras defined by quivers with monomial relations, the computation of the entries of the Cartan matrix amounts to counting nonzero paths in the quivers, leading naturally to a combinatorial setting. For derived module categories, the invariant factors, and hence the determinant, of the Cartan matrix are preserved by derived equivalences.

In the generalization called q-Cartan matrices (the classical Cartan matrix corresponding to $q = 1$), each nonzero path is weighted by a power of an indeterminate q according to its length. We study q-Cartan matrices for gentle and skewed-gentle algebras, which occur naturally in representation theory, especially in the context of derived categories. We determine normal forms for these matrices in the skewed-gentle case, giving explicit combinatorial formulae for the invariant factors and the determinant. As an application, we show how to use our formulae for the difficult problem of distinguishing derived equivalence classes.

1. Introduction

This paper deals with combinatorial aspects in the representation theory of algebras. For certain classes of algebras, which are defined purely combinatorially by directed graphs and homogeneous relations, we will characterize important representation-theoretic invariants in a combinatorial way. This leads to new explicit invariants for the derived module categories of the algebras involved.

Our starting point is that the unimodular equivalence class of the Cartan matrix of a finite-dimensional algebra is invariant under derived equivalence. Hence, being able to determine normal forms of Cartan matrices yields invariants of the derived category.

The class of algebras we study are the gentle algebras and the related skewed-gentle algebras. Gentle algebras are defined purely combinatorially in terms of a

MSC2000: 16G10, 18E30, 05E99, 05C38, 05C50.
Keywords: Cartan matrices, derived categories, skewed-gentle algebras.

quiver with relations (for details, see Section 2); the more general skewed-gentle algebras (introduced in [Geiß and de la Peña 1999]) are then defined from gentle algebras by specifying special vertices which are split for the quiver of the skewed-gentle algebra (see Section 4). These algebras occur naturally in the representation theory of finite-dimensional algebras, especially in the context of derived categories. For instance, the algebras which are derived equivalent to hereditary algebras of type $\mathbb{A}$ are precisely the gentle algebras whose underlying undirected graph is a tree [Assem and Happel 1981]. The algebras which are derived equivalent to hereditary algebras of type $\tilde{\mathbb{A}}$ are certain gentle algebras whose underlying graph has exactly one cycle [Assem and Skowroński 1987]. Remarkably, the class of gentle algebras is closed under derived equivalence [Schröer and Zimmermann 2003]; however, the class of skewed-gentle algebras is not.

A fundamental distinction in the representation theory of algebras is given by the representation type, which can be either finite, tame or wild. In the modern context of derived categories, also derived representation types have been defined. Again, gentle algebras occur naturally in this context. D. Vossieck [2001] showed that an algebra A has a discrete derived category if and only if either A is derived equivalent to a hereditary algebra of type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$ or A is gentle with underlying quiver (Q, I) having exactly one (undirected) cycle and the number of clockwise and of counterclockwise paths of length 2 in the cycle that belong to I are different. Skew-gentle algebras are known to be of derived tame representation type (for a definition of derived tameness, see [Geiß and Krause 2002]).

It is a long-standing open problem to classify gentle algebras up to derived equivalence. A complete answer has only been obtained for the derived discrete case [Bobiński et al. 2004]. The main problem is to find good invariants of the derived categories.

In this paper we provide easy-to-compute invariants of the derived categories of skewed-gentle algebras which are of a purely combinatorial nature. Our results are obtained from a detailed computation of the q-Cartan matrices of gentle and skewed-gentle algebras.

The following notion will be crucial throughout the paper. Let (Q, I) be a (gentle) quiver with relations. An oriented path $p = p_0 p_1 \dots p_{k-1}$ with arrows $p_0, \dots, p_{k-1}$ in Q is called an oriented k-cycle with full zero relations if p has the same start and end point, and if $p_i p_{i+1} \in I$ for all $i = 0, \dots, k-2$ and also $p_{k-1} p_0 \in I$. Such a cycle is called minimal if the arrows $p_0, p_1, \dots, p_{k-1}$ on p are pairwise different.

We call two matrices C, D with entries in a polynomial ring $\mathbb{Z}[q]$ unimodularly equivalent (over $\mathbb{Z}[q]$) if there exist matrices P, Q over $\mathbb{Z}[q]$ of determinant 1 such that $D = PCQ$.

We can now state our main result on gentle algebras.

Theorem 3.2. *Let (Q, I) be a gentle quiver and $A = KQ/I$ the corresponding gentle algebra. Denote by c_k the number of minimal oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_A(q)$ is unimodularly equivalent (over $\mathbb{Z}[q]$) to a diagonal matrix with entries $(1-(-q)^k)$, with multiplicity c_k, $k \geq 1$, and all further diagonal entries equal to* 1.

Corollary 3.3. *Let (Q, I) be a gentle quiver and $A = KQ/I$ the corresponding gentle algebra. Denote by c_k the number of minimal oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_A(q)$ has determinant*

$$\det C_A(q) = \prod_{k\geq 1} (1-(-q)^k)^{c_k}.$$

The following consequence of this corollary was first proved in [Holm 2005]. For a gentle quiver (Q, I) we denote by $oc(Q, I)$ the number of minimal oriented cycles of odd length in Q having full zero relations, and by $ec(Q, I)$ the number of analogous cycles of even length.

Corollary 3.5. *Let (Q, I) be a gentle quiver, and $A = KQ/I$ the corresponding gentle algebra. Then the determinant of the Cartan matrix C_A satisfies*

$$\det C_A = \begin{cases} 0 & \text{if } ec(Q, I) > 0, \\ 2^{oc(Q,I)} & \text{else}. \end{cases}$$

The most important application of Theorem 3.2 is the next corollary, which gives for gentle algebras easy-to-check combinatorial invariants of the derived category.

Corollary 3.6. *Let (Q, I) and (Q', I') be gentle quivers, and let $A = KQ/I$ and $A' = KQ'/I'$ be the corresponding gentle algebras. If A and A' are derived equivalent, then $ec(Q, I) = ec(Q', I')$ and $oc(Q, I) = oc(Q', I')$.*

As an illustration we give in Section 3 a complete derived equivalence classification of gentle algebras with two simple modules and of gentle algebras with three simple modules and Cartan determinant 0.

Our main result on skewed-gentle algebras determines the normal form of their q-Cartan matrices.

Theorem 4.1. *Let $\hat{A} = K\hat{Q}/\hat{I}$ be a skewed-gentle algebra, arising from choosing a suitable set of special vertices in the gentle quiver (Q, I). Denote by c_k the number of minimal oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_{\hat{A}}(q)$ is unimodularly equivalent to a diagonal matrix with entries $1-(-q)^k$, with multiplicity c_k, $k \geq 1$, and all further diagonal entries equal to* 1.

As an immediate consequence we obtain that the Cartan determinant of any skewed-gentle algebra is the same as the Cartan determinant for the underlying gentle algebra.

Corollary 4.3. *Let $\hat{A} = K\hat{Q}/\hat{I}$ be a skewed-gentle algebra, arising from choosing a suitable set of special vertices in the gentle quiver (Q, I), with corresponding gentle algebra $A = KQ/I$. Then* $\det C_{\hat{A}}(q) = \det C_A(q)$, *and thus in particular, the determinants of the ordinary Cartan matrices coincide*: $\det C_{\hat{A}} = \det C_A$.

The paper is organized as follows. Section 2 collects the necessary background and definitions about quivers with relations and q-Cartan matrices. In Section 3 we prove all the main results about q-Cartan matrices for gentle algebras, and give some extensive examples to illustrate our results. Section 4 contains the analogous main results for skewed-gentle algebras.

2. Quivers, q-Cartan matrices and derived invariants

Algebras can be defined naturally from a combinatorial viewpoint by using directed graphs. A finite directed graph Q is called a *quiver*. For any arrow α in Q we denote by $s(\alpha)$ its start vertex and by $t(\alpha)$ its end vertex. An oriented path p in Q of length r is a sequence $p = \alpha_1\alpha_2\dots\alpha_r$ of arrows α_i such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $i = 1, \dots, r-1$; its start vertex is $s(p) := s(\alpha_1)$ and its end vertex is $t(p) := t(\alpha_r)$. (For each vertex v in Q we allow a trivial path e_v of length 0, having v as its start and end vertex.)

The path algebra KQ, where K is any field, has as a basis the set of all oriented paths in Q. Multiplication is defined by concatenation of paths: the product of two paths p and q is defined to be the concatenated path pq if $t(p) = s(q)$, and zero otherwise.

More general algebras can be obtained by introducing relations on a path algebra. An ideal $I \subset KQ$ is called admissible if $I \subseteq J^2$, where J is the ideal of KQ generated by the arrows of Q.

The pair (Q, I), where Q is a quiver and $I \subset KQ$ is an admissible ideal, is called a *quiver with relations*.

For any quiver with relations (Q, I), we can consider the factor algebra $A = KQ/I$, where K is a field. We identify paths in the quiver Q with their cosets in A. Let Q_0 denote the set of vertices of Q. For any $i \in Q_0$ there is a path e_i of length zero. These are primitive orthogonal idempotents in A; the sum $\sum_{i\in Q_0} e_i$ is the unit element in A. In particular we get $A = 1 \cdot A = \bigoplus_{i\in Q_0} e_iA$, so the (right) A-modules $P_i := e_iA$ are the indecomposable projective A-modules.

The *Cartan matrix* $C = (c_{ij})$ of an algebra $A = KQ/I$ is the $|Q_0| \times |Q_0|$-matrix defined by setting $c_{ij} := \dim_K \mathrm{Hom}_A(P_j, P_i)$.

Recall that when I is generated by monomials, $A = KQ/I$ is called a monomial algebra. For monomial algebras, computing entries of the Cartan matrix reduces to counting paths in the quiver Q which are nonzero in A. In fact, any homomorphism $\varphi : e_jA \to e_iA$ of right A-modules is uniquely determined by $\varphi(e_j) \in e_iAe_j$, the

K-vector space generated by all paths in Q from vertex i to vertex j, which are nonzero in $A = KQ/I$. In particular, $c_{ij} = \dim_K e_i A e_j$.

This is the key viewpoint in this paper, enabling us to obtain results on the representation-theoretic Cartan invariants by combinatorial methods. It allows us to study a refined version of the Cartan matrix, which we call the q-Cartan matrix. (This also occurs in the literature as the filtered Cartan matrix; see [Fuller 1992], for instance.)

Let Q be a quiver and assume that the relation ideal I is generated by homogeneous relations, i.e., by linear combinations of paths having the same length (actually, for the algebras considered in this paper, the ideal I will always be generated by monomials and commutativity (mesh) relations). The path algebra KQ is a graded algebra, with grading given by path lengths. Since I is homogeneous, the factor algebra $A = KQ/I$ inherits this grading. So the morphism spaces $\mathrm{Hom}_A(P_j, P_i) \cong e_i A e_j$ become graded vector spaces. Recall that the dimensions of these vector spaces are the entries of the (ordinary) Cartan matrix.

Definition. Let $A = KQ/I$ be a finite-dimensional algebra, and assume that the ideal I is generated by homogeneous relations. For any vertices i and j in Q let $e_i A e_j = \bigoplus_n (e_i A e_j)_n$ be the graded components.

Let q be an indeterminate. The q-Cartan matrix $C_A(q) = (c_{ij}(q))$ of A is defined as the matrix with entries $c_{ij}(q) := \sum_n \dim_K (e_i A e_j)_n \, q^n \in \mathbb{Z}[q]$.

In other words, the entries of the q-Cartan matrix are the Poincaré polynomials of the graded homomorphism spaces between projective modules. Loosely speaking, when counting paths in the quiver of the algebra, each path is weighted by some power of q according to its length.

Clearly, specializing to $q = 1$ gives back the usual Cartan matrix C_A (that is, we forget the grading). Even if we are mainly interested in the ordinary Cartan matrix, the point of view of q-Cartan matrices provides some new insights as we take a closer look at the invariants of the Cartan matrix.

Example 2.1. We consider the following two quivers:

$$Q_1 \qquad \begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\delta"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\gamma"] \end{tikzcd} \qquad\qquad Q_2 \qquad \begin{tikzcd} 1 \arrow[rr, "\alpha"] & & 2 \arrow[dl, "\alpha"] \\ & 3 \arrow[ul, "\alpha"] & \end{tikzcd}$$

Let $A = KQ_1/I_1$, where the ideal I_1 is generated by $\alpha\beta$, $\gamma\delta$ and $\delta\alpha - \beta\gamma$. The q-Cartan matrix of A has the form

$$C_A(q) = \begin{pmatrix} 1+q^2 & q & 0 \\ q & 1+q^2 & q \\ 0 & q & 1+q^2 \end{pmatrix}.$$

The second algebra $B = KQ_2/I_2$ is defined by the quiver Q_2, subject to the generating relations α^4 (all paths of length four are zero). The q-Cartan matrix of B has the form

$$C_B(q) = \begin{pmatrix} 1+q^3 & q & q^2 \\ q^2 & 1+q^3 & q \\ q & q^2 & 1+q^3 \end{pmatrix}.$$

Cartan matrices provide invariants which are preserved under derived equivalences and thus improve our understanding of derived module categories; this is our main motivation to study normal forms, invariant factors and determinants of Cartan matrices in this paper. The following result is contained in the proof of [Bocian and Skowroński 2005, Proposition 1.5].

Theorem 2.2. *Let A be a finite-dimensional algebra. The unimodular equivalence class of the Cartan matrix C_A is invariant under derived equivalence.*

In particular, the determinant of the Cartan matrix is invariant under derived equivalence.

Remark 2.3. This theorem only deals with ordinary Cartan matrices $C_A = C_A(1)$. The determinant of the q-Cartan matrix is in general not invariant under derived equivalence. As an example, consider the algebras A and B from Example 2.1: we have $\det C_A(q) = 1+q^2+q^4+q^6$ and $\det C_B(q) = 1+q^3+q^6+q^9$, but the algebras A and B are derived equivalent — they are Brauer tree algebras for trees with the same number of edges and the same exceptional multiplicity [Rickard 1989a]. Note that when specializing to $q = 1$ we do get the same determinants for the ordinary Cartan matrices, as predicted by Theorem 2.2.

However, the natural setting when dealing with q-Cartan matrices is that of graded derived categories. Indeed, the determinant of the q-Cartan matrix (which is defined so as to take the grading into account) is invariant under graded derived equivalences. We are very grateful to the referee for pointing this out to us. We do not discuss this aspect in this paper further, but shall address the topic of graded derived equivalences for gentle algebras in detail in a subsequent publication.

For instance, the algebra B above is graded derived equivalent to the algebra A, where the grading on A is chosen so that α and β are of degree 2, and δ and γ of degree 1. Then Rickard's derived equivalence [1989a] lifts to a graded derived equivalence.

3. Gentle algebras

In this section we prove the previously quoted Theorem 3.2 on the unimodular equivalence class of the q-Cartan matrix of an arbitrary gentle algebra.

We first recall the definition of special biserial algebras and of gentle algebras, as these details will be crucial for what follows.

Let Q be a quiver and I an admissible ideal in the path algebra KQ. We call the pair (Q, I) a special biserial quiver (with relations) if it satisfies the following properties.

(i) Each vertex of Q is starting point of at most two arrows, and end point of at most two arrows.

(ii) For each arrow α in Q there is at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.

A finite-dimensional algebra A is called special biserial if it has a presentation as $A = KQ/I$, where (Q, I) is a special biserial quiver.

Gentle quivers form a subclass of the class of special biserial quivers. A pair (Q, I) as above is called a *gentle quiver* if it is special biserial and moreover the following holds.

(iii) The ideal I is generated by paths of length 2.

(iv) For each arrow α in Q there is at most one arrow β' with $t(\alpha) = s(\beta')$ such that $\alpha\beta' \in I$, and there is at most one arrow γ' with $t(\gamma') = s(\alpha)$ such that $\gamma'\alpha \in I$.

A finite-dimensional algebra A is called gentle if it has a presentation as $A = KQ/I$, where (Q, I) is a gentle quiver.

The following lemma will be very useful. It holds not only for gentle algebras but for those where we have dropped condition (iv) above.

Recall that two matrices C, D with entries in $\mathbb{Z}[q]$ are called unimodularly equivalent (over $\mathbb{Z}[q]$) if there exist matrices P, Q over $\mathbb{Z}[q]$ of determinant 1 such that $D = PCQ$.

Lemma 3.1. *Let (Q, I) be a special biserial quiver, and assume that I is generated by paths of length 2. Let $A = KQ/I$ be the corresponding special biserial algebra. Let α be an arrow in Q, not a loop, such that there is no arrow β with $s(\alpha) = t(\beta)$ and $\beta\alpha \in I$, or there is no arrow γ with $t(\alpha) = s(\gamma)$ and $\alpha\gamma \in I$. Let Q' be the quiver obtained from Q by removing the arrow α, let I' be the corresponding relation ideal and set $A' = KQ'/I'$. Then the q-Cartan matrices $C_A(q)$ and $C_{A'}(q)$ are unimodularly equivalent (over $\mathbb{Z}[q]$).*

Proof. We consider the case where α is an arrow in Q such that there is no arrow β with $s(\alpha) = t(\beta)$ and $\beta\alpha \in I$; the second case is dual.

Let $\alpha = p_0 : v_0 \to v_1$. Since (Q, I) is special biserial, there is a unique maximal nonzero path starting with p_0, say $p = p_0 p_1 \dots p_t$, where $p_i : v_i \to v_{i+1}$ for $i = 1, \dots, t$. Because A is finite-dimensional, the condition on $\alpha = p_0$ guarantees that $v_i \neq v_0$ for all $i > 0$, but we may have $v_i = v_j$ for some $i > j > 0$. Now any nonzero path of length j, say, ending at v_0 can uniquely be extended to a nonzero path of length $j+i$ ending at v_i, by concatenation with $p_0 \dots p_{i-1}$. Conversely, any nonzero path ending at v_i and involving p_0 arises in this way.

Now denote the column corresponding to a vertex v in the q-Cartan matrix $C_A(q)$ by s_v. We perform column transformations on $C_A(q)$ by replacing the columns s_{v_i} by $s_{v_i} - q^i s_{v_0}$, for $i = 1, \dots, t+1$ (if $v_i = v_j$ for some $i > j$, the column $s_{v_i} = s_{v_j}$ will then be replaced by $s_{v_i} - (q^i + q^j)s_{v_0}$). The resulting matrix $\tilde{C}(q)$ is then exactly the Cartan matrix $C_{A'}(q)$ to the algebra A' corresponding to the quiver Q' where $\alpha = p_0$ has been removed. □

For any vertex in a quiver Q, its *valency* is defined as the number of arrows attached to it, i.e., the number of incoming arrows plus the number of outgoing arrows (in particular, any loop contributes twice to the valency).

Theorem 3.2. *Let (Q, I) be a gentle quiver and $A = KQ/I$ the corresponding gentle algebra. Denote by c_k the number of minimal oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_A(q)$ is unimodularly equivalent (over $\mathbb{Z}[q]$) to a diagonal matrix with entries $(1 - (-q)^k)$, with multiplicity c_k, $k \geq 1$, and all further diagonal entries equal to* 1.

Proof. We work by double induction on the number of vertices and the number of arrows. Clearly the result holds if Q has no arrows or if it consists of one vertex with a loop.

If Q has a vertex v of valency 1 or 3, or of valency 2 but with no zero relation at v, we can use Lemma 3.1 to remove an arrow from Q; by the conditions in Lemma 3.1, the removed arrow is not involved in any oriented cycle with full zero relations. Hence $C_A(q)$ is unimodularly equivalent to $C_{A'}(q)$, where the corresponding quiver has one arrow less but the same number of oriented cycles with full zero relations, and hence the result holds by induction.

Hence we may now assume that all vertices are of valency 0, 2 or 4, and if a vertex is of valency 2, there is a zero relation at the vertex. Also, if Q is not connected, we may use induction on the number of vertices to obtain the result for the components and thus for the whole quiver; hence we may assume that Q is connected. We now only have vertices v of valency 2 with a nonloop zero relation at v, and vertices of valency 4. As we do not have paths of arbitrary lengths, not all vertices can be of valency 4 (see also [Holm 2005, Lemma 3]).

Take a vertex $v = v_1$ of valency 2, with incoming arrow $p_0 : v_0 \to v_1$ and outgoing arrow $p_1 : v_1 \to v_2$ with $p_0 p_1 = 0$ (here, $v_0 \neq v \neq v_2$).

Since (Q, I) is gentle, there is a unique maximal path p in Q with nonrepeating arrows starting in v_0 with p_0, such that the product of any two consecutive arrows is zero in A; in our present situation this path is an oriented cycle $\mathscr{C}$ with full zero relations returning to v_0. We denote the vertices on this path by v_0, $v_1 = v$, $v_2, \dots,$

v_s, $v_{s+1} = v_0$, and the arrows by $p_i : v_i \to v_{i+1}$, for $i = 0, \dots, s$ (also $p_s p_0 = 0$); note that the arrows on p are distinct, but the vertices are not necessarily distinct (but we do have $v_i \neq v_1$ for all $i \neq 1$).

Denote by z_w the row of the q-Cartan matrix $C_A(q)$ corresponding to vertex w. In $C_A(q)$, we now replace the row z_v by the linear combination

$$Z = \sum_{i=1}^{s+1} (-q)^{i-1} z_{v_i}$$

to obtain a new matrix $\tilde{C}(q)$ (this is a unimodular transformation over $\mathbb{Z}[q]$).

The careful choice of the coefficients is just made so that we can refine the argument in [Holm 2005]. We recall some of the notation there. For any arrow α in Q let $\mathcal{P}(\alpha)$ be the set of paths starting at α which are nonzero in A. At each vertex v_i there is at most one outgoing arrow $r_i \neq p_i$; for this arrow we have $p_{i-1} r_i \neq 0$, since (Q, I) is gentle.

Hence, canceling p_i induces a natural bijection $\phi : \mathcal{P}(p_i) \to \{e_{v_{i+1}}\} \cup \mathcal{P}(r_{i+1})$, for $i = 1, \dots, s-1$, such that a path of q-weight q^j is mapped to a path of q-weight q^{j-1} (if there is no arrow r_i, we set $\mathcal{P}(r_i) = \varnothing$).

As v is of valency 2, with a zero-relation at v, we also have the trivial bijection $\mathcal{P}(p_0) = \{p_0\} \to \{e_{v_1}\}$, again with a weight reduction by q. Now almost everything cancels in Z, apart from the one term $1 - (-q)^{s+1}$ that we obtain as the entry in the column corresponding to v.

In the next step, we use the dual (counterclockwise) operation on the columns labeled by the vertices on the cycle $\mathcal{C}$, i.e., we set $v_{s+2} = v = v_1$ and replace the column s_v by the linear combination

$$S = \sum_{i=1}^{s+1} (-q)^{s+1-i} s_{v_{i+1}}.$$

Ordering vertices so that v corresponds to the first row and column of the Cartan matrix, we have thus unimodularly transformed $C_A(q)$ to a matrix of the form

$$\begin{pmatrix} 1-(-q)^{s+1} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & C'(q) & \\ 0 & & & \end{pmatrix}$$

where $C'(q)$ is the q-Cartan matrix of the gentle algebra A' for the quiver Q' obtained from Q by removing v and the arrows incident with v. Note that in comparison with Q, the quiver Q' has one vertex less and one cycle with full zero relations of length $s+1$ less; now by induction, the result holds for $C'(q) = C_{A'}(q)$, and hence the result for $C_A(q)$ follows immediately. $\square$

This result has several immediate nice consequences.

Corollary 3.3. *Let (Q, I) be a gentle quiver, and let $A = KQ/I$ be the corresponding gentle algebra. Denote by c_k the number of minimal oriented k-cycles in Q with full zero relations. Then the q-Cartan matrix $C_A(q)$ has determinant*

$$\det C_A(q) = \prod_{k \geq 1} (1 - (-q)^k)^{c_k}.$$

Remark 3.4. Let (Q, I) be a gentle quiver, with set of vertices Q_0. Then, as a direct consequence of Theorem 3.2, there are at most $|Q_0|$ minimal oriented cycles with full zero relations in the quiver (this could also be proved directly by induction).

Note that the property of being gentle is invariant under derived equivalence [Schröer and Zimmermann 2003], and we now have some invariants to distinguish the derived equivalence classes. For a gentle quiver (Q, I), recall that $ec(Q, I)$ and $oc(Q, I)$ denote the number of minimal oriented cycles in Q with full zero relations of even and odd length, respectively. As an immediate consequence of Corollary 3.3 we obtain the following formula for the Cartan determinant which was the main result in [Holm 2005]:

Corollary 3.5. *Let (Q, I) be a gentle quiver, and let $A = KQ/I$ be the corresponding gentle algebra. Then the determinant of the Cartan matrix C_A satisfies*

$$\det C_A = \begin{cases} 0 & \text{if } ec(Q, I) > 0, \\ 2^{oc(Q,I)} & \text{else.} \end{cases}$$

In combination with Remark 3.4, this implies that the Cartan determinant of a gentle algebra $A = KQ/I$ is at most $2^{l(A)}$, where $l(A) = |Q_0|$ is the number of simple modules of A.

The most important application of Theorem 3.2 is the following corollary, which gives for gentle algebras new, combinatorial and easy-to-check invariants of the derived category.

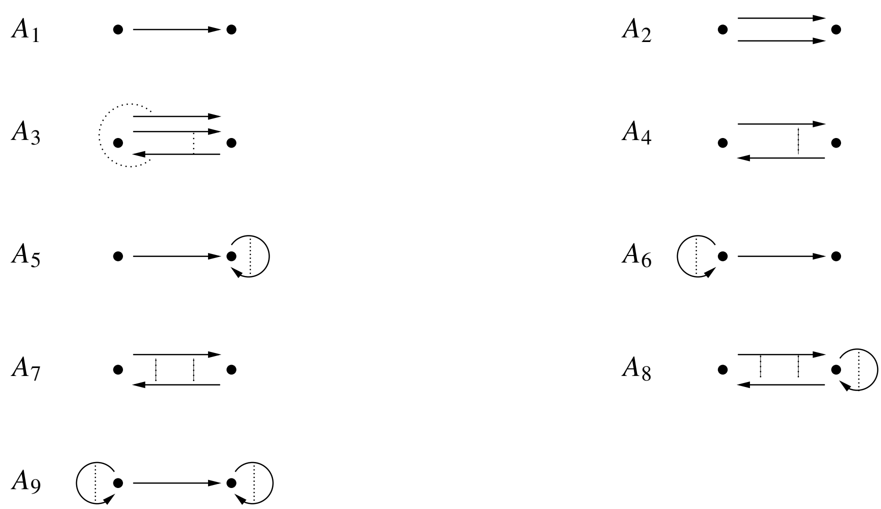
Corollary 3.6. *Let (Q, I) and (Q', I') be gentle quivers, and let $A = KQ/I$ and $A' = KQ'/I'$ be the corresponding gentle algebras. If A and A' are derived equivalent, then $ec(Q, I) = ec(Q', I')$ and $oc(Q, I) = oc(Q', I')$.*

Proof. Since A and A' are derived equivalent, their (ordinary) Cartan matrices C_A and $C_{A'}$ are unimodularly equivalent over $\mathbb{Z}$. By specializing to $q = 1$ in Theorem 3.2, representatives for the equivalence classes are given by diagonal matrices with entries equal to 2 for each minimal oriented cycle with full zero relations of odd length, an entry 0 for each such cycle of even length, and remaining entries 1. These are precisely the elementary divisors over $\mathbb{Z}$. The elementary divisors of an integer matrix are uniquely determined, and the diagonal matrices in Theorem 3.2 are actually the Smith normal forms of C_A and $C_{A'}$ over $\mathbb{Z}$. But by Theorem 2.2 the unimodular equivalence class, and hence the Smith normal form, is invariant under derived equivalence.

Hence, the diagonal entries in the above normal forms for C_A and $C_{A'}$ must occur with exactly the same multiplicities. Thus we get the same number of minimal oriented cycles with full zero relations of even length and of odd length; that is, $ec(Q, I) = ec(Q', I')$ and $oc(Q, I) = oc(Q', I')$. □

We now illustrate our results and apply them to derived equivalence classifications of gentle algebras.

Example 3.7 (Gentle algebras with two simple modules). There are nine connected gentle quivers (Q, I) with two vertices, as given in the following list. The dotted lines indicate the zero relations generating the admissible ideal I.



Bekkert and Drozd [2003] showed that these are precisely the basic connected algebras with two simple modules that are derived tame. As a direct illustration of our results we show how to classify these algebras up to derived equivalence.

Recall that the property of being gentle is invariant under derived equivalence [Schröer and Zimmermann 2003]. Moreover, the number of simple modules of an algebra is a derived invariant [Rickard 1989b]. Thus we will be able to describe the complete derived equivalence classes.

We have shown above that the numbers $oc(Q, I)$ and $ec(Q, I)$ are derived invariants. In addition we look at two classical invariants, the center and the first Hochschild cohomology group HH^1. Recall that the center of an algebra (and more generally the Hochschild cohomology ring) is invariant under derived equivalence

[Rickard 1991]. If the quiver contains a loop, then the dimension of HH^1 depends on the characteristic being 2 or not. We indicate the dimension in characteristic 2 in parentheses in the table below. These values can be computed using a method based on work of M. Bardzell [1997] on minimal projective bimodule resolutions for monomial algebras; a very nice explicit combinatorial description is given by C. Strametz [2001, Proposition 2.6].

Algebra	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9
$oc(Q, I)$	0	0	0	0	1	1	0	1	2
$ec(Q, I)$	0	0	0	0	0	0	1	1	0
$\dim Z(A)$	1	1	1	2	1	1	1	2	1
$\dim \mathrm{HH}^1(A)$	0	3	2	1	1(2)	1(2)	1	2(3)	3(5)

The algebras A_1, A_2, A_3, A_4 are pairwise not derived equivalent. This can be deduced directly from the above table, since the dimensions of the first Hochschild cohomology groups are different.

The algebras A_5 and A_6 are derived equivalent. (This can be shown by explicitly constructing a suitable tilting complex, similar to the detailed example given in the Appendix.)

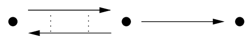
The algebras A_7 and A_8 with Cartan determinant 0 are not derived equivalent, since their centers have different dimensions.

In summary, there are exactly eight derived equivalence classes of connected gentle algebras with two simple modules. They are separated by the vertical lines in the above table.

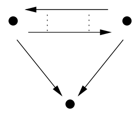
Example 3.8 (Gentle algebras with three simple modules). Let (Q, I) be a connected gentle quiver with three vertices, with corresponding gentle algebra $A = KQ/I$. By Remark 3.4 and Corollary 3.5 we deduce that $\det C_A \in \{0, 1, 2, 4, 8\}$. By Theorem 2.2, algebras with different Cartan determinant cannot be derived equivalent.

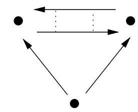
As an illustration, we shall give a complete derived equivalence classification of those algebras with Cartan determinant 0. By Corollary 3.5, a gentle algebra has Cartan determinant 0 if and only if the quiver contains an even oriented cycle with full zero relations. There are 18 connected gentle quivers with three vertices having Cartan determinant 0, as listed on the next page.

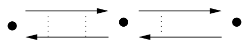
The main tool will be Corollary 3.6, which states that the numbers $ec(Q, I)$ and $oc(Q, I)$ of minimal oriented cycles with full zero relations of even and odd length, respectively, are invariants of the derived category. This will already settle large parts of the classification. In addition we will need to look at the centers and at the first Hochschild cohomology group. The table on the next page collects all the

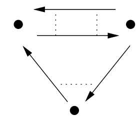
Λ_1 

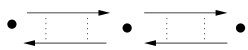
Λ_2 

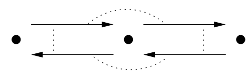
Λ_3 

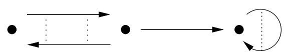
Λ_4 

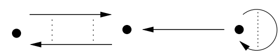
Λ_5 

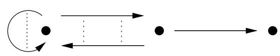
Λ_6 

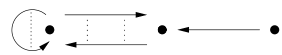
Λ_7 

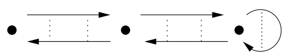
Λ_8 

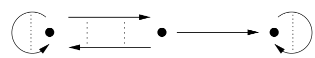
Λ_9 

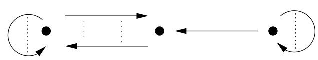
Λ_{10} 

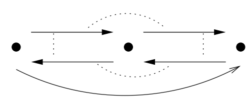
Λ_{11} 

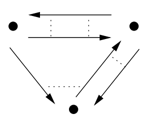
Λ_{12} 

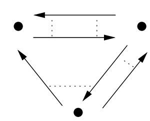
Λ_{13} 

Λ_{14} 

Λ_{15} 

Λ_{16} 

Λ_{17} 

Λ_{18} 

Algebra	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5	Λ_6	Λ_7	Λ_8
$oc(Q, I)$	0	0	0	0	0	0	0	0
$ec(Q, I)$	1	1	1	1	1	1	2	1
$\dim Z(\Lambda)$	1	1	1	1	2	2	1	3
$\dim \mathrm{HH}^1(\Lambda)$	1	1	4	4	2	2	2	2

Λ_9	Λ_{10}	Λ_{11}	Λ_{12}	Λ_{13}	Λ_{14}	Λ_{15}	Λ_{16}	Λ_{17}	Λ_{18}	Algebra
1	1	1	1	1	2	2	0	0	0	$oc(Q, I)$
1	1	1	1	2	1	1	1	1	1	$ec(Q, I)$
1	1	1	1	2	1	1	1	1	1	$\dim Z(\Lambda)$
2(3)	2(3)	2(3)	2(3)	3(4)	4(6)	4(6)	4	3	3	$\dim \mathrm{HH}^1(\Lambda)$

necessary invariants. Again, in the cases where the quiver has loops, the dimension of HH^1 depends on the characteristic being 2 or not, and in these cases the dimension in characteristic 2 is given in parentheses. For the derived equivalence classification, it only remains to consider those algebras having the same invariants. In the cases where the algebras are in fact derived equivalent, we leave out the details of the construction of a suitable tilting complex; in the appendix a detailed example is provided which serves to indicate the strategy which also works in all other cases.

The algebras Λ_1 and Λ_2 are derived equivalent; so are Λ_3 and Λ_4. But Λ_1 and Λ_3 represent different derived equivalence classes since their first Hochschild cohomology groups have different dimensions.

The algebras Λ_5 and Λ_6 are derived equivalent. (The details for this case are provided in the appendix.)

Similarly, the algebras Λ_9, Λ_{10}, Λ_{11} and Λ_{12} are derived equivalent, as are Λ_{14} and Λ_{15} and also Λ_{17} and Λ_{18}.

The case of Λ_{16} is more subtle. This algebra has exactly the same invariants as Λ_3 and Λ_4. However, we claim that Λ_{16} is not derived equivalent to Λ_4. In fact, the Lie algebra structures on HH^1 are not isomorphic. Note that with the Gerstenhaber bracket, the first Hochschild cohomology becomes a Lie algebra. By a result of B. Keller [2003], this Lie algebra structure on HH^1 is invariant under derived equivalence. As mentioned before, by work of M. Bardzell [1997] there is an explicit way of computing HH^1 for a gentle algebra, and a nice combinatorial version due to C. Strametz ([2001, Proposition 2.6] for the additive structure and [2001, Theorem 2.7] for the Lie algebra structure). With this method one can compute that the four-dimensional Lie algebras on $\mathrm{HH}^1(\Lambda_{16})$ and on $\mathrm{HH}^1(\Lambda_4)$ are not isomorphic. In fact, the Lie algebra center of $\mathrm{HH}^1(\Lambda_{16})$ is two-dimensional, whereas the Lie algebra center of $\mathrm{HH}^1(\Lambda_4)$ has dimension 1.

This completes the derived equivalence classification of connected gentle algebras with three simple modules and Cartan determinant 0. The ten classes are separated in the table of the previous page by vertical lines.

4. Skew-gentle algebras

Skew-gentle algebras were introduced in [Geiß and de la Peña 1999]; for the notation and definition we follow here mostly [Bekkert et al. 2003], but we try to explain how the construction works rather than repeating the technical definition from the latter reference.

We start with a gentle pair (Q, I). A set Sp of vertices of the quiver Q is an admissible set of *special* vertices if the quiver with relations obtained from Q by adding loops with square zero at these vertices is again gentle; we denote this gentle pair by $(Q^{\mathrm{sp}}, I^{\mathrm{sp}})$. The triple (Q, Sp, I) is then called *skewed-gentle*.

The admissibility of the set Sp of special vertices is both a local as well as a global condition. Let v be a vertex in the gentle quiver (Q, I); then we can only add a loop at v if v is of valency 1 or 0 or if it is of valency 2 with a zero relation, but not one coming from a loop. Hence only vertices of this type are potential special vertices. But for the choice of an admissible set of special vertices we also have to take care of the global condition that after adding all loops, the pair $(Q^{\mathrm{sp}}, I^{\mathrm{sp}})$ still does not have paths of arbitrary lengths.

Given a skewed-gentle triple (Q, Sp, I), we now construct a new quiver with relations $(\hat{Q}, \hat{I})$ by doubling the special vertices, introducing arrows to and from these vertices corresponding to the previous such arrows and replacing a previous zero relation at the vertices by a mesh relation.

More precisely, we proceed as follows. The nonspecial (or ordinary) vertices in Q are also vertices in the new quiver; any arrow between nonspecial vertices as well as corresponding relations are also kept. Any special vertex $v \in \mathrm{Sp}$ is replaced by two vertices v^+ and v^- in the new quiver. An arrow a in Q from a nonspecial vertex w to v (or from v to w) will be doubled to arrows $a^{\pm} : w \to v^{\pm}$ (or $a^{\pm} : v^{\pm} \to w$, respectively) in the new quiver; an arrow between two special vertices v, w will correspondingly give four arrows between the pairs $v^{\pm}$ and $w^{\pm}$. We say that these new arrows lie over the arrow a. Any relation $ab = 0$ where $t(a) = s(b)$ is nonspecial gives a corresponding zero relation for paths of length 2 with the same start and end points lying over ab. If v is a special vertex of valency 2 in Q, then the corresponding zero relation at v, say $ab = 0$ with $t(a) = v = s(b)$, is replaced by mesh commutation relations saying that any two paths of length 2 lying over ab, having the same start and end points but running over v^+ and v^-, respectively, coincide in the factor algebra to the new quiver with relations $(\hat{Q}, \hat{I})$.

We will speak of $(\hat{Q}, \hat{I})$ as a skewed-gentle quiver *covering* the gentle pair (Q, I).

A K-algebra is then called *skewed-gentle* if it is Morita equivalent to a factor algebra $K\hat{Q}/\hat{I}$, where $(\hat{Q}, \hat{I})$ comes from a skewed-gentle triple (Q, Sp, I) as above.

Remark 4.1. Let $A = KQ/I$ be gentle. In a gentle quiver, there is at most one nonzero cyclic path starting and ending at a given vertex; hence the diagonal entries in the q-Cartan matrix $C_A(q)$ are 1 or of the form $1 + q^j$, for some $j \in \mathbb{N}$.

If a vertex v in Q can be chosen as a special vertex for a covering skewed-gentle quiver $\hat{Q}$, then the corresponding diagonal entry in $C_A(q)$ is 1, as otherwise we have paths of arbitrary lengths in Q^{sp}; hence in the corresponding q-Cartan matrix for the skewed-gentle algebra $\hat{A}$ we have $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ on the diagonal for the two split vertices $v^{\pm}$ in $\hat{Q}$.

Theorem 4.2. *Let (Q, I) be a gentle quiver and $(\hat{Q}, \hat{I})$ a covering skewed-gentle quiver. Let $\hat{A} = K\hat{Q}/\hat{I}$ be the corresponding skewed-gentle algebra. Denote by*

c_k the number of oriented k-cycles in (Q, I) with full zero relations. Then the q-Cartan matrix $C_{\hat{A}}(q)$ is unimodularly equivalent (over $\mathbb{Z}[q]$) to a diagonal matrix with entries $1-(-q)^k$, with multiplicity c_k, $k \geq 1$, and all further diagonal entries equal to 1.

Proof. Again, we argue by induction on the number of vertices and arrows. We let $A = KQ/I$ be the gentle algebra and $C_A(q)$ the q-Cartan matrix as before.

If Q has no arrows, then $\hat{Q}$ is just obtained by doubling the special vertices, and this still has no arrows, so the result clearly holds.

If Q has an arrow α as in Lemma 3.1, with a nonspecial $s(\alpha)$ in the first case, and a nonspecial $t(\alpha)$ in the second case, respectively, then we can argue as in the proof of Lemma 3.1 to remove α. Let us consider again the situation of the first case, so here $s(\alpha) = v_0$ is nonspecial.

Note that a maximal nonzero path p starting from v_0 with α or $\alpha^{\pm}$ (if v_1 is special) will end on a nonspecial vertex (and hence this maximal path is unique in $\hat{A}$); in general, this path will be longer than the one taken in A.

In the column transformations, we only have to be careful at doubled vertices on the path p; here we replace both corresponding columns $\hat{s}_{v_i^{\pm}}$ of $C_{\hat{A}}(q)$ by

$$\hat{s}_{v_i^{\pm}} - q^i \hat{s}_{v_0}.$$

This leads to the Cartan matrix for the skewed-gentle algebra where α or $\alpha^{\pm}$, respectively, has been removed from $\hat{Q}$, which is a skewed-gentle cover for the quiver obtained from Q by deleting α. The claim follows by induction.

Now assume Q has a source v — say the first vertex — which is special. (The case of a sink is dual.) Then the q-Cartan matrix for $\hat{A}$ has the form

$$C_{\hat{A}}(q) = \begin{pmatrix} 1 & 0 & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & \hat{C}'(q) & \\ 0 & 0 & & & \end{pmatrix} \sim \tilde{\hat{C}}(q) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \hat{C}'(q) & \\ 0 & 0 & & & \end{pmatrix},$$

where $\hat{C}'(q)$ is the q-Cartan matrix of the skewed-gentle algebra $\hat{A}'$ for the quiver $\hat{Q}'$ obtained from $\hat{Q}$ by removing v^+, v^- and the arrows incident with $v^{\pm}$. Note that $\hat{Q}'$ is the skewed-gentle cover for the quiver Q' which is obtained from Q by removing v and the arrow incident with v, and the choice $\mathrm{Sp}' = \mathrm{Sp} \setminus \{v\}$ as the set of special vertices; in short, we write this as $\hat{Q}' = \widehat{Q'}$. Again, using induction the claim follows immediately.

Thus again, we may now assume that Q has only vertices of valency 2 with a (nonloop) zero relation or vertices of valency 4; note that any special vertex in Q

has to be of valency 2. As before, we may also assume that Q (and hence also $\hat{Q}$) is connected.

If there are no nonspecial vertices, or if all nonspecial vertices are of valency 4, then Q^{sp} is not gentle. Hence Q has a nonspecial vertex v of valency 2 with a zero relation at v. Let $p_0 : v_0 \to v$ be the (unique) incoming arrow.

Again we consider the unique maximal path p in Q with nonrepeating arrows starting in v_0 with p_0, such that the product of any two consecutive arrows is zero in A; as before, in our current situation p has to be a cycle $\mathscr{C} = p_0 p_1 \dots p_s$, where $p_i : v_i \to v_{i+1}$, $i = 0, \dots, s$, and $v_{s+1} = v_0$. As in the previous situation, the arrows are distinct, but vertices $v_i \neq v_0$ may be repeated.

For a vertex w in Q we denote by z_w the row of the q-Cartan matrix $C_A(q)$ corresponding to w.

If w is nonspecial, we denote by $\hat{z}_w$ the corresponding row in the q-Cartan matrix $\hat{C}(q) = C_{\hat{A}}(q)$. If w is special, then for the two vertices $w^{\pm}$ we have two corresponding rows $\hat{z}_{w^{\pm}}$ in the Cartan matrix $\hat{C}(q)$, and we then set $\hat{z}_w = \hat{z}_{w^+} + \hat{z}_{w^-}$.

Before, we transformed C by replacing z_v with

$$Z = \sum_{i=1}^{s+1} (-q)^{i-1} z_{v_i}$$

and obtained a matrix $\tilde{C}(q)$. We now do a parallel transformation on $\hat{C}(q)$, that is, we replace $\hat{z}_v$ by

$$\hat{Z} = \sum_{i=1}^{s+1} (-q)^{i-1} \hat{z}_{v_i},$$

and we obtain a matrix $\tilde{\hat{C}}(q)$. We have to compare the differences and check that everything stays under control for the induction argument.

If a vertex v_i, $1 \leq i \leq s$, is special, note that the doubled contribution in $\hat{z}_{v_i} = \hat{z}_{v_i^+} + \hat{z}_{v_i^-}$ is needed on the one hand for the cancellation with the previous row, and on the other hand to continue around the cycle $\mathscr{C}$. As v is nonspecial and of valency 2 with a zero-relation, we note that as before, in $\hat{Z}$ we only have the contribution $1 - (-q)^{s+1}$ at v.

Following this with the parallel operation to the previous column operation we then replace the column $\hat{s}_v$ by the linear combination

$$\hat{S} = \sum_{i=1}^{s+1} (-q)^{s+1-i} \hat{s}_{v_{i+1}},$$

where we use analogous conventions as before.

With v corresponding to the first row and column of the Cartan matrix, we have thus unimodularly transformed $\hat{C}(q)$ to a matrix of the form

$$\begin{pmatrix} 1-(-q)^{s+1} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \hat{C}'(q) & \\ 0 & & & \end{pmatrix}$$

where $\hat{C}'(q)$ is the Cartan matrix of the skewed-gentle algebra $\hat{A}'$ for the quiver $\hat{Q}'$ obtained from $\hat{Q}$ by removing v and the arrows incident with v. Note that in fact, $\hat{Q}' = \widehat{Q'}$ in the notation of our previous proof; that is, as explained earlier, $\hat{Q}'$ is the skewed-gentle cover for the quiver Q' and the choice $\mathrm{Sp}' = \mathrm{Sp}\setminus\{v\}$ as the set of special vertices.

Thus the result follows by induction. $\square$

Remark 4.3. By comparing Theorems 3.2 and 4.2 we observe that the q-Cartan matrix $C_A(q)$ for the gentle algebra A to (Q, I), and the q-Cartan matrix $C_{\hat{A}}(q)$ for a skewed-gentle cover $\hat{A}$ are unimodularly equivalent to diagonal matrices which only differ by adding as many further 1's on the diagonal as there are special vertices chosen in Q. In particular, with notation as above,

$$\det C_{\hat{A}}(q) = \det C_A(q) = \prod_{k\geq 1} (1-(-q)^k)^{c_k}.$$

This observation has an immediate consequence upon specializing to $q = 1$:

Corollary 4.4. *Let (Q, I) be a gentle quiver, and $(\hat{Q}, \hat{I})$ a covering skewed-gentle quiver. Then the determinant of the ordinary Cartan matrix of the skewed-gentle algebra $\hat{A} = K\hat{Q}/\hat{I}$ is the same as the one for the gentle algebra $A = KQ/I$, i.e.,* $\det C_{\hat{A}} = \det C_A$.

Remark 4.5. A gentle algebra and a (proper) skewed-gentle algebra may have the same q-invariants but they cannot be derived equivalent by [Schröer and Zimmermann 2003, Corollary 1.2].

Appendix. Tilting complexes and derived equivalences: a detailed example

This appendix is aimed at providing enough background on tilting complexes and explicit computations of their endomorphism rings so that the interested reader can fill in the details in the derived equivalence classifications of Examples 3.7 and 3.8. We explained there in detail how to distinguish derived equivalence classes (since this is the main topic of this paper), but have been fairly short on indicating why certain algebras in the lists are actually derived equivalent. In this section we will go through one example in detail; this will indicate the main strategy which also works in all other cases.

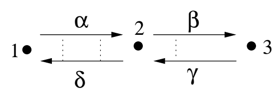
For an algebra A denote by $D^b(A)$ the bounded derived category and by $K^b(P_A)$ the homotopy category of bounded complexes of finitely generated projective A-modules.

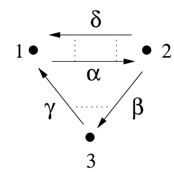
Two algebras A and B are called derived equivalent if $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. By J. Rickard's theorem [1989b], this happens if and only there exists a tilting complex T for A such that the endomorphism ring $\mathrm{End}_{K^b(P_A)}(T)$ in the homotopy category is isomorphic to B. A bounded complex T of projective A-modules is called a tilting complex if the following conditions are satisfied.

(i) $\mathrm{Hom}_{K^b(A)}(T, T[i]) = 0$ for $i \neq 0$ (where [.] denotes the shift operator)

(ii) $\mathrm{add}(T)$, the full subcategory of $K^b(P_A)$ consisting of direct summands of direct sums of copies of T, generates $K^b(P_A)$ as a triangulated category.

In Example 3.8 we stated that the algebras Λ_5 and Λ_6 are derived equivalent. For the convenience of the reader we recall the definition of these algebras.

Λ_5 

Λ_6 

Recall from Section 2 our conventions to deal with right modules and to read paths from left to right. In particular, left multiplication by a nonzero path from vertex j to vertex i gives a homomorphism $P_i \to P_j$.

We define a bounded complex $T := T_1 \oplus T_2 \oplus T_3$ of projective Λ_5-modules. Let $T_1 : 0 \to P_3 \to 0$ and $T_3 : 0 \to P_1 \to 0$ be stalk complexes concentrated in degree 0. Moreover, let

$$T_2 : 0 \to P_1 \oplus P_3 \xrightarrow{(\delta, \beta)} P_2 \to 0$$

(in degrees 0 and -1). We claim that T is a tilting complex. Property (i) above is obvious for all $|i| \geq 2$ since we are dealing with two-term complexes.

Let $i = -1$ and consider possible maps $T_2 \to T_j[-1]$, where $j \in \{1, 2, 3\}$. This is given by a map of complexes of the form

$$\begin{tikzcd} 0 \arrow[r] & P_1 \oplus P_3 \arrow[r, "(\delta, \beta)"] & P_2 \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & Q \arrow[r] & \cdots \end{tikzcd}$$

where Q could be either of P_1, P_3, or $P_1 \oplus P_3$. But since we are dealing with gentle algebras, no nonzero map can be zero when composed with both δ and β. So the only homomorphism of complexes $T_2 \to T_j[-1]$ is the zero map, as desired. Directly from the definition we see that $\mathrm{Hom}(T_1, T_j[-1]) = 0$ and $\mathrm{Hom}(T_3, T_j[-1]) = 0$ (since they are stalk complexes).

Thus we have shown that $\mathrm{Hom}(T, T[-1]) = 0$.

Now let $i = 1$. We have to consider maps $T_j \to T_2[1]$; these are given by

$$\begin{tikzcd} & 0 \arrow[rr] & & Q \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & P_1 \oplus P_3 \arrow[rr, "{(\delta,\beta)}"] & & P_2 \arrow[r] & 0 \end{tikzcd}$$

where Q again can be either of P_1, P_3, or $P_1 \oplus P_3$. Now there certainly exist nonzero homomorphisms of complexes. But they are all homotopic to zero. In fact, every path in the quiver of Λ_5 from vertex 2 to vertex 1 or 3 either starts with δ or with β. Accordingly, every homomorphism $Q \to P_2$ can be factored through the map $(\delta, \beta) : P_1 \oplus P_3 \to P_2$.

It follows that $\mathrm{Hom}_{K^b(P_A)}(T, T[1]) = 0$ (in the homotopy category).

It remains to show that the complex T also satisfies property (ii) of the definition of a tilting complex. It suffices to show that the projective indecomposable modules P_1, P_2 and P_3, viewed as stalk complexes, can be generated by $\mathrm{add}(T)$. This is clear for P_1 and P_3 since they occur as summands of T. For P_2, consider the map of complexes $\Psi : T_2 \to T_3 \oplus T_1$ given by the identity map on $P_1 \oplus P_3$ in degree 0. Then the stalk complex $P_2[0]$ with P_2 in degree 0 can be shown to be homotopy equivalent (i.e. isomorphic in $K^b(P_A)$) to the mapping cone of Ψ. Thus we have a distinguished triangle

$$\underbrace{T_2}_{\in\,\mathrm{add}(T)} \to \underbrace{T_3 \oplus T_1}_{\in\,\mathrm{add}(T)} \to P_2[0] \to \underbrace{T_2[1]}_{\in\,\mathrm{add}(T)} .$$

By definition, $\mathrm{add}(T)$ is triangulated, so it follows that also the stalk complex $P_2[0]$ lies in $\mathrm{add}(T)$, which proves (ii).

Hence, T is indeed a tilting complex for Λ_5.

By Rickard's theorem, the endomorphism ring of T in the homotopy category is derived equivalent to Λ_5. We need to show that $E := \mathrm{End}_{K^b(P_A)}(T)$ is isomorphic to Λ_6. Note that the vertices of the quiver of E correspond to the summands of T.

For explicit calculations, the following formula is useful, which gives a general method for computing the Cartan matrix of an endomorphism ring of a tilting complex from the Cartan matrix of A.

Alternating sum formula. For a finite-dimensional algebra A, let $Q = (Q^r)_{r\in\mathbb{Z}}$ and $R = (R^s)_{s\in\mathbb{Z}}$ be bounded complexes of projective A-modules. Then

$$\sum_i (-1)^i \dim \mathrm{Hom}_{K^b(P_A)}(Q, R[i]) = \sum_{r,s} (-1)^{r-s} \dim \mathrm{Hom}_A(Q^r, R^s).$$

In particular, if Q and R are direct summands of a tilting complex then

$$\dim \mathrm{Hom}_{K^b(P_A)}(Q, R) = \sum_{r,s} (-1)^{r-s} \dim \mathrm{Hom}_A(Q^r, R^s).$$

The Cartan matrix of Λ_5 has the form $\begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. From that, using the alternating sum formula, we can compute the Cartan matrix of E to be $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Note that this is actually the Cartan matrix of Λ_6.

Now we have to define maps of complexes between the summands of T, corresponding to the arrows of the quiver of Λ_6. The final step then is to show that these maps satisfy the defining relations of Λ_6, up to homotopy.

We define $\tilde{\alpha} : T_1 \to T_2$ by the map $(\alpha\beta, 0) : P_3 \to P_1 \oplus P_3$ in degree 0. This is indeed a homomorphism of complexes, since $\delta\alpha = 0$ in Λ_5. Moreover, we define $\tilde{\beta} : T_2 \to T_3$ and $\tilde{\delta} : T_2 \to T_1$ by the projection onto the first and second summand in degree 0, respectively. Finally, we define $\tilde{\gamma} : T_3 \to T_1$ by $\gamma\delta : P_1 \to P_3$.

We now have to check the relations, up to homotopy. We write compositions from left to right (as in the relations of the quiver of E). Clearly, $\tilde{\alpha}\tilde{\delta} = 0$. The composition $\tilde{\beta}\tilde{\gamma} : T_2 \to T_1$ is given in degree 0 by

$$(\gamma\delta, 0) : P_1 \oplus P_3 \to P_3.$$

So it is not the zero map, but is homotopic to it via the homotopy map $\gamma : P_2 \to P_3$ (use that $\gamma\beta = 0$ in Λ_5). Finally, consider $\tilde{\delta}\tilde{\alpha}$ on T_2. It is given by

$$\begin{pmatrix} 0 & \alpha\beta \\ 0 & 0 \end{pmatrix}$$

in degree 0 and the zero map in degree -1. It is indeed homotopic to zero via the homotopy map $(\alpha, 0) : P_2 \to P_1 \oplus P_3$. (Note that here we use that $\alpha\delta = 0$ and $\delta\alpha = 0$ in Λ_5.)

Thus, we have defined maps between the summands of T, corresponding to the arrows of the quiver of Λ_6. We have shown that they satisfy the defining relations of Λ_6, and that the Cartan matrices of E and Λ_6 coincide. From this we can conclude that $E \cong \Lambda_6$. Hence, Λ_5 and Λ_6 are derived equivalent, as desired.

All the other derived equivalences stated in Examples 3.7 and 3.8 can be verified exactly along these lines. In particular, they can also be realized by tilting complexes with nonzero entries in only two degrees.

Acknowledgment

We thank the referee for very helpful and insightful comments, and in particular for pointing out the importance of graded derived equivalences in our context of q-Cartan matrices.

References

[Assem and Happel 1981] I. Assem and D. Happel, "Generalized tilted algebras of type A_n", *Comm. Algebra* **9**:20 (1981), 2101–2125. MR 83a:16023a Zbl 0481.16009

[Assem and Skowroński 1987] I. Assem and A. Skowroński, "Iterated tilted algebras of type $\tilde{\mathbf{A}}_n$", *Math. Z.* **195**:2 (1987), 269–290. MR 88m:16033 Zbl 0601.16022

[Bardzell 1997] M. J. Bardzell, "The alternating syzygy behavior of monomial algebras", *J. Algebra* **188**:1 (1997), 69–89. MR 98a:16009 Zbl 0885.16011

[Bekkert and Drozd 2003] V. Bekkert and Y. Drozd, "Derived categories for some classes of algebras", preprint, 2003, Available at http://www.mat.ufmg.br/∼bekkert/pub.html.

[Bekkert et al. 2003] V. Bekkert, E. N. Marcos, and H. A. Merklen, "Indecomposables in derived categories of skewed-gentle algebras", *Comm. Algebra* **31**:6 (2003), 2615–2654. MR 2005b:16024 Zbl 1061.16025

[Bobiński et al. 2004] G. Bobiński, C. Geiß, and A. Skowroński, "Classification of discrete derived categories", *Cent. Eur. J. Math.* **2**:1 (2004), 19–49. MR 2005e:16020 Zbl 1036.18007

[Bocian and Skowroński 2005] R. Bocian and A. Skowroński, "Weakly symmetric algebras of Euclidean type", *J. Reine Angew. Math.* **580** (2005), 157–199. MR 2005m:16016 Zbl 02157768

[Fuller 1992] K. R. Fuller, "The Cartan determinant and global dimension of Artinian rings", pp. 51–72 in *Azumaya algebras, actions, and modules* (Bloomington, IN, 1990), edited by D. Haile and J. Osterburg, Contemp. Math. **124**, Amer. Math. Soc., Providence, RI, 1992. MR 92m:16035 Zbl 0746.16013

[Geiß and Krause 2002] C. Geiss and H. Krause, "On the notion of derived tameness", *J. Algebra Appl.* **1**:2 (2002), 133–157. MR 2003c:16019 Zbl 1034.16021

[Geiß and de la Peña 1999] C. Geiß and J. A. de la Peña, "Auslander-Reiten components for clans", *Bol. Soc. Mat. Mexicana* (3) **5**:2 (1999), 307–326. MR 2001g:16035 Zbl 0959.16013

[Holm 2005] T. Holm, "Cartan determinants for gentle algebras", *Arch. Math.* (*Basel*) **85**:3 (2005), 233–239. MR 2006e:16022 Zbl 1084.16009

[Keller 2003] B. Keller, "Derived invariance of higher structures on the Hochschild complex", preprint, 2003.

[Rickard 1989a] J. Rickard, "Derived categories and stable equivalence", *J. Pure Appl. Algebra* **61**:3 (1989), 303–317. MR 91a:16004 Zbl 0685.16016

[Rickard 1989b] J. Rickard, "Morita theory for derived categories", *J. London Math. Soc.* (2) **39**:3 (1989), 436–456. MR 91b:18012 Zbl 0642.16034

[Rickard 1991] J. Rickard, "Derived equivalences as derived functors", *J. London Math. Soc.* (2) **43**:1 (1991), 37–48. MR 92b:16043 Zbl 0683.16030

[Schröer and Zimmermann 2003] J. Schröer and A. Zimmermann, "Stable endomorphism algebras of modules over special biserial algebras", *Math. Z.* **244**:3 (2003), 515–530. MR 2004i:16024 Zbl 1036.16004

[Strametz 2001] C. Strametz, "The Lie algebra structure of the first Hochschild cohomology group of a monomial algebra", preprint, 2001. Announced in *C. R. Math. Acad, Sci. Paris* **334**:9 (2002), 733–738. math.RT/0111060

[Vossieck 2001] D. Vossieck, "The algebras with discrete derived category", *J. Algebra* **243**:1 (2001), 168–176. MR 2002f:16025 Zbl 1038.16010

Received May 1, 2005. Revised September 22, 2005.

CHRISTINE BESSENRODT
INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISKRETE MATHEMATIK
FAKULTÄT FÜR MATHEMATIK UND PHYSIK
LEIBNIZ UNIVERSITÄT HANNOVER
WELFENGARTEN 1
D-30167 HANNOVER
GERMANY

bessen@math.uni-hannover.de
http://www-ifm.math.uni-hannover.de/~bessen/

THORSTEN HOLM
DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF LEEDS
LEEDS LS2 9JT
ENGLAND
Current address:
INSTITUT FÜR ALGEBRA UND GEOMETRIE
OTTO-VON-GUERICKE-UNIVERSITÄT MAGDEBURG
POSTFACH 4120
D-39016 MAGDEBURG
GERMANY

tholm@maths.leeds.ac.uk
http://www.maths.leeds.ac.uk/~tholm