POLYNOMIAL GROWTH SOLUTIONS TO HIGHER-ORDER LINEAR ELLIPTIC EQUATIONS AND SYSTEMS

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For an equation or system of equations $Lu = 0$, where $L$ is a uniformly elliptic operator of order $2m$ and $u$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^N$, we prove that the dimension of the space of polynomial growth solutions of degree at most $d$ is bounded by $Cd^{2mnN}$, where $C$ is a constant. If the system is in divergence form, we prove that this dimension is in fact bounded by $Cd^{mnN}$.

Introduction

We consider an equation or a system of equations of the form

$$Lu = 0,$$

where $L$ is a uniformly elliptic operator of order $2m$, with $m > 1$, defined on $\mathbb{R}^n$. We want to estimate the dimension of the following space of solutions to $Lu = 0$.

Definition 0.1. For each nonnegative number $d$ we denote by

$$\mathcal{H}_d^L(\mathbb{R}^n) = \left\{ u \mid Lu = 0 \text{ and } |u|(x) = O(r_p^d(x)) \right\}$$

the space of polynomial growth solutions of degree at most $d$, where $r_p(x)$ is the Euclidean distance from a fixed point $p$ to $x$ in $\mathbb{R}^n$. We denote the dimension of $\mathcal{H}_d^L(\mathbb{R}^n)$ by

$$h_d^L(\mathbb{R}^n) = \dim \mathcal{H}_d^L(\mathbb{R}^n).$$

When $L = \Delta$ is the Laplacian, this subject has been studied extensively for a variety of open manifolds $M$ (meaning noncompact and without boundary). Let $n$ be the dimension of $M$. Yau conjectured that $h_d^\Delta(M) < \infty$ for all $d \geq 1$. For $M = \mathbb{R}^n$ this is easy to see; in fact $h_d^\Delta(\mathbb{R}^n)$ equals

$$\left( \begin{array}{c} n+d-1 \\ d \end{array} \right) + \left( \begin{array}{c} n+d-2 \\ d-1 \end{array} \right) \sim \frac{2}{(n-1)!} d^{n-1} \quad \text{as } d \to \infty.$$

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Yau’s conjecture was partially confirmed for the case $d = 1$ by Li and Tam [1989], who proved that under the same conditions, if the volume growth of $M$ satisfies

$$V_p(r) = O(r^k_p)$$

for some $k > 0$, then

$$h_d^A(M) \leq k + 1 = h_d^A(\mathbb{R}^k).$$

The conjecture was then proved in full by Colding and Minicozzi [1997], who showed that for a complete open manifold $M$ of nonnegative Ricci curvature, there exists $C > 0$ depending only on the dimension $n$ and such that

$$h_d^A(M) \leq C d^{n-1}.$$ 

In view of the formula (0-1) for $h_d^A(\mathbb{R}^n)$, this estimate is sharp in the order of $d$ as $d \to \infty$. The authors also proved that if a complete open manifold $M$ satisfies a Poincaré inequality and a volume doubling property, then $h_d^A(M)$ is finite and can be estimated in terms of a constant depending on the manifold and $d$. However, in this case, the order in $d$ is not sharp.

Soon thereafter, Li [1997] proved a more general estimate with a substantially simpler proof. Namely, if $M$ (open, complete) satisfies a mean value inequality and a volume comparison condition, then

$$h_d^A(M) \leq C d^{n-1}.$$ 

Later Li and Wang [1999a] showed that the finiteness of $h_d^A(M)$ is actually valid in a much bigger class of manifolds. In particular, they proved that if $M$ satisfies a weak mean value inequality and has polynomial volume growth, then $h_d^A(M)$ must be finite for all $d \geq 1$. However, in this case, the estimate on $h_d^A(M)$ is exponential in $d$ as $d \to \infty$.

Recently, Li and Wang [1999b] showed that if $M$ is a complete manifold satisfying the Sobolev inequality $\mathcal{F}(B, \nu)$, the space $\mathcal{H}^A_d(M)$ is finite-dimensional, and its dimension $h_d^A$ satisfies

$$h_d^A(M) \leq C(B, \nu) d^{\nu}$$

for all $d \geq 1$. They proved that if $M$ is a complete $n$-dimensional open manifold with nonnegative sectional curvature, then

$$\liminf_{d \to \infty} d^{-(n-1)} h_d^A(M) \leq \frac{2}{(n-1)!},$$

and the equality

$$\liminf_{d \to \infty} d^{-(n-1)} h_d^A(M) = \frac{2}{(n-1)!}$$

holds if and only if $M = \mathbb{R}^n$. 
In this note we extend some of the preceding results to higher-order operators. To simplify the presentation, we restrict ourselves to Euclidean space. So we assume that $L$ is of higher order $2m$ with $m > 1$ and try to estimate $h^L_d(\mathbb{R}^n)$.

In Section 1 we show that if $Lu = 0$ is a uniformly elliptic equation or a uniformly elliptic system of equations of order $2m$ in nondivergence form, then

$$h^L_d(\mathbb{R}^n) \leq C d^{2mnN},$$

where $N$ is the number of equations in the system $Lu = 0$.

In Section 2 we consider the case where $Lu = 0$ is a uniformly elliptic equation or a uniformly elliptic system of equations of order $2m$ in divergence form. Then

$$h^L_d(\mathbb{R}^n) \leq C d^{mnN},$$

where $N$ is the number of equations in the system $Lu = 0$.

## 1. Equations in nondivergence form

In Euclidean space $\mathbb{R}^n$ with rectangular coordinates $x_1, \ldots, x_n$, we consider the differential operator

$$Lu \equiv \sum_{|\alpha| = 2m} a_\alpha(x) \, D^\alpha u(x),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha = \frac{\partial^{2m}}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Throughout the section, we impose the following condition on the operator $L$.

**Condition L.** The coefficients $a_\alpha$ in the equation $Lu = 0$ are uniformly continuous and satisfy the uniform ellipticity condition; that is, there exists a constant $\Lambda > 0$ such that

$$\Lambda |\Xi|^{2m} \geq \sum_{|\alpha| = 2m} a_\alpha(x) \Xi^\alpha \geq \Lambda^{-1} |\Xi|^{2m}$$

for all $x, \Xi \in \mathbb{R}^n$.

The assumptions imply that there exists a constant $C > 0$ such that, for any function $w \in C^\infty_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla^{2m} w|^2(x) \, dx \leq C \int_{\mathbb{R}^n} |Lw|^2(x) \, dx$$

(see [Agmon et al. 1959; 1964]). We establish some preliminary lemmas before we prove our first main result.
Lemma 1.1 [Li and Wang 1999b]. Let $V$ be a $k$-dimensional subspace of a vector space $W$. Assume that $W$ is endowed with an inner product $I$ and a bilinear form $\Phi$. Then for any given linearly independent set of vectors $\{w_1, \ldots, w_{k-1}\} \subset W$, there exists an orthonormal basis $\{v_1, \ldots, v_k\}$ of $V$ with respect to $I$ such that $\Phi(v_i, w_j) = 0$ for all $1 \leq j < i \leq k$.

Let $\phi$ be a positive function defined on a fixed geodesic ball $B_p(r)$. We introduce two inner products $I_r$ and $\Phi_r$ on the space $W = L^2(B_p(r), dx) \cap L^2(B_p(r), \phi \, dx)$:

$$I_r(f, g) = \int_{B_p(r)} f(x) g(x) \, dx, \quad \Phi_r(f, g) = \int_{B_p(r)} f(x) g(x) \phi(x) \, dx.$$ 

For $i = 1, 2, \ldots$, let $\lambda_i(r)$ be the $i$-th Dirichlet eigenvalue of $B_p(r)$ arranged in nondecreasing order.

Lemma 1.2. Let $V$ be a $k$-dimensional subspace of $\mathfrak{h}_r^2(\mathbb{R}^n)$. For any fixed number $\theta > 1$, let $\operatorname{tr}_{I_{\theta r}} I_r(V)$ denote the trace of the bilinear form $I_r$ with respect to the inner product $I_{\theta r}$ on $V$. Then

$$\operatorname{tr}_{I_{\theta r}} I_r(V) \leq \sum_{i=1}^k \frac{C_8 \theta^{4m-2}}{\lambda_i(\theta r)(\theta - 1)^{4m} r^2},$$

where $C_8$ is a constant.

Proof. Set $\bar{\theta} = \frac{1}{2} (1 + \theta)$ and let $\phi \in C_0^{2m}(B_p(\bar{\theta}r))$ be a nonnegative function defined on $B_p(\bar{\theta}r)$ satisfying $\phi = 1$ on $B_p(r)$, $0 \leq \phi \leq 1$ on $B_p(\bar{\theta}r)$, $\phi = 0$ on $\partial B_p(\bar{\theta}r)$, and

$$|\nabla^j \phi| \leq \frac{C}{(\theta - 1)^j r^j}$$

for some constant $C = C(n, m)$, and $1 \leq j \leq 2m$. By unique continuation, $V \subset H^{2m}(B_p(\bar{\theta}r), dx) \cap H^{2m}(B_p(r), \phi \, dx)$ is a $k$-dimensional subspace. Applying Lemma 1.1 with $w_1, \ldots, w_k$ the Dirichlet eigenfunctions of $B_p(\bar{\theta}r)$ corresponding to the eigenvalues $\lambda_1(\bar{\theta}r), \ldots, \lambda_k(\bar{\theta}r)$, we get an orthonormal basis $\{v_1, \ldots, v_k\}$ of $V$ with respect to the inner product $I_{\bar{\theta}r}$ and

$$\Phi_{\bar{\theta}r}(v_i, w_j) = \int_{B_p(\bar{\theta}r)} v_i(x) w_j(x) \phi(x) \, dx = 0$$

for $0 \leq j < i \leq k$. Thus, for any $1 \leq i \leq k$, the variational principle implies that

$$\lambda_i(\bar{\theta}r) \int_{B_p(\bar{\theta}r)} (\phi v_i)^2 \leq \int_{B_p(\bar{\theta}r)} |\nabla (\phi v_i)|^2.$$
Hence,
\begin{equation}
(1-1) \quad \text{tr}_L, I_r (V) = \sum_{i=1}^{k} \int_{B_{p}(r)} v_i^2 \leq \sum_{i=1}^{k} \frac{1}{\lambda_i(\theta r)} \int_{B_{p}(\theta r)} |\nabla (\phi v_i)|^2
\end{equation}
\begin{equation}
\leq C \sum_{i=1}^{k} \frac{1}{\lambda_i(\theta r)} \int_{B_{p}(\theta r)} |\nabla^2 (\phi v_i)|^2 \bar{\theta}^2 r^2
\end{equation}
\begin{equation}
\leq C^{2m-1} \sum_{i=1}^{k} \frac{1}{\lambda_i(\theta r)} \int_{B_{p}(\theta r)} |\nabla^{2m} (\phi v_i)|^2 \bar{\theta}^{4m-2} r^{4m-2}
\end{equation}
\begin{equation}
\leq C_1 \sum_{i=1}^{k} \frac{\bar{\theta}^{4m-2} r^{4m-2}}{\lambda_i(\theta r)} \int_{B_{p}(\theta r)} |L(\phi v_i)|^2
\end{equation}
\begin{equation}
\leq C_2 \sum_{i=1}^{k} \sum_{j=0}^{2m-1} \frac{\bar{\theta}^{4m-2} r^{4m-2}}{\lambda_i(\theta r)} \int_{B_{p}(\theta r)} \frac{|\nabla^j v_i|^2}{(\theta - 1)^{4m-2j} r^{4m-2}},
\end{equation}
where $C_1$ and $C_2$ are constants.

Let $\eta \in C_0^{2m}(B_p(\theta r))$ be a nonnegative function defined on $B_p(\theta r)$ satisfying $\eta = 1$ on $B_p(\theta r)$, $0 \leq \eta \leq 1$, $\eta = 0$ on $\partial B_p(\theta r)$, and
\begin{equation}
|\nabla^j \eta| \leq \frac{\tilde{C}}{\theta - 1)^{2m}}, \quad 1 \leq j \leq 2m.
\end{equation}

Note that
\begin{equation}
(1-2) \quad \int_{B_p(\theta r)} |\nabla^{2m} v_i|^2 \leq \int_{B_p(\theta r)} |\nabla^{2m} (\eta v_i)|^2 \leq C_3 \int_{B_p(\theta r)} |L(\eta v_i)|^2
\end{equation}
\begin{equation}
\leq C_4 \int_{B_p(\theta r)} \sum_{j=0}^{2m-1} \frac{|\nabla^j v_i|^2}{(\theta - 1)^{4m-2j} r^{4m-2j}}.
\end{equation}

Introduce the weighted seminorms
\begin{equation}
\Psi_k = \sup_{1 < \sigma < \theta < \theta} (\sigma - 1)^{2k} r^{2k} \int_{B_{p}(\sigma r)} |\nabla^k v_i|^2
\end{equation}
for each $0 \leq k \leq 2m$. In terms of these seminorms, (1-2) implies that
\begin{equation}
(1-3) \quad \Psi_{2m} \leq C_5 \sum_{k=0}^{2m-1} \Psi_k.
\end{equation}

For each $1 \leq k \leq 2m - 1$, we apply an interpolation inequality to get
\begin{equation}
(1-4) \quad \Psi_k \leq \epsilon \Psi_{2m} + C(k) \epsilon^{k/(k-2m)} \Psi_0
\end{equation}
for any \( \epsilon > 0 \), where \( C(k) \) is a constant. Putting (1-4) into (1-3), and arranging \( \epsilon > 0 \) to be small, we conclude that \( \Psi_{2m} \leq C_6 \Psi_0 \). In particular, we have

\[
\int_{B_p(\theta r)} |\nabla^{2m} v| \leq C_6 \frac{\rho}{(\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v^2.
\]

Therefore,

\[
(1-5) \int_{B_p(\theta r)} \sum_{j=0}^{2m-1} \frac{|\nabla^{j} v|^2}{(\theta - 1)^{4m-2j} r^{4m-2j}} \leq \sum_{j=0}^{2m-1} \frac{1}{(\theta - 1)^{4m-2j} r^{4m-2j}} \times \left( \epsilon_j \int_{B_p(\theta r)} |\nabla^{2m} v|^2 + C(j) \epsilon_j^{j/(j-2m)} \int_{B_p(\theta r)} v^2 \right) \leq 2m \int_{B_p(\theta r)} |\nabla^{2m} v|^2 + \frac{C}{(\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v^2 \leq \frac{C_7}{(\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v^2,
\]

where we have set \( \epsilon_j = (\theta - 1)^{4m-2j} r^{4m-2j} \). Substituting (1-5) into (1-1), we get

\[
\text{tr}_{\theta r} I_r(V) \leq \sum_{i=1}^{k} \frac{2m-1}{\lambda_i(\theta r)} \frac{|\nabla^{j} v|}{(\theta - 1)^{4m-2j} r^{4m-2j}} \int_{B_p(\theta r)} v^2 \leq \sum_{i=1}^{k} \frac{C_7 \theta^{4m-2} r^{4m-2}}{\lambda_i(\theta r) (\theta - 1)^{4m} r^{4m}} \int_{B_p(\theta r)} v^2 = \sum_{i=1}^{k} \frac{C_8 \theta^{4m-2} r^{4m}}{\lambda_i(\theta r) (\theta - 1)^{4m} r^{4m}}.
\]

\[\square\]

**Lemma 1.3 [Li 1997].** Let \( K \) be a \( k \)-dimensional linear space of functions defined on \( \mathbb{R}^n \). Suppose that each function \( u \in K \) is of polynomial growth of at most degree \( d \). Then for any \( \theta > 1, \delta > 0, \) and \( r_0 > 0 \), there exists \( r > r_0 \) such that if \( \{u_i\}_{i=1}^{k} \) is an orthonormal basis of \( K \) with respect to the inner product

\[
I_{\theta r}(u, v) = \int_{B_p(\theta r)} u(x) v(x) \, dx,
\]

then

\[
\text{tr}_{\theta r} I_r = \sum_{i=1}^{k} \int_{B_p(\theta r)} u_i^2(x) \, dx \geq k \theta^{-(2d+n+\delta)}.
\]

**Proof.** We reproduce Li’s argument. Let \( \text{tr}_{\rho} I_r \) denote the trace of the bilinear form \( I_r \) with respect to \( I_{\rho} \), and let \( \det_{\rho} I_r \) be the determinant of \( I_r \) with respect to \( I_{\rho} \). Assume that the lemma is false. Then, for \( r > r_0 \), we have

\[
\text{tr}_{\theta r} I_r < k \theta^{-(2d+n+\delta)}.
\]
The arithmetic-geometric mean inequality asserts that
\[(\det_{\theta_r} I_r)^{1/k} \leq k^{-1}(\text{tr}_{\theta_r} I_r).\]

This implies that
\[\det_{\theta_r} I_r \leq \theta^{-k(2d+n+\delta)}\]
for all \(r > r_0\). Setting \(r = r_0 + 1\) and iterating the inequality \(j\) times yields
\[(1-6) \det_{\theta_r} I_r \leq \theta^{-jk(2d+n+\delta)}.
\]
However, for a fixed \(I_r\)-orthonormal basis \(\{u_i\}_{i=1}^k\) of \(K\), the assumption on \(K\) implies that there exists a constant \(C > 0\), depending on \(K\), such that
\[\int_{B_p(r)} u_i^2(x) \, dx \leq C(1+r^{2d+n})\]
for all \(1 \leq i \leq k\). In particular, this implies that
\[\det_{\theta_r} I_{\theta/r} \leq kC \theta^{jk(2d+n+\delta)} r^{k(2d+n+\delta)}.
\]
This contradicts (1-6) as \(j \to \infty\).

**Theorem 1.4.** Assume that Condition L holds. Let \(n > 2\). Then the space \(H^1_d(\mathbb{R}^n)\) is finite-dimensional, and its dimension \(h^1_d(\mathbb{R}^n)\) satisfies the estimate
\[h^1_d(\mathbb{R}^n) \leq C_{10} d^{2mn}\]
for all \(d \geq 1\), where \(C_{10}\) is a constant.

**Proof.** It is well known that the \(k\)-th Dirichlet eigenvalue of \(B_p(r) \subset \mathbb{R}^n\) satisfies
\[\lambda_k(r) \geq Cr^{-2/k^{2/n}}\]
for all \(k\) and \(r > 0\), where \(C\) is a constant depending only on \(n\). In particular,
\[\sum_{i=1}^k \lambda_i^{-1}(\theta r) \leq C \theta^2 r^2 k^{1-(2/n)}.
\]
Lemma 1.3 yields that for any \(k\)-dimensional subspace \(V\) of \(H^1_d(\mathbb{R})\) and any \(\theta > 1\), there exists \(R > 0\) such that
\[\text{tr}_{I_{\theta r}} I_R(V) \geq k \theta^{-(2d+n+1)}.
\]
Applying Lemma 1.2, we conclude that
\[k \theta^{-(2d+n+1)} \leq \text{tr}_{I_{\theta r}} I_R(V) \leq \frac{C_8 \theta^{4m-2}}{(\theta-1)^{4m} R^2} \sum_{i=1}^k \lambda_i^{-1}(\theta R) \leq C_9 \theta^{4m} (\theta - 1)^{-4m} k^{1-(2/n)}.
\]
Choosing $\theta = 1 + d^{-1}$, we obtain $k \leq C_{10} d^{2mn}$. This shows that $h^L_d(\mathbb{R}^n) \leq C_{10} d^{2mn}$ for all $d \geq 1$.

Consider the system of partial differential equations

$$(Lu)_i \equiv \sum_{j=1}^{N} \sum_{|\alpha|=2m} a_{ij}^\alpha(x) D^\alpha u_j(x),$$

where $i = 1, \ldots, N$, and $u = (u_1, \ldots, u_N): \mathbb{R}^n \to \mathbb{R}^N$.

**Condition $L$.** The coefficient matrix $a_{ij}^\alpha(x)$ is uniformly continuous and satisfies the ellipticity condition that there exists a constant $\Lambda > 0$ such that

$$\Lambda |\Xi|^{2mn} |\eta|^2 \geq \sum_{|\alpha|=2m} \sum_{i,j=1}^{N} a_{ij}^\alpha(x) \Xi^\alpha \eta_i \eta_j \geq \Lambda^{-1} |\Xi|^{2mn} |\eta|^2$$

for all $x, \Xi \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^N$.

**Definition 1.5.** For each nonnegative number $d$ we denote by

$$\mathcal{H}^L_d(\mathbb{R}^n) = \{ u \mid Lu = 0 \text{ and } |u(x)| = O(r^d_u(x)) \}$$

the space of polynomial growth $L$-harmonic functions of degree at most $d$.

By modifying our previous argument, we have the following theorem.

**Theorem 1.6.** Assume that **Condition $L$** holds. Then the space $\mathcal{H}^L_d(\mathbb{R}^n)$ is finite-dimensional, and its dimension $h^L_d(\mathbb{R}^n)$ satisfies

$$h^L_d(\mathbb{R}^n) \leq C_{11} d^{2mn}$$

for all $d \geq 1$, where $C_{11}$ is a constant.

## 2. Equations in divergence form

In this section, we consider the differential equation

$$Lu \equiv \sum_{|\alpha|=|\beta|=m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x))$$

on $\mathbb{R}^n$. We assume that the coefficients of $L$ satisfy the following condition.

**Condition $L_d$.** The coefficients $a_{\alpha\beta}$ are measurable, bounded, and there exists a constant $\delta_0 > 0$ such that for all $u \in \mathcal{H}^m_d(\mathbb{R}^n)$,

$$\mathcal{D}(u, u) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha u(x) \, dx \geq \delta_0 \| \nabla^m u \|_2^2.$$
It is easy to see that if \( L \) satisfies the uniform ellipticity condition (meaning that \( \Lambda |\Xi|^{2m} \geq \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) \Xi^\alpha \Xi^\beta \geq \Lambda^{-1} |\Xi|^{2m} \) for some \( \Lambda > 0 \) and all \( x, \Xi \in \mathbb{R}^n \)), then Condition \( L_d \) holds.

**Lemma 2.1.** Let \( V \) be a \( k \)-dimensional subspace of \( \mathcal{H}_d^1(\mathbb{R}^n) \). For any fixed number \( \theta > 1 \), let \( \text{tr}_{I_\theta} I_r(V) \) denote the trace of the bilinear form \( I_r \) with respect to the inner product \( I_{\theta r} \) on \( V \). Then for any fixed integer \( m \), we have

\[
\text{tr}_{I_\theta} I_r(V) \leq C_{20} \sum_{i=1}^{k} \frac{\tilde{\theta}^{2m-2}}{\lambda_i(\tilde{\theta} r)(\theta - 1)^{2m-2}},
\]

where \( C_{20} \) is a constant.

**Proof.** Set \( \tilde{\theta} = \frac{1}{2}(1+\theta) \) and let \( \phi \in C^m_0(B_p(\tilde{\theta} r)) \) be a nonnegative function defined on \( B_p(\tilde{\theta} r) \) satisfying \( \phi = 1 \) on \( B_p(r) \), \( 0 \leq \phi \leq 1 \) on \( B_p(\tilde{\theta} r) \), \( \phi = 0 \) on \( \partial B_p(\tilde{\theta} r) \), and

\[
|\nabla^j \phi| \leq \frac{C}{(\theta - 1) j r^j}
\]

for some constant \( C = C(n, m) \), and \( 1 \leq j \leq m \). Observe that by unique continuation, \( V \subset H^m(B_p(\tilde{\theta} r), dx) \cap H^m(B_p(r), \phi \, dx) \) is a \( k \)-dimensional subspace. Applying Lemma 1.1, with \( w_1, \ldots, w_k \) the Dirichlet eigenfunctions of \( B_p(\tilde{\theta} r) \) corresponding to the eigenvalues \( \lambda_1(\tilde{\theta} r), \ldots, \lambda_k(\tilde{\theta} r) \), we get an orthonormal basis \( \{v_1, \ldots, v_k\} \) of \( V \) with respect to the inner product \( I_{\tilde{\theta} r} \), and

\[
\Phi_{\tilde{\theta} r}(v_i, w_j) = \int_{B_p(\tilde{\theta} r)} v_i(x) w_j(x) \phi(x) \, dx = 0
\]

for \( 0 \leq j < i \leq k \). Thus, for any \( 1 \leq i \leq k \), the variational principle implies that

\[
\lambda_i(\tilde{\theta} r) \int_{B_p(\tilde{\theta} r)} (\phi v_i)^2 \leq \int_{B_p(\tilde{\theta} r)} |\nabla(\phi v_i)|^2.
\]

Hence,

\[
(2-1) \quad \text{tr}_{I_\theta} I_r(V) = \sum_{i=1}^{k} \int_{B_p(r)} v_i^2 \leq \sum_{i=1}^{k} \frac{1}{\lambda_i(\tilde{\theta} r)} \int_{B_p(\tilde{\theta} r)} |\nabla(\phi v_i)|^2
\]

\[
\leq C \sum_{i=1}^{k} \frac{1}{\lambda_i(\tilde{\theta} r)} \int_{B_p(\tilde{\theta} r)} |\nabla^2(\phi v_i)|^2 \tilde{\theta}^2 r^2
\]

\[
\leq C^{m-1} \sum_{i=1}^{k} \frac{\tilde{\theta}^{2m-2} r^{2m-2}}{\lambda_i(\tilde{\theta} r)} \int_{B_p(\tilde{\theta} r)} |\nabla^m(\phi v_i)|^2
\]

\[
\leq C^{m-1} 2^m \sum_{i=1}^{k} \sum_{j=0}^{m} \frac{\tilde{\theta}^{2m-2} r^{2m-2}}{\lambda_i(\tilde{\theta} r)} \int_{B_p(\tilde{\theta} r)} |\nabla^j v_i|^2 \bar{\theta}^{2m-2} r^{2m-2j}.
\]
Let $\eta \in C^n_0(B_p(\theta r))$ be a nonnegative function defined on $B_p(\theta r)$ satisfying $\eta = 1$ on $B_p(\theta r)$, $0 \leq \eta \leq 1$, $\eta = 0$ on $\partial B_p(\theta r)$, and

$$|\nabla^j \eta| \leq \frac{\tilde{C}}{(\theta - 1)^{jr}}$$

for some constant $\tilde{C} = \tilde{C}(n, m)$, and $1 \leq j \leq 2m$. Note that

$$(2-2) \quad \int_{B_p(\theta r)} |\nabla^m(\eta v_1)|^2 \leq \frac{1}{\delta_0} \int_{B_p(\theta r)} a_{\alpha\beta} D^\alpha(\eta v_1) D^\beta(\eta v_1)$$

$$= \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \left( (D^\alpha(\eta v_1) - \eta D^\alpha v_1, D^\beta(\eta v_1)) + (\eta D^\alpha v_1, D^\beta(\eta v_1)) \right)$$

$$= \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \left( D^\alpha(\eta v_1) - \eta D^\alpha v_1, D^\beta(\eta v_1) \right)$$

$$+ \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \left( D^\alpha v_1, \eta D^\beta(\eta v_1) - D^\beta(\eta^2 v_1) \right),$$

where we have used the Gårding inequality from the first to the second line, and inserted the term $\mathcal{E}(v_1, \eta v_1) = 0$ into the last equality. For any $\epsilon_1 > 0$, we have

$$(2-3) \quad \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \left( D^\alpha(\eta v_1) - \eta D^\alpha v_1, D^\beta(\eta v_1) \right)$$

$$\leq \frac{C_{12}}{\delta_0} \left( \frac{1}{2\epsilon_1} \int_{B_p(\theta r)} |\nabla^m(\eta v_1) - \eta \nabla^m v_1|^2 + \frac{\epsilon_1}{2} \int_{B_p(\theta r)} |\nabla^m(\eta v_1)|^2 \right)$$

$$\leq C_{13} \left( \frac{\epsilon_1}{2} \int_{B_p(\theta r)} |\nabla^m(\eta v_1)|^2 + \frac{1}{2\epsilon_1} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_1|^2}{(\theta - 1)^{2m-2j} r^{2m-2j}} \right).$$

Also,

$$\frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \left( D^\alpha v_1, \eta D^\beta(\eta v_1) - D^\beta(\eta^2 v_1) \right)$$

$$\leq C_{14} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} |\nabla^m v_1| |\nabla^{m-j} \eta^2| |\nabla^j v_1|$$

$$\leq C_{15} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^m v_1| \eta |\nabla^j v_1|}{(\theta - 1)^{m-j} r^{m-j}} = C_{15} \int_{B_p(\theta r)} \sum_{j=0}^{m-1} \frac{|\nabla^m v_1| |\nabla^j v_1|}{(\theta - 1)^{m-j} r^{m-j}} \frac{1}{\delta_0} \int_{B_p(\theta r)} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \left( D^\alpha v_1, \eta D^\beta(\eta v_1) - D^\beta(\eta^2 v_1) \right).$$
\[
\begin{align*}
&\leq \frac{\epsilon_1}{2} \int_{B_p(\delta r)} |\eta \nabla^m v_i|^2 + \frac{C_{15}^2}{2\epsilon_1} \int_{B_p(\delta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta - 1)^{2m-2j} r^{2m-2j}} \\
&\leq \frac{\epsilon_1}{2} \int_{B_p(\delta r)} |\nabla^m (\eta v_i)|^2 + \frac{C_{16}}{2\epsilon_1} \int_{B_p(\delta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta - 1)^{2m-2j} r^{2m-2j}}.
\end{align*}
\]

Choosing \( \epsilon_1 \) to be sufficiently small and substituting this estimate and (2-3) into (2-2), we get

\[
(2-4) \int_{B_p(\delta r)} |\nabla^m v_i|^2 \leq \int_{B_p(\delta r)} |\nabla^m (\eta v_i)|^2 \leq C_{17} \int_{B_p(\delta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta - 1)^{2m-2j} r^{2m-2j}}.
\]

In terms of the weighted seminorms

\[
\Psi_k = \sup_{1 < \theta < \sigma < \theta} (\sigma - 1)^{2k} \eta^{2k} \int_{B_p(\sigma r)} |\nabla^k v_i|^2,
\]

Equation (2-4) can be written into

\[
(2-5) \quad \Psi_m \leq C_{17} \sum_{k=0}^{m-1} \Psi_k.
\]

For each \( 1 \leq k \leq 2m - 1 \), we have the interpolation inequality

\[
(2-6) \quad \Psi_k \leq \epsilon \Psi_{2m} + C(k) \epsilon^{k/(k-2m)} \Psi_0
\]

for any \( \epsilon > 0 \). Thus, by substituting (2-6) into (2-5), with a properly chosen \( \epsilon \), we get \( \Psi_m \leq C_{18} \Psi_0 \). In particular, we conclude that

\[
\int_{B_p(\delta r)} |\nabla^m v_i|^2 \leq \frac{C_{18}}{(\theta - 1)^{2m-2m} r^{2m}} \int_{B_p(\delta r)} v_i^2.
\]

Hence,

\[
\begin{align*}
&\int_{B_p(\delta r)} \sum_{j=0}^{m-1} \frac{|\nabla^j v_i|^2}{(\theta - 1)^{2m-2j} r^{2m-2j}} \\
&\leq \sum_{j=0}^{m-1} \frac{1}{(\theta - 1)^{2m-2j} r^{2m-2j}} \left( \epsilon_j \int_{B_p(\delta r)} |\nabla^m v_i|^2 + C(j) \epsilon_j^{j/(j-m)} \int_{B_p(\delta r)} v_i^2 \right) \\
&\leq m \delta^2 \int_{B_p(\delta r)} |\nabla^m v_i|^2 + \frac{C(m)}{(\theta - 1)^{2m} r^{2m}} \int_{B_p(\delta r)} v_i^2 \\
&\leq \frac{C_{19}}{(\theta - 1)^{2m} r^{2m}} \int_{B_p(\delta r)} v_i^2,
\end{align*}
\]

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where $\epsilon_j = (\theta - 1)^{2m-2j}r^{2m-2j}$. Substituting this inequality into (2-1), we get

$$\text{tr}_{I_{\theta r}} I_{\theta r} (V) \leq C_{p}^{m-1} 2^{m} \sum_{i=1}^{k} \sum_{j=0}^{m} \frac{\theta^{2m-2j}r^{2m-2j}}{\lambda_j(\theta r)} \int_{B_{p}(\theta r)} \frac{|\nabla j v_i|^2}{(\theta - 1)^{2m-2j}r^{2m-2j}} \leq C_{20} \sum_{i=1}^{k} \frac{\theta^{2m-2j}}{\lambda_i(\theta r)(\theta - 1)^{2m}r^{2m}}.$$ 

Now the next theorem may be proved in a similar fashion to Theorem 1.4 by using Lemmas 1.3 and 2.1.

**Theorem 2.2.** Assume that Condition $L_d$ holds. Then the space $\mathcal{H}^{L_d}(\mathbb{R}^n)$ is finite-dimensional, and its dimension $h^{L_d}(\mathbb{R}^n)$ satisfies the estimate

$$h^{L_d}(\mathbb{R}^n) \leq C d^{mn}$$

for all $d \geq 1$.

Theorem 2.2 can be generalized to the case of systems of partial differential equations. More specifically, for the system

$$(\mathcal{L}u)_i \equiv \sum_{|\alpha|=m} (-1)^m D^\alpha \left( \sum_{|\beta|=m} a_{ij}^{\alpha \beta}(x) D^\beta u_j(x) \right),$$

where $1 \leq i, j \leq N$ and $u = (u_1, \ldots, u_N) : \mathbb{R}^n \to \mathbb{R}^N$, assume that the coefficients of $\mathcal{L}$ satisfy the following condition.

**Condition $\mathcal{L}_d$.** The coefficient matrix $(a_{ij}^{\alpha \beta}(x))$ is measurable, bounded, and there exists a constant $\delta_0 > 0$ such that for all $u \in \mathcal{H}^{m}_0(\mathbb{R}^n, \mathbb{R}^N)$,

$$\int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} \sum_{i, j=1}^{N} a_{ij}^{\alpha \beta}(x) D^\beta u_i(x) D^\alpha u_j(x) \, dx \geq \delta_0 \sum_{j=1}^{N} \|\nabla^m u_j\|^2_2.$$ 

It is easy to see that if $L$ satisfies the uniform ellipticity condition, that is, there exists a constant $\Lambda > 0$ such that

$$\Lambda |\mathbb{E}|^{2mN} |\eta|^2 \geq \sum_{|\alpha|=|\beta|=m} \sum_{i, j=1}^{N} a_{ij}^{\alpha \beta}(x) \mathbb{E}^\alpha \mathbb{E}^\beta \eta_i \eta_j \geq \Lambda^{-1} |\mathbb{E}|^{2mN} |\eta|^2$$

for all $x, \mathbb{E} \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^N$, then Condition $\mathcal{L}_d$ holds.
Theorem 2.3. Assume that Condition $\mathcal{L}_f$ holds. Then the space $\mathcal{H}^f_\vartheta(\mathbb{R}^n)$ is finite dimensional, and its dimension $h^f_\vartheta(\mathbb{R}^n)$ satisfies the estimate
$$h^f_\vartheta(\mathbb{R}^n) \leq C d^{mnN}$$
for all $d \geq 1$.

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References


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