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TWISTED MODULES FOR VERTEX ALGEBRAS ASSOCIATED WITH VERTEX ALGEBROIDS

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We continue the work in our earlier paper, "On certain vertex algebras and their modules associated with vertex algebroids", *J. Algebra* 283 (2005), 367–398, to construct and classify graded simple twisted modules for the $\mathbb{N}$-graded vertex algebras constructed by Gorbounov, Malikov and Schechtman from vertex algebroids. In addition, we determine the full automorphism groups of those $\mathbb{N}$-graded vertex algebras in terms of the automorphism groups of the corresponding vertex algebroids.

1. Introduction

For most of the important examples of vertex operator algebras $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ graded by the $L(0)$-weight [Frenkel et al. 1988], the $\mathbb{Z}$-grading satisfies the condition $V_{(n)} = 0$ for $n < 0$ and $V_{(0)} = \mathbb{C}\mathbf{1}$, where $\mathbf{1}$ is the vacuum vector. For a vertex operator algebra V with this special property, the homogeneous subspace $V_{(1)}$ has a natural Lie algebra structure with $[u, v] = u_0 v$ for $u, v \in V_{(1)}$ and the product $u_1 v$ ($\in V_{(0)}$) defines a symmetric invariant bilinear form on $V_{(1)}$.

A series of articles on gerbes of chiral differential operators [Gorbounov et al. 2004] and on chiral de Rham complexes [Malikov et al. 1998; 1999] have investigated $\mathbb{N}$-graded vertex algebras $V = \coprod_{n\in\mathbb{N}} V_{(n)}$ with $V_{(0)}$ not necessarily 1-dimensional. In this case, the bilinear operations $(u, v) \mapsto u_i v$ for $i \geq 0$ are closed on $V_{(0)} \oplus V_{(1)}$:

$$u_i v \in V_{(0)} \oplus V_{(1)} \quad \text{for } u, v \in V_{(0)} \oplus V_{(1)},\ i \geq 0.$$

The skew symmetry and the Jacobi identity for the vertex algebra V give rise to several compatibility relations. Such algebraic structures on $V_{(0)} \oplus V_{(1)}$ are summarized in the notion of what became known as a 1-truncated conformal algebra. The subspace $V_{(0)}$ equipped with the product $(a, b) \mapsto a_{-1} b$ is a commutative associative algebra with the vacuum vector $\mathbf{1}$ as the identity and $V_{(0)}$ as a nonassociative algebra acts on $V_{(1)}$ by $a \cdot u = a_{-1} u$ for $a \in V_{(0)}$, $u \in V_{(1)}$. All these structures on

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$V_{(0)} \oplus V_{(1)}$ are further summarized in the notion known as a vertex A-algebroid, where A is a (unital) commutative associative algebra.

Gorbounov, Malikov and Schechtman also constructed in [Gorbounov et al. 2004], starting from any vertex A-algebroid, an $\mathbb{N}$-graded vertex algebra

$$V = \coprod_{n \in \mathbb{N}} V_{(n)}$$

such that $V_{(0)} = A$ and the vertex A-algebroid $V_{(1)}$ is isomorphic to the given one. All the $\mathbb{N}$-graded vertex algebras constructed are generated by $V_{(0)} \oplus V_{(1)}$ with a spanning property of PBW type. That work demonstrated that such $\mathbb{N}$-graded vertex algebras are natural and important to study. For example, the vertex (operator) algebra associated with a $\beta\gamma$ system, which plays a central role in free field realization of affine Lie algebras (see [Wakimoto 1986; Feĭgin and Frenkel 1988; 1990a; 1990b, Frenkel and Ben-Zvi 2001]), is such an $\mathbb{N}$-graded vertex algebra. The vertex (operator) algebras constructed from toroidal Lie algebras are also of this type [Berman et al. 2002a; 2002b].

In [Li and Yamskulna 2005], we revisited those $\mathbb{N}$-graded vertex algebras and we classified all the $\mathbb{N}$-graded simple modules in terms of simple modules for certain Lie algebroids. In the theory of vertex algebras, in addition to the notion of module we have the notion of twisted module and twisted modules play a very important role, especially in the study of the so-called orbifold theory. Certainly, twisted modules also play an important role in other studies. In this paper, we continue to study the twisted modules for the $\mathbb{N}$-graded vertex algebras associated with vertex algebroids.

Let B be a vertex A-algebroid and let V_B be the associated $\mathbb{N}$-graded vertex algebra. In this paper, we define a notion of automorphism of the vertex A-algebroid B and we prove that any automorphism of the vertex A-algebroid B can be extended uniquely to an automorphism of the $\mathbb{N}$-graded vertex algebra V_B and that the full automorphism group of the $\mathbb{N}$-graded vertex algebra V_B is naturally isomorphic to the full automorphism group of the vertex A-algebroid B. Let g be an automorphism of the vertex A-algebroid B of order T (finite). Then the g-fixed point A^0 is a subalgebra of A and the g-fixed point B^0 is a vertex A^0-algebroid. Furthermore, $B^0/A^0\partial A^0$ is a Lie A^0-algebroid. It is proved that the category of $\frac{1}{T}\mathbb{N}$-graded simple g-twisted V_B-modules is equivalent to a subcategory of simple modules for the Lie A^0-algebroid $B^0/A^0\partial A^0$.

This paper is organized as follows: In the next section we review the construction of vertex algebras associated with vertex algebroids and we identify their automorphism groups with the automorphism groups of the vertex algebroids. In Section 3 we classify graded simple twisted modules.

2. Preliminaries

We recall the notions of 1-truncated conformal algebra, vertex algebroid and Lie algebroid, and we review the construction of the $\mathbb{N}$-graded vertex algebra V_B associated with a vertex A-algebroid B. We also define notions of (endomorphism) automorphism of a 1-truncated conformal algebra and of a vertex A-algebroid B. We then identify the group of grading-preserving automorphisms of V_B with the group of automorphisms of the vertex A-algebroid B.

First, we recall from [Gorbounov et al. 2004] (compare [Bressler 2002; 2003]) the notions of 1-truncated conformal algebra, vertex algebroid and Lie algebroid.

Definition 2.1. A 1-*truncated conformal algebra* is a graded vector space $C = C_0 \oplus C_1$ equipped with a linear map $\partial : C_0 \to C_1$ and bilinear operations $(u, v) \mapsto u_i v$ for $i = 0, 1$ of degree $-i-1$ on C such that the following axioms hold:

1. (Derivation) for $a \in C_0$, $u \in C_1$,
$$(\partial a)_0 = 0; \quad (\partial a)_1 = -a_0; \quad \partial(u_0 a) = u_0 \partial a$$
2. (Commutativity) for $a \in C_0$, $u, v \in C_1$,
$$u_0 a = -a_0 u; \quad u_0 v = -v_0 u + \partial(v_1 u); \quad u_1 v = v_1 u$$
3. (Associativity) for $\alpha, \beta, \gamma \in C_0 \oplus C_1$,
$$\alpha_0 \beta_i \gamma = \beta_i \alpha_0 \gamma + (\alpha_0 \beta)_i \gamma.$$

Remark 2.2. Let $C = C_0 \oplus C_1$ be a 1-truncated conformal algebra and let ℓ be any nonzero complex number. Set $C[\ell] = C_0 \oplus C_1$ as a vector space. We retain all the structures on C except that we change the bilinear operation $C_1 \times C_1 \to C_0 : u \times v \mapsto u_1 v$ by multiplying $1/\ell$ and change the linear operator ∂ by multiplying ℓ. Then one can show that $C[\ell]$ is a 1-truncated conformal algebra.

Definition 2.3. Let A be a unital commutative associative algebra over $\mathbb{C}$. A *vertex A-algebroid* is a $\mathbb{C}$-vector space Γ equipped with

1. a $\mathbb{C}$-bilinear map
$$A \times \Gamma \to \Gamma; \quad (a, v) \mapsto a * v$$
such that $1 * v = v$ for $v \in \Gamma$,
2. a Leibniz $\mathbb{C}$-algebra structure $[\cdot, \cdot] : \Gamma \otimes_{\mathbb{C}} \Gamma \to \Gamma$,
3. a homomorphism of Leibniz $\mathbb{C}$-algebras $\pi : \Gamma \to \mathrm{Der}(A)$,
4. a symmetric $\mathbb{C}$-bilinear pairing $\langle \cdot, \cdot \rangle : \Gamma \otimes_{\mathbb{C}} \Gamma \to A$.
5. a $\mathbb{C}$-linear map $\partial : A \to \Gamma$ such that $\pi \circ \partial = 0$.

All the following conditions are assumed to hold, for $a, a' \in A$, $u, v, v_1, v_2 \in \Gamma$:

$$
\begin{aligned}
a * (a' * v) - (aa') * v &= \pi(v)(a) * \partial(a') + \pi(v)(a') * \partial(a),\\
[u, a * v] &= \pi(u)(a) * v + a * [u, v],\\
[u, v] + [v, u] &= \partial(\langle u, v\rangle),\\
\pi(a * v) &= a\pi(v),\\
\langle a * u, v\rangle &= a\langle u, v\rangle - \pi(u)(\pi(v)(a)),\\
\pi(v)(\langle v_1, v_2\rangle) &= \langle [v, v_1], v_2\rangle + \langle v_1, [v, v_2]\rangle,\\
\partial(aa') &= a * \partial(a') + a' * \partial(a),\\
[v, \partial(a)] &= \partial(\pi(v)(a)),\\
\langle v, \partial(a)\rangle &= \pi(v)(a),
\end{aligned}
$$

Proposition 2.4 [Li and Yamskulna 2005]. *Let A be a unital commutative associative algebra and let B be a module for A as a nonassociative algebra. Then a vertex A-algebroid structure on B is equivalent to a* 1*-truncated conformal algebra structure on $C = A \oplus B$ with*

$$
\begin{aligned}
a_i a' &= 0,\\
u_0 v &= [u, v],\\
u_1 v &= \langle u, v\rangle,\\
u_0 a &= \pi(u)(a),\\
a_0 u &= -u_0 a = -\pi(u)(a),
\end{aligned}
$$

for $a, a' \in A,\ u, v \in B,\ i = 0, 1,$ *such that*

$$
\begin{aligned}
a(a'u) - (aa')u &= (u_0 a)\partial a' + (u_0 a')\partial a,\\
u_0(av) - a(u_0 v) &= (u_0 a)v,\\
u_0(aa') &= a(u_0 a') + (u_0 a)a',\\
a_0(a'v) &= a'(a_0 v),\\
(au)_1 v &= a(u_1 v) - u_0 v_0 a,\\
\partial(aa') &= a\partial(a') + a'\partial(a).
\end{aligned}
$$

Definition 2.5. Let A be a unital commutative associative algebra. A *Lie A-algebroid* is a Lie algebra $\mathfrak{g}$ equipped with an A-module structure and a module action on A by derivation such that

$$
\begin{aligned}
[u, av] &= a[u, v] + (ua)v,\\
a(ub) &= (au)b
\end{aligned}
$$

for $u, v \in \mathfrak{g}$ and $a, b \in A$. A *module* for a Lie A-algebroid $\mathfrak{g}$ is a vector space W equipped with a $\mathfrak{g}$-module structure and an A-module structure such that

$$u(aw) - a(uw) = (ua)w,$$
$$a(uw) = (au)w$$

for $a \in A$, $u \in \mathfrak{g}$, $w \in W$.

Lemma 2.6 [Bressler 2003]. *Let A be a unital commutative associative algebra (over $\mathbb{C}$) and let B be a vertex A-algebroid. Then $B/A\partial A$ is naturally a Lie A-algebroid.*

Next, we recall the construction of vertex algebras associated with vertex algebroids, following the exposition of [Li and Yamskulna 2005].

First, starting with a 1-truncated conformal algebra $C = A \oplus B$ we construct a Lie algebra. Set

$$L(A \oplus B) = (A \oplus B) \otimes \mathbb{C}[t, t^{-1}].$$

In the obvious way we define the subpaces $L(A)$ and $L(B)$. Set

$$\hat{\partial} = \partial \otimes 1 + 1 \otimes d/dt : \ L(A) \to L(A \oplus B).$$

We define

$$\deg(a \otimes t^n) = -n - 1 \quad \text{for } a \in A, \ n \in \mathbb{Z},$$
$$\deg(b \otimes t^n) = -n \qquad \text{for } b \in B, \ n \in \mathbb{Z},$$

making $L(A \oplus B)$ a $\mathbb{Z}$-graded vector space. The linear map $\hat{\partial}$ is homogeneous of degree 1. Set

$$\mathcal{L} = L(A \oplus B)/\hat{\partial} L(A).$$

Define a bilinear product $[\,\cdot\,,\cdot\,]$ on $L(A \oplus B)$ such that for $a, a' \in A$, $b, b' \in B$, $m, n \in \mathbb{Z}$,

$$[a \otimes t^m, a' \otimes t^n] = 0, \tag{2-1}$$
$$[a \otimes t^m, b \otimes t^n] = a_0 b \otimes t^{m+n}, \tag{2-2}$$
$$[b \otimes t^n, a \otimes t^m] = b_0 a \otimes t^{m+n}, \tag{2-3}$$
$$[b \otimes t^m, b' \otimes t^n] = b_0 b' \otimes t^{m+n} + m(b_1 b') \otimes t^{m+n-1}. \tag{2-4}$$

Proposition 2.7 [Li and Yamskulna 2005]. *Let $C = A \oplus B$ be a 1-truncated conformal algebra. The subspace $\hat{\partial} L(A)$ of the nonassociative algebra $\big(L(A \oplus B), [\,\cdot\,,\cdot\,]\big)$ is a two-sided ideal. The quotient nonassociative algebra $\mathcal{L}$ is a $\mathbb{Z}$-graded Lie algebra.*

Let ρ be the projection map from $L(A \oplus B)$ to $\mathscr{L}$. For $u \in A \oplus B$ and $n \in \mathbb{Z}$, set

$$u(n) = \rho(u \otimes t^n) = u \otimes t^n + \hat{\partial} L(A) \in \mathscr{L}.$$

We have graded Lie subalgebras

$$\mathscr{L}^{\geq 0} = \rho((A \oplus B) \otimes \mathbb{C}[t]),$$
$$\mathscr{L}^{<0} = \rho((A \oplus B) \otimes t^{-1}\mathbb{C}[t^{-1}])$$

and we have $\mathscr{L} = \mathscr{L}^{\geq 0} \oplus \mathscr{L}^{<0}$ as a vector space.

Considering $\mathbb{C}$ as a trivial $\mathscr{L}^{\geq 0}$-module we form the induced module

$$V_{\mathscr{L}} = U(\mathscr{L}) \otimes_{U(\mathscr{L}^{\geq 0})} \mathbb{C}.$$

We assign $\deg \mathbb{C} = 0$, making $V_{\mathscr{L}}$ naturally an $\mathbb{N}$-graded $\mathscr{L}$-module:

$$V_{\mathscr{L}} = \coprod_{n \in \mathbb{N}} (V_{\mathscr{L}})_{(n)}. \tag{2-5}$$

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers. Set

$$\mathbf{1} = 1 \otimes 1 \in V_{\mathscr{L}}.$$

By the P-B-W theorem, we have $V_{\mathscr{L}} = U(\mathscr{L}^{<0}) = S(\mathscr{L}^{<0})$. In view of this, we can and we do consider $A \oplus B$ as a subspace:

$$A \oplus B \to V_{\mathscr{L}}; \quad u \mapsto u(-1)\mathbf{1}.$$

Theorem 2.8 ([Li and Yamskulna 2005]; compare [Dong et al. 2002]). *There exists a unique vertex algebra structure on $V_{\mathscr{L}}$ with $\mathbf{1}$ as the vacuum vector and with $Y(u, x) = \sum_{n \in \mathbb{Z}} u(n)x^{-n-1}$ for $u \in A \oplus B$. Moreover, the vertex algebra $V_{\mathscr{L}}$ is naturally an $\mathbb{N}$-graded vertex algebra and is generated by the subspace $A \oplus B$ with A of degree* 0 *and B of degree* 1.

Remark 2.9. For $n \in \mathbb{Z}$, set

$$A(n) = \{a(n) \mid a \in A\}, \quad B(n) = \{b(n) \mid b \in B\} \subset \mathscr{L},$$

and we set

$$B(-) = \coprod_{n=1}^{\infty} B(-n) \subset \mathscr{L}.$$

Both $A(-1)$ and $B(-)$ are Lie subalgebras of $\mathscr{L}^{<0}$ and we have $\mathscr{L}^{<0} = A(-1) \oplus B(-)$ as a vector space. Then

$$V_{\mathscr{L}} = U(\mathscr{L}^{<0}) = S(\mathscr{L}^{<0}) = S(A(-1) \oplus B(-)) = S(B(-)) \otimes S(A(-1)).$$

Consequently, $(V_{\mathscr{L}})_{(n)} = S(B(-))_{(n)} \otimes S(A(-1))$ for $n \in \mathbb{N}$. In particular, $(V_{\mathscr{L}})_{(0)} = S(A(-1))$.

Now, we assume that A is a unital commutative associative algebra with identity e and that B is a vertex A-algebroid. Thus $C = A \oplus B$ is a 1-truncated conformal algebra. We set

$$E = \mathrm{span}\{e - \mathbf{1},\ a(-1)a' - aa',\ a(-1)b - ab \mid a, a' \in A,\ b \in B\} \subset V_{\mathscr{L}},$$
$$I_B = U(\mathscr{L})\mathbb{C}[\mathscr{D}]E.$$

It was proved in [Li and Yamskulna 2005] that the $\mathscr{L}$-submodule I_B of $V_{\mathscr{L}}$ is a two-sided graded ideal of the $\mathbb{N}$-graded vertex algebra $V_{\mathscr{L}}$. The $\mathbb{N}$-graded vertex algebra V_B associated with the vertex A-algebroid B is defined to be the quotient vertex algebra

$$V_B = V_{\mathscr{L}}/I_B. \tag{2-6}$$

Proposition 2.10 [Gorbounov et al. 2004; Li and Yamskulna 2005]. *Let A be a unital commutative associative algebra with identity e and let B be a vertex A-algebroid. Then V_B is an $\mathbb{N}$-graded vertex algebra such that $(V_B)_{(0)} = A$, $(V_B)_{(1)} = B$ and for $n \geq 1$,*

$$(V_B)_{(n)} = \mathrm{span}\{b_1(-n_1)\cdots b_k(-n_k)\mathbf{1} \mid b_i \in B,\ n_1 \geq n_2 \geq \cdots \geq n_k \geq 1,\ n_1 + \cdots + n_k = n\}.$$

In particular, V_B is generated by the subspace $A \oplus B$.

Next, we discuss homomorphisms and automorphisms for 1-truncated conformal algebras, vertex A-algebroids and for the $\mathbb{N}$-graded vertex algebras V_B.

Definition 2.11. Let $C = A \oplus B$ and $C' = A' \oplus B'$ be 1-truncated conformal algebras. A *homomorphism* from C to C' is a linear map $f : C \to C'$ such that $f(A) \subset A'$, $f(B) \subset B'$, $f\partial = \partial f$, and such that

$$f(u_i v) = f(u)_i f(v)$$

for $u, v \in C$ and $i = 0, 1$.

Lemma 2.12. *Let f be an endomorphism of a 1-truncated conformal algebra $C = A \oplus B$. Then the linear endomorphism of $L(A \oplus B)$ defined by*

$$\hat{f}(u \otimes t^n) = f(u) \otimes t^n \tag{2-7}$$

for $u \in A \oplus B$ and $n \in \mathbb{Z}$ gives rise to an endomorphism of $\mathscr{L}$, which we denote by $\hat{f}$ again. Furthermore, $\hat{f}$ preserves the $\mathbb{Z}$-grading of $\mathscr{L}$.

Proof. Using the property that $f\partial = \partial f$, we have $\hat{f}\hat{\partial} = \hat{\partial}\hat{f}$. For $u, v \in C = A \oplus B$, as $f(u_i v) = f(u)_i f(v)$ for $i = 0, 1$, from (2-1)–(2-4) we have

$$\hat{f}\big([u \otimes t^m, v \otimes t^n]\big) = [f(u) \otimes t^m, f(v) \otimes t^n] = [\hat{f}(u \otimes t^m), \hat{f}(v \otimes t^n)].$$

Thus $\hat{f}$ gives rise to an endomorphism of the Lie algebra $\mathscr{L}$. It is clear that $\hat{f}$ preserves the $\mathbb{Z}$-grading. □

Definition 2.13. Let A and A' be unital commutative associative algebras and let B be a vertex A-algebroid, B' a vertex A'-algebroid. A *vertex algebroid homomorphism* from B to B' is a linear map $f : A \oplus B \to A' \oplus B'$ such that $f(A) \subset A'$, and $f(B) \subset B'$ and such that

1. $f|_A$ is an associative algebra homomorphism.
2. $f|_B$ is a Leibniz algebra homomorphism.
3. $f(ab) = f(a)f(b)$ for $a \in A$ and $b \in B$.
4. $\langle f(u), f(v)\rangle = f(\langle u, v\rangle)$ for $u, v \in B$.
5. $f \circ \partial = \partial \circ f$.
6. $f(b_0 a) = f(b)_0 f(a)$ for $a \in A$ and $b \in B$.

An *automorphism of a vertex A-algebroid B* is a bijective vertex algebroid endomorphism of the vertex A-algebroid B.

Let $(V, Y, \mathbf{1})$ be a vertex algebra. An *endomorphism* of V is a linear map $g : V \to V$ such that

$$g(\mathbf{1}) = \mathbf{1}, \tag{2-8}$$

$$g(Y(u, x)v) = Y(g(u), x)g(v) \tag{2-9}$$

for $u, v \in V$. An *automorphism* of V is a bijective endomorphism of V. The group of automorphisms of V is denoted by $\mathrm{Aut}(V)$. If $V = \coprod_{m\in\mathbb{Z}} V_{(m)}$ is a $\mathbb{Z}$ (or $\mathbb{N}$)-graded vertex algebra, we denote by $\mathrm{Aut}^0(V)$ the group of grading-preserving automorphisms of V.

Lemma 2.14. *Let B be a vertex A-algebroid and let g be a grading-preserving automorphism of the vertex algebra V_B. Then g restricted to $A \oplus B$ is an automorphism of the vertex A-algebroid B.*

Proof. As $(V_B)_{(0)} = A$ and $(V_B)_{(1)} = B$, g is a linear bijection on $A \oplus B$ that preserves the subspaces A and B. For $a, a' \in A$ and $b, b' \in B$, we have

$$\begin{aligned}
&g(aa') = g(a(-1)a') = g(a)_{-1}g(a') = g(a)g(a'),\\
&g(ab) = g(a(-1)b) = g(a)_{-1}g(b) = g(a)g(b),\\
&g([b, b']) = g(b_0 b') = g(b)_0 g(b') = [g(b), g(b')],\\
&g(\langle b, b'\rangle) = g(b_1 b') = g(b)_1 g(b') = \langle g(b), g(b')\rangle,\\
&g(b_0 a) = g(b)_0 g(a),\\
&g(\partial(a)) = g(a(-2)\mathbf{1}) = g(a)_{-2}\mathbf{1} = \partial(g(a)).
\end{aligned}$$

Thus g is an automorphism of vertex A-algebroid B. □

On the other hand, we are going to prove that any automorphism of a vertex A-algebroid B extends uniquely to an automorphism of the $\mathbb{N}$-graded vertex algebra V_B. First we have:

Lemma 2.15. *Let $C = A \oplus B$ be a* 1*-truncated conformal algebra and let g be an endomorphism of C. Then g extends uniquely to an endomorphism of the $\mathbb{N}$-graded vertex algebra $V_{\mathscr{L}}$. Furthermore, if g is an automorphism, then the extension is an automorphism.*

Proof. Since $A \oplus B$ generates $V_{\mathscr{L}}$ as a vertex algebra, the uniqueness is clear. It remains to prove the existence. By Lemma 2.12, we have a grading-preserving endomorphism $\hat{g}$ of the Lie algebra $\mathscr{L}$, hence a grading-preserving endomorphism of the universal enveloping algebra $U(\mathscr{L})$. Clearly, $\hat{g}$ preserves the Lie subalgebra $\mathscr{L}^{<0}$ and its universal enveloping algebra $U(\mathscr{L}^{<0})$. It follows from the construction of $V_{\mathscr{L}}$ that there exists a linear endomorphism $\bar{g}$ of $V_{\mathscr{L}}$ such that $\bar{g}(\mathbf{1}) = \mathbf{1}$ and

$$\bar{g}(u_n v) = g(u)_n \bar{g}(v)$$

for $u \in A \oplus B$, $v \in V_{\mathscr{L}}$, $n \in \mathbb{Z}$. Since $V_{\mathscr{L}}$ is generated by $A \oplus B$, it follows form [Lepowsky and Li 2004] that $\bar{g}$ is an endomorphism of $V_{\mathscr{L}}$. Clearly $\bar{g}$ extends g.

If g is an automorphism of the 1-truncated conformal algebra $C = A \oplus B$, from the first assertion we have vertex algebra endomorphisms $\bar{g}$ and $\overline{g^{-1}}$ of $V_{\mathscr{L}}$, extending g and g^{-1}, respectively. Since $gg^{-1} = g^{-1}g = 1$ on $A \oplus B$ and since $A \oplus B$ generates $V_{\mathscr{L}}$ as a vertex algebra, we have $\bar{g}\overline{g^{-1}} = \overline{g^{-1}}\bar{g} = 1$. Thus, $\bar{g}$ is an automorphism of $V_{\mathscr{L}}$. □

Proposition 2.16. *Let g be an endomorphism of a vertex A-algebroid B. Then g extends uniquely to an endomorphism of V_B as an $\mathbb{N}$-graded vertex algebra. Furthermore, if g is an automorphism, then the extension is an automorphism.*

Proof. The uniqueness is clear, as $A \oplus B$ generates V_B as a vertex algebra. For the existence, first by Lemma 2.15, we have a grading-preserving endomorphism $\bar{g}$ of the vertex algebra $V_{\mathscr{L}}$, extending g. Now we show that $\bar{g}$ reduces to an endomorphism of V_B. Recall that $V_B = V_{\mathscr{L}}/I_B$, where I_B is the two-sided ideal of $V_{\mathscr{L}}$, generated by

$$E = \operatorname{span}\{e - \mathbf{1},\ a(-1)a' - aa',\ a(-1)b - ab \mid a, a' \in A,\ b \in B\}.$$

Now, we must prove $\bar{g}(I_B) \subset I_B$. As E generates I_B as a two-sided ideal, it suffices to prove that $\bar{g}(E) \subset E$. Let $a, a' \in A,\ b \in B$. We have

$$\begin{aligned}
&\bar{g}(e - \mathbf{1}) = e - \mathbf{1} \in E,\\
&\bar{g}(a(-1)a' - aa') = g(a)(-1)g(a') - g(a)g(a') \in E,\\
&\bar{g}(a(-1)b - ab) = g(a)(-1)g(b) - g(a)g(b) \in E.
\end{aligned}$$

This proves $\bar{g}(E) \subset E$. Therefore, $\bar{g}$ reduces to an endomorphism of the $\mathbb{N}$-graded vertex algebra V_B. The second assertion follows immediately from the proof of the second assertion of Lemma 2.15. □

Combining Lemma 2.14 with Proposition 2.16, we have:

Theorem 2.17. *Let A be a unital commutative associative algebra and let B be a vertex A-algebroid. The group* $\mathrm{Aut}^0(V_B)$ *of grading-preserving automorphisms of the $\mathbb{N}$-graded vertex algebra V_B is isomorphic to the group of automorphisms of vertex A-algebroid B with the restriction map as an isomorphism.*

3. Classification of graded simple twisted V_B-modules

In this section we construct and classify graded simple twisted V_B-modules by exploiting a twisted analogue of the Lie algebra $\mathscr{L}$. First, we recall the definition of the notion of twisted module for a vertex algebra and we discuss several properties of twisted modules.

Let V be a vertex algebra and let g be an automorphism of V of order $T < \infty$. Decompose V into eigenspaces of g:

$$V = \coprod_{r=0}^{T-1} V^r, \quad \text{where } V^r = \{v \in V \mid g(v) = e^{2r\pi\sqrt{-1}/T} v\}.$$

A *g-twisted V-module* [Lepowsky 1985; Frenkel et al. 1988; Feingold et al. 1991; Dong 1994] is a vector space M equipped with a linear map

$$Y_M : V \to (\mathrm{End}\, M)[[x^{1/T}, x^{-1/T}]],$$
$$u \mapsto Y_M(u, x) = \sum_{n \in (1/T)\mathbb{Z}} u_n x^{-n-1}$$

satisfying the following conditions:

1. For $u \in V$, $w \in M$, we have $u_n w = 0$ for $n \in \frac{1}{T}\mathbb{Z}$ sufficiently large.
2. $Y_M(\mathbf{1}, x) = 1_M$ (the identity operator on M).
3. For $u \in V^r$ with $0 \le r \le T-1$,

$$Y_M(u, x) = \sum_{n \in (r/T)+\mathbb{Z}} u_n x^{-n-1} \in x^{-r/T} (\mathrm{End}\, M)[[x, x^{-1}]]. \tag{3-1}$$

4. For $u \in V^r$ with $0 \le r \le T-1$, $v \in V$,

$$x_0^{-1}\delta\Big(\frac{x_1-x_2}{x_0}\Big) Y_M(u, x_1) Y_M(v, x_2) - x_0^{-1}\delta\Big(\frac{x_2-x_1}{-x_0}\Big) Y_M(v, x_2) Y_M(u, x_1) = x_2^{-1}\Big(\frac{x_1-x_0}{x_2}\Big)^{-r/T} \delta\Big(\frac{x_1-x_0}{x_2}\Big) Y_M(Y(u, x_0)v, x_2) \tag{3-2}$$

(the *twisted Jacobi identity*).

Remark 3.1. Let (M, Y_M) be a g-twisted V-module and let U be any vertex subalgebra of V^0. Then M is a U-module. Thus, if g is taken to be the identity map, the notion of g-twisted V-module reduces to that of V-module while the twisted Jacobi identity reduces to the ordinary (untwisted) Jacobi identity.

Lemma 3.2 ([Dong et al. 1998]; compare [Dong et al. 1997]). *Let (M, Y_M) be a g-twisted V-module. Then*

$$Y_M(\mathcal{D}v, x) = \frac{d}{dx} Y_M(v, x) \tag{3-3}$$

for $v \in V$, where $\mathcal{D}v = v_{-2}\mathbf{1}$.

Remark 3.3. For $v \in V$, $u \in V^r$, $p \in \mathbb{Z}$ and $s, t \in \mathbb{Q}$, extracting the coefficients of $z_0^{-p-1} z_1^{-s-1} z_2^{-t-1}$ from the twisted Jacobi identity (3-2) we get

$$\begin{aligned} \sum_{m\geq 0} \binom{s}{m} (u_{p+m}v)_{s+t-m} \\ = \sum_{m\geq 0} (-1)^m \binom{p}{m} \{u_{p+s-m}v_{t+m} - (-1)^p v_{p+t-m}u_{s+m}\}. \end{aligned} \tag{3-4}$$

By taking Res_{x_0} of (3-2), we obtain the *twisted commutator formula*:

$$\begin{aligned} [Y_M(u, x_1), Y_M(v, x_2)] \\ = \operatorname{Res}_{x_0} x_2^{-1} \left(\frac{x_1 - x_0}{x_2}\right)^{-r/T} \delta\left(\frac{x_1 - x_0}{x_2}\right) Y_M(Y(u, x_0)v, x_2). \end{aligned} \tag{3-5}$$

Multiplying (3-2) by $\left(\frac{x_1-x_0}{x_2}\right)^{r/T}$ and then taking Res_{x_1}, we obtain the *twisted iterate formula*:

$$Y_M(Y(u, x_0)v, x_2) = \operatorname{Res}_{x_1} \left(\frac{x_1 - x_0}{x_2}\right)^{r/T} \cdot X,$$

where

$$X = x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y_M(u, x_1)Y_M(v, x_2) - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y_M(v, x_2)Y_M(u, x_1).$$

From the twisted Jacobi identity one has the following *twisted weak associativity*: For $u \in V^r$ with $0 \leq r \leq T - 1$ and for $v \in V$, $w \in W$,

$$(x_0 + x_2)^{k+(r/T)} Y_M(u, x_0 + x_2)Y_M(v, x_2)w = (x_2 + x_0)^{k+(r/T)} Y_M(Y(u, x_0)v, x_2)w$$

where k is a nonnegative integer such that $x^{k+(r/T)} Y_M(u, x)w \in M[[x]]$. One can prove, as in [Li 1996, Lemma 2.8] that the twisted Jacobi identity is equivalent to the twisted commutator formula and the twisted weak associativity.

Let M be a g-twisted V-module. For a subset U of M, denote the smallest g-twisted V-submodule containing U by $\langle U\rangle$, which is called the g-twisted V-submodule generated by U. Just as with untwisted modules, from the twisted weak associativity, we have

$$\langle U\rangle = \mathrm{span}\{v_n w \mid v \in V,\ n \in \tfrac{1}{T}\mathbb{Z},\ w \in U\}.$$

We define

$$\mathrm{Ann}_V(U) = \{v \in V \mid Y(v, x)w = 0 \quad \text{for } w \in U\}, \tag{3-6}$$

the *annihilator of U in V*.

Proposition 3.4. *For any subset U of a g-twisted V-module M, the annihilator* $\mathrm{Ann}_V(U)$ *is an ideal of V. Moreover,*

$$\mathrm{Ann}_V(U) = \mathrm{Ann}_V(\langle U\rangle).$$

Proof. This follows from the proof of [Lepowsky and Li 2004, Proposition 4.5.11], with weak associativity and weak commutativity replaced by twisted associativity and twisted commutativity. □

Let S be a subset of V. Define

$$\mathrm{Ann}_M(S) = \{w \in M \mid Y_M(v, x)w = 0 \quad \text{for } v \in S\},$$

the *annihilator of S in M*. We follow [Lepowsky and Li 2004]. By suitably modifying the proof of Proposition 4.5.14 of that reference, replacing weak commutativity by twisted commutativity and replacing Proposition 4.5.11 by Proposition 3.4 of the present work, we have:

Proposition 3.5. *For a subset S of V, the annihilator* $\mathrm{Ann}_M(S)$ *is a g-twisted V-submodule of M. Furthermore,*

$$\mathrm{Ann}_M(S) = \mathrm{Ann}_M(\langle S\rangle).$$

Here $\langle S\rangle$ is the ideal of V generated by S.

Lemma 3.6 [Li 1996, Lemma 2.11]. *Let V be a vertex algebra with an automorphism g of order T and let $a \in V^k$ and $b, u^0, \dots, u^r \in V$ with $0 \le k \le T-1$. If*

$$[Y(a, x_1), Y(b, x_2)] = \sum_{j=0}^{r} \frac{1}{j!} Y(u^j, x_2) \left(\frac{\partial}{\partial x_2}\right)^j x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)$$

acting on V, then for any g-twisted V-module (M, Y_M) we have

$$[Y_M(a, x_1), Y_M(b, x_2)] = \sum_{j=0}^{r} \frac{1}{j!} Y_M(u^j, x_2) \left(\frac{\partial}{\partial x_2}\right)^j x_1^{-1}\delta\left(\frac{x_2}{x_1}\right)\left(\frac{x_2}{x_1}\right)^{k/T}$$

acting on M. On the other hand, the converse is also true for any faithful g-twisted V-module (M, Y_M).

Definition 3.7. Let $V = \coprod_{m\in\mathbb{Z}} V_{(m)}$ be a $\mathbb{Z}$-graded vertex algebra. A $\frac{1}{T}\mathbb{N}$-*graded g-twisted V-module* is a g-twisted V-module M equipped with a $\frac{1}{T}\mathbb{N}$-grading

$$M = \coprod_{n\in(1/T)\mathbb{N}} M(n)$$

such that

$$v_m M(n) \subset M(n + p - m - 1)$$

for $v \in V_{(p)}$ and $m, n \in \frac{1}{T}\mathbb{Z}$ with $p \in \mathbb{Z}$.

Next, we study $\frac{1}{T}\mathbb{N}$-graded g-twisted modules for the $\mathbb{N}$-graded vertex algebra V_B associated to a vertex A-algebroid B, where g is an automorphism of order $T < \infty$ of the $\mathbb{N}$-graded vertex algebra V_B and of the vertex A-algebroid B (compare Theorem 2.17).

Noticing that $A \oplus B$ is a 1-truncated conformal algebra, we start with a general 1-truncated conformal algebra $C = C_0 \oplus C_1$ with an automorphism g of C of order $T < \infty$. Associated with the 1-truncated conformal algebra $C = C_0 \oplus C_1$ we have the Lie algebra $\mathscr{L}(C)$ and the vertex algebra $V_{\mathscr{L}(C)}$ with C as a generating subspace. In view of Lemma 2.15 g is an order-T automorphism of the vertex algebra $V_{\mathscr{L}(C)}$.

Lemma 3.8 ([Dong et al. 1998]; compare [Borcherds 1986]). *Let V be a vertex algebra and let T be a positive integer. Set*

$$L_T(V) = V \otimes \mathbb{C}[t^{1/T}, t^{-1/T}], \tag{3-7}$$

a vector space, and set

$$\hat{\partial} = \mathscr{D} \otimes 1 + 1 \otimes \frac{d}{dt},$$

a linear operator on $L_T(V)$. The bilinear (multiplicative) operation on $L_T(V)$, defined by

$$[u \otimes t^m, v \otimes t^n] = \sum_{i\geq 0} \binom{m}{i} (u_i v \otimes t^{m+n-i}) \tag{3-8}$$

for $u, v \in V$, $m, n \in \frac{1}{T}\mathbb{Z}$, gives rise to a Lie algebra structure on $L_T(V)/\hat{\partial}L_T(V)$, which is denoted by $\mathscr{L}(V, T)$. Furthermore, any order-T automorphism g of C gives rise to an order-T automorphism, also denoted by g, of $\mathscr{L}(V, T)$, where

$$g(v \otimes t^n) = e^{-2n\pi\sqrt{-1}}(gv \otimes t^n) \tag{3-9}$$

for $v \in V$ and $n \in \frac{1}{T}\mathbb{Z}$.

Specializing Lemma 3.8 with $V = V_{\mathscr{L}(C)}$, we have a Lie algebra $\mathscr{L}(V_{\mathscr{L}(C)}, T)$ and an automorphism g. For $u \in C$ and $m \in \frac{1}{T}\mathbb{Z}$, denote by $u(m)$ the canonical image of $u \otimes t^m$ in $\mathscr{L}(V_{\mathscr{L}(C)}, T)$. We have

$$(\partial a)(m) = -ma(m-1),$$
$$[u(m), v(n)] = \sum_{i=0}^{1} \binom{m}{i} (u_i v)(m+n-i)$$

for $a \in C_0$, $u, v \in C$ and $m, n \in \frac{1}{T}\mathbb{Z}$. Because $u_i v \in C$ for $u, v \in C$ and $i \geq 0$, we see that the $u(m)$, for $u \in C$ and $m \in \frac{1}{T}\mathbb{Z}$, span a Lie subalgebra $\mathscr{L}(C, T)$ of $\mathscr{L}(V_{\mathscr{L}(C)}, T)$. Denote by $\mathscr{L}(C, g)$ the g-fixed point Lie subalgebra:

$$\mathscr{L}(C, g) = \mathscr{L}(C, T)^g.$$

Using Lemma 3.8 we immediately have:

Proposition 3.9. *Let $C = C_0 \oplus C_1$ be a 1-truncated conformal algebra and let g be an order-T automorphism of C. Then*

$$\mathscr{L}(C, g) = L(C, g)/\hat{\partial} L(C_0, g),$$

as a vector space, where

$$L(C, g) = \coprod_{r=0}^{T-1} C^r \otimes t^{r/T} \mathbb{C}[t, t^{-1}],$$

$L(C_0, g)$ is a subspace defined in the obvious way, and $\hat{\partial}$ is given by

$$\hat{\partial} = \partial \otimes 1 + 1 \otimes d/dt : L(C_0, g) \to L(C, g).$$

For $u \in C^r$ with $0 \leq r \leq T-1$ and for $n \in \mathbb{Z}$, denote by $u(n + r/T)$ the canonical image of $u \otimes t^{n+r/T}$ in $\mathscr{L}(C, g)$. Then the following relations hold for $a \in C_0^r$, $a' \in C_0^{r'}$, $b \in C_1^s$, $b' \in C_1^{s'}$, $m, n \in \mathbb{Z}$:

$$(\partial a)\Big(m + \frac{r}{T}\Big) = -\Big(m + \frac{r}{T}\Big) a\Big(m - 1 + \frac{r}{T}\Big),$$

$$\Big[a\Big(m + \frac{r}{T}\Big), a'\Big(n + \frac{r'}{T}\Big)\Big] = 0, \tag{3-10}$$

$$\Big[a\Big(m + \frac{r}{T}\Big), b\Big(n + \frac{s}{T}\Big)\Big] = (a_0 b)\Big(m + n + \frac{r+s}{T}\Big), \tag{3-11}$$

$$\Big[b\Big(m + \frac{s}{T}\Big), b'\Big(n + \frac{s'}{T}\Big)\Big] = (b_0 b')\Big(m + n + \frac{s+s'}{T}\Big) + \Big(m + \frac{s}{T}\Big)(b_1 b')\Big(m + n + \frac{s+s'}{T} - 1\Big). \tag{3-12}$$

We define

$$\begin{aligned}\deg a(n+r/T) &= -n-1 \quad \text{for } a\in C_0^r,\ n\in\mathbb{Z},\\ \deg b(n+r/T) &= -n \qquad\quad \text{for } b\in C_1^r,\ n\in\mathbb{Z},\end{aligned}$$

making $\mathcal{L}(C,g)$ a $\frac{1}{T}\mathbb{Z}$-graded Lie algebra. For $n\in\frac{1}{T}\mathbb{Z}$, denote by $\mathcal{L}(C,g)_{(n)}$ the degree-n subspace. We have the following triangular decomposition

$$\mathcal{L}(C,g)=\mathcal{L}(C,g)_+\oplus\mathcal{L}(C,g)_{(0)}\oplus\mathcal{L}(C,g)_-,$$

where $\mathcal{L}(C,g)_\pm=\coprod_{0<n\in\frac{1}{T}\mathbb{Z}}\mathcal{L}(C,g)_{(\pm n)}$. Notice that $\mathcal{L}(C,g)_{(0)}$ is spanned by the elements $a(-1), b(0)$ for $a\in C_0^0$ and $b\in C_1^0$.

For $u\in C^r$ with $0\le r\le T-1$, form the generating function

$$u(x)=\sum_{n\in\frac{r}{T}+\mathbb{Z}}u(n)x^{-n-1}\in\mathcal{L}(C,g)[[x^{1/T},x^{-1/T}]]. \tag{3-13}$$

For any $\mathcal{L}(C,g)$-module W, consider $u(x)$ as an element of $(\operatorname{End}W)[[x^{1/T},x^{-1/T}]]$, which we denote by $u_W(x)$:

$$u_W(x)=u(x)=\sum_{n\in(1/T)\mathbb{Z}}u(n)x^{-n-1}\in(\operatorname{End}W)[[x^{1/T},x^{-1/T}]]. \tag{3-14}$$

Lemma 3.10. *The commutation relations* (3-10)–(3-12) *amount to the following relations in terms of generating functions*:

$$[a(x_1),a'(x_2)]=0, \tag{3-15}$$

$$[a(x_1),b'(x_2)]=x_2^{-1}\left(\frac{x_1}{x_2}\right)^{-r/T}\delta\left(\frac{x_1}{x_2}\right)(a_0b')(x_2), \tag{3-16}$$

$$\begin{aligned}[b(x_1),b'(x_2)]&=x_2^{-1}\left(\frac{x_1}{x_2}\right)^{-r/T}\delta\left(\frac{x_1}{x_2}\right)(b_0b')(x_2)\\&\quad+(b_1b')(x_2)\frac{\partial}{\partial x_2}x_2^{-1}\left(\frac{x_1}{x_2}\right)^{-r/T}\delta\left(\frac{x_1}{x_2}\right)\end{aligned} \tag{3-17}$$

for $a\in C_0^r$, $b\in C_1^r$, $a'\in C_0$, and $b'\in C_1$. Moreover, we have

$$(\partial a)(x)=\frac{d}{dx}a(x)\quad \textit{for } a\in C_0.$$

From these relations we immediately have:

Corollary 3.11. *For $a,a'\in C_0$ and $b,b'\in C_1$,*

$$\begin{aligned}[a(x_1),a'(x_2)]&=0,\\ (x_1-x_2)[a(x_1),b(x_2)]&=0,\\ (x_1-x_2)^2[b(x_1),b'(x_2)]&=0.\end{aligned}$$

Definition 3.12. An $\mathscr{L}(C, g)$-module W is said to be *restricted* if for any $w \in W$ and $u \in C^r$ with $0 \leq r \leq T-1$, $u(n+r/T)w = 0$ for $n \in \mathbb{Z}$ sufficiently large, that is, $u_W(x) \in \mathrm{Hom}(W, W((x^{1/T})))$ for $u \in C$.

The following result is analogous to a result of [Li 1996] for twisted affine Lie algebras:

Proposition 3.13. *Let $C = C_0 \oplus C_1$ be a 1-truncated conformal algebra and let g be an automorphism of $V_{\mathscr{L}(C)}$ of order T, which is extended from an automorphism of C. Every g-twisted $V_{\mathscr{L}(C)}$-module W is naturally a restricted $\mathscr{L}(C, g)$-module with $u_W(x) = Y_W(u, x)$ for $u \in C$. Moreover, the set of g-twisted $V_{\mathscr{L}(C)}$-submodules of W is precisely the set of $\mathscr{L}(C, g)$-submodules of W. On the other hand, for any restricted $\mathscr{L}(C, g)$-module W, there exists a unique g-twisted $V_{\mathscr{L}(C)}$-module structure Y_W on W such that*

$$Y_W(u, x) = u_W(x) \quad \textit{for } u \in C = C_0 \oplus C_1 \subset V_{\mathscr{L}(C)}. \tag{3-18}$$

Proof. On the vertex algebra $V_{\mathscr{L}(C)}$, the following relations hold for $a, a' \in C_0$ and $b, b' \in C_1$:

$$[Y(a, x_1), Y(a', x_2)] = 0,$$
$$[Y(a, x_1), Y(b', x_2)] = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) Y(a_0 b', x_2),$$
$$[Y(b, x_1), Y(b', x_2)] = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) Y(b_0 b', x_2) + Y(b_1 b', x_2)\frac{\partial}{\partial x_2} x_2^{-1}\delta\left(\frac{x_1}{x_2}\right).$$

From Lemmas 3.6 and 3.10, every g-twisted $V_{\mathscr{L}(C)}$-module W is naturally a restricted $\mathscr{L}(C, g)$-module with $u_W(x) = Y_W(u, x)$ for $u \in C$. As C generates $V_{\mathscr{L}(C)}$ as a vertex algebra, the set of g-twisted $V_{\mathscr{L}(C)}$-submodules of W is precisely the set of $\mathscr{L}(C, g)$-submodules of W.

Let $S = \mathrm{span}\{u_W(x) \mid u \in C\}$. In view of Corollary 3.11, S is a local subspace of $\mathrm{Hom}(W, W((x^{1/T})))$. Note that $\mathrm{Hom}(W, W((x^{1/T})))$ is naturally $\mathbb{Z}/T\mathbb{Z}$-graded:

$$\mathrm{Hom}(W, W((x^{1/T}))) = \coprod_{r=0}^{T-1} x^{r/T}\, \mathrm{Hom}(W, W((x))),$$

and that S is a graded subspace.

Let σ_T be the linear automorphism of $\mathrm{Hom}(W, W((x^{1/T})))$ defined by

$$\sigma_T(\alpha(x)) = e^{-2r\pi\sqrt{-1}/T}\alpha(x)$$

for $\alpha(x) \in x^{r/T}\, \mathrm{Hom}(W, W((x)))$ with $0 \leq r \leq T-1$; compare (3-1).

S generates a vertex algebra $\langle S\rangle$ inside $\mathrm{Hom}(W, W((x^{1/T})))$ [Li 1996] with the identity operator 1_W as the vacuum vector and with σ_T as an automorphism. W is naturally a faithful σ_T-twisted $\langle S\rangle$-module with $Y_W(\alpha(x), x_0) = \alpha(x_0)$. With the relations (3-15)-(3-17), by Lemma 3.6, we have

$$
\begin{aligned}
&[Y(a_W(x), x_1), Y(a'_W(x), x_2)] = 0,\\
&[Y(a_W(x), x_1), Y(b'_W(x), x_2)] = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) Y((a_0b')_W(x), x_2),\\
&[Y(b_W(x), x_1), Y(b'_W(x), x_2)] = x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) Y((b_0b')_W(x), x_2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + Y((b_1b')_W(x), x_2)\frac{\partial}{\partial x_2}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)
\end{aligned}
$$

for $a \in C_0^r$, $b \in C_1^r$, $a' \in C_0$, and $b' \in C_1$. We also have

$$Y((\partial a)_W(x), x_1) = Y\left(\frac{d}{dx}a_W(x), x_1\right) = \frac{\partial}{\partial x_1}Y(a_W(x), x_1)$$

for $a \in A$. By Lemmas 3.6 and 3.10, $\langle S\rangle$ is naturally an $\mathcal{L}(C)$-module with $u_{\langle S\rangle}(x_1) = Y(u_W(x), x_1)$ for $u \in C$. Furthermore, $\langle S\rangle$ as an $\mathcal{L}(C)$-module is generated by 1_W and we have $u_W(x)_n 1_W = 0$ for $u \in C$, $n \geq 0$. From the construction of $V_{\mathcal{L}(C)}$ as an $\mathcal{L}(C)$-module, there exists a unique $\mathcal{L}(C)$-module homomorphism ψ from $V_{\mathcal{L}(C)}$ to $\langle S\rangle$, sending $\mathbf{1}$ to 1_W. As $V_{\mathcal{L}(C)}$ as a vertex algebra is generated by C, ψ is a vertex algebra homomorphism. We have

$$\psi(u) = \psi(u(-1)\mathbf{1}) = u_W(x)_{-1}1_W = u_W(x)$$

for $u \in C$. It is clear that $\psi(C^r) \subset S^r$ for $0 \leq r \leq T-1$. As C generates $V_{\mathcal{L}(C)}$ as a vertex algebra, ψ preserves the $\mathbb{Z}/T\mathbb{Z}$-gradings, i.e., $\sigma_T\psi = \psi g$. Consequently, W is a g-twisted $V_{\mathcal{L}(C)}$-module. □

For the rest of this paper, we assume that *A is a unital commutative associative algebra whose identity is denoted by e and B is a vertex A-algebroid and we assume that $g \in \mathrm{Aut}^0(V_B)$ with $o(g) = T < \infty$.* Recall that $C = A \oplus B$ is naturally a 1-truncated conformal algebra. An $\mathcal{L}(C, g)$-module of *level* $k \in \mathbb{C}$ is an $\mathcal{L}(C, g)$-module on which $e(-1)$ acts as scalar k.

Immediately from Proposition 3.13 we have:

Proposition 3.14. *Every g-twisted V_B-module is naturally a restricted $\mathcal{L}(C, g)$-module of level* 1. *Moreover, the set of g-twisted V_B-submodules is precisely the set of $\mathcal{L}(C, g)$-submodules.*

We have the following decompositions into g-eigenspaces:

$$V_B = \coprod_{r=0}^{T-1} V_B^r, \quad A = \coprod_{r=0}^{T-1} A^r, \quad B = \coprod_{r=0}^{T-1} B^r.$$

Clearly, A^0 is a subalgebra of A, containing the identity, and B^0 is a vertex A^0-algebroid. Furthermore, by Lemma 2.6, $B^0/A^0\partial A^0$ is a Lie A^0-algebroid. Set

$$I=\sum_{r=1}^{T-1} A^r\cdot A^{T-r}\subset A^0.$$

It is clear that I is a two-sided ideal of A^0, so that A^0/I is a unital commutative associative algebra. Furthermore, $B^0/(I\cdot B^0+A^0\partial A^0)$ is a Lie A^0/I-algebroid.

Proposition 3.15. *Let $M=\coprod_{n\in(1/T)\mathbb{N}} M(n)$ be a $\frac{1}{T}\mathbb{N}$-graded g-twisted V_B-module. Then $M(0)$ is a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$ with*

$$a\cdot w=a_{-1}w\quad \text{for } a\in A^0,\ w\in M(0),$$
$$b\cdot w=b_0w\quad \text{for } b\in B^0,\ w\in M(0).$$

Furthermore, for $a\in A^r$, $a'\in A^{T-r}$, $b\in B^{T-r}$ with $0<r\le T-1$ and for $w\in M(0)$, we have $(aa')\cdot w=0$ and $(ab)\cdot w=(1-\frac{r}{T})(a_0b)\cdot w$.

Proof. Let U be the vertex subalgebra of V_B generated by $A^0\oplus B^0$. As $A^0\oplus B^0\subset V_B^0$, U is actually a vertex subalgebra of V_B^0. From Remark 3.1, M is a U-module. With $(V_B)_{(0)}=A$ and $(V_B)_{(1)}=B$, we have $(V_B^0)_{(0)}=A^0$ and $(V_B^0)_{(1)}=B^0$. Consequently, we have $U_{(0)}=A^0$ and $U_{(1)}=B^0$. It follows from the construction of V_{B^0} that U is a homomorphic image of the vertex algebra V_{B^0}, so that W is naturally a V_{B^0}-module. By [Li and Yamskulna 2005, Proposition 4.8], $W(0)$ is naturally a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$.

Let $a\in A^r$, $a'\in A^{T-r}$, $b\in B^{T-r}$, $w\in M(0)$ with $0<r\le T-1$. By substituting $u=a$, $v=a'$, $p=-1$, $s=-1+\frac{r}{T}$, $t=-\frac{r}{T}$ in (3-4), we get

$$\begin{aligned}(aa')\cdot w&=(aa')_{-1}w=(a(-1)a')_{-1}w\\&=\sum_{m\ge 0} a_{-2+(r/T)-m}a'_{-(r/T)+m}w+a'_{-1-(r/T)-m}a_{-1+(r/T)+m}w=0.\end{aligned}$$

Similarly, by substituting $u=a$, $v=b$, $p=-1$, $s=-1+\frac{r}{T}$ and $t=1-\frac{r}{T}$ in (3-4), we have

$$\begin{aligned}(ab)\cdot w&=(ab)_0w=(a(-1)b)_0w\\&=\Big(1-\frac{r}{T}\Big)(a_0b)_{-1}w\\&\qquad+\sum_{m\ge 0}\{a_{-2+(r/T)-m}b_{1-(r/T)+m}+b_{-(r/T)-m}a_{-1+(r/T)+m}\}w\\&=\Big(1-\frac{r}{T}\Big)(a_0b)\cdot w,\end{aligned}$$

completing the proof. □

Let U be a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$ such that

$$(aa')\cdot u = 0 \quad \text{and} \quad (ab)\cdot u = \Big(1-\frac{r}{T}\Big)(a_0 b)\cdot u$$

for $a \in A^r$, $a' \in A^{T-r}$, $b \in B^{T-r}$, $u \in U$, and $0 < r \le T-1$. We construct a $\frac{1}{T}\mathbb{N}$-graded g-twisted V_B-module $M = \coprod_{n\in(n/T)\mathbb{N}} M(n)$ with $M(0) = U$ as a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$.

First, U is a module for the Lie algebra $A^0 \oplus B^0/\partial A^0$. Recall that $\mathscr{L}(C, g)_{(0)} = A^0 \oplus B^0/\partial A^0$. For convenience, we set $\mathscr{L}(C, g)_{\le 0} = \mathscr{L}(C, g)_{(0)} \oplus \mathscr{L}(g)_-$. Then U is an $\mathscr{L}(C, g)_{\le 0}$-module under the actions

$$a\Big(n+\frac{r}{T}-1\Big)\cdot u = \delta_{n+r/(T),0}\, au,$$
$$b\Big(n+\frac{r}{T}\Big)\cdot u = \delta_{n+r/(T),0}\, bu$$

for $a \in A^r$, $b \in B^r$, and $n \ge 0$. Next, we form the induced $\mathscr{L}(C, g)$-module

(3-19) $$M_g(U) = \mathrm{Ind}_{\mathscr{L}(C,g)_{\le 0}}^{\mathscr{L}(C,g)} U = U(\mathscr{L}(C, g)) \otimes_{U(\mathscr{L}(C,g)_{\le 0})} U.$$

We endow U with degree 0, making $M_g(U)$ a $\frac{1}{T}\mathbb{N}$-graded restricted $\mathscr{L}(C, g)$-module. By Proposition 3.13, $M_g(U)$ is naturally a g-twisted $V_{\mathscr{L}(C)}$-module. In view of the P-B-W theorem, we may and we should consider U as the degree-zero subspace of $M_g(U)$.

We set

(3-20) $$W_g(U) = \mathrm{span}\{v_n u \mid v \in E,\ n \in \tfrac{1}{T}\mathbb{Z},\ u \in U\} \subset M_g(U)$$

and define

(3-21) $$M_B(U) = M_g(U)/U(\mathscr{L}(C, g))W_g(U).$$

Since $U(\mathscr{L}(C, g))W_g(U)$ is an $\mathscr{L}(C, g)$-submodule of $M_g(U)$, by Proposition 3.13 $U(\mathscr{L}(C, g))W_g(U)$ is a g-twisted $V_{\mathscr{L}(C)}$-submodule. Then $M_B(U)$ is a g-twisted $V_{\mathscr{L}(C)}$-module. Clearly, $M_B(U)$ is generated by $\bar{U}$ the image of U in $M_B(U)$. In fact, $M_B(U)$ is a g-twisted V_B-module by the following:

Lemma 3.16. *Let (M, Y_M) be a g-twisted $V_{\mathscr{L}}$-module. Suppose that for $a \in A^r$, $a' \in A$, $b \in B$ with $0 \le r \le T-1$, we have*

$$Y_M(e, x)w = w,$$
$$Y_M(a(-1)a', x)w = Y_M(aa', x)w,$$
$$Y_M(a(-1)b, x)w = Y_M(ab, x)w$$

for all $w \in K$, where K is a generating subspace of M. Then M is naturally a g-twisted V_B-module.

Proof. Recall that

$$E = \text{span}\{e - \mathbf{1},\ a(-1)a' - aa',\ a(-1)b - ab \mid a, a' \in A, b \in B\} \subset V_{\mathscr{L}(C)}.$$

By the assumptions on a, a', b, we have $K \subset \text{Ann}_M(E)$. Using Proposition 3.5, we obtain $\text{Ann}_M(I_B) = \text{Ann}_M(E)$. Since $\text{Ann}_M(I_B)$ is a g-twisted $V_{\mathscr{L}(C)}$ submodule of M and M is generated by K, we have $\text{Ann}_M(I_B) = M$. This implies that M is a g-twisted V_B-module. □

One can see that Lemma 3.16 indeed implies that $M_B(U)$ is naturally a g-twisted V_B-module. Furthermore we have:

Theorem 3.17. *Let U be a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$ such that*

$$(aa') \cdot u = 0 \quad \textit{and } (ab) \cdot u = (1 - \frac{r}{T})(a_0 b) \cdot u$$

for $a \in A^r$, $a' \in A^{T-r}$, $b \in B^{T-r}$, $u \in U$, $r \neq 0$. Then $M_B(U)$ is naturally a g-twisted V_B-module such that $M_B(U)(0) = U$.

Proof. To show that $M_B(U)(0) = U$, we must prove that $(U(\mathscr{L}(g))W_g(U))(0) = 0$. First we show that $W_g(U)(0) = 0$. Notice that for $v \in (V_{\mathscr{L}(C)})^r_{(m)}$ with $m \in \mathbb{Z}$, we have $\deg v_{k+r/T} = m - k - r/T - 1$ for $k \in \mathbb{Z}$. Then from the definition of $W_g(U)$, $W_g(U)(0)$ is spanned by the vectors

$$(e - \mathbf{1})_{-1}u, \quad (a(-1)a')_{-1}u - (aa')_{-1}u, \quad (a(-1)b)_0 u - (ab)_0 u$$

for $u \in U$, $a \in A^r$, $a' \in A^{T-r}$, $b \in B^{T-r}$ with $0 \leq r \leq T - 1$. Since e_{-1} acts as e (the identity of A^0) on U, we have $(e - \mathbf{1})_{-1}u = 0$ for $u \in U$. If $r = 0$, by (3-4),

$$(a_{-1}a')_{-1}u = a(-1)a'(-1)u = a(a'u) = (aa')u = (aa')_{-1}u,$$

and

$$(a(-1)b)_0 u = a(-1)b(0)u = a(bu) = (ab)u = (ab)_0 u.$$

Next, we assume that $r > 0$. By (3-4), we have

$$\begin{aligned}
&(a(-1)a')_{-1}u \\
&\quad = \sum_{i=0}^{\infty} a\Big(-1 - i + \frac{r}{T}\Big)a'\Big(i - 1 - \frac{r}{T}\Big)u + \sum_{i=0}^{\infty} a'\Big(-2 - i - \frac{r}{T}\Big)a\Big(i + \frac{r}{T}\Big)u \\
&\quad = a\Big(-1 + \frac{r}{T}\Big)a'\Big(-1 - \frac{r}{T}\Big)u = a'\Big(-1 - \frac{r}{T}\Big)a\Big(-1 - \frac{r}{T}\Big)u \\
&\quad = 0 \\
&\quad = (aa')_{-1}u
\end{aligned}$$

and

$$
\begin{aligned}
&(a(-1)b)_0u\\
&\quad=\sum_{i=0}^{\infty}a\Big(-1-i+\frac{r}{T}\Big)b\Big(i-\frac{r}{T}\Big)u+\sum_{i=0}^{\infty}b\Big(-i-1-\frac{r}{T}\Big)a\Big(i+\frac{r}{T}\Big)u-\frac{r}{T}(a_0b)_{-1}u\\
&\quad=a\Big(-1+\frac{r}{T}\Big)b\Big(-\frac{r}{T}\Big)u-\frac{r}{T}(a_0b)\cdot u\\
&\quad=b\Big(-\frac{r}{T}\Big)a\Big(-1+\frac{r}{T}\Big)u+(a_0b)_{-1}u-\frac{r}{T}(a_0b)_{-1}u\\
&\quad=\Big(1-\frac{r}{T}\Big)(a_0b)\cdot u\\
&\quad=(ab)\cdot u.
\end{aligned}
$$

Hence, $W_g(U)(0)=0$.

We now show that

$$\mathscr{L}(C,g)_{\leq 0}W_g(U)\subset W_g(U).$$

Recall from [Li and Yamskulna 2005, Lemma 4.2] that

$$v_iE\subset E\quad\text{for } v\in C=A\oplus B,\ i\geq 0.$$

Since $M_g(U)$ is a $\frac{1}{T}\mathbb{N}$-graded $\mathscr{L}(C,g)$-module with U as the degree-zero subspace, we have $\mathscr{L}(C,g)_{\leq 0}U\subset U$. For $v\in C=A\oplus B$, $c\in E$, $m,t\in\frac{1}{T}\mathbb{Z}$, $u\in U$, from the twisted commutator formula (3-5) (cf. (3-8)), we have

$$v_mc_tu=c_tv_mu+\sum_{i\geq 0}\binom{m}{i}(v_ic)_{m-t-i}u.$$

These immediately imply that $\mathscr{L}(C,g)_{\leq 0}W_g(U)\subset W_g(U)$. Then

$$
\begin{aligned}
U(\mathscr{L}(C,g))W_g(U)&=U(\mathscr{L}(C,g)_+)U(\mathscr{L}(C,g)_{\leq 0})W_g(U)\\
&=U(\mathscr{L}(C,g)_+)W_g(U)\\
&=W_g(U)+\mathscr{L}(C,g)_+U(\mathscr{L}(C,g)_+)W_g(U),
\end{aligned}
$$

which implies that $(U(\mathscr{L}(C,g))W_g(U))(0)=0$. This completes the proof. □

Next, we continue with Theorem 3.17 to construct and classify $\frac{1}{T}\mathbb{N}$-graded simple g-twisted V_B-modules. Let U be a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$ as in Theorem 3.17. Let $J(U)$ be the sum of all graded $\mathscr{L}(C,g)$-submodules of $M_g(U)$ with trivial degree-zero subspaces. Then $J(U)$ is the unique maximal graded $\mathscr{L}(C,g)$-submodule of $M_g(U)$ with the property that $J(U)\cap U=0$. Set

$$L_g(U)=M_g(U)/J(U),\tag{3-22}$$

a $\frac{1}{T}\mathbb{N}$-graded g-twisted V_B-module.

Lemma 3.18. *Let U be a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$ as in Theorem 3.17. Then $L_g(U)$ is a $\frac{1}{T}\mathbb{N}$-graded g-twisted V_B-module such that $L_g(U)(0) = U$ as a module for the Lie A^0-algebroid $B^0/A^0\partial A^0$ and such that for any nonzero graded submodule W of $L_g(U)$, we have $W(0) \neq 0$. Furthermore, if U is a simple $B^0/A^0\partial A^0$-module, $L_g(U)$ is a $\frac{1}{T}\mathbb{N}$-graded simple g-twisted V_B-module.*

Proof. Similar to the proof of Theorem 4.12 in [Li and Yamskulna 2005]. □

Lemma 3.19. *Let $W = \coprod_{n\in(1/T)\mathbb{Z}} W(n)$ be a $\frac{1}{T}\mathbb{N}$-graded simple g-twisted V_B-module with $W(0) \neq 0$. Then $W \cong L_g(W(0))$.*

Proof. Similar to the proof of Lemma 4.13 in [Li and Yamskulna 2005]. □

To summarize we have:

Theorem 3.20. *Let H be a complete set of equivalence class representatives of simple modules for the Lie A^0-algebroid $B^0/A^0\partial A^0$ satisfying the condition that*

$$(aa')U = 0, \quad \left((ab) - \left(1 - \frac{r}{T}\right)(a_0 b)\right)U = 0$$

for $a \in A^r, a' \in A^{T-r}, b \in B^{T-r}$ with $0 < r < T$. Then $\{L_g(U) \mid U \in H\}$ is a complete set of equivalence class representatives of $\frac{1}{T}\mathbb{N}$-graded simple g-twisted V_B-modules.

Proof. Similar to the proof of Theorem 4.14 in [Li and Yamskulna 2005]. □

Finally, we remark that by taking $g = 1$, the identity map of V_B, we recover Theorem 4.14 of [Li and Yamskulna 2005]:

Corollary 3.21. *If H is a complete set of equivalence class representatives of simple modules for the Lie A-algebroid $B/A\partial A$, then $\{L_1(U) \mid U \in H\}$ is a complete set of equivalence class representatives of $\mathbb{N}$-graded simple V_B-modules.*

References

[Berman et al. 2002a] S. Berman, Y. Billig, and J. Szmigielski, “Vertex operator algebras and the representation theory of toroidal algebras”, pp. 1–26 in *Recent developments in infinite-dimensional Lie algebras and conformal field theory*, edited by S. Berman et al., Contemp. Math. **297**, Amer. Math. Soc., Providence, RI, 2002. MR 2003j:17037 Zbl 1018.17017

[Berman et al. 2002b] S. Berman, C. Dong, and S. Tan, “Representations of a class of lattice type vertex algebras”, *J. Pure Appl. Algebra* **176**:1 (2002), 27–47. MR 2003k:17034 Zbl 1043.17016

[Borcherds 1986] R. E. Borcherds, “Vertex algebras, Kac–Moody algebras, and the Monster”, *Proc. Nat. Acad. Sci. U.S.A.* **83**:10 (1986), 3068–3071. MR 87m:17033 Zbl 0613.17012

[Bressler 2002] P. Bressler, “Vertex algebroids, I”, preprint, 2002. math.AG/0202185

[Bressler 2003] P. Bressler, “Vertex algebroids, II”, preprint, 2003. math.AG/0304115

[Dong 1994] C. Dong, “Twisted modules for vertex algebras associated with even lattices”, *J. Algebra* **165**:1 (1994), 91–112. MR 95i:17032 Zbl 0807.17023

[Dong et al. 1997] C. Dong, H. Li, and G. Mason, "Regularity of rational vertex operator algebras", *Adv. Math.* **132**:1 (1997), 148–166. MR 98m:17037 Zbl 0902.17014

[Dong et al. 1998] C. Dong, H. Li, and G. Mason, "Twisted representations of vertex operator algebras", *Math. Ann.* **310**:3 (1998), 571–600. MR 99d:17030 Zbl 0890.17029

[Dong et al. 2002] C. Dong, H.-S. Li, and G. Mason, "Vertex Lie algebra, vertex Poisson algebras and vertex algebras", pp. 69–96 in *Recent developments in infinite-dimensional Lie algebras and conformal field theory* (Charlottesville, VA, 2000), edited by S. Berman et al., Contemp. Math. **297**, Amer. Math. Soc., Providence, RI, 2002. MR 2003h:17035

[Feĭgin and Frenkel 1988] B. L. Feĭgin and È. V. Frenkel, "A family of representations of affine Lie algebras", *Uspekhi Mat. Nauk* **43**:5 (1988), 227–228. In Russian; translated in *Russian Math. Surveys* **43** (1988), 221–222. MR 89k:17016 Zbl 0668.17015

[Feĭgin and Frenkel 1990a] B. L. Feĭgin and E. V. Frenkel, "Affine Kac–Moody algebras and semi-infinite flag manifolds", *Comm. Math. Phys.* **128** (1990), 161–189. MR 92f:17026 Zbl 0722.17019

[Feigin and Frenkel 1990b] B. L. Feigin and E. V. Frenkel, "Representations of affine Kac–Moody algebras and bosonization", pp. 271–316 in *Physics and mathematics of strings*, edited by L. Brink et al., World Sci., Teaneck, NJ, 1990. MR 92d:17025 Zbl 0734.17011

[Feingold et al. 1991] A. J. Feingold, I. B. Frenkel, and J. F. X. Ries, *Spinor construction of vertex operator algebras, triality, and* $E_8^{(1)}$, Contemporary Mathematics **121**, Amer. Math. Soc., Providence, RI, 1991. MR 92k:17041 Zbl 0743.17029

[Frenkel and Ben-Zvi 2001] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs **88**, Amer. Math. Soc., Providence, RI, 2001. MR 2003f:17036 Zbl 0981.17022

[Frenkel et al. 1988] I. Frenkel, J. Lepowsky, and A. Meurman, *Vertex operator algebras and the Monster*, Pure and Applied Mathematics **134**, Academic Press, Boston, 1988. MR 90h:17026 Zbl 0674.17001

[Gorbounov et al. 2004] V. Gorbounov, F. Malikov, and V. Schechtman, "Gerbes of chiral differential operators, II: Vertex algebroids", *Invent. Math.* **155**:3 (2004), 605–680. MR 2005e:17047 Zbl 1056.17022

[Lepowsky 1985] J. Lepowsky, "Calculus of twisted vertex operators", *Proc. Nat. Acad. Sci. U.S.A.* **82**:24 (1985), 8295–8299. MR 88f:17030 Zbl 0579.17010

[Lepowsky and Li 2004] J. Lepowsky and H. Li, *Introduction to vertex operator algebras and their representations*, Progress in Mathematics **227**, Birkhäuser, Boston, 2004. MR 2004k:17050 Zbl 1055.17001

[Li 1996] H.-S. Li, "Local systems of twisted vertex operators, vertex operator superalgebras and twisted modules", pp. 203–236 in *Moonshine, the Monster, and related topics* (South Hadley, MA, 1994), edited by C. Dong and G. Mason, Contemp. Math. **193**, Amer. Math. Soc., Providence, RI, 1996. MR 96m:17050 Zbl 0844.17022

[Li and Yamskulna 2005] H. Li and G. Yamskulna, "On certain vertex algebras and their modules associated with vertex algebroids", *J. Algebra* **283**:1 (2005), 367–398. MR 2006h:17035 Zbl 1066.17017

[Malikov and Schechtman 1999] F. Malikov and V. Schechtman, "Chiral de Rham complex, II", preprint, 1999. math.AG/9901065

[Malikov et al. 1998] F. Malikov, V. Schechtman, and A. Vaintrob, "Chiral de Rham complex", preprint, 1998. math.AG/9803041

[Wakimoto 1986] M. Wakimoto, "Fock representations of the affine Lie algebra $A_1^{(1)}$", *Comm. Math. Phys.* **104**:4 (1986), 605–609. MR 87m:17011 Zbl 0587.17009

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