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A SPECTRAL DECOMPOSITION FOR SINGULAR-HYPERBOLIC SETS

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We extend the Spectral Decomposition Theorem for hyperbolic sets to singular-hyperbolic sets on 3-manifolds. We prove that an attracting singular-hyperbolic set with dense periodic orbits and a unique equilibrium of a C^r vector field, where $r \geq 1$, is a finite union of transitive sets; the union is disjoint or the set contains finitely many distinct homoclinic classes. If the vector field is C^r-generic, the union is in fact disjoint.

1. Introduction and statement of the results

The *Spectral Decomposition Theorem for hyperbolic systems* plays a central role in dynamics [Smale 1967]. In the case of an attracting hyperbolic set in which the periodic orbits are dense it asserts that the set is a finite disjoint union of homoclinic classes. Here we present a version of this result in the context of *singular-hyperbolic systems* [Morales et al. 2004], proving that an attracting singular-hyperbolic set with dense periodic orbits and a unique equilibrium is a finite union of transitive sets. Moreover, the union is disjoint or the set contains finitely many distinct homoclinic classes. If the flow is C^r-generic, the union is in fact disjoint. Let us state our results in a precise way.

Throughout, M denotes a compact 3-manifold and X denotes a C^r vector field in M, where $r \geq 1$. The flow of X will be denoted by X_t, $t \in \mathbb{R}$. The *omega-limit set* of a point $p \in M$ is the set $\omega_X(p)$ defined by

$$\omega_X(p) = \left\{x \in M : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty\right\}.$$

A compact invariant set A is *transitive* if $A = \omega_X(p)$ for some $p \in A$. We say that A is *attracting* if there is a compact neighborhood U of A such that

$$A = \bigcap_{t\geq 0} X_t(U).$$

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An *attractor* is a transitive attracting set. (Note that many authors, such as Milnor [1985], define an attractor to be what we call an attracting set.) A *homoclinic class* of X is the closure of the transverse homoclinic points associated to a hyperbolic periodic orbit [Palis and Takens 1993]. It follows from the Birkhoff–Smale Theorem that any homoclinic class is a transitive set with dense periodic orbits.

Definition 1. A compact invariant set Λ of X is *partially hyperbolic* if there are an invariant splitting $T\Lambda = E^s \oplus E^c$ and positive constants K, λ such that:

1. *E^s is contracting:*

$$\|DX_t/E^s_x\| \le Ke^{-\lambda t} \quad \text{for all } x \in \Lambda \text{ and } t > 0.$$

2. *E^s dominates E^c*; that is, $E^s_x \neq 0$, $E^c_x \neq 0$ and

$$\|DX_t/E^s_x\| \cdot \|DX_{-t}/E^c_{X_t(x)}\| \le Ke^{-\lambda t} \quad \text{for all } x \in \Lambda \text{ and } t > 0.$$

We say that the central subbundle E^c of a partially hyperbolic set Λ is *volume-expanding* if the constants K, λ above satisfy

$$|J(DX_t/E^c_x)| \ge Ke^{\lambda t},$$

for every $x \in \Lambda$ and $t > 0$, where $J(\,\cdot\,)$ is the jacobian.

Definition 2. Let Λ be a compact invariant set of a vector field X on a 3-manifold. We say that Λ is *singular-hyperbolic* if all its singularities are hyperbolic, and it is partially hyperbolic with a volume-expanding central subbundle [Bonatti et al. 2005; Morales et al. 2004].

A *singular-hyperbolic attractor* is an attractor that is also a singular-hyperbolic set. The most important examples of singular-hyperbolic attractors are nontrivial hyperbolic attractors and the *geometric Lorenz attractor* [Afraĭmovich et al. 1982; Guckenheimer and Williams 1979]. More examples can be found in [Morales et al. 2000; 2005; Morales and Pujals 1997]. See [Bonatti et al. 2005, Chapter 9] for background concerning singular-hyperbolic sets.

As already mentioned, an attracting hyperbolic set with dense periodic orbits is a finite *disjoint* union of homoclinic classes. A natural candidate for a singular-hyperbolic version of this result can be obtained replacing hyperbolic by singular-hyperbolic in its statement. However, the resulting version is false; we describe a counterexample. Start with the modification of the geometric Lorenz attractor [Guckenheimer and Williams 1979] obtained by adding two singularities to the flow located at $W^u(\sigma)$, as indicated in Figure 1. This modification is done in such a way that the new flow restricted to the cross section S has a C^∞ invariant stable foliation and the quotient map in the leaf space is piecewise expanding with a single discontinuity c, as in the Lorenz case [Guckenheimer and Williams 1979, p. 63]. Then the resulting attracting set can be proved to be a homoclinic class, just as

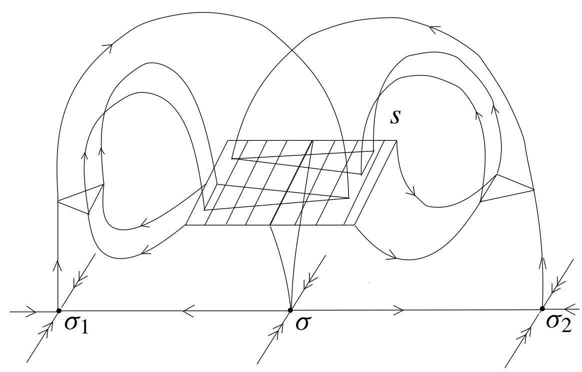

Figure 1

in the Lorenz case recently done in [Bautista 2004]. In particular, such a set is transitive with *dense periodic orbits* and is also singular-hyperbolic by construction. Afterward we glue together in a C^∞ fashion two copies of this flow along the unstable manifold of the singularity σ, thus obtaining the flow depicted in Figure 2. In this way we obtain an attracting singular-hyperbolic set with dense periodic orbits and three equilibria which is not the *disjoint* union of homoclinic classes (although it is the union of two transitive sets). This completes the construction of the counterexample.

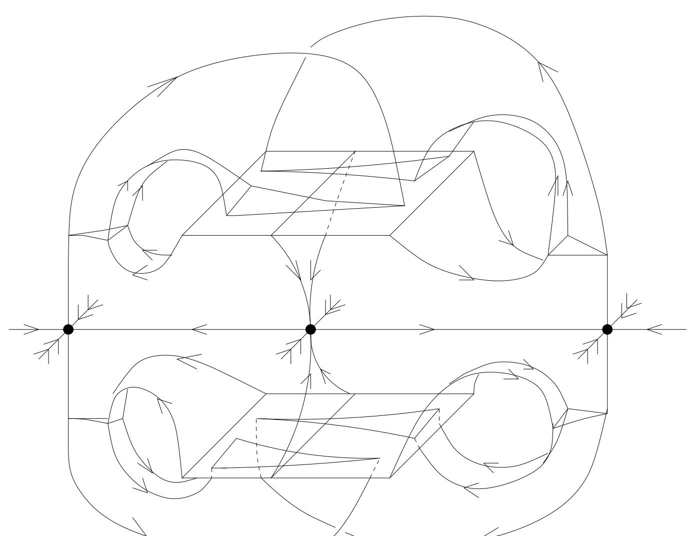

Figure 2

Although this counterexample has *three* equilibria, it is possible to construct one with a *unique* equilibrium. The construction in this case is much more elaborate than the one just described above [Bautista et al. 2005].

These counterexamples illustrate the situations that can appear when we consider the spectral decomposition for singular-hyperbolic sets instead of hyperbolic sets. In particular, it is possible to obtain a *finite union of transitive sets* rather than a finite disjoint union of homoclinic classes. It turns out that the former situation always occurs in the presence of a sole equilibrium. More precisely, we shall prove the following result.

Theorem 3. *An attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite union of transitive sets.*

The most common example of a transitive set is a homoclinic class. Every known example of a singular-hyperbolic attractor is a homoclinic class [Bautista 2004], and it has been conjectured that this is always the case [Morales 2004; 2005]. If this conjecture were true, we would be able to strengthen Theorem 3, obtaining a finite union of homoclinic classes instead of transitive sets.

It is natural to ask whether the union in Theorem 3 is disjoint. To answer this question we recall that a vector field is *Kupka–Smale* if all its closed orbits are hyperbolic and their associated invariant manifolds are in general position [Smale 1967].

Theorem 4. *For a Kupka–Smale vector field, an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite* ***disjoint*** *union of transitive sets.*

Theorem 4 implies that the union in Theorem 3 is disjoint for most vector fields on closed 3-manifolds. Indeed, denote by $\chi^r(M)$ the set of all C^r vector fields on a compact 3-manifold M endowed with the C^r-topology, $r \geq 1$. A subset of $\chi^r(M)$ is *residual* if it is a countable intersection of open, dense subsets of $\chi^r(M)$.

Denote by $\mathcal{R}^r(M)$ the subset of all vector fields $X \in \chi^r(M)$ for which every attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of X is a finite *disjoint* union of transitive sets. Standard C^1-generic arguments [Bonatti et al. 2005] imply that $\mathcal{R}^r(M)$ is residual in $\chi^r(M)$ when $r = 1$. The following corollary proves this assertion for all $r \geq 1$. The proof follows combining Theorem 4 with the classical Kupka–Smale Theorem [Palis and de Melo 1982].

Corollary 5. $\mathcal{R}^r(M)$ *is residual in* $\chi^r(M)$ *for every* $r \geq 1$.

Next we investigate what can happen outside the residual subset $\mathcal{R}^r(M)$ in Corollary 5. If Λ is a compact invariant set of a vector field X we define

$$\mathcal{H}_X(\Lambda) = \{H : H \text{ is a homoclinic class of } X \text{ contained in } \Lambda\}.$$

An interesting question is to give sufficient conditions for

$$\#\mathcal{H}_X(\Lambda) < \infty,$$

where # denotes cardinality. For instance, $\mathcal{H}_X(\Lambda)$ is finite if

- Λ is a homoclinic class and $X \in \chi^r(M)$ is C^1-generic [Bonatti et al. 2005], or if
- Λ is hyperbolic.

Problem 9.32 (p. 283) in [Bonatti et al. 2005] asks whether $\mathcal{H}_X(\Lambda)$ is finite for *every* singular-hyperbolic set Λ. The next result give a partial positive answer for this question.

Theorem 6. *Let Λ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of $X \in \chi^r(M)$. If Λ is **not** a disjoint union of transitive sets, then $\#\mathcal{H}_X(\Lambda) < \infty$.*

To conclude this section we point out that Theorem 3 applies to the class of singular-hyperbolic vector fields introduced in [Bautista 2005]. By definition, a vector field X is *singular-hyperbolic* if its nonwandering set $\Omega(X)$ is the closure of its closed orbits and, denoting by $S(X)$ the union of the attracting and repelling closed orbits, there is a *disjoint* union

$$\Omega(X) \setminus S(X) = \Omega_1(X) \cup \Omega_2(X),$$

where $\Omega_1(X)$ is a singular-hyperbolic set for X and $\Omega_2(X)$ is a singular-hyperbolic set for $-X$.

The class of singular-hyperbolic vector fields contains Axiom A vector fields and the geometric Lorenz attractor. If the conjecture in [Morales 2004] mentioned above were true, this class would contain also the singular-Axiom A vector fields defined in [Morales and Pacifico 2003]. In any case there are many singular-hyperbolic vector fields which are also Kupka–Smale. An example of a singular-hyperbolic vector field in S^3 which is not Kupka–Smale can be derived from the example described before. An example of a singular-hyperbolic vector field in S^3 satisfying the hypotheses of the next corollary can be found in [Morales and Pacifico 2003].

The following is a direct consequence of Theorems 3 and 4.

Corollary 7. *Let X be a singular-hyperbolic vector field with a unique singularity on a compact 3-manifold. If $\Omega_1(X)$ is attracting and $\Omega_2(X)$ is repelling, then $\Omega(X)$ is a finite union of transitive sets. If X is Kupka–Smale, then such an union is disjoint. In particular, the union is disjoint for a residual subset of vector fields in $\chi^r(M)$, $r \geq 1$.*

2. Proofs

We start with some preliminary results from [Morales and Pacifico 2004] to be used in the proof of the theorems.

Let X be a C^r vector field on a compact boundaryless 3-manifold $r \geq 1$. Denote by $Cl(A)$ the closure of a set A. In the statement of the following two theorems we let Λ be a compact invariant set of X satisfying the following properties:

(1) Λ is connected.

(2) Λ is an attracting singular-hyperbolic set.

(3) The periodic orbits of X contained in Λ are dense in Λ.

(4) Λ has a unique singularity σ.

Combining Lemma 2.1 and Theorem 2.8 in [Morales and Pacifico 2004] we obtain the following result.

Theorem 8. Λ *is the union of transitive sets. More precisely, it is itself transitive or is the union of two homoclinic classes.*

It follows from [Morales et al. 1999] that the singularity σ above has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

In particular, σ has a two-dimensional *stable manifold* $W^s(\sigma)$ and a one-dimensional *unstable manifold* $W^u(\sigma)$, each one tangent at σ to the eigenspaces associated to the eigenvalue sets $\{\lambda_2, \lambda_3\}$ and $\{\lambda_1\}$, respectively. It turns on also that σ has a *strong stable manifold* $W^{ss}(\sigma)$ contained in $W^s(\sigma)$ and tangent at σ to the set of eigenvalues $\{\lambda_2\}$. In particular, $W^{ss}(\sigma)$ divides $W^s(\sigma)$ in two connected components, denoted by $W^{s,+}$ and $W^{s,-}$.

Theorem 9 [Morales and Pacifico 2004, Theorem 2.8]. *If* Λ *is not transitive then for all* $a \in W^u_X(\sigma) \setminus \{\sigma\}$ *there is a periodic orbit* O *of* X *with a* ***positive*** *expanding eigenvalue and such that* $a \in W^s_X(O)$.

Proof of Theorem 3. Let Λ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity. Split Λ into finitely many connected components. Such components are clearly attracting with dense periodic orbits and the nonsingular ones are hyperbolic hence transitive by the Spectral Theorem [Smale 1967]. On the other hand, the singular component satisfies properties (1)–(4) above, so it is the union of transitive sets, by Theorem 8. Then Λ, which is the union of its components, must be a finite union of transitive sets. □

As already noted, it has been conjectured that every singular-hyperbolic attractor is a homoclinic class. If this conjecture were true, we would be able to strengthen Theorem 3, obtaining a finite union of homoclinic classes instead of transitive sets.

Remark 10. The proof above implies that Theorem 3 holds for an arbitrary number of singularities as long as they belong to different connected components of Λ. It also implies that the union in Theorem 3 can be chosen to be disjoint or formed by homoclinic classes.

Proof of Theorem 4. Let X a Kupka–Smale vector field in a compact 3-manifold and let Λ be an attracting singular-hyperbolic set of X with dense periodic orbits and a unique singularity σ. It suffices to prove that the component of Λ containing σ is transitive. Suppose, for a contradiction, that this is not so. Applying Theorem 9 to the (nontransitive) component containing σ, we would obtain $a \neq \sigma$ in the unstable manifold $W^u_X(\sigma)$ such that $\omega_X(a)$ is a periodic orbit O. But σ has a one-dimensional unstable manifold $W^u_X(\sigma)$, so the vector field X would exhibit a nontransversal intersection between $W^u(\sigma)$ and $W^s(O)$, a contradiction since X is Kupka–Smale. □

Proof of Theorem 6. Take $X \in \chi^r(M)$ and let Λ be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of X. If z belongs to a hyperbolic periodic orbit of X, denote by $H_X(z)$ the homoclinic class of X associated to z. Clearly one has $H_X(z) = H_X(z')$ whenever z, z' belong to the same hyperbolic periodic orbit.

Assume that Λ is *not* a disjoint union of transitive sets. Split Λ into finitely many connected components as before. It suffices to prove that each such component Λ_0 satisfies

$$\#\mathcal{H}_X(\Lambda_0) < \infty.$$

The nonsingular ones are hyperbolic [Morales et al. 1999], so they satisfy this requirement. In addition, these components are also transitive.

Now consider the singular component Λ_0. It clearly contains the sole equilibrium σ of Λ. As before, σ has a one-dimensional unstable manifold $W^u_X(\sigma)$ [Morales et al. 1999]. Then $W^u_X(\sigma) \setminus \{\sigma\}$ consists of two regular orbits. Fix a, a' in each orbit.

Observe that Λ_0 cannot be transitive, for otherwise Λ would be a disjoint union of transitive sets, contrary to the hypothesis. Then Theorem 9 applied to Λ_0 implies that $\omega_X(a) = O_0$ and $\omega_X(a') = O'_0$, where O_0 and O'_0 are periodic orbits with positive eigenvalues of X.

Now assume for a contradiction that $\#\mathcal{H}_X(\Lambda_0) = \infty$. There is an infinite sequence of periodic orbits $O_n \subset \Lambda_0$ and a infinite sequence $z_n \in O_n$ such that

$$H_X(z_n) \neq H_X(z_m) \quad \text{for } n \neq m. \tag{1}$$

Define $A = Cl\big(\bigcup_n H_X(z_n)\big)$. If $\sigma \notin A$ then A is hyperbolic, yielding $\mathcal{H}_X(A) < \infty$, which contradicts (1). We conclude that $\sigma \in A$, that is,

$$\sigma \in Cl\big(\bigcup_n H_X(z_n)\big).$$

Thus there is a sequence $x_n \in H_X(z_n)$ such that $x_n \to \sigma$. The Birkhoff–Smale Theorem [Palis and Takens 1993] implies that x_n is an accumulation point of periodic orbits homoclinically related to O_n. Hence, we can assume that $x_n = z_n$ without loss of generality.

Since O_n is periodic and z_n lies in O_n, we have $z_n \notin W^s_X(\sigma)$. So, O_n accumulates on either a_0 or a'_0. We shall assume the first case since the proof for the second case is analogous.

Because the expanding eigenvalue of O is positive, O divides its own unstable manifold $W^u_X(O)$ into two connected components $W^{u,+}$, $W^{u,-}$ labeled according to the following rule: Let $W^{s,+}$, $W^{s,-}$ be the two connected components of $W^s(\sigma)\setminus W^{ss}(\sigma)$. If $I^{\pm}$ is an interval with boundary point a and pointing to the side of $W^{s,\pm}$, then the positive orbit of $I^{\pm}$ accumulates on $W^{u,\pm}$, by the Inclination Lemma [Palis and de Melo 1982]. For details we refer the reader to [Morales and Pacifico 2004, Definition 2.9, p. 335]. The main property of $W^{u,\pm}$ is that if z lies in a periodic orbit and is sufficiently close to some point in $W^{u,\pm}$ then

$$H_X(z) = Cl(W^{u,\pm}). \tag{2}$$

This property is described in [Morales and Pacifico 2004, Proposition 2.13, p. 336].

Now we obtain the desired contradiction. Since $\omega_X(a_0) = O$ and O_n accumulates at a_0 we can find $z'_n \in O_n$ passing close to O as indicated in Figure 3. In particular, by the Inclination Lemma [Palis and de Melo 1982], we can assume that z'_n converges to a point in either $W^{u,+}$ or $W^{u,-}$. Again we assume the first case since the second one is analogous.

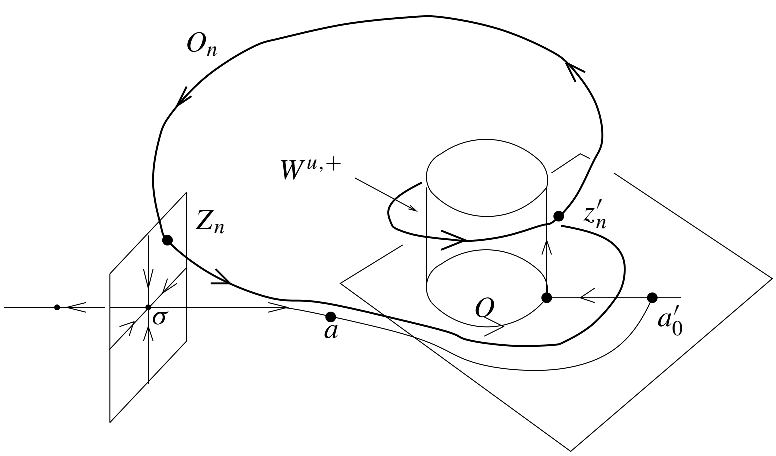

Figure 3

Then, (2) implies

$$H_X(z_n') = Cl(W^{u,+}) \quad \text{for all } n.$$

But z_n' is in the orbit of z_n so

$$H_X(z_n) = H_X(z_n').$$

Then, $H_X(z_n) = Cl(W^{u,+})$ and so

$$H_X(z_n) = H_X(z_m), \quad \text{for all } n, m.$$

However, this is a contradiction by (1). □

By Remark 10, the set Λ in Theorem 6 is also a finite union of homoclinic classes.

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