HANDLE ADDITIONS PRODUCING ESSENTIAL SURFACES

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We construct a small, hyperbolic 3-manifold $M$ with the property that, for any integer $g \geq 2$, there are infinitely many separating slopes $r$ in $\partial M$ such that the 3-manifold $M(r)$ obtained by attaching a 2-handle to $M$ along $r$ contains an essential separating closed surface of genus $g$. The resulting manifolds $M(r)$ are still hyperbolic. This contrasts sharply with known finiteness results on Dehn filling and with the known finiteness result on handle addition for the cases $g = 0, 1$. Our 3-manifold $M$ is the complement of a hyperbolic, small knot in a handlebody of genus 3.

1. Introduction

All manifolds in this paper are orientable and all surfaces $F$ in 3-manifolds $M$ are embedded and proper, unless otherwise specified. A surface $F \subset M$ is proper if $F \cap \partial M = \partial F$.

Let $M$ be a compact 3-manifold. An incompressible, $\partial$-incompressible surface $F$ in $M$ is essential if it is not parallel to $\partial M$. A 3-manifold $M$ is simple if $M$ is irreducible, $\partial$-irreducible, anannular and atoroidal. In this paper, a compact 3-manifold $M$ is said to be hyperbolic if $M$ with its toroidal boundary components removed admits a complete hyperbolic structure with totally geodesic boundary. By Thurston’s theorem, a Haken 3-manifold is hyperbolic if and only if it is simple. A knot $K$ in $M$ is hyperbolic if $M_K$, the complement of $K$ in $M$, is hyperbolic. A 3-manifold $M$ is small if $M$ contains no essential closed surface. A knot $K$ in $M$ is small if $M_K$ is small.

A slope $r$ in $\partial M$ is an isotopy class of unoriented essential simple closed curves in $F$. We denote by $M(r)$ the manifold obtained by attaching a 2-handle to $M$ along a regular neighborhood of $r$ in $\partial M$ and then capping off the possible spherical component with a 3-ball. If $r$ lies in a toroidal component of $\partial M$, this operation is known as Dehn filling.


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Essential surfaces are a basic tool in the study of 3-manifolds, and handle addition is a basic method to construct 3-manifolds. A central question connecting those two topics is the following:

**Question 1.** Let $M$ be a hyperbolic 3-manifold with nonempty boundary, containing no essential closed surface of genus $g$. How many slopes $r \subset \partial M$ are there such that $M(r)$ contains an essential closed surface of genus $g$? (The question is asked only for hyperbolic 3-manifolds to avoid possibly infinitely many slopes produced by Dehn twists along essential discs or annuli. The mapping class group of a hyperbolic 3-manifold is finite.)

The main result of this paper shows that there can be many such slopes:

**Theorem 1.** There is a small, hyperbolic knot $K$ in a handlebody $H$ of genus 3 such that, for any given integer $g \geq 2$, there are infinitely many separating slopes $r$ in $\partial H$ such that $H_K(r)$ contains an essential separating closed surface of genus $g$. Moreover the resulting manifolds $H_K(r)$ are still hyperbolic.

**Remarks.** Let $M$ be a hyperbolic 3-manifold with nonempty boundary.

1. Suppose $\partial M$ is a torus. W. Thurston’s pioneer result [1982] asserts that there are at most finitely many slopes on $\partial M$ such that $M(r)$ is not hyperbolic; hence the number of slopes in Question 1 is finite when $g = 0$ or 1. Sharp upper bounds for this number were given by Gordon and Luecke for $g = 0$, and by Gordon for $g = 1$; see the survey paper [Gordon 1997]. Hatcher [1982] proved that the number is finite for any $g$.

2. Suppose $\partial M$ has genus at least 2. Scharlemann and Wu [1993] have shown that if $g = 0$ or 1, there are only finitely many separating slopes $r$ such that $M(r)$ contains an essential closed surface of genus $g$. Recently Lackenby [2002] generalized Thurston’s finiteness result to handlebody attaching, proving that, for a hyperbolic 3-manifold $M$, there is a finite set $C$ of exceptional curves on $\partial M$ such that attaching a handlebody to $M$ yields a hyperbolic-like manifold if none of those curves bounds a meridian disc of the handlebody.

3. In [Qiu and Wang 2005] we proved Theorem 1 for $g$ even.

Theorem 1 and the finiteness results just cited give a global view about the answer of Question 1.

**Outline of the proof of Theorem 1 and organization of the paper.** In Section 2 we first construct a knot $K$ in the handlebody $H$ of genus 3 for Theorem 1, then we construct infinitely many surfaces $S_{g,l}$ of genus $g$ for each $g \geq 2$ such that

1. all those surfaces are disjoint from the given $K$, hence contained in $H_K$; and
2. for fixed $g$, all the $\partial S_{g,l}$ are connected and provide infinitely many slopes in $\partial H$ as $l$ varies. Those $\partial S_{g,l}$ will serve as the slopes $r$ in Theorem 1. We denote
by $\hat{S}_{g,l} \subset H_K(\partial S_{g,l})$ the closed surface of genus $g$ obtained by capping off the boundary of $S_{g,l}$ with a disk. We will prove in Section 3 that $\hat{S}_{g,l}$ is incompressible in $H_K(\partial S_{g,l})$. In Sections 4 and 5 we prove that the knot $K$ is hyperbolic and small.

2. Construction of the knot $K$ and the surfaces $S_{g,l}$ in $H$

Let $H$ be a handlebody of genus 3. Suppose that $B_1$, $B_2$ and $B_3$ are basis disks of $H$, and $E_1$, $E_2$ are disks in $H$ that separate $H$ into three solid tori $J_1$, $J_2$ and $J_3$. See Figure 1.

Let $c$ be a closed curve in $\partial H$ as in Figure 2. The boundary of $E_1 \cup E_2$ separates $c$ into 10 arcs $c_1, \ldots, c_{10}$, where $c_1, c_3, c_9 \subset J_1$ meet $B_1$ in two, one, one points respectively; $c_2, c_4, c_6, c_8, c_{10} \subset J_2$ meet $B_2$ in one, one, two, zero, one points respectively; $c_5, c_7 \subset J_3$ meet $B_3$ in one, three points respectively.

Let $u_1, \ldots, u_{2g}, v_1, \ldots, v_{2g}$ be $4g$ points located on $\partial E_1$ in the cyclic order $u_1$, $u_3$, $u_5$, $u_7$, $u_9$, $u_{2g-1}$, $u_{2g-2}$, $u_4$, $u_2$, $v_1$, $v_3$, $v_{2g-1}$, $v_{2g-2}$, $v_4$, $v_2$ as in Figure 3. In view of the order of these points, $C$ can be
isotoped so that $\partial c_1 = \{u_1, v_1\}$, $\partial c_2 = \{u_1, v_2\}$, $\partial c_{10} = \{v_1, u_2\}$, $\partial c_3 = \{v_2, u_3\}$, $\partial c_9 = \{u_2, v_3\}$. Now suppose $u_{2i+1}v_{2i}$ and $v_{2i+1}u_{2i}$, for $1 \leq i \leq g - 1$, are arcs in $\partial J_1 - \hat{E}_1$ parallel to $c_3$ and $c_9$, and that $u_2v_1 = c_{10}$, $v_2u_1 = c_2$, and $u_{2i}v_{2i-1} = v_{2i}u_{2i-1}$, for $2 \leq i \leq g$, are parallel arcs in $\partial (J_2 \cup J_3) - \hat{E}_1$, each of which intersects $B_2$ in one point and $B_3$ in $l$ points (see Figure 3, where $l = 2$). Finally define $\alpha_1 = u_1v_1$, and let $\alpha_k$ be the union of $v_{k-1}u_k$, $\alpha_{k-1}$ and $u_{k-1}v_k$, for $k = 2, \ldots, 2g$. Then $\alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_{2g}$ is an increasing sequence of arcs.

Let $\alpha \subset \partial H$ be an arc which meets $\partial S$ exactly at its two endpoints for a proper separating surface $S \subset H$. The surface resulting from tubing $S$ along $\alpha$ in $H$, denoted by $S(\alpha)$, is obtained by first attaching a 2-dimensional 1-handle $N(\alpha) \subset \partial H$ to $S$, then making the surface $S \cup N(\alpha)$ to be proper, that is, pushing its interior into the interior of $H$. The image of $N(\alpha)$ after the pushing is still denoted by $N(\alpha)$. In fact, $S \cup N(\alpha)$ is a once punctured torus. Since $S$ is orientable and separating, $S(\alpha)$ is still separating and orientable.

Since $\alpha_1$ meets $E_1$ exactly in its two endpoints, we do tubing on $E_1$ along $\alpha_1$ to get a proper surface $E_1(\alpha_1)$. Now $\alpha_2$ meets $E_1(\alpha_1)$ exactly in its two endpoints. We do tubing on $E_1(\alpha_1)$ along $\alpha_2$ to get $E_1(\alpha_1, \alpha_2) = E_1(\alpha_1)(\alpha_2)$, where the tube $N(\alpha_2)$ is thinner and closer to $\partial H$ so that it goes over the tube $N(\alpha_1)$. Hence $E_1(\alpha_1, \alpha_2)$ is a properly embedded surface (indeed, a one-punctured torus). By the same argument, we do tubing along $\alpha_3, \ldots, \alpha_{2g}$ to get a proper embedded surface $E_1(\alpha_1, \ldots, \alpha_{2g})$ in $H$, denoted by $S_{g, l}$. This surface is orientable and separating.

Since $S_{g, l}$ is obtained from the disc $E_1$ by attaching $2g$ 1-handles to $E_1$ such that the ends of any two handles are alternating, $S_{g, l}$ is a once punctured orientable surface of genus $g$. We summarize the facts just discussed:

**Lemma 2.1.** $S_{g, l}$ is a once punctured surface of genus $g$ and is separating in $H$. 


Now let $K$ be a knot in $\hat{H}$ obtained by first pushing $c_6$ into $\hat{H}$ deeply and then pushing $C - c_6$ into $\hat{H}$ so that it stays between $N(\alpha_3)$ and $N(\alpha_4)$. The following fact is clear:

**Lemma 2.2.** $K$ is disjoint from $S_{g,l}$ for all $g,l$.

3. **Proof of Theorem 1 assuming that $K$ is hyperbolic and small**

We denote by $\hat{S}_{g,l} \subset H_K(\partial S_{g,l}) \subset H(\partial S_{g,l})$ the surface obtained by capping off the boundary of $S_{g,l}$ with a disk. Then $\hat{S}_{g,l}$ is a closed surface of genus $g$.

From the definition of $S_{g,l}$ for a given genus $g$, the boundary $\partial S_{g,l}$ provides infinitely many boundary slopes as $l$ varies from 1 to infinity. Then Theorem 1 follows from the next two propositions (apart from the last assertion, which follows directly from [Scharlemann and Wu 1993]).

**Proposition 3.0.** $K \subset H$ is a hyperbolic, small knot.

**Proposition 3.1.** $\hat{S}_{g,l}$ is incompressible in $H_K(\partial S_{g,l})$.

We postpone the proof of the first of these results and prove the second here. Recall that a surface $F$ in a 3-manifold is compressible if either $F$ is a 2-sphere that bounds a 3-ball, or there is an essential simple closed curve in $F$ that bounds a disk in $M$; otherwise, $F$ is incompressible. Hence Proposition 3.1 is a consequence of the following result:

**Proposition 3.2.** $\hat{S}_{g,l}$ is incompressible in $H(\partial S_{g,l})$.

We choose the center of $E_1$ as the common base point for the fundamental groups of $H$ and of all surfaces $S_{g,l}$.

Now $\pi_1(S_{g,l})$ is a free group of rank $2n$ generated by $(x_1, \ldots, x_{2n})$, where $x_i$ is the generator given by the centerline of the tube $N(\alpha_i)$; and $\pi_1(H)$ is a free group of rank three generated by curves $y_1, y_2, y_3$ corresponding to $B_1, B_2, B_3$, as in Figure 1. Let $i : S_{g,l} \to H$ be the inclusion. One can read $i_*(x_i)$ directly as words in $y_1$, $y_2$, $y_3$:

\[
i_*(x_1) = y_1^2, \]
\[
i_*(x_2) = y_2 y_1^2 y_2, \]
\[
i_*(x_3) = y_1 y_2 y_1^2 y_2 y_1, \]
\[
i_*(x_4) = y_2 y_3^2 y_1 y_2 y_1^2 y_2 y_1 y_2 y_3^2, \]

and in general, for $2 \leq i \leq g$,

\[
i_*(x_{2i-1}) = y_1 (y_2 y_3^2 y_1)^{i-2} y_2 y_1^2 y_2 (y_1 y_2 y_3)^{i-2} y_1, \]
\[
i_*(x_{2i}) = (y_2 y_3^2 y_1)^{i-1} y_2 y_1^2 y_2 (y_1 y_2 y_3)^{i-1}. \]

**Lemma 3.3.** $S_{g,l}$ is incompressible in $H$. 

The proof is the same as that in [Qiu 2000].

Now $S_{g,l}$ separates $H$ into two components $P_1$ and $P_2$ with $\partial P_1 = T_1 \cup S_{g,l}$ and $\partial P_2 = T_2 \cup S_{g,l}$, where $T_1 \cup T_2 = \partial H$ and $\partial T_1 = \partial T_2 = \partial S_{g,l}$.

**Lemma 3.4.** $T_1$ and $T_2$ are incompressible in $H$.

*Proof.* We have $H_1(H) = \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$, with the three generators $\gamma_1, \gamma_2$ and $\gamma_3$. By the preceding argument, $i_*(H_1(S_{g,l}))$ is a subgroup of $H_1(H)$ generated by $2\gamma_1, 2\gamma_2$ and $2\gamma_3$. Thus $H_1(H)/i_*(H_1(S_{g,l})) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a finite group.

Suppose $T_1$ or $T_2$ is compressible. Then it bounds a compressing disk $D_1$ in $H$. Since $\partial D \cap \partial S_{g,l} = \emptyset$ and $S_{g,l}$ is incompressible in $H$, by a standard argument in 3-manifold topology, we may assume that $D_1 \cap S_{g,l} = \emptyset$. Since $H$ is a handlebody, we may also assume that $D_1$ is nonseparating in $H$. Thus there are two properly embedded disks $D_2$ and $D_3$ in $H$ such that $\{D_1, D_2, D_3\}$ is a basis of disks of $H$. Let $z_1, z_2, z_3$ be generators of $\pi_1(H)$ corresponding to $D_1, D_2, D_3$. Since $S_{g,l}$ misses $D_1$, we have $i_*(\pi_1(S_{g,l})) \subset G$, where $G$ is a subgroup of $\pi_1(H)$ generated by $z_2$ and $z_3$. Then $H_1(H)/i_*(H_1(S_{g,l}))$ is an infinite group, a contradiction. \qed

*Proof of Proposition 3.2.* Since $H$ is a handlebody and $S_{g,l}$ is incompressible in $H$, $P_1$ and $P_2$ are handlebodies. By Lemmas 3.3, 3.4 and the Handle Addition Lemma [Jaco 1984], $\hat{S}_{g,l}$ is incompressible in $P_i(\partial S_{g,l})$ for $i = 1, 2$. Since $H(\partial S_{g,l}) = P_1(\partial S_{g,l}) \cup \hat{S}_{g,l}, P_2(\partial S_{g,l})$, the surface $\hat{S}_{g,l}$ is incompressible in $H(\partial S_{g,l})$. \qed

4. $H_4$ is irreducible, $\partial$-irreducible and annular

By construction, $K$ is cut by $E_1 \cup E_2$ into ten arcs $a_1, \ldots, a_{10}$, where $a_i$ arises from pushing $c_i$ into $H$. Now let $N(K) = K \times D$ be a regular neighborhood of $K$ in $H$, where the product structure has been adjusted so that $\bigcup_{i=1}^{10} \partial a_i \times D$ is contained in $E_1 \cup E_2$. Let $H_K = H - \hat{N}(K)$ and $F_i = E_i - \hat{N}(K)$; also set $M_i = H_K \cap J_i$, for $i = 1, 2, 3$, and $T = \partial (K \times D)$. Then $F_1 \cup F_2$ separates $T$ into ten annuli $A_1, \ldots, A_{10}$ such that $A_i = a_i \times \partial D$.

$K$ and $C$ bound a nonembedded annulus $A_*$, which is cut by $E_1 \cup E_2$ into ten disk $D_{1*}, \ldots, D_{10*}$ in $H$. Note that $D_* = \bigcup_{i \neq 6} D_{i*}$ is still a disk. Let $D_i = D_{i*} \cap H_K$ for $i \neq 6$. Then $D_i$ is a proper disk in some $M_i$ and $\bigcup_{i \neq 6} D_i$ is still a disk; see Lemma 4.1. Now we number the $\partial A_i$ such that $\partial_1 A_i = \partial_2 A_{i-1}$ and $\partial_2 A_i = \partial_1 A_{i+1}$. For $i \neq 6$, let $W_i = \partial \hat{N}(D_i \cup A_1) - \partial M_i$. Then $W_i$ is a proper separating disk in $M_i$. Each $W_i$ intersects $F_1 \cup F_2$ in two arcs $l_i$ and $l_{i+1}$. Note that $W = \bigcup_{i \neq 6} W_i$ is a disk. Thus $\partial W$ is a union of two arcs in $\partial H$ and $l_6 \cup l_7$; see Figure 4. Since $c_3, c_9$ are parallel in $\partial J_1 - \hat{E}_1$, there are two arcs parallel to $c_3$ in $\partial J_1 - \hat{E}_1$, say $l', l''$, and two arcs in $F_1$, say $l_1, l_2$, such that $l' \cup l'' \cup l_1 \cup l_2$ bounds a disk $W'$ that separates $M_1$ into two handlebodies of genus two $H^1, H^2$ with $A_1 \subset H^1$ and $A_3, A_9 \subset H^2$.

We denote by $\mu$ the meridian slope on $T$ and by $\tau$ the longitude slope on $T$.
Lemma 4.0. (1) \( K \neq 1 \) in \( \pi_1(H) \).

(2) Suppose \( a_i \subset J_m \) where \( i \neq 4, 8 \). Let \( b_i \subset E_1 \cup E_2 \) be a given arc with \( \partial b_i = \partial a_i \) and let \( B \subset J_m \) be a nonseparating proper disk. Then \( a_i \cup b_i \) intersects \( \partial B \) in at least one point for all \( i \), in at least three points when \( i = 7 \), and in at least two points when \( i = 1, 6 \).

(3) There is no relative homotopy on \( (J_m, E_1 \cup E_2) \) sending \( a_i \) to \( E_1 \cup E_2 \).

Recall that a 3-manifold \( M \) is irreducible if it contains no essential 2-spheres. \( M \) is \( \partial \)-irreducible if \( \partial M \) is incompressible. \( M \) is atoroidal if it contains no essential tori. \( M \) is anannular if it contains no essential annuli.

Lemma 4.1. \( H_K \) is irreducible.

Proof. Suppose that \( H_K \) is reducible, so there is an essential 2-sphere \( S \) in \( H_K \). Since \( H \) is irreducible, \( S \) bounds a 3-ball \( B^3 \) in \( H \) and \( K \subset B^3 \), which contradicts Lemma 4.0(1).

Lemma 4.2. \( F_1 \cup F_2 \) is incompressible and \( \partial \)-incompressible in \( H_K \).

Proof. Suppose first that \( F_1 \cup F_2 \) is compressible in \( H_K \). Then there is a disk \( D \) in \( M \) such that \( \partial D \subset F_1 \) and \( \partial D \) is an essential circle on \( F_1 \). Without loss of generality, we assume that \( \partial D \subset F_1 \) and \( \partial D \subset M_2 \). Denote by \( B' \) the disk bounded by \( \partial B \) in \( E_1 \). Then \( B \cup B' \) is a 2-sphere \( S \subset J_2 \), and it follows easily from Lemma 4.1 that \( S \) bounds a 3-ball \( B^3 \) in \( J_2 \). Since \( \partial B \) is essential in \( F_1 \), \( B' \) contains at least one component of \( \partial a_i \). Since \( S \) is separating and \( a_i \) is connected, we must have \( (a_i, \partial a_i) \subset (B^3, B') \), which provides a relative homotopy on \( (J_2, E_1) \) sending \( a_i \) to \( E_1 \). This contradicts Lemma 4.0(2).
Now suppose $F_1 \cup F_2$ is $\partial$-compressible in $H$. There is an essential arc $a$ in $F_1 \cup F_2$ which, with an arc $b$ in $\partial H$, bounds a disk $B$ in $H_K$ with $B \cap (F_1 \cup F_2) = a$. Without loss of generality, we assume that $a \subset F_2$ and $B \subset M_2$. There are two cases: $b \subset T$. Then $b$ is a proper arc in one of $A_4$, $A_6$, and $A_8$, say $A_6$. If $b$ is not essential in $A_6$, then $a$ and an arc $b'$ in $\partial A_6$ form an essential circle in $F_2$ bounding a disc in $M_2$. This contradicts the incompressibility of $F_2$ we just proved. If $b$ is essential in $A_6$, the disk $B$ provides a relative homotopy on $(J_2, E_2)$ sending $a_6$ to $E_2$, which contradicts Lemma 4.0(2).

$b \subset \partial H$. If $B$ is nonseparating in $J_2$, then $b_6$ can be chosen so that $a_6 \cup b_6$ intersects $\partial B$ in at most one point, where $b_6$ is an arc in $E_2$ connecting the endpoints of $a_6$; this contradicts Lemma 4.0(2). If $B$ is separating in $J_2$, then $B$ separates $J_2$ into a 3-ball $B^3$ and a solid torus $J$. We denote by $D_1$, $D_2$ the two components of $E_2 - a$. Since $a$ is essential in $F_2$, each of $\hat{D}_1$ and $\hat{D}_2$ contains at least one endpoint of $a_4$, $a_6$ and $a_8$.

Suppose that $D_1 \subset B^3$ and $D_2 \cup E_1 \subset J$. By construction, $\partial_1 a_4$, $\partial_1 a_8 \subset E_1$, $\partial_2 a_4$, $\partial_2 a_8 \subset E_2$, and $\partial a_6 \subset E_2$. Since $a_4$, $a_6$ and $a_8$ are disjoint from $B$, we have $a_4$, $a_8 \subset J$ and $a_6 \subset B^3$. This contradicts Lemma 4.0(2).

Suppose that $D_1 \subset J$ and $D_2 \cup E_1 \subset B^3$. Then $a_2$, $a_{10} \subset B^3$. This contradicts Lemma 4.0(2). \hfill \Box

Lemma 4.3. $H_K$ is $\partial$-irreducible.

Proof. Suppose $H_K$ is $\partial$-reducible. Let $B$ be a compressing disk of $\partial H_K$. If $\partial B \subset T$, then $H_K$ contains an essential 2-sphere, which contradicts Lemma 4.1. Below we assume that $\partial B \subset \partial H$. Since $F_1 \cup F_2$ is incompressible and $\partial$-incompressible in $H_K$ (Lemma 4.2), by a standard cut and paste argument, we may assume that $B \cap (F_1 \cup F_2) = \emptyset$. We assume that $B \subset M_2$. (The other cases are similar.) Then $B$ misses $b_6$. If $B$ is nonseparating in $J_2$, by Lemma 4.0(2), $B$ intersects $a_6$, a contradiction. If $B$ is separating, then $B$ separates a 3-ball $B^3$ from $J_2$. Since $\partial B$ is essential in $\partial H_K$, there are two cases: Either $B^3$ contains only one of $E_1$ and $E_2$, say $E_1$, in which case $a_8 \cap B \neq \emptyset$, a contradiction; or $B^3$ contains both $E_1$ and $E_2$, in which case there is a relative homotopy on $(J_2, E_2)$ sending $a_6$ to $E_2$, in contradiction with Lemma 4.0(2). \hfill \Box

Lemma 4.4. $M$ is annular.

Proof. Suppose $H_K$ contains an essential annulus $A$. We can choose $A$ so that $|A \cap (F_1 \cup F_2)|$ is minimal among all essential annuli in $H_K$. This condition, together with Lemma 4.2 and the proof of Lemma 4.3, implies that each component of $A \cap (F_1 \cup F_2)$ is essential in both $A$ and $F_1 \cup F_2$. There are three cases: 

Case 1: $\partial A \subset T$. Here $A$ is separating in $H_k$; otherwise, $H$ contains either a nonseparating 2-sphere or a nonseparating torus. Hence the union of $A$ and an
annulus $A'$ on $T$ makes a separating torus $T'$, cutting off a manifold with boundary $T \cup T'$. Since $M$ is irreducible, $T'$ is incompressible, so by Lemma 5.5 $T'$ is parallel to $T$, which implies that $A$ is inessential. (The arguments in Section 5 are independent of those in Section 4.)

Case 2: $\partial_1A \subset T$ and $\partial_2A \subset \partial H$. By Lemma 4.3, both $\partial H$ and $T$ are incompressible in $H_K$. Clearly $H_K$ is not homeomorphic to $T \times I$. Since Dehn fillings along $\mu$ and $\partial A_1$ both compress $\partial H$, by an important theorem in Dehn filling, $\Delta(\partial_1A, \mu) \leq 1$. See [Culler et al. 1987, 2.4.3].

We first suppose that $\partial_1A$ is the meridian slope $\mu$. Then $\partial_1A$ is disjoint from $F_1 \cup F_2$. We claim that $A$ is disjoint from $F_1 \cup F_2$.

Suppose, to the contrary, that $A \cap (F_1 \cup F_2) \neq \emptyset$. Since $F_1 \cup F_2$ is incompressible and $\partial$-incompressible in $H_K$ (Lemma 4.2), by a standard cut and paste argument, we may assume that $\partial_2 A \cap (F_1 \cup F_2) = \emptyset$. Now each component of $A \cap (F_1 \cup F_2)$ is an essential simple closed curve in $A$. Let $a$ be an outermost circle in $A \cap (F_1 \cup F_2)$. Then $a$ and $\partial_1 A$ bound an annulus $A^*$ in $A$ such that $A^*$ is disjoint from $F_1 \cup F_2$.

We may assume that $a \subset F_1$ and $\partial_1 A \subset A_i$ for some $i$. Let $B^*$ be the disk bounded by $a$ on $E_1$ and let $D$ be the meridian disk of $N(K)$ bounded by $\partial_1 A$. Since $a$ is essential on $F_1$, $B^*$ contains at least one component of $\partial D$. In $H$, $B^* \cup A^* \cup D$ is a separating 2-sphere $S^2$ that bounds a 3-ball $B^3$. For $j \neq i$, if $\partial_1 A_j \subset B^*$, then $\partial_2 a_j \subset B^*$ and $a_j \subset B^3$. This possibility is ruled out by Lemma 4.0(2). Note also that $\partial_1 a_i \subset B^*$ and that $\partial_2 a_i$ is not contained in $B^*$. Now let $A'$ be the annulus bounded by $a$ and $\partial_1 a_i \times \partial D = \partial_1 A_i$ in $F_1$. Then $A^* \cup A'$ is isotopic to an annulus disjoint from $F_1 \cup F_2$. By the preceding argument, $A^* \cup A'$ is inessential. Thus we can properly isotope $A$ by pushing the annulus $A^*$ to the other side of $F_1$ to reduce $|A \cap (F_1 \cup F_2)|$, contradicting our choice of $A$ at the beginning of the proof.

We may assume that $A$ is contained in $M_2$. Let $D$ be the meridian disk of $N(K)$ bounded by $\partial_1 A$ and set $B = A \cup \partial_1 A$. Then $B$ is a proper disk in $J_2$, meeting $K$ in exactly one point; hence $B$ is a meridian disk of $J_2$. Let $b_6$ be an arc on $E_2$ connecting the two endpoints of $c_6$. Then $c_6 \cup b_6$ would be a closed curve of winding number 2 in the solid torus $J_2$ intersecting $B$ at most once, which is absurd.

Next we suppose that $\Delta(\partial_1A, \mu) = 1$. Then $A$ is cut by $(F_1 \cup F_2)$ into ten squares $S_i$, $i = 1, \ldots, 10$, each of which has two opposite sides in $F_1 \cup F_2$, the other two sides being the longitude arc $a_i$ in $A_i$ and $a^*_i \subset \partial H$. Let $b^*_z$ be the arc connecting the two endpoints of $a^*_z$ in $E_1$ and let $b_6^*$ be the arc connecting the two endpoints of $a^*_6$ in $E_2$. The two simple closed curves $b^*_z \cup a^*_z$ and $b^*_6 \cup a^*_6$ on $\partial J_2$ are disjoint. But in $\pi_1(J_2)$, we have $b^*_z \cup a^*_z = y_2$ and $b^*_6 \cup a^*_6 = y^2_2$, a contradiction.

Case 3: $\partial A \subset \partial H$. Suppose first that $A \cap (F_1 \cup F_2) = \emptyset$. Then $A$ is contained in one of $M_1$, $M_2$ and $M_3$. Since $A$ is essential and $H_K$ is $\partial$-irreducible, $A$ is disjoint from
$D_i$ for $i \neq 6$. Since each component of $\partial H \cap J_1 - c_1 \cup c_3$ and $\partial H \cap J_3 - c_5 \cup c_7$ is a disc, $A \subset M_2$. Since $A$ is disjoint from $c_2, c_4, c_8, c_{10}$, each component of $\partial A$ intersects $B_2$ in only one point in $J_2$ (see Figure 2). Thus $A$ is isotopic to each component of $\partial J_2 - \partial A$ in $J_2$. This means that $A$ is not essential in $M_2$, a contradiction.

Now suppose that $A \cap (F_1 \cup F_2) \neq \emptyset$. There are two subcases:

Case 3a: Each component of $A \cap (F_1 \cup F_2)$ is an essential circle. Let $a$ be an outermost component of $A \cap (F_1 \cup F_2)$. That means that $\partial_1 A$, together with $a$, bounds an annulus $A^*$ in $A$ such that $A^* \cap (F_1 \cup F_2) = a$. Then $A^* \subset M_i$. We denote by $B^*$ the disk bounded by $a$ in $E_1 \cup E_2$. Let $D^* = A^* \cup B^*$. Then $D^*$ is a disk. Let $D$ be the disk obtained from $D^*$ by pushing $B^*$ slightly into $J_l$. Then $D$ is a properly embedding disk in $J_l$ such that $D$ intersects each $a_i$ in at most two points. Furthermore, if $D$ intersects $a_i$ in two points for some $i$, the two endpoints of $a_i$ lie in $B^*$. Thus, in this case, the algebraic intersection number of $a_i$ and $D$ is 0. By Lemma 4.0, $A^*$ is separating in $J_l$.

Suppose that $A^*$ is contained in one of $J_1$ and $J_3$, say $J_1$. Then $\partial_1 A$ is parallel to $\partial E_1$. We denote by $A'$ the annulus bounded by $\partial_1 A$ and $a$ in $\partial J_1$. Since $a$ is essential in $F_1$, $B_2$ contains at least one endpoint of $a_1, a_3, a_9$. Furthermore, $\partial_1 a_i \subset B^*$ if and only if $\partial_2 a_i \subset B^*$. Now if $\partial_1 a_j \subset A'$ for some $j$, then $\partial_2 a_j \subset A'$. This means that $a_j$ is disjoint from $B_1$ as in Figure 1, a contradiction. Thus for each $i, j$, we have $\partial_j a_i \subset B^*$, which means that $a$ is parallel to $\partial E_1$ in $F_1$. Now $\partial D_i$, for $i = 1, 3, 9$, intersects each component of $\partial A^*$ in two points, which means that $D_i$ intersects $A^*$ in two arcs each of which has its two endpoints in distinct components of $\partial A^*$. (Otherwise, since $\partial_1 A$ is isotopic to $\partial E_1$, we would have $a_i \cup b_i = -1$ in $\tau_1(J_i)$, where $b_i$ is an arc in $\partial E_1$ connecting the two endpoints of $a_i$, a contradiction.) Thus we can push $\partial_1 A$ into $M_2$ to reduce $|A \cap (F_1 \cup F_2)|$, contradicting our assumption on $A$.

Suppose instead that $A^* \subset M_2$. Without loss of generality, we assume that $a \subset F_1$. We denote by $A'$ the annulus bounded by $\partial E_1$ and $a$ in $E_1$. Then $A'$ and $B^*$ lie on distinct sides of $J_2 - A^*$. If $\partial_1 A$ is isotopic to $\partial E_2$, then $a_8 \cup b_6 = -1$ in $\tau_1(J_2)$ where $b_6$ is an arc in $E_2$ connecting the two endpoints of $a_6$, a contradiction. If $\partial_1 A$ bounds a disk $D$ in $\partial J_2$ such that $E_1, E_2 \subset D$, then $a_4 \cup a_8 \cup b^1 \cup b^2 = 1$ in $\tau_1(J_2)$, where $b^i$ is an arc in $E_i$ connecting the endpoints of $a_{4i}$ and $a_8$, a contradiction. Now $\partial_1 A$ is isotopic to $\partial E_1$. Then $D_4$ intersects $A^*$ in an arc. By the preceding argument, we can push $\partial_1 A$ into $M_1$ to reduce $|A \cap (F_1 \cup F_2)|$.

Case 3b: Each component of $A \cap (F_1 \cup F_2)$ is an essential arc. Then $F_1 \cup F_2$ cuts $A$ into proper squares $S_l$ in $M_l$ for $l = 2$ or 3, each $S_l$ having two opposite sides in $F_1 \cup F_2$ and the remaining two sides in $\partial H$. If $S_l \subset J_l$ for $l = 2$ or 3, then $S_l$ is a separating disc in $J_l$. Otherwise, say $S_l$ is a nonseparating disc in $J_2$. By the same reason as that at the end of the proof of Lemma 4.3, the fact that $\tau_2 \cap (F_1 \cup F_2)$ consists of
two proper arcs in $E_1 \cup E_2$ implies that $b_6$ can be chosen so as to intersect $\partial S_i$ in at most two points; furthermore, if $b_6$ intersects $\partial S_i$ in two points then $S_i \cap F_1 = \emptyset$ and $S_i \cap b_2 = \emptyset$, where $b_i$ is an arc in $E_1 \cup E_2$ connecting the two endpoints of $a_i$. This means that $S_i$ meets $a_2$ or $a_6$ by Lemma 4.0(1), a contradiction. Now each $S_i$ cuts off a 3-ball $B^3_i$ from $J_l$ for $l = 2$ or 3 as in Figure 5. Let $S^1_i$ and $S^2_i$ be the two disks of $B^3_i \cap (E_1 \cup E_2)$ and $S_i \subset J_l$ where $l = 2$ or 3. By Lemma 4.0(2), we have:

(i) $\partial_1 a_j \subset S^1_i$ if and only if $\partial_2 a_j \subset S^2_i$.

(ii) If $a_j$ is contained in $B^3_i$, then $a_l$ is not contained in $B^3_i$.

This means that for each $i$, there is only one boundary component of $F_1 \cup F_2$ lying in each of $S^1_i$ and $S^2_i$. Thus if $S_i$ lies in $M_1$ for some $i$, then $S_i$ is also separating in $J_1$. Otherwise, say $S_i$ is nonseparating in $J_1$. By (i) and (ii), the three circles $a_1 \cup b_1$, $a_3 \cup b_3$, $a_9 \cup b_9$ intersect $S_i$ in two points, a contradiction. It follows that $S_i$ is also as in Figure 5 and $A$ cuts off a solid torus $P$ from $H$. Thus $D_{i_0}$ can be chosen to be disjoint from $A$ even if $i = 6$. This means that $K$ and a component of $\partial A$ bound an annulus, which has been ruled out in Case 2. □

5. $H_K$ contains no closed essential surface

Suppose $H_K$ contains essential closed surfaces. Let $W$, $W'$ and $W_i$ be the disks defined in Section 4. Denote by $X(F)$ the union of the components of $F \cap M_1$ isotopic to $\partial H \cap M_1$. We define the complexity on the essential closed surfaces $F$ in $H_K$ by the quadruple

$$C(F) = (|F \cap W|, |F \cap F_2|, |(F \cap M_1 - X(F)) \cap W'|, |F \cap F_1|).$$

We rank complexities in lexicographic order. Suppose $F$ minimizes $C(F)$. By a standard argument in 3-manifold topology, we derive the following facts:

**Lemma 5.0.** (1) Each component of $F \cap (F_1 \cup F_2)$ is an essential circle in both $F$ and $F_1 \cup F_2$.  

![Figure 5](image-url)
(2) Each component of $F \cap W$ is an arc in $W$ one of whose endpoints lies in $l_6$ and the other in $l_7$. Similarly each component of $F \cap W'$ is an arc in $W'$ one of whose endpoints lies in $l'_1$ and the other in $l'_2$. Hence $|F \cap l_i| = |F \cap l_j|$ for all $i, j$ and $|F \cap l^1| = |F \cap l^2|$ as in Figure 6.

(3) Each component of $F \cap (F_1 \cup F_2)$ isotopic to $\partial A_i$ is disjoint from $W \cup W'$.

For two surfaces $P_1$ and $P_2$ in a 3-manifold, a pattern of $P_1 \cap P_2$ is a set of disjoint arcs and circles representing isotopy classes of $P_1 \cap P_2$. For each isotopy class $s$, we denote by $\nu(s)$ the number of components of $P_1 \cap P_2$ in the isotopy class $s$.

The proof of the next lemma is similar to that of [Qiu and Wang 2004, Lemma 4.3].

**Lemma 5.1.** Each component of $F \cap M_3$ is isotopic to one of $\partial H \cap M_3$, $A_5$ and $A_7$.

**Proof.** The four arcs $l_5, l_6, l_7, l_8$ separate $F_2$ into four annuli $A^5, A^6, A^7, A^8$ and a disk $D$. By the minimality of $|F \cap W|$, the pattern of $F \cap A^j$ is as in Figure 7, left, and the pattern of $F \cap D$ is as in Figure 7, right. Since $|F \cap l_i|$ is a constant, $\nu(d_5) = 0$. If $\nu(d_i) \neq 0$ for $1 \leq i \leq 4$, then $F \cap F_2$ contains $\min(\nu(d_1), \ldots, \nu(d_4))$.
components parallel to a disk on $\partial E_2$. Now if $\nu(d_1) = 0$, then $\nu(d_3) = 0$. Similarly, if $\nu(d_2) = 0$, then $\nu(d_4) = 0$. Thus according to the order of $l_5, l_6, l_7, l_8$ in $F_2$, the pattern of $F \cap F_2$ is as in one of the diagrams in Figure 8, with $\nu(m_2) = \nu(m_3)$. Note that $W_5$ and $W_7$ separate $M_3$ into three solid tori $J_1, J_2, J_3$. Without loss of generality, we assume that $A_5 \subset J_1$, $A_7 \subset J_2$. Let $S = F \cap M_3$ and $S'$ be a component of $S$.

Now we claim that if one of component of $\partial S'$ is isotopic to $\partial E_2$, then $S'$ is isotopic to $\partial H \cap M_3$.

Let $\partial_1 S$ be the outermost component of $\partial S$ isotopic to $\partial E_2$. Now $\partial_1 S$ intersects $l_i$ as in Figure 8. Without loss of generality, we assume that $\partial_1 S \subset \partial S'$. We denote by $e_i$ the arc $\partial_1 S \cap A_i$. Now let $S_1 = S' \cap J_1$, then $S_1$ is an incompressible surface in $J_1$. Note that $\partial S_1 = e_5 \cup e_6 \cup (S \cap W_5)$ bounds a disk in $J_1$ parallel to a disk on $\partial M_3$. Similarly $S_2$ is a disk in $J_2$ parallel to a disk on $\partial M_3$ bounded by $e_7 \cup e_8 \cup (S \cap W_7)$. $\partial S_3$ also has one component which is trivial in $\partial M_3$, as in Figure 9, left. Hence one component of $S_3$ is a disk in $J_3$ parallel to $\partial J_3$. Thus $S' = S_1 \cup_{\partial J_1} S_3 \cup_{\partial J_2} S_2$ is isotopic to $M_3 \cap \partial H$.

Now we claim that $\nu(m_2) = \nu(m_3) = 0$ in both parts of Figure 8.

Let $S_0 = S - X'$, where $X'$ is a subset of $S$ each of whose components is isotopic to $\partial H \cap M_3$. Then no component of $\partial S_0$ is isotopic to $\partial E_2$. Let $P_3 = S_0 \cap J_3$. If
$\nu(m_2) \neq 0$, then $P_3$ is incompressible in $J^3$ and $\partial P_3$ contains $2\nu(m_2) = 2\nu(m_3)$ components $c$, as in Figure 9, right. Since $a_7$ intersects a basis disk $B_3$ of $J_3$ in three points and $a_5$ intersects $B_3$ in one point, $c$ does not bound a disk in $J^3$. Since $J^3$ is a solid torus, each component of $P_3$ is a $\partial$-compressible annulus. Let $D^*$ be a $\partial$-compressing disk of an outermost component of $P_3$. This disk can be isotoped so that $D^* \cap \partial J^3 \subset E_2 \cap J^3$. Then, back in $J_3$, $D^*$ is isotopic to one of $D_1, D_2, D_3$ as in Figure 10. In the case of $D_1$ or $D_2$, one can push $F$ along the disc to reduce $|F \cap W|$; in the case of $D_3$, one can push $F$ along the disc to reduce $|F \cap F_2|$, without increasing $|F \cap W|$. Either way, the minimality of $C(F)$ is contradicted.

Now let $P$ be a component of $S = F \cap M_3$. If one component of $\partial P$ is isotopic to $\partial E_2$, then $P$ is isotopic to $M_3 \cap \partial H$. If not, each component of $\partial P$ is isotopic to one component of $\partial A_5 \cup \partial A_7$. By the minimality of $C(F)$, $P$ is contained in $J^1$ or $J^2$. It is easy to see that $P$ is isotopic to one of $A_5$ and $A_7$. □

Now we consider $S = F \cap M_1$. Note that $W_1$ and $W'$ separate $M_1$ into two solid tori $J^1, J^2$ and a handlebody of genus two $H'$ such that $A_1 \subset J^1$ and $A_3, A_9 \subset H'$; moreover $l_1, l_2, l^1, l^2$ separate $F_1$ into two annuli and two planar surfaces with three boundary components and a disk $D$ such that $\partial J^2 \cap F_1 = D$. See Figure 11. Let $k_1$ be a component of $F \cap W_1$, $k_2$ a component of $F \cap W'$, and $k_i', i = 1, 2$, an arc in $D$ connecting the two endpoints of $k_i$. Let $\alpha = k_1 \cup k_1'$ and $\beta = k_2 \cup k_2'$. Note that $k_1'$ and $k_2'$ can be chosen so that $\beta$ intersects $\alpha$ in one point. Furthermore, by construction, $\alpha$ intersects a basis disk of $J^2$ in two points and $\beta$ intersects a basis disk of $J^2$ in one point. Now we fix the orientations of $\alpha$ and $\beta$ so that $\alpha = y^2$ and $\beta = y$, where $y$ is a generator of $\pi_1(J^2)$. Then $\alpha \beta^{-2}$ is an essential circle in $\partial J^2$ and null homotopic in $J^3$.

The next lemma follows immediately from the proof of Lemma 5.1.
Lemma 5.2. Let $P$ be a component of $S = F \cap M_1$. If one component of $\partial P$ is isotopic to $\partial E_1$. Then $P$ is isotopic to $M_1 \cap \partial H$.

By the construction and Lemma 5.0, the pattern of $\partial S \cap (F_1 \cap (J^1 \cup H'))$ is as in one of the diagrams in Figure 11, and moreover

1. in Figure 11, left, we have $v(f_1) = v(f_2)$, $v(f_3) = v(f_5)$, $v(f_4) = v(f_6)$ and $v(f_3) + v(f_4) = v(f_1)$;
2. in Figure 11, right, we have $v(f_1) = v(f_2) = v(f_3) + v(f_4)$, $v(f_3) = v(f_6)$ and $v(f_4) = v(f_5) = v(f_7) = v(f_8) \neq 0$.

Lemma 5.3. If the pattern of $S \cap (F_1 \cap (J^1 \cup H'))$ is as in Figure 11, left, the pattern of $S \cap F_1$ is as in Figure 12 with $v(n_2) = v(n_3) = v(n_4)$. 
Proof. If \( v(f_3) = 0 \), the pattern of \( S \cap F_1 \) is as in Figure 12 with \( v(n_2) = v(n_3) = v(n_4) \) and \( v(n_1) = 0 \).

Suppose instead that \( v(f_3) \neq 0 \). Since \( v(f_3) = v(f_5) \leq v(f_1) = v(f_2) \), the pattern of \( S \cap D \) is as in Figure 13, where \( v(d_1) = v(d_3) \) and \( v(d_2) = v(d_4) \). If \( v(d_1), v(d_2) \neq 0 \), then \( S \cap F_1 \) contains \( \min(v(d_1), v(d_2)) \) components isotopic to \( \partial E_1 \). Thus if \( v(d_1) = v(d_2) \), then \( S \cap F_1 \) is as in Figure 12 with \( v(n_2) = v(n_3) = v(n_4) \). Now without loss of generality, we assume that \( v(d_1) < v(d_2) \). Let \( k = v(d_2) - v(d_1) \).

By Lemmas 5.0(2) and 5.2, \( \partial(S \cap J^2) \) contains \( n = \gcd(k, k + v(d_5)) \) components \( c \) isotopic to \( \alpha \beta^q \), where \( |p| = (k + v(d_5))/n \) and \( |q| = k/n \). Since \( y + v(d_5) \geq y, c \) is not null homotopic in \( J^2 \). Moreover, \( c \) intersects both \( d_2 \) and \( d_4 \); if \( v(d_5) \neq 0 \), then \( c \) also intersects \( d_5 \). Thus these curves separate \( \partial J^2 \) into \( m \) annuli \( A^1, \ldots, A^m \) such that, for each \( j \), there is an arc in \( D \cap A^j \) connecting the two boundary components of \( A^j \). Since \( J^2 \) is a solid torus, each component of \( (S - X(F)) \cap J^2 \) is an annulus. Let \( D^* \) be a \( \partial \)-compressing disk of \( (S - X(F)) \cap J^2 \). Then \( D^* \) can be moved so that \( D^* \cap \partial J^2 = D^* \cap D = a \). Thus there are three possibilities:

1. The two endpoints of \( a \) lie in one of \( d_2, d_4, d_5 \). Then \( D^* \) is one of \( D^1, D^3 \) as in Figure 14, left. In each case, one can push \( F \) along the disc to reduce \( |F \cap W| \), a contradiction.

2. One endpoint of \( a \) lies in \( d_2 \cup d_4 \) and the other lies in \( d_5 \). Then \( D^* \) is \( D^2 \) as in Figure 14, left. This case is similar to the previous case.

3. One endpoint of \( a \) lies in \( d_2 \) and the other lies in \( d_4 \). In this case, \( v(d_5) = 0 \). By Lemma 5.0(2), we have \( v(f_4) = v(f_6) = 0 \) in Figure 11, left. Now the pattern of \( S \cap F_1 \) is as in Figure 14, right, and \( D^* \) is also as in the same figure. By doing a surgery on \( F \) along \( D^* \), we obtain a surface \( F' \) isotopic to \( F \) such that \( |F' \cap W| = |F \cap W|, |F' \cap F_2| = |F \cap F_2| \) and \( |(F' \cap M_1 - X(F)) \cap W'| < |(F \cap M_1 - X(F)) \cap W'| \) (by Lemma 5.2), contradicting minimality. \( \square \)
Lemma 5.4. If the pattern of $S \cap (F_1 \cap (J_1 \cup H'))$ is as in Figure 11, right, then the pattern of $S \cap F_1$ is as in Figure 15.

Proof. We have $\nu(f_1) = \nu(f_2) = \nu(f_3) + \nu(f_4) = \nu(f_5) + \nu(f_7)$. Thus the pattern of $S \cap D$ is as in Figure 16, where $\nu(d_1) = \nu(d_3)$, $\nu(d_2) = \nu(d_4)$, and $\nu(d_5) = 2\nu(f_5)$. Therefore $\nu(d_5) \neq 0$. Referring to Figure 11, right, we distinguish two cases: $\nu(f_3) = \nu(f_6) = 0$ and $\nu(f_3) = \nu(f_6) = 0$.

If $\nu(f_3) = \nu(f_6) = 0$, we have $\nu(d_5) = \nu(d_1) + \nu(d_2)$. There are three subcases:

Suppose first that $\nu(d_1) = \nu(d_2)$. Since $\nu(d_5) \neq 0$, $\partial(S \cap J^2)$ contains $\nu(d_1)$ trivial components in $\partial J^2$ bounding some disks in $S$ as in Figure 9, left, and $\nu(d_5)$ components isotopic to $\beta$. Since $\beta$ intersects a basis disk of $J^2$ in one point, each nontrivial component of $S \cap J^2$, say $A^*$, is an annulus parallel to each component.
completes the analysis when \( \nu(\partial E) \) is homotopic in \( J \). Thus there is a \( \partial \)-compressing disk \( D^* \) of \( S \cap J^2 \) as in Figure 16. By doing a surgery on \( F \) along \( D^* \), we can obtain a surface \( F' \) isotopic to \( F \) such that \( |F' \cap W| = |F \cap W| \), \( |F' \cap F_2| = |F \cap F_2| \), and \( |(F' \cap M_1 - X(F')) \cap W'| < |(F \cap M_1 - X(F)) \cap W'| \), a contradiction.

Suppose instead that \( \nu(d_1) < \nu(d_2) \). Set \( k = |v(d_2) - v(d_1)| \) and \( n = \gcd(k, k + v(d_5)) \). Then \( \partial(S \cap J^2) \) contains \( v(d_1) \) trivial components and \( n \) components \( c \) isotopic to \( \alpha^p \beta^q \), where \( |q| = (k + v(d_5))/n \) and \( |p| = k/n \). By construction, \( p > 0 \) if and only if \( q > 0 \). (See Figure 2.) That means that \( c \) is not null homotopic in \( J^2 \). By the proof of Lemma 5.3, we can obtain a surface \( F' \) isotopic to \( F \) such that \( C(F') < C(F) \), a contradiction.

Finally, suppose that \( \nu(d_1) > \nu(d_2) \), and define \( k \) as in the previous case. By the preceding argument, \( \partial(S \cap J^2) \) contains \( v(d_2) \) trivial components and \( n \) components \( c \) isotopic to \( \alpha^p \beta^q \), where \( |q| = (k + v(d_5))/n \) and \( |p| = k/n \). If \( c \) is not null-homotopic in \( J_2 \), then by the preceding argument, we can obtain a surface \( F' \) isotopic to \( F \) so that \( C(F') < C(F) \), a contradiction. Assume that \( q = -2p \). Then \( v(d_5) = v(d_1) - v(d_2) \). Since \( v(d_5) = v(d_1) + v(d_2) \), \( v(d_2) = 0 \) and \( v(d_5) = v(d_1) \). Thus \( F_1 \cap F \) is as in Figure 15 with \( v(n_2) = v(n_3) = v(n_4) \) and \( v(n_1) = 0 \). This completes the analysis when \( v(f_5) = v(f_6) = 0 \).

If \( v(f_3) = v(f_6) \neq 0 \) in Figure 11, right, there are two subcases:

Suppose first that \( \nu(d_1) \leq \nu(d_2) \). Then \( S \cap F_1 \) contains \( \min(v(d_1), v(f_3)) \) components isotopic to \( \partial E_1 \). If \( \nu(d_1) \geq v(f_3) \), we can obtain, by the same argument as in the preceding case, a surface \( F' \) isotopic to \( F \) such that \( C(F') < C(F) \), a contradiction. Assume that \( \nu(d_1) < v(f_3) \), then \( S \cap F_1 \) contains \( v(d_1) \) components isotopic to \( \partial E_1 \). Now \( 2v(f_1) = v(d_1) + v(d_2) \). By assumption, \( v(f_1) = v(f_3) + v(d_4) \). Thus \( \nu(d_1) < v(d_2) \). Then, by the proof of Lemma 5.3, \( \partial(S \cap J^2) \) contains \( \gcd(k, k + v(d_5)) \) components each of which is isotopic to \( \alpha^p \beta^q \), where

![Figure 16](image-url)
\(|q| = (k + \nu(d_5))/n\) and \(|p| = k/n\) (here again we have set \(k = |v(d_2) - v(d_1)|\) and \(n = \gcd(k, k + v(d_5))\). If \(q \neq -2p\), then by the proof of Lemma 5.3, there is in \(H_K\) an essential closed surface \(F'\) isotopic to \(F\) such that \(C(F') < C(F)\), a contradiction. Since \(y = v(d_2) - v(d_1) = 2(v(f_1) - v(d_1)) > 2(v(f_1) - v(f_3)) = 2v(f_3)\), we conclude that \(v(d_5) = 2v(f_5)\). Thus \(q \neq -2p\).

If, on the other hand, \(v(d_1) > v(d_2)\), then \(S \cap F_1\) contains \(\min(v(d_2), v(f_3))\) components isotopic to \(\partial E_1\). If \(v(d_2) \geq v(f_3)\), then by the same argument as before the pattern of \(F \cap F_1\) is as in Figure 15, with \(v(n_1) = v(f_3)\) and \(v(n_2) = v(n_3) = n(n_4)\). But then we see that it is impossible to have \(v(d_1) < v(f_3)\). □

**Lemma 5.5.** \(H_K\) contains no closed essential surface.

**Proof.** Suppose, to the contrary, that \(H_K\) contains an essential closed surface \(F\) such that the complexity \(C(F)\) is minimal among all surfaces isotopic to \(F\). By Lemma 5.1, the pattern of \(F \cap F_2\) is as in one of the diagrams of Figure 8. Furthermore, \(v(m_2) = v(m_3) = 0\) for any case. By Lemmas 5.3 and 5.4, the pattern of \(F \cap F_1\) is as in one of Figures 12 and 15. Furthermore, \(v(n_2) = v(n_3) = v(n_4)\). By Lemma 5.5, \(v(n_1) = v(n_2) = v(m_1)\).

In \(M_2\), the pattern of \(F \cap F_1\) can be labeled as in one of the diagrams on the top row of Figure 17, and the pattern of \(F \cap F_2\) can be labeled as in Figure 17, bottom.

![Figure 17](image-url)
Note that $W_2, W_4, W_8, W_{10}$ separate $M_2$ into four solid tori $J^1, J^2, J^4, J^5$ and a handlebody of genus two $H'$ such that $A_{2i} \subset J^i$ for $i = 1, 2, 4, 5$ and $A_6 \subset H'$. Let $S = F \cap H'$.

Now we claim that $\nu(n_2) = \nu(n_3) = \nu(n_4) = 0$. There are two cases:

Case 1. The pattern of $F \cap F_1$ is as in Figure 17, top left. Now each component of $\partial S$ is contained in one of the eight families $x_1, \ldots, x_8$ as in Figures 18 and 19, where the boundary components of $\partial S$ contained in $\bigcup_{i=1}^4 x_i$ are produced by cutting along the arcs in $F \cap (W_2 \cup W_4 \cup W_8 \cup W_{10})$ whose endpoints lie in $m_1 \cup n_1$ and the components of $\partial S$ contained in $x_7 \cup x_8$ are produced by cutting along the arcs whose endpoints lie in $n_2 \cup n_3 \cup n_4 \cup m_1$, and each component in $x_5 \cup x_6$ is isotopic to one component of $\partial A_6$. Each component lying in $x_3 \cup x_4$ is trivial in $\partial H'$. By observation, there are two disks $D^1$ and $D^2$ in $\partial H'$ such that $\partial D^i = b_i \cup b'_i$, where $b_i \subset F_1$ and $b'_i \subset S$ as in Figure 19. Back to $M_2$, $D^1$ and $D^2$ are as in Figure 12. Thus by doing surgeries on $F$ along $D^1$ and $D^2$, we can obtain a surface $F'$ isotopic to $F$ such that $|F' \cap W| = |F \cap W|$, $|F' \cap F_2| = |F \cap F_2|$ and $|(F' \cap M_1 - X(F')) \cap W'| < |(F \cap M_1 - X(F)) \cap W'|$, contradicting minimality.

Case 2. The pattern of $F \cap F_1$ is as in Figure 17, top right. This is similar to Case 1.
Now $v(n_2) = v(n_3) = v(n_4) = 0$ and $\partial S$ is as in Figure 18. By construction, there is a disk $B^* = H' \cap D_{6\ast}$ in $H'$ such that $\partial B^*$ intersects each component in $x_1 \cup x_2 \cup x_3 \cup x_6$ in only one point as in Figure 18. Thus $S \cap B^*$ offers a $\partial$-compressing disk $D^*$ of $S$ such that $D^*$ is disjoint from $\bar{A}_6$. We denote by $A$ the annulus bounded by an outermost component of $x_1$, say $e_1$, and an outermost component of $x_2$, say $e_2$, in $\partial H'$, and $T_1$ the punctured torus bounded by an outermost component of $x_1$ and an outermost component of $x_2$ in $\partial H'$ as in Figure 18. Now if $\partial D^* \cap \partial H' = a \subset A$, then $e_1 \cup e_2$ bounds an annulus in $S$ parallel to $A$. This means that one component of $F \cap M_2$ is parallel to $\partial H \cap M_2$.

Let $X_0(F)$ be a union of components in $F \cap M_2$ parallel to $\partial H \cap M_2$ or $A_6$, and set $S = (F \cap M_2 - X_0(F)) \cap H'$. Then $(F \cap M_2 - X_0(F)) \cap H' \cap B^*$ offers a $\partial$-compressing disk, also denoted by $D^*$, of $S$ such that $\partial D^* \cap \partial H' = a$.

We claim each component of $S$ is isotopic to one component of $\partial A_6$. There are five possibilities:

1. **The two endpoints of a lies in $x_5(x_6)$**. Then $D^*$ can be moved to be $D^1$ as in Figure 20(a). Thus by doing a surgery on $F$ along $D^1$, we can obtain a surface $F'$ isotopic to $F$ such that $|F' \cap W| = |F \cap W|$, $|F' \cap F_2| < |F \cap F_2|$, a contradiction.

2. **The two endpoints of a lies in $x_{1}(x_2)$**. Then $D^*$ can be moved to be $D^2$ as in Figure 20(b), contradicting the minimality of $|F \cap W|$.

3. **One endpoint of a lies in $x_5$ and the other lies in $x_6$**. Since $\partial B^*$ intersects $\bigcup_{i=1}^{6} x_i$ in the order $x_6$, $x_3$, $x_1$, $x_2$, $x_4$, $x_5$, there is by the argument in (1) an outermost component of $S \cap B^*$ in $B^*$, say $b$, which, together with an arc $b^*$ in $\partial H'$, bounds an outermost disk $D$ such that $\partial b$ is contained in $x_5$, $\partial b^*$ is contained in $x_6$ and $b^*$ intersects $A_6$ in an arc. Since $S$ is incompressible, by the standard argument, the component of $S$ containing $b$ is parallel to $A_6$, a contradiction.

4. **One endpoint of a lies in $x_1$ and the other lies in $x_2$**. Then $\partial_1 a \subset c_1$ and $\partial_2 a \subset c_2$, where $c_1$ is a component of $x_1$ and $c_2$ is a component of $x_2$. We denote again by $A$ the annulus bounded by $c_1, c_2$ in $\partial H'$ and by $T_1$ the punctured torus bounded by...
c_1, c_2 in \partial H'. Note that a is disjoint from \hat{A}_6 and A_6 \subset T_1. Hence a \subset A. By the preceding argument, the component of F \cap M_2 consisting of c_1 and c_2 is parallel to \partial H \cap M_2. By the definition of S, this is impossible.

(5) One endpoint of a lies in x_1 \cup x_2 and the other lies in x_5 \cup x_6. Since S is incompressible, each component c of x_3 \cup x_4 bounds a disk D_c in S parallel to a disk D'_c on \partial H'; see Figure 18. Let S^* = S - \bigcup_{c \in x_3 \cup x_4} D_c. Note that \partial B^* intersects \bigcup_{i=1}^6 x_i in the order x_6, x_3, x_1, x_2, x_4, x_5. Hence each component of S \cap B^* is an arc b such that \partial_1 b \subset x_1 \cup x_2 and \partial_2 b \subset x_5 \cup x_6. Otherwise there would be an outermost component b^* of S^* \cap B^* in B^* such that \partial b^* is as in one of the above four cases, a contradiction.

Each component of S \cap B^* is an arc b such that \partial_1 b \subset x_1 \cup x_2 and \partial_2 \subset x_5 \cup x_6. Set H^* = H' - B^* \times (0, 1) and S^{**} = S^* - B^* \times (0, 1), where B^* \times I is a regular neighborhood of B^* in H'. Then H^* is a solid torus. Since each component of x_1 \cup x_2 \cup x_5 \cup x_6 intersects \partial B^* in one point, each component h of \partial S^{**} is obtained by doing a band sum of one component h_1 of x_5 \cup x_6 and one component h_2 of x_1 \cup x_2 along a component of S^* \cap B^*. Since h_1 = 1 \in \pi_1(H), we have h_2 \neq 1 \in \pi_1(H^*), so h \neq 1 \in \pi_1(H^*). Recall the disk B_2 in H defined in Section 2. The intersection B_2 \cap H' is a planar surface P such that one component of \partial P, say \partial_1 P, is disjoint from A_6, and the other components of \partial P lie in \hat{A}_6. Furthermore, \partial_1 P intersects each component in x_1 \cup x_2 in one point. Hence P - B^* \times (0, 1) is a properly embedded disk in H^* intersecting each component of \partial S^{**} in one point. This means that each component of S^{**} is an annulus A parallel to each component of \partial H^* - \partial A.

Suppose that D is a \partial-compressing disk of A in H^* such that the arc \alpha = D \cap \partial H^* lies on the annulus A^* on \partial H^* which contains the disk A_6 - B^* \times (0, 1). Then D is disjoint from x_3 \cup x_4. Since the disk D^* = B^* \times \{0, 1\} \cup (A_6 - B^* \times (0, 1)) intersects \partial A^* in two arcs, D can be moved to have the arc \alpha lying on A^* - D^*. Furthermore, since each component h of \partial S^{**} is obtained by doing a band sum of one component h_1 of x_5 \cup x_6 and one component h_2 of x_1 \cup x_2, we may assume that \partial \alpha \subset x_1 \cup x_2. Hence D is also a \partial-compressing disk of S^* in H'. By the preceding argument, this is impossible.

Also by the preceding argument, if one component of F \cap (F_1 \cup F_2) is parallel to \partial E_1 or \partial E_2 then it is parallel to \partial H. Suppose that each component of F \cap (F_1 \cup F_2) is isotopic to one component of \partial A_i. By the minimality of C(F), F is disjoint from W_i for i \neq 6 and F is also disjoint from \partial N(B^* \cup A_6) - \partial H' in H'. Thus each component of F \cap M_j is an annulus parallel to A_i for some i. That means that F is isotopic to T, a contradiction. □

Proof of Proposition 3.0. The proposition follows immediately from Lemmas 4.1, 4.3, 4.4 and 5.5 and [Scharlemann and Wu 1993, Theorem 1]. □
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