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A submanifold M_r^n of pseudo-Euclidean space E_s^4 is said to have harmonic mean curvature vector if $\Delta\vec{H} = \vec{0}$, where $\vec{H}$ denotes the mean curvature vector field and Δ the Laplacian of the induced pseudo-Riemannian metric. We prove that every nondegenerate Lorentz hypersurface M_1^3 of E_1^4 with harmonic mean curvature vector is minimal.

1. Introduction

A submanifold M of a Riemannian manifold N is called *minimal* if its mean curvature vector field $\vec{H}$ vanishes. Of particular interest are minimal submanifolds of Euclidean spaces E^m. They are special cases of larger classes of submanifolds such as submanifolds of finite type, or submanifolds with harmonic mean curvature vector field. The study of such submanifolds was initiated by B.-Y. Chen [1993; 1996] in the context of the theory of submanifolds of finite type.

Let M be an n-dimensional connected submanifold of Euclidean space E^m. Denote by $\vec{x}$, $\vec{H}$, and Δ the position vector field of M, the mean curvature vector field of M, and the Laplace operator of M with respect to the induced Riemannian metric of M. It is well known (see [Chen 1984], for instance) that

$$\Delta\vec{x} = -n\vec{H}. \tag{1}$$

This equation shows that M is a minimal submanifold of E^m if and only if its coordinate functions are harmonic. We also observe that every minimal submanifold of E^m satisfies

$$\Delta\vec{H} = \vec{0}. \tag{2}$$

Submanifolds of E^m that satisfy condition (2) are said to have *harmonic mean curvature vector field*. These submanifolds are often called *biharmonic* since, in view of (1), condition (2) is equivalent to $\Delta^2\vec{x} = \vec{0}$. The question that naturally

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arises is whether the class of biharmonic submanifolds is larger than that of minimal submanifolds. For a survey of progress on this problem for E^m, see [Chen 1991].

Conjecture (B.-Y. Chen). *The only biharmonic submanifolds of Euclidean spaces are the minimal submanifolds.*

This conjecture is supported by the work of several researchers. Chen himself [1984] proved that every biharmonic surface in E^3 is minimal. I. Dimitrić [1989; 1992], generalizing Chen's result, proved that any biharmonic submanifold M of a Euclidean space E^m is minimal if it is a curve, a submanifold with constant mean curvature, a hypersurface with at most two distinct principal curvatures, a pseudo-umbilical submanifold of dimension $n \neq 4$, or a submanifold of finite type.

Th. Hasanis and Th. Vlachos [1995] proved that every biharmonic hypersurface in E^4 is minimal. Their work used coordinates and required lengthy computer calculations. In [Defever 1998] one of us gave a coordinate-free and more concise proof of the same theorem, using purely analytical arguments that afford greater insight into the structure of the hypersurface.

In contrast to the Euclidean case, the conjecture generally fails for submanifolds in a pseudo-Euclidean space E_s^m. This is not unexpected: a problem formulated in Euclidean spaces often appears considerably different when considered in pseudo-Euclidean spaces. B.-Y. Chen and S. Ishikawa [1991] gave examples of nonminimal biharmonic space-like surfaces with constant mean curvature in pseudo-Euclidean spaces E_s^4 ($s = 1, 2$). The same authors [1998] classified pseudo-Riemannian biharmonic surfaces of signature $(1, 1)$ with constant nonzero mean curvature and flat normal connection in E_s^4.

However, biharmonicity implies minimality in some special cases. It was shown in [Chen and Ishikawa 1998] that any biharmonic surface in E_s^3 ($s = 1, 2$) is minimal, and in [Defever et al. 2006] that every biharmonic hypersurface M_r^3 of E_s^4 ($s = 1, 2, 3$) whose shape operator is diagonal is minimal.

Here we address the same question for biharmonic Lorentz hypersurfaces of pseudo-Euclidean space E_1^4, where no restriction for the shape operator is imposed. By work of A. Z. Petrov [1969] and M. Magid [1984; 1985], besides the diagonal form of the shape operator, there are three additional canonical forms. We prove that for each such canonical form of the shape operator of a biharmonic Lorentz hypersurface in E_1^4, the mean curvature is zero (see Propositions 1, 2 and 3). This is done by considering two possibilities, namely whether or not H is a constant. In the first case, we prove that H must be zero. In the second case, we look at the vector ∇H, and show that this can be either space-like or light-like. Hence:

Theorem. *Every nondegenerate biharmonic Lorentz hypersurface of four-dimensional pseudo-Euclidean space E_1^4 is minimal.*

2. Preliminaries

Biharmonic submanifolds. Let M_1^3 be a Lorentz hypersurface of pseudo-Euclidean space E_1^4. Let $\vec{\xi}$ denote a unit normal vector field with $\langle\vec{\xi},\vec{\xi}\rangle = 1$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M_1^3 and E_1^4 respectively. For any vector fields X, Y tangent to M_1^3, the Gauss formula is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\vec{\xi},$$

where h is the scalar-valued second fundamental form. If we denote by S the shape operator of M_1^3 associated to $\vec{\xi}$, the Weingarten formula is given by

$$\tilde{\nabla}_X \vec{\xi} = -S(X),$$

where $\langle S(X), Y\rangle = h(X, Y)$. If $H = \frac{1}{3}\operatorname{tr} S$, the mean curvature vector $\vec{H} = H\vec{\xi}$ is a well defined normal vector field to M_1^3 in E_1^4. The Codazzi equation is given by

$$(\nabla_X S)Y = (\nabla_Y S)X, \tag{3}$$

and the Gauss equation by

$$R(X, Y)Z = \langle S(Y), Z\rangle S(X) - \langle S(X), Z\rangle S(Y); \tag{4}$$

see [O'Neill 1983]. A hypersurface M_1^3 of E_1^4 is said to have harmonic mean curvature vector field if

$$\Delta\vec{H} = 0.$$

This condition is equivalent to

$$\Delta\vec{H} = 2S(\nabla H) + 3H(\nabla H) + \{\Delta H + H\operatorname{tr} S^2\}\vec{\xi} = \vec{0};$$

see [Chen and Ishikawa 1991]. Therefore we have the following necessary and sufficient conditions for a hypersurface M_1^3 of E_1^4 to be biharmonic:

$$S(\nabla H) = -\frac{3H}{2}(\nabla H), \tag{5}$$

$$\Delta H + H\operatorname{tr} S^2 = 0, \tag{6}$$

where the Laplace operator Δ acting on scalar-valued function f is given by

$$\Delta f = -\sum_{i=1}^{3} \varepsilon_i (e_i e_i f - \nabla_{e_i} e_i f); \tag{7}$$

see [Chen and Ishikawa 1991], for example. Here $\{e_1, e_2, e_3\}$ is a local orthonormal frame of $T_p M_1^3$ with $\langle e_i, e_i\rangle = \varepsilon_i = \pm 1$.

Hypersurfaces in pseudo-Euclidean spaces. Consider the real vector space $\mathbb{R}^4$ with the standard basis $\{e_1, e_2, e_3, e_4\}$. Let $\langle\ ,\ \rangle$ denote the indefinite inner product on $\mathbb{R}^4$ whose matrix with respect to the standard basis is $\mathrm{diag}(-1, 1, 1, 1)$. This is called the Lorentz metric on $\mathbb{R}^4$. The space $\mathbb{R}^4$ with this metric is called 4-*dimensional pseudo-Euclidean space*, and is denoted by E^4_1.

A vector $X \in E^4_1$ is called *time-like*, *space-like*, or *light-like* according to whether $\langle X, X\rangle$ is negative, positive, or zero. A nondegenerate hypersurface M^3_r of pseudo-Euclidean space E^4_1 can itself be endowed with a Riemannian or a Lorentzian metric structure, according to whether the metric induced on M^3_r from the Lorentzian metric on E^4_1 is (positive) definite or indefinite. In the former case a normal vector to M^3_r is time-like, and in the latter case a normal vector to M^3_r is space-like.

The shape operator of a Riemannian submanifold is always diagonalizable, but this is not the case for the shape operator of a Lorentzian submanifold. We know from [Petrov 1969, 50–55] that a symmetric endomorphism of a vector space with a Lorentzian inner product can be put into one of four possible canonical forms. In particular, the matrix representation G of the induced metric on M^3_1 is of Lorentz type, so the shape operator S of M^3_1 can be put into one of the following four forms with respect to frames $\{e_1, e_2, e_3\}$ at $T_pM^3_1$ [Magid 1984; 1985]:

(I) $$S = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(II) $$S = \begin{pmatrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(III) $$S = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 1 & 0 & \lambda \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(IV) $$S = \begin{pmatrix} \mu & -\nu & 0 \\ \nu & \mu & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \nu \neq 0.$$

The matrices G for cases (I) and (IV) are with respect to an orthonormal basis of $T_pM^3_1$, whereas for cases (II) and (III) are with respect to a *pseudo-orthonormal basis*. This is a basis $\{e_1, e_2, e_3\}$ of $T_pM^3_1$ satisfying $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0$, and $\langle e_1, e_2\rangle = \langle e_3, e_3\rangle = 1$. In this work we examine cases (II), (III) and (IV), where the hypersurface M^3_1 of E^4_1 has nondiagonal shape operator. Case (I) has been studied in [Defever et al. 2006].

3. Biharmonic hypersurfaces

Let M_1^3 be a biharmonic hypersurface in E_1^4. Then conditions (5) and (6) are satisfied. In order to prove the Theorem we need to show that the mean curvature H vanishes. We will consider each case for the shape operator S separately.

Assume the shape operator S has the canonical form (II).

Case 1. H is constant. Then equation (6) implies that

$$H \operatorname{tr} S^2 = 0.$$

If H is zero the result follows. Otherwise, $\operatorname{tr} S^2 = 0$ implies that $2\lambda^2 + \lambda_3^2 = 0$, so $\lambda = \lambda_3 = 0$, and since $2\lambda + \lambda_3 = 3H$, we also obtain that $H = 0$.

Case 2. H is not constant. Hence $\nabla H \neq \vec{0}$. As the shape operator has the canonical form (II) (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$), then $S(e_1) = \lambda e_1 + e_2$, $S(e_2) = \lambda e_2$, and $S(e_3) = \lambda_3 e_3$. Therefore, by using (5) we conclude that ∇H can be considered either in the direction of e_3, or in the direction of e_2. In the first case, ∇H is space-like (it cannot be time-like since $\langle e_3, e_3\rangle = 1$), and $\lambda_3 = -\frac{3}{2}H$. In the second case ∇H is light-like, and $\lambda = -\frac{3}{2}H$. These two cases need to be examined separately.

Proposition 1. *Let M_1^3 be a Lorentz biharmonic hypersurface of pseudo-Euclidean space E_1^4 with shape operator of type (II), and suppose ∇H is space-like. Then M_1^3 is minimal.*

Proof. We assume that $H \neq 0$ and we will end up with a contradiction. Since ∇H is nonzero, the vector equation (5) shows that ∇H is an eigenvector of S with eigenvalue $-\frac{3}{2}H$. We write

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k.$$

We take into account the action of S on the basis $\{e_1, e_2, e_3\}$, and use the Codazzi equations (3). The relations

$$\begin{aligned}
&\langle(\nabla_{e_1}S)e_2, e_1\rangle = \langle(\nabla_{e_2}S)e_1, e_1\rangle, && \langle(\nabla_{e_2}S)e_3, e_3\rangle = \langle(\nabla_{e_3}S)e_2, e_3\rangle,\\
&\langle(\nabla_{e_1}S)e_3, e_3\rangle = \langle(\nabla_{e_3}S)e_1, e_3\rangle, && \langle(\nabla_{e_2}S)e_3, e_2\rangle = \langle(\nabla_{e_3}S)e_2, e_2\rangle,\\
&\langle(\nabla_{e_1}S)e_2, e_3\rangle = \langle(\nabla_{e_2}S)e_1, e_3\rangle, && \langle(\nabla_{e_1}S)e_3, e_2\rangle = \langle(\nabla_{e_3}S)e_1, e_2\rangle,\\
&\langle(\nabla_{e_2}S)e_3, e_1\rangle = \langle(\nabla_{e_3}S)e_2, e_1\rangle
\end{aligned}$$

imply that $\omega_{21}^1 = \omega_{22}^2$, $\omega_{32}^3 = \omega_{31}^3 = \omega_{23}^1 = 0$, $\omega_{12}^3 = \omega_{21}^3$, $e_3(\lambda) = (\lambda_3 - \lambda)\omega_{13}^1$, $e_3(\lambda) = (\lambda_3 - \lambda)\omega_{23}^2$. From the last two equations we obtain that $\omega_{13}^1 = \omega_{23}^2$, and from $\operatorname{tr} S = 3H = 2\lambda + \lambda_3$, it follows that $\lambda = \frac{9}{4}H \neq \lambda_3$.

Further, the conditions

$$\nabla_{e_p}\langle e_1, e_1\rangle = \nabla_{e_p}\langle e_2, e_2\rangle = \nabla_{e_p}\langle e_3, e_3\rangle = \nabla_{e_p}\langle e_1, e_3\rangle = \nabla_{e_p}\langle e_2, e_3\rangle = 0$$

for $p = 1, 2, 3$ imply that $\omega_{p1}^2 = \omega_{p2}^1 = \omega_{p3}^3 = 0$, and $\omega_{p1}^3 = -\omega_{p3}^2$, $\omega_{p2}^3 = -\omega_{p3}^1$. As a consequence, we also obtain that $\omega_{33}^1 = \omega_{33}^2 = \omega_{22}^3 = 0$. Therefore, the covariant derivatives $\nabla_{e_i}e_j$ simplify to

$$\begin{aligned}
&\nabla_{e_1}e_1 = \omega_{11}^1 e_1, &&\nabla_{e_1}e_2 = \omega_{12}^2 e_2 + \omega_{12}^3 e_3, &&\nabla_{e_1}e_3 = \omega_{13}^1 e_1 + \omega_{13}^2 e_2\\
&\nabla_{e_2}e_1 = \omega_{21}^3 e_3, &&\nabla_{e_2}e_2 = 0, &&\nabla_{e_2}e_3 = \omega_{23}^2 e_2\\
&\nabla_{e_3}e_1 = 0, &&\nabla_{e_3}e_2 = \omega_{32}^2 e_2, &&\nabla_{e_3}e_3 = 0.
\end{aligned}$$

Next, we construct an orthonormal basis $\{X_1, X_2, X_3\}$ from the pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ such that

$$X_1 = \frac{e_1 + e_2}{\sqrt{2}}, \quad X_2 = \frac{e_1 - e_2}{\sqrt{2}}, \quad X_3 = e_3.$$

The shape operator S with respect to this new basis takes the form

$$S = \begin{pmatrix} \lambda + \frac{1}{2} & \frac{1}{2} & 0\\ -\frac{1}{2} & \lambda - \frac{1}{2} & 0\\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Note that X_3 is still in the direction of ∇H, and that $\lambda_3 = -\frac{3}{2}H$. Therefore, since $\nabla(H) = X_1(H)X_1 + X_2(H)X_2 + X_3(H)X_3$, we have

$$X_1(H) = X_2(H) = 0, \quad X_3(H) \neq 0. \tag{8}$$

Since M_1^3 is a Lorentz hypersurface, $\operatorname{tr} S = 3H$, $\lambda = \frac{9}{4}H$, and $\operatorname{tr} S^2 = \frac{99}{8}H^2$. By expressing the Laplace operator (7) in terms of the basis $\{X_1, X_2, X_3\}$, we reduce Equation (6) to

$$-\big(X_1X_1(H) - \nabla_{X_1}X_1(H)\big) + \big(X_2X_2(H) - \nabla_{X_2}X_2(H)\big) \\ -\big(X_3X_3(H) - \nabla_{X_3}X_3(H)\big) + H\Big(\frac{99H^2}{8}\Big) = 0,$$

which by the use of (8) becomes

$$\nabla_{X_1}X_1(H) - \nabla_{X_2}X_2(H) - e_3e_3(H) + \frac{99H^3}{8} = 0. \tag{9}$$

On the other hand, an easy computation shows that

$$\nabla_{X_1}X_1 = \tfrac{1}{2}\big(\omega_{11}^1 e_1 + \omega_{11}^3 e_3 + \omega_{12}^2 e_2 + \omega_{12}^3 e_3 + \omega_{21}^3 e_3\big)$$

and similarly for $\nabla_{X_2} X_2$, so we obtain

$$\nabla_{X_1} X_1(H) = \tfrac{1}{2}\big(\omega_{11}^3 + \omega_{12}^3 + \omega_{21}^3\big) e_3(H),$$
$$\nabla_{X_2} X_2(H) = \tfrac{1}{2}\big(\omega_{11}^3 - \omega_{12}^3 - \omega_{21}^3\big) e_3(H).$$

Hence equation (9) simplifies to

$$e_3 e_3(H) - 2\omega_{12}^3 e_3(H) - \frac{99H^3}{8} = 0. \tag{10}$$

Substituting $\lambda = \frac{9}{4}H$ into $e_3(\lambda) = (\lambda_3 - \lambda)\omega_{13}^1$ we obtain

$$e_3(H) = -\frac{5H}{3}\omega_{13}^1 = \frac{5H}{3}\omega_{12}^3. \tag{11}$$

We evaluate the Gauss equation (4) for $\langle R(e_3, e_1)e_2, e_3\rangle$ and equate the left-hand side by using the definition of the curvature tensor to obtain

$$e_3(\omega_{12}^3) = (\omega_{12}^3)^2 - \frac{27H^2}{8}. \tag{12}$$

Applying e_3 to both sides of (11) and using (12) we get

$$e_3 e_3(H) = \frac{40H}{9}(\omega_{12}^3)^2 - \frac{45H^3}{8}.$$

Substituting this into (10) and using (11) we obtain

$$\tfrac{5}{9}(\omega_{12}^3)^2 - 9H^2 = 0, \tag{13}$$

since we have assumed that $H \neq 0$. Acting with e_3 on (13) and using expressions (11) and (12), we simultaneously obtain that

$$\frac{5}{9}(\omega_{12}^3)^2 - \frac{255H^2}{8} = 0.$$

Therefore, H must be zero. □

Proposition 2. *Let M_1^3 be a Lorentz biharmonic hypersurface of pseudo-Euclidean space E_1^4 with shape operator of type (II), and suppose ∇H is light-like. Then M_1^3 is minimal.*

Proof. By hypothesis ∇H is along the vector e_2, and $\lambda = -\frac{3}{2}H$. Since $\operatorname{tr} S = 3H$, we have $\lambda_3 = 6H$. Because the basis $\{e_1, e_2, e_3\}$ is pseudo-orthonormal, it follows that $\nabla(H) = e_2(H)e_1 + e_1(H)e_2 + e_3(H)e_3$. Therefore,

$$e_2(H) = e_3(H) = 0, \quad e_1(H) \neq 0. \tag{14}$$

By writing $\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k$, we obtain

$$\begin{aligned}
0 &= \nabla_{e_i} \langle e_j, e_k\rangle = \langle \nabla_{e_i} e_j, e_k\rangle + \langle e_j, \nabla_{e_i} e_k\rangle \\
&= \omega_{ij}^1 \langle e_1, e_k\rangle + \omega_{ij}^2 \langle e_2, e_k\rangle + \omega_{ij}^3 \langle e_3, e_k\rangle + \omega_{ik}^1 \langle e_j, e_1\rangle + \omega_{ik}^2 \langle e_j, e_2\rangle + \omega_{ik}^3 \langle e_j, e_3\rangle.
\end{aligned}$$

By assigning i, j, k any values from $\{1, 2, 3\}$, certain of the ω_{ij}^k vanish, and others satisfy simple relations. In particular, we obtain

$$\nabla_{e_1} e_3 = -\omega_{12}^3 e_1 + \omega_{13}^2 e_2, \qquad \nabla_{e_3} e_1 = \omega_{31}^1 e_1 + \omega_{31}^3 e_3, \tag{15}$$

$$\nabla_{e_2} e_3 = -\omega_{22}^3 e_1 - \omega_{21}^3 e_2, \qquad \nabla_{e_3} e_2 = -\omega_{31}^1 e_2 + \omega_{32}^3 e_3. \tag{16}$$

Using relations (14) we get $[e_2, e_3](H) = e_2 e_3(H) - e_3 e_2(H) = 0$. Also, since $[e_2, e_3](H) = \nabla_{e_2} e_3(H) - \nabla_{e_3} e_2(H)$, it follows that $\omega_{22}^3 = 0$, so relations (16) simplify to

$$\nabla_{e_2} e_3 = -\omega_{21}^3 e_2, \quad \nabla_{e_3} e_2 = -\omega_{31}^1 e_2 + \omega_{32}^3 e_3. \tag{17}$$

We use the Codazzi equations to obtain

$$\langle (\nabla_{e_1} S) e_3, e_3\rangle = \langle (\nabla_{e_3} S) e_1, e_3\rangle, \quad \langle (\nabla_{e_2} S) e_3, e_3\rangle = \langle (\nabla_{e_3} S) e_2, e_3\rangle,$$

which, combined with (15) and (17), imply respectively that

$$e_1(\lambda_3) = \omega_{32}^3, \tag{18}$$

$$e_2(\lambda_3) = (\lambda - \lambda_3)\omega_{32}^3. \tag{19}$$

Using (14) and that $\lambda_3 = 6H$, relation (19) implies that $(\lambda - \lambda_3)\omega_{32}^3 = 0$. If $\omega_{32}^3 = 0$, then from (18) it follows that $e_1(\lambda_3) = 0$, which contradicts (14). If $\lambda = \lambda_3$, then $-\frac{3}{2}H = 6H$, i.e. $H = 0$. □

***Assume the shape operator S has the canonical form* (III).**

Case 1. H is constant. Then Equation (6) implies that

$$H \operatorname{tr} S^2 = 0.$$

If H is zero the result follows. Otherwise, $\operatorname{tr} S^2 = 0$ implies that $\lambda = 0$. But since $\operatorname{tr} S = 3\lambda = 3H$, it also follows that $H = 0$.

Case 2. H is not constant. Then $\nabla H \neq \vec{0}$, and the vector equation (5) shows that ∇H is an eigenvector of S with eigenvalue $-\frac{3}{2}H$. Since the shape operator has the canonical form (III) (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$), then $S(e_1) = \lambda e_1 + e_3$, $S(e_2) = \lambda e_2$, and $S(e_3) = e_2 + \lambda e_3$ (with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ of $T_p M_1^3$). Hence, ∇H is in the direction of e_2, i.e., it is light-like, and $\lambda = -\frac{3}{2}H$. We will prove the following:

Proposition 3. *Let M_1^3 be a Lorentz biharmonic hypersurface of pseudo-Euclidean space E_1^4 with shape operator of type* (III), *and suppose ∇H is light-like. Then M_1^3 is minimal.*

Proof. The shape operator S, with respect to the orthonormal basis $\{X_1, X_2, X_3\}$ of $T_pM_1^3$ considered in Proposition 1, takes the form

$$S = \begin{pmatrix} \lambda & 0 & \frac{1}{\sqrt{2}} \\ 0 & \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \lambda \end{pmatrix}.$$

Since $\operatorname{tr} S = 3H$, it follows that $3\lambda = -\frac{9}{2}H = 3H$, so $H = 0$. □

***Assume the shape operator S has the canonical form* (IV).**

Case 1. H is constant. Then the scalar equation (6) becomes

$$H \operatorname{tr} S^2 = 0.$$

If H is zero, the result follows. Otherwise, $\operatorname{tr} S^2 = 2\mu^2 - 2\nu^2 + \lambda_3^2 = 0$, and from the form of the shape operator, we also have $2\mu + \lambda_3 = 3H \neq 0$. We apply the Codazzi equations (3) to the relations

$$\langle (\nabla_{e_i} S)e_j, e_k \rangle = \langle (\nabla_{e_j} S)e_i, e_k \rangle$$

for each triplet (i, j, k) in the set

$$\{(1,2,1), (1,2,2), (1,3,1), (2,3,2), (1,3,3), (2,3,3), (1,2,3), (1,3,3), (2,3,1)\},$$

and obtain the following linear system of nine equations (we denote λ_3 by λ from now on):

$$\begin{aligned}
-e_1(\nu) &= e_2(\mu) + \nu(1 - \varepsilon_1\varepsilon_2)\omega_{21}^2, \\
e_2(\nu) &= e_1(\mu) + \nu(\varepsilon_1\varepsilon_2 - 1)\omega_{12}^1, \\
(\lambda - \mu)\omega_{13}^1 + \nu\omega_{13}^2 &= e_3(\mu) + \nu(1 - \varepsilon_1\varepsilon_2)\omega_{32}^1, \\
(\lambda - \mu)\omega_{23}^2 - \nu\omega_{23}^1 &= e_3(\mu) - \nu(1 - \varepsilon_1\varepsilon_2)\omega_{31}^2, \\
e_1(\lambda) &= (\mu - \lambda)\omega_{31}^3 + \nu\omega_{32}^3, \\
e_2(\lambda) &= -\nu\omega_{31}^3 + (\mu - \lambda)\omega_{32}^3, \\
-\nu\omega_{11}^3 + (\mu - \lambda)\omega_{12}^3 &= \nu\omega_{22}^3 + (\mu - \lambda)\omega_{21}^3, \\
e_3(\nu) &= (\lambda - \mu)\omega_{13}^2 - \nu\omega_{13}^1, \\
-e_3(\nu) &= (\lambda - \mu)\omega_{23}^1 + \nu\omega_{23}^2.
\end{aligned}$$

We assume that $\varepsilon_1\varepsilon_2 = 1$ (the case $\varepsilon_1\varepsilon_2 = -1$ can be treated accordingly) so $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_3 = -1$. Applying e_i $(i = 1, 2, 3)$ to the equation $2\mu^2 - 2\nu^2 + \lambda^2 = 0$ we conclude that

$$e_i(\nu) = \frac{\mu - \lambda}{\nu} e_i(\mu),$$

and taking into account the relation $\omega_{ij}^k = -\varepsilon_j\varepsilon_k\omega_{ik}^j$, the system above simplifies to the following linear system of seven equations in six unknowns ω_{13}^1, ω_{13}^2, ω_{23}^2, ω_{23}^1, ω_{31}^3, ω_{32}^3:

$$\begin{aligned} (\lambda-\mu)\omega_{13}^1 + \nu\omega_{13}^2 &= e_3(\mu),\\ (\lambda-\mu)\omega_{23}^2 - \nu\omega_{23}^1 &= e_3(\mu),\\ (\mu-\lambda)\omega_{31}^3 + \nu\omega_{32}^3 &= e_1(\lambda),\\ -\nu\omega_{31}^3 + (\mu-\lambda)\omega_{32}^3 &= e_2(\lambda),\\ -\nu\omega_{13}^1 + (\mu-\lambda)\omega_{13}^2 - \nu\omega_{23}^2 + (\lambda-\mu)\omega_{23}^1 &= 0,\\ -\nu\omega_{13}^1 + (\lambda-\mu)\omega_{13}^2 &= e_3(\nu),\\ (\lambda-\mu)\omega_{23}^1 + \nu\omega_{23}^2 &= -e_3(\nu). \end{aligned} \tag{20}$$

We wish to show that this system has no (nonzero) solution. If $\mu = \lambda$, the conditions $\operatorname{tr} S^2 = 0$ and $\operatorname{tr} S = 3H$ imply that μ, λ, ν are constants, so the system simplifies to a homogeneous system with only the trivial solution. If $\mu \neq \lambda$, it follows that the ranks of the coefficient matrix and the augmented matrix of system (20) are always different, with only one possible exception, namely

$$(\lambda-\mu)^2 e_3(\nu) - \nu^2 e_3(\nu) + 2(\lambda-\mu)\nu e_3(\mu) = 0, \tag{21}$$

which we wish to exclude. Taking into account the relations $2\mu^2 - 2\nu^2 + \lambda^2 = 0$ and $2\mu + \lambda = 3H$, equation (21) is equivalent to the possibilities, either $9H^2 e_3(\nu) = 0$, or $(8\nu^2 - 9H^2)e_3(\nu) = 0$.

If $e_3(\nu) = 0$, then ν is constant (recall that $H \neq 0$). Therefore $e_3(\mu) = 0$, so μ is constant as well, which implies that λ is constant, so $e_1(\lambda) = e_2(\lambda) = 0$. Hence the linear system (20) becomes a homogeneous system with the trivial solution only. If $8\nu^2 - 9H^2 = 0$, then ν is constant, so argue similarly. To summarize, the possibility that H is a nonzero constant and $\operatorname{tr} S^2 = 0$, is excluded.

Case 2. H is not constant. Since the shape operator, with respect to an orthonormal basis $\{e_1, e_2, e_3\}$ of $T_pM_1^3$, has the canonical form (IV), we have $S(e_1) = \mu e_1 + \nu e_2$, $S(e_2) = -\nu e_1 + \mu e_2$, and $S(e_3) = \lambda_3 e_3$. This means that ∇H is in the direction of e_3, i.e., space-like.

The following proposition is proved along the same lines as Proposition 1.

Proposition 4. *Let M_1^3 be a Lorentz biharmonic hypersurface of pseudo-Euclidean space E_1^4 with shape operator of type (IV), and suppose ∇H is space-like. Then M_1^3 is minimal.*

Proof. We assume that $H \neq 0$ and we will end up to a contradiction. Then $\nabla H \neq \vec{0}$ and the vector equation (5) shows that ∇H is an eigenvector of S with eigenvalue $-\frac{3}{2}H$. Then $\lambda_3 = -\frac{3}{2}H$, and

$$e_1(H) = e_2(H) = 0, \quad e_3(H) \neq 0.$$

From the equation $\operatorname{tr} S = 3H$, it follows that $\mu = \frac{9}{4}H$. Next, we try to obtain simplified expressions for $\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k$. We apply the Codazzi equations (3) for

$$\langle(\nabla_{e_1} S)e_3, e_1\rangle, \quad \langle(\nabla_{e_2} S)e_3, e_2\rangle, \quad \langle(\nabla_{e_1} S)e_3, e_2\rangle, \quad \langle(\nabla_{e_1} S)e_3, e_3\rangle, \quad \langle(\nabla_{e_2} S)e_3, e_3\rangle$$

and obtain

$$e_3(H) = -\frac{5H}{3}\omega_{13}^1, \quad e_3(H) = -\frac{5H}{3}\omega_{23}^2, \quad e_3(\nu) = -\nu\omega_{13}^1$$
$$\frac{15H}{4}\omega_{31}^3 + \nu\omega_{32}^3 = 0, \quad \frac{15H}{4}\omega_{32}^3 - \nu\omega_{31}^3 = 0.$$

Therefore $\omega_{13}^1 = \omega_{23}^2$, and since H and ν are not zero, $\omega_{31}^3 = \omega_{32}^3 = 0$. Taking into account the condition $\omega_{ij}^k = -\varepsilon_j \varepsilon_k \omega_{ik}^j$, the previous relations give $\omega_{33}^1 = \omega_{33}^2 = 0$. Finally, since $[e_1, e_2](H) = 0$, it follows that $\nabla_{e_1} e_2(H) - \nabla_{e_2} e_1(H) = 0$; thus $\omega_{12}^3 = \omega_{21}^3 = 0$.

Next, we use the Gauss equation (4) and the definition of the curvature tensor for $\langle R(e_1, e_3)e_1, e_3\rangle$ and $\langle R(e_3, e_2)e_3, e_2\rangle$ to obtain

$$e_3(\omega_{11}^3) = -(\omega_{13}^1)^2 + \frac{27H^2}{8}, \quad e_3(\omega_{23}^2) = -(\omega_{23}^2)^2 + \frac{27H^2}{8}. \tag{22}$$

Hence, in view of (7), and taking into account the relations $\omega_{11}^3 = -\varepsilon_1\varepsilon_3\omega_{13}^3 = \omega_{13}^3$, $\omega_{22}^3 = -\varepsilon_2\varepsilon_3\omega_{23}^2 = -\omega_{23}^2$, and $\omega_{13}^1 = \omega_{23}^2$, equation (6) reduces to

$$e_3 e_3(H) + 2\omega_{13}^1 e_3(H) - H\left(\frac{99H^2}{8} - 2\nu^2\right) = 0. \tag{23}$$

Applying e_3 to both sides of the equality $e_3(H) = -\frac{5}{3}H\omega_{13}^1$ and using (22), we get

$$e_3 e_3(H) = \frac{40H}{9}(\omega_{13}^1)^2 - \frac{45H^3}{8},$$

so (23) becomes

$$\tfrac{10}{9}(\omega_{13}^1)^2 + 2\nu^2 - 18H^2 = 0. \tag{24}$$

Acting with e_3 on (24) we obtain

$$\tfrac{10}{9}(\omega_{13}^1)^2 + 2\nu^2 - \frac{135H^2}{4} = 0.$$

The two last equations imply that $H = 0$, which is a contradiction. □

The Theorem stated in the introduction now follows from Propositions 1–4.

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