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**FULL EXTENSIONS AND APPROXIMATE UNITARY
EQUIVALENCE**

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Let A be a unital separable amenable C^* -algebra and let C be a unital C^* -algebra with a certain infinite property. We show that two full monomorphisms $h_1, h_2 : A \rightarrow C$ are approximately unitarily equivalent if and only if $[h_1] = [h_2]$ in $KL(A, C)$. Let B be a nonunital but σ -unital C^* -algebra for which $M(B)/B$ has a certain infinite property. We prove that two full essential extensions are approximately unitarily equivalent if and only if they induce the same element in $KL(A, M(B)/B)$. The set of approximately unitarily equivalence classes of full essential extensions forms a group. If A satisfies the Universal Coefficient Theorem, the group can be identified with $KL(A, M(B)/B)$.

1. Introduction

The study of C^* -algebra extensions originated in the study of essentially normal operators on the infinite-dimensional separable Hilbert space. The original Brown–Douglas–Fillmore theory gives a classification of essential normal operators via certain Fredholm related indices (see [Brown et al. 1973b]). Later the theory was expanded to yield a classification of essential extensions of $C(X)$ by compact operators [Brown et al. 1973a; Brown 1984]. The study of C^* -algebra extensions developed into Kasparov’s KK -theory and its applications can be found not only in operator theory and operator algebras but also in differential geometry and non-commutative geometry.

Let

$$0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$$

be an essential extension of A by B . The extension is determined by a monomorphism $\tau : A \rightarrow M(B)/B$, the Busby invariant. When B is a σ -unital stable C^* -algebra, $KK^1(A, B)$ gives a complete classification of such essential extensions, up to stable unitary equivalence. However, $KK^1(A, B)$ gives little information, if any, about unitary equivalence classes of these mentioned extensions when $B \neq \mathcal{K}$ in

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general. There are examples in which $KK^1(A, B) = \{0\}$ but inequivalent nontrivial extensions exist [Lin 1998, Example 0.6]. There are also examples in which there are infinitely many inequivalent classes of trivial extensions [Lin 1995b, 7.4 and 7.5]. When B is not stable, $KK^1(A, B)$ certainly should not be used to understand unitary equivalence classes of the essential extensions mentioned above.

There are a number of results in classification of essential extensions (up to unitary equivalence or approximate unitary equivalence) when $B \neq \mathcal{K}$. Kirchberg's results [1996] on extensions in which B is a nonunital purely infinite simple C^* -algebra show that $KK^1(A, B)$ can be used to compute unitary equivalence classes of those extensions. If B is a nonunital but σ -unital simple C^* -algebra with continuous scale (see item (6) on page 391), then $M(B)/B$ is simple. A classification of essential extensions of a separable amenable C^* -algebra A by B (up to approximate unitary equivalence) was obtained in [Lin 2005b] (for some special cases in which $A = C(X)$, a classification up to unitary equivalence was obtained in [Lin 1995a; 1995b; 1997]). In this case, B may not be stable, so $KK^1(A, B)$ is not used as an invariant for essential extensions. Results about extensions of AF-algebras can be found in [Brown and Elliott 1982; Goodearl and Handelman 1982; Elliott and Handelman 1989].

Here we study full essential extensions. These are essential extensions $\tau : A \rightarrow M(B)/B$ such that $\tau(a)$ is a full element for each nonzero element $a \in A$. Since the Calkin algebra $M(\mathcal{K})/\mathcal{K}$ is simple, all essential extensions by \mathcal{K} are full. If B is a nonunital but σ -unital purely infinite simple C^* -algebra, $M(B)/B$ is also simple. Therefore essential extensions by those C^* -algebras are all full. The homogeneous extensions of A by $C(X) \otimes \mathcal{K}$ studied by Pimsner, Popa and Voiculescu [Pimsner et al. 1979; 1980] are all full extensions. In these three cases, B is stable. There are nonstable, nonunital but σ -unital C^* -algebras that have continuous scale. In that case essential extensions by these C^* -algebras are also full. Furthermore, if A is a unital simple C^* -algebra and if the monomorphism $\tau : A \rightarrow M(B)/B$ is unital, then the essential extension induced by τ is always full for any nonunital C^* -algebra B .

With a technical condition on $M(B)/B$, we show that two full essential extensions are approximately unitarily equivalent if they induce the same element in $KL(A, M(B)/B)$ (see Theorem 2.5), provided that A is amenable and separable. When A is assumed to satisfy the (Approximate) Universal Coefficient Theorem, we show that there is a bijective correspondence between approximate unitary equivalence classes of essential and full extensions and $KL(A, M(B)/B)$. The advantage of studying these full extensions is that full extensions (in these cases) are “approximately absorbing”. For stable B , we show that $KK^1(A, B)$ classifies the unitary equivalence classes of full essential extensions. In this case, full extensions are “purely large” in the sense of [Elliott and Kucerovsky 2001].

Section 2 describes the main results in this paper and introduces the technical conditions P1, P2 and P3. Section 3 shows that the corona algebras $M(B)/B$ of many stable C^* -algebras satisfy these conditions. In Section 4, we show that there are examples of nonstable, nonunital and σ -unital C^* -algebras B for which $M(B)/B$ has properties P1, P2 and P3. In Section 5 we give a few modified versions of some known results concerning amenable contractive completely positive linear maps. Section 6 discusses certain commutants in the ultrapower of corona algebras. In Section 7 we prove Theorem 2.5, mentioned above. In Section 8 we prove other main results described in Section 2.

Conventions and definitions. (1) Ideals are always closed and two-sided.

(2) Let A be a C^* -algebra and let $p, q \in A$ be two projections. We write $p \sim q$ if there exists $v \in A$ such that $v^*v = p$ and $vv^* = q$.

(3) Let A and B be C^* -algebras and let $L_1, L_2 : A \rightarrow B$ be linear maps. Take $\mathcal{F} \subset A$ and $\varepsilon > 0$. We write $L_1 \sim_\varepsilon L_2$ on \mathcal{F} if

$$\|L_1(a) - L_2(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

(4) Let A and B be C^* -algebras. A contractive completely positive linear map $L : A \rightarrow B$ is said to be *amenable* if for $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists an integer $n > 0$ and two contractive completely positive linear maps $\phi : A \rightarrow M_n$ and $\psi : M_n \rightarrow A$ such that

$$\psi \circ \phi \sim_\varepsilon L \quad \text{on } \mathcal{F}.$$

(5) A C^* -algebra A is said to be *amenable* (or nuclear) if id_A is amenable.

(6) Let B be a nonunital but σ -unital simple C^* -algebra. B is said to have continuous scale if there exists an approximate identity $\{e_n\}$ of B with $e_{n+1}e_n = e_n$ such that, for each nonzero element $b \in B$, there exists an integer $n > 0$ for which $e_{n+m} - e_n \lesssim b$ for all m ; see [Lin 2004b].

Let $e \in B$ be a nonzero projection and $T_e(B)$ the set of all traces t on B for which $t(e) = 1$. Let B be a separable nonunital simple C^* -algebra with real rank 0, stable rank 1 and weakly unperforated $K_0(B)$. If $\sup_n \{t(e_n)\}$ is a continuous function on $T_e(B)$, then B has continuous scale.

(7) Let $\{A_n\}$ be a sequence of C^* -algebras. Denote by $c_0(\{A_n\})$ the C^* direct sum of $\{A_n\}$ and by $l^\infty(\{A_n\})$ the C^* - product of $\{A_n\}$. Let $q_\infty(\{A_n\})$ be the quotient

$$q_\infty(\{A_n\}) = l^\infty(\{A_n\})/c_0(\{A_n\}).$$

When $A = A_n$ for all n , we write $c_0(A)$, $l^\infty(A)$ and $q_\infty(A)$ for simplicity.

(8) For each integer $n > 0$, define $f_n \in C_0((0, \infty))$ by

$$(1-1) \quad f_n(t) = \begin{cases} 1 & \text{if } t \geq 1/n; \\ \text{linear} & \text{if } 1/(n+1) \leq t < 1/n; \\ 0 & \text{if } 0 \leq t < 1/(n+1). \end{cases}$$

(9) An element a in a C^* -algebra A is said to be *full* if the ideal generated by a is A itself. Let A and B be C^* -algebras and let $h : A \rightarrow B$ be a monomorphism. The monomorphism h is said to be *full* if $h(a)$ is full for every nonzero $a \in A$.

(10) Let $a \in \underline{A}_+$ be a nonzero element, we write $\text{Her}(a)$ for the hereditary C^* -subalgebra aAa generated by a .

2. Main results

Property P1. Let B be a unital C^* -algebra. We say that B has property P1 if for every full element $b \in B$ there exist $x, y \in B$ such that $xbx = 1$. If b is positive, it is easy to see that $xbx = 1$ implies that there is $z \in B$ such that $z^*bz = 1$.

It is obvious that an element b is full if and only if b^*b is full. It follows that B has property P1 if and only if for every full element $0 \leq b \leq 1$, there exists $x \in B$ such that $x^*bx = 1$.

Every unital purely infinite simple C^* -algebra has property P1.

Many other unital C^* -algebras have property P1. Let A be a unital C^* -algebra and $B = A \otimes \mathcal{K}$. In the next section we will show that $M(B)$ and $M(B)/B$ have property P1 for many such B . In Section 3 we will show that, for some nonstable (but σ -unital) C^* -algebra C , $M(C)$ and $M(C)/C$ can also have property P1.

Property P2. Let B be a unital C^* -algebra. We say that B has property P2 if 1 is proper infinite, that is, if there is a projection $p \neq 1$ and partial isometries $w_1, w_2 \in B$ such that $w_1^*w_1 = 1$, $w_1w_1^* = p$, $w_2^*w_2 = 1$ and $w_2w_2^* \leq 1 - p$.

In this case it is easy to see that, for each integer $n \geq 2$, there are mutually orthogonal and mutually equivalent projections $s_{11}, s_{22}, \dots, s_{nn}$ such that $1_B \geq \sum_{i=1}^n s_{ii}$ and there exists an isometry $Z \in B$ such that $Z^*Z = 1_B$ and $ZZ^* = s_{11}$. Let $C = s_{11}Bs_{11}$. Then we may write $M_n(C) \subset B$.

It is clear that if B is stable then $M(B)$ and $M(B)/B$ have property P2.

Proposition 2.1. *Let B be a unital C^* -algebra having property P1. If B contains two mutually orthogonal full elements, B has property P2.*

Proof. Let $0 \leq a, b \leq 1$ be two mutually orthogonal full elements in B . Since B has property P1, there are $x, y \in B$ such that $x^*ax = 1$ and $y^*by = 1$. Set $v_1 = a^{1/2}x$ and $v_2 = b^{1/2}y$. Then $v_i^*v_i = 1$ and $s_{11} = v_1v_1^*$ and $s_{22} = v_2v_2^*$ are two projections. Thus B has property P2. \square

Every purely infinite C^* -algebra (not necessary simple; see [Kirchberg and Rørdam 2000]) has properties P1 and P2.

Property P3. Let B be a unital C^* -algebra. We say that B has property P3 if, for any separable C^* -subalgebra $A \subset B$, there exists a sequence of sequences of elements $\{ \{ a_n^{(i)} : i = 1, 2, \dots \} : n \in \mathbb{N} \}$ in B satisfying these properties:

- (a) $0 \leq a_n^{(i)} \leq 1$ for all i and n ;
- (b) $\lim_{n \rightarrow \infty} \| a_n^{(i)} c - c a_n^{(i)} \| = 0$ for all i and all $c \in A$;
- (c) $\lim_{n \rightarrow \infty} \| a_n^{(i)} a_n^{(j)} \| = 0$ if $i \neq j$;
- (d) $\{ a_n^{(i)} \}$ is a full element in $l^\infty(B)$, for all i .

Even though property P3 looks more complicated than P1 and P2, it will be shown in Proposition 3.13 that $M(B)/B$ has property P3 for all $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra, for all B having continuous scale, and for many other nonunital σ -unital C^* -algebras B .

Proposition 2.2. *Let $B = C \otimes C_1$, where C_1 is a unital separable amenable purely infinite simple C^* -algebra. Then B has properties P1, P2 and P3.*

Let B be a nonunital but σ -unital C^* -algebra and let A be a unital separable amenable C^* -algebra. We study essential extensions of the following form:

$$(2-1) \quad 0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0.$$

Using the Busby invariant, we study monomorphisms $\tau : A \rightarrow M(B)/B$. We will only consider the case in which the corona algebra $M(B)/B$ has properties P1, P2 and P3.

Definition 2.3. An essential extension $\tau : A \rightarrow M(B)/B$ is said to be *full* if τ is a full monomorphism. An extension τ is *weakly unital* if τ is unital monomorphism. If A is a unital simple C^* -algebra then every weakly unital essential extension is full. If $M(B)/B$ is simple, every essential extension is full.

Definition 2.4. Let A be a unital separable C^* -algebra and C a unital C^* -algebra. Let $h_1, h_2 : A \rightarrow C$ be homomorphisms. We say h_1 and h_2 are *approximately unitarily equivalent* if there exists a sequence of partial isometries $u_n \in C$ such that $u_n^* h_1(1_A) u_n = h_2(1_A)$, $u_n h_2(1_A) u_n^* = h_1(1_A)$ and

$$\lim_{n \rightarrow \infty} \| \text{ad } u_n \circ h_1(a) - h_2(a) \| = 0 \quad \text{for all } a \in A.$$

Note that if both h_1 and h_2 are unital, the u_n can be chosen to be unitaries.

Let B be a nonunital but σ -unital C^* -algebra. Two essential extensions of A by B are said to be *approximately unitarily equivalent* if the corresponding Busby invariants $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are approximately unitarily equivalent.

Recall that $\tau : A \rightarrow M(B)/B$ is trivial if there is a monomorphism $h : A \rightarrow M(B)$ such that $\pi \circ h = \tau$, where $\pi : M(B) \rightarrow M(B)/B$ is the quotient map. In the case that $B = C \otimes \mathcal{K}$, where C is a σ -unital C^* -algebra, the invariants τ_1 and τ_2 are *stably* unitarily equivalent if there exists a trivial extension $\tau_0 : A \rightarrow M(B)/B$ and a unitary $u \in M_2(M(B)/B)$ such that $\text{ad } u \circ (\tau_1 \oplus \tau_0) = \tau_2 \oplus \tau_0$.

Let $\mathbf{Ext}(A, B)$ be the set of *stable* unitary equivalence classes of extensions of the form (2-1). When A is a separable amenable C^* -algebra, $\mathbf{Ext}(A, B)$ can be identified with $KK^1(A, B)$. When A satisfies the Universal Coefficient Theorem, $KK^1(A, B)$ is computable. However, as mentioned in the introduction, $KK^1(A, B)$ may not provide any useful information about unitary equivalence of extensions in general. In particular, when B is not stable, $KK^1(A, B)$ should not be used to describe unitary equivalence classes of essential extensions.

The first main result of this paper is the following:

Theorem 2.5. *Let A be a unital separable amenable C^* -algebra and let B be a nonunital but σ -unital C^* -algebra such that $M(B)/B$ has properties P1, P2 and P3. Two full monomorphisms $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \quad \text{in } KL(A, M(B)/B).$$

We will describe $KL(A, C)$ in Definition 7.1. Theorem 2.5 is an easy corollary of the next theorem.

Theorem 2.6. *Let A be a unital separable amenable C^* -algebra and let B be a unital C^* -algebra having properties P1, P2 and P3. Two full monomorphisms $h_1, h_2 : A \rightarrow B$ are approximately unitarily equivalent, i.e., there exists a sequence of partial isometries $u_n \in B$ such that $u_n^*u_n = h_1(1_A)$, $u_nu_n^* = h_2(1_A)$ and*

$$\lim_{n \rightarrow \infty} \text{ad } u_n \circ h_1(a) = h_2(a) \quad \text{for all } a \in A,$$

if and only if $[h_1] = [h_2]$ in $KL(A, B)$.

Corollary 2.7. *Let A be a unital separable amenable simple C^* -algebra and B a nonunital but σ -unital C^* -algebra such that $M(B)/B$ has properties P1, P2 and P3. Suppose that $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are two weakly unital essential extensions. Then τ_1 and τ_2 are approximately unitarily equivalent if and only if*

$$[\tau_1] = [\tau_2] \quad \text{in } KL(A, M(B)/B).$$

Definition 2.8. Let A be a unital separable amenable C^* -algebra and B a unital C^* -algebra having property P2. Fix a full monomorphism $j_o : A \rightarrow \mathcal{O}_2 \rightarrow B$. (Note that P2 implies such full monomorphisms do exist.) Let $h_1, h_2 : A \rightarrow B \otimes \mathcal{K}$ be homomorphisms. We write $h_1 \sim h_2$ if $h_1 \oplus j_o$ is approximately unitarily equivalent

to $h_2 \oplus j_o$. Denote by $H(A, B)$ the set of \sim -equivalence classes of homomorphisms $A \rightarrow B \otimes \mathcal{K}$.

Proposition 2.9. *Let A be a unital separable amenable C^* -algebra and B a unital C^* -algebra having property P2. Then $H(A, B)$ is a group with zero element $[j_o]$.*

Corollary 2.10. *Let A be a unital separable amenable C^* -algebra and B a unital C^* -algebra having properties P1, P2 and P3. Denote by $H_f(A, B)$ the set of approximate unitary equivalence classes of full monomorphisms from A to $B \otimes \mathcal{K}$. Then $H_f(A, B)$ is a group with zero element $[j_o]$.*

Definition 2.11. Let A be a unital separable amenable C^* -algebra and B a nonunital but σ -unital C^* -algebra. Denote by $\mathbf{Ext}_{ap}^f(A, B)$ the approximate unitary equivalence classes of full essential extensions. Denote by $\tau_o : A \rightarrow M(B)/B$ an essential extension that factors through \mathbb{O}_2 . Note that $[\tau_o] = 0$ in $KL(A, M(B)/B)$. Suppose that $M(B)/B$ has properties P1, P2 and P3. By Corollary 2.7, τ_o is unique up to approximately unitary equivalence, Let $\tau_1, \tau_2 : A \rightarrow M(B)/B$ be full essential extensions. Since $M(B)/B$ has property P2, there are partial isometries $z_1, z_2 \in M(B)/B$ such that $z_1^* z_1 = 1_{M(B)/B}$, $z_1 z_1^* = \tau_1(1_A)$, $z_2^* z_2 = 1_{M(B)/B}$ and $z_2 z_2^* = \tau_2(1_A)$. Define $[\tau_1] + [\tau_2] = [\text{ad } z_1 \circ \tau_1 \oplus \text{ad } z_2 \circ \tau_2]$.

Note this is well defined, since $[\tau_o] = 0$ in $KL(A, M(B)/B)$ and $\text{ad } z_1 \circ \tau \oplus \text{ad } z_2 \circ \tau_o$ is approximately unitarily equivalent to τ by Corollary 2.7. With this addition $\mathbf{Ext}_{ap}^f(A, B)$ forms a semigroup.

By Corollary 2.10, we have:

Corollary 2.12. *Let A be a unital separable amenable C^* -algebra and B a nonunital but σ -unital C^* -algebra for which $M(B)/B$ has properties P1, P2 and P3. Then $\mathbf{Ext}_{ap}^f(A, B)$ is a group with zero element $[\tau_o]$, where $\tau_o : A \rightarrow M(B)/B$ is a full monomorphism that factors through \mathbb{O}_2 .*

If, furthermore, A satisfies the so-called Approximate Universal Coefficient Theorem (see Definition 7.1 below), we can say more:

Theorem 2.13. *Let A be a unital separable amenable C^* -algebra satisfying the Approximate Universal Coefficient Theorem and let B be a nonunital but σ -unital C^* -algebra such that $M(B)/B$ has properties P1, P2 and P3. Then there is a bijection Γ from $\mathbf{Ext}_{ap}^f(A, B)$ onto $KL(A, M(B)/B)$.*

The Approximate Universal Coefficient Theorem will be briefly reviewed in Definitions Definition 7.1 and Definition 8.1. Note that, when B is not stable, $K_i(M(B)/B)$ is very different from $K_i(SB)$ [Lin 2005b, 1.7].

In the special case that $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra, we have:

Theorem 2.14. *Let A be a unital separable amenable C^* -algebra and set $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra such that $M(B)/B$ has the property P1.*

Two full essential extensions $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are unitarily equivalent if and only if

$$[\tau_1] = [\tau_2] \quad \text{in } KK^1(A, B).$$

If $x \in KK^1(A, B)$, there is a full essential extension $\tau : A \rightarrow M(B)/B$ such that $[\tau] = x$.

Theorem 2.15. *Let A be a unital separable amenable C^* -algebra and set $B = C \otimes \mathcal{K}$, where C is a unital C^* -algebra for which the tracial state space $T(C)$ is nonempty. Suppose that there is $d > 0$ for which C satisfies the following:*

- (1) *If $p, q \in B$ are two projections, the condition $t(p) > d + t(q)$ for all $t \in T(C)$ implies $q \sim p$ in B .*
- (2) *If $b \in M_k(C)$ is such that $1 \geq b \geq 0$ and $\tau(b) > \alpha + d$ for all $\tau \in T(A)$, there is a projection $e \in \overline{bM_k(A)b}$ such that $\tau(e) > \alpha$ for all $\tau \in T(A)$.*

Then two full essential extensions $\tau_1, \tau_2 : A \rightarrow M(B)/B$ are unitarily equivalent if and only if

$$[\tau_1] = [\tau_2].$$

Remark 2.16. In the case that $B = \mathcal{K}$, Theorem 2.14 is the classical Brown–Douglas–Fillmore theorem, and $M(\mathcal{K})/\mathcal{K}$ is a purely infinite simple C^* -algebra. It has property P1 (as well as P2 and P3) and every essential extension is full. Let X be a compact metric space with finite dimension d . When $B = C(X) \otimes \mathcal{K}$, $M(B)/B$ has property P1 (Corollary 3.9). Theorems 2.14 and 2.15 deal with the extensions studied by Pimsner, Popa and Voiculescu [Pimsner et al. 1979; 1980]. The case where B is a nonunital purely infinite simple C^* -algebra was proved by Kirchberg.

Theorem 2.14 is closely related to a result of Elliott and Kucerovsky [2001]; see Remark 8.7 for discussion.

3. C^* -algebras have properties P1, P2 and P3

Let A be a unital C^* -algebra. Denote by $T(A)$ (or T if no confusion exits) the set of tracial states on A . If $t \in T(A)$, we extend t to a trace $(t \otimes Tr)$ on $A \otimes M_n$ by defining $t((a_{ij})) = \sum_{i=1}^n t(a_{ii})$. We further use t for the trace defined on a dense set of $A \otimes \mathcal{K}$. If $a \in A \otimes \mathcal{K}_+$, then $t(a)$ is well defined (although it could be infinity). Suppose that $h_n \in A \otimes \mathcal{K}_+$ is such that $h_n \nearrow h \in A^{**}$. Then $t(h) = \lim_{n \rightarrow \infty} t(h_n)$. These conventions will be used in this section.

Lemma 3.1. *Let A be a unital C^* -algebra and I a σ -unital ideal of A . If $a \in (A/I)_+$ is a full element, there exists a full element $b \in A_+$ such that $\pi(b) = a$, where $\pi : A \rightarrow A/I$ is the quotient map.*

Proof. This result is certainly known, but we prove it for completeness. Since $a \in (A/I)_+$ is full, there are $x_1, x_2, \dots, x_m \in A/I$ such that

$$\sum_{i=1}^m x_i^* a x_i = 1.$$

Hence there are $c \in A_+$ and $y_1, y_2, \dots, y_m \in A$ such that $\pi(c) = a$ and $1 - \sum_{i=1}^m y_i^* c y_i \in I$. Let e be a strictly positive element of I . Put $b = c + e$. Denote by J the ideal generated by b . Since $b \geq c$ and $b \geq e$, both c and e are in J . It follows that $I \subset J$. Since $\sum_{i=1}^m y_i^* c y_i \in J$, it follows that $1 \in J$. Thus $J = A$, and b is full. \square

Corollary 3.2. *Let A be a unital C^* -algebra and I a σ -unital ideal of A . If A has property P1, then so does A/I .*

Lemma 3.3. *Let A be a unital C^* -algebra and let $B = A \otimes \mathcal{K}$. Suppose that $a \in M(B)$ is an element for which $b = \pi(a)$ is full in $M(B)/B$, where $\pi : M(B) \rightarrow M(B)/B$ is the quotient map. Then a is full in $M(B)$. If $M(B)/B$ has property P1, so does $M(B)$.*

Proof. There are $x_1, x_2, \dots, x_m, y_1, \dots, y_m \in M(B)/B$ such that $\sum_{i=1}^m x_i b y_i = 1$. Then there are $w_1, w_2, \dots, w_m, z_1, z_2, \dots, z_m \in M(B)$ such that

$$1 - \sum_{i=1}^m w_i a z_i \in B.$$

Let $\{e_{ij}\}$ be a system of matrix units for \mathcal{K} . Put $E_n = \sum_{i=1}^n e_{ii}$. Then $\{E_n\}$ is an approximate identity consisting of projections. It follows that there exists $n > 0$ such that

$$\left\| \sum_{i=1}^m (1 - E_n) w_i a z_i (1 - E_n) - (1 - E_n) \right\| < \frac{1}{2}.$$

Thus there exists $s \in (1 - E_n)M(B)(1 - E_n)$ such that

$$\sum_{i=1}^m s^* (1 - E_n) w_i a z_i (1 - E_n) s = 1 - E_n.$$

But there exists $V \in M(B)$ such that $V^*(1 - E_n)V = 1$. Therefore a is full.

For the last statement, we take $m = 1$ in the argument above. \square

Proposition 3.4. *Let B be a unital purely infinite simple C^* -algebra. Then $M(B \otimes \mathcal{K})$ and $M(B \otimes \mathcal{K})/A \otimes \mathcal{K}$ have property P1.*

Proof. It follows from [Zhang 1992] that $M(B \otimes \mathcal{K})/(B \otimes \mathcal{K})$ is purely infinite and simple. Therefore $M(B \otimes \mathcal{K})/B \otimes \mathcal{K}$ has property P1. Now Lemma 3.3 implies that $M(B \otimes \mathcal{K})$ has property P1. \square

Theorem 3.5. *Let $B = A \otimes \mathcal{K}$, where A is a unital separable C^* -algebra for which $T(A) \neq \emptyset$. Let $d > 0$. Suppose A satisfies the following:*

- (1) *If $p, q \in B$ are two projections then $t(p) > d + t(q)$ for all $t \in T(A)$ implies $p \precsim q$ in B .*
- (2) *If $1 \geq b \geq 0$ in $M_k(A)$ such that $\tau(b) > \alpha + d$ for all $\tau \in T(A)$ (and some $\alpha > 0$), then there is a projection $e \in \overline{bM_k(A)b}$ such that $\tau(e) > \alpha$ for all $\tau \in T(A)$.*

Then $M(B)$ and $M(B)/B$ have property P1.

Proof. Let $b \in M(B)$ be a full element. Without loss of generality, we may assume that $0 \leq b \leq 1$. Let $\{e_{ij}\}$ be a system of matrix units for \mathcal{K} and set $E_n = \sum_{k=1}^n e_{ii}$. Then $E_n b E_n$ converges to b in the strict topology. Furthermore $b^{1/2} E_n b^{1/2}$ increasingly converges to b in the strict topology.

Since b is full, there are $x_1, x_2, \dots, x_m \in M(B)$ such that

$$\sum_{k=1}^m x_i^* b x_i = 1.$$

Let $\tau \in T(A)$ be a tracial state. We extend τ to B_+ and then to $M(B)_+$ in a usual way. Let T be the set of all (densely defined) traces on $M(B)_+$ whose restrictions to A are tracial states. With the usual weak $*$ -topology, T is a compact convex set.

Because $b^{1/2} x_i^* x_i b^{1/2} \leq \|x_i\|^2 b$, one has

$$\tau(x_i^* b x_i) = \tau(b^{1/2} x_i^* x_i b^{1/2}) \leq \|x_i\|^2 \tau(b)$$

for all $\tau \in T(A)$ Therefore

$$\sum_{i=1}^m \tau(x_i^* b x_i) \leq \left(\sum_{i=1}^m \|x_i\|^2 \right) \tau(b)$$

for all $\tau \in T(A)$. Since $\tau(1) = \infty$, it follows that $\tau(b) = \infty$. Because $b^{1/2} E_n b^{1/2} \nearrow b$ and because T is compact, by Dini's theorem, we have $\tau(b^{1/2} E_n b^{1/2}) \rightarrow \infty$ uniformly on T . Since $\tau(E_n b E_n) = \tau(b^{1/2} E_n b^{1/2})$ for all $\tau \in T$, we have $\tau(E_n b E_n) \nearrow \infty$ uniformly on T . There is $n(1) \geq 1$ such that

$$\tau(E_{n(1)} b E_{n(1)}) > 1 + 2d \quad \text{for all } \tau \in T.$$

Let A_1 be the hereditary C^* -subalgebra of B generated by $E_{n(1)} b E_{n(1)}$. It follows from assumption (2) that there is a projection $p_1 \in A_1$ such that $\tau(p_1) > 1 + d$ for all $\tau \in T$. Thus there is $v_1 \in B$ such that $v_1^* v_1 \leq p_1$ and $v_1 v_1^* = E_1$. There are nonnegative continuous functions $f, g \in C_0((0, 2\|b\|])$ such that $gf = f$ and

$$\|f(E_{n(1)} b E_{n(1)}) p_1 f(E_{n(1)} b E_{n(1)}) - p_1\| < \frac{1}{4}.$$

It follows by [Effros 1981, A8] that there is a projection

$$q_1 \in \overline{f(E_{n(1)}bE_{n(1)})Bf(E_{n(1)}bE_{n(1)})}$$

such that q_1 is unitarily equivalent to p_1 . We conclude that $g(E_{n(1)}bE_{n(1)})q_1 = q_1$, since $gf = f$. By functional calculus, we see that there is $f_1 \in A_1$ such that

$$f_1 E_{n(1)} b E_{n(1)} f_1 = g.$$

Thus we obtain $z_1 \in E_{n(1)} B E_{n(1)}$ such that

$$z_1^* b z_1 = z_1^* E_{n(1)} b E_{n(1)} z_1 = E_1.$$

Note that $\tau((1 - E_{n(1)})bE_{n(1)}) = \tau(bE_{n(1)}(1 - E_{n(1)})) = 0$. It follows that

$$\tau((1 - E_{n(1)})b(1 - E_{n(1)})) = \tau((1 - E_{n(1)})b).$$

Since $\tau(E_{n(1)}bE_{n(1)}) < \infty$ for all $\tau \in T$, we conclude that

$$\tau((1 - E_{n(1)})b(1 - E_{n(1)})) = \infty \quad \text{for all } \tau \in T.$$

By the earlier argument we obtain $n(2) > n(1)$ and $z_2 \in (E_{n(2)} - E_{n(1)})B(E_{n(2)} - E_{n(1)})$ such that

$$z_2^* b z_2 = z_2^* (E_{n(2)} - E_{n(1)}) b (E_{n(2)} - E_{n(1)}) z_2 = E_2 - E_1.$$

Continuing this fashion, we obtain a sequence $\{n(k)\}$ with $n(k+1) > n(k)$ and $z_k \in (E_{n(k+1)} - E_{n(k)})B(E_{n(k+1)} - E_{n(k)})$ such that

$$z_k^* b z_k^* = z_k^* (E_{n(k+1)} - E_{n(k)}) b (E_{n(k+1)} - E_{n(k)}) z_k = E_{k+1} - E_k,$$

for $k = 1, 2, \dots$. Hence $z = \sum_{k=1}^{\infty} z_k \in M(B)$, since the sum converges in the strict topology. Furthermore,

$$z^* b z = 1.$$

This shows that $M(B)$ has property P1. By Corollary 3.2, $M(B)/B$ also has property P1. □

Corollary 3.6. *Let A be a unital AF-algebra and take $B = A \otimes \mathcal{K}$. Then $M(B)$ and $M(B)/B$ have property P1.*

Proof. Clearly A satisfies (1) in Theorem 3.5 with any $d > 0$. To see that it satisfies (2), we let $1 \geq b \geq 0$ be an element in $M_n(A)$ such that $\tau(b) > \alpha + d$ for all $\tau \in T$. Set $C = \overline{bM_n(A)b}$ and let $\{e_n\}$ be an approximate identity for C consisting of projections. Then $\|e_n b e_n - b\| \rightarrow 0$ as $n \rightarrow \infty$. Since $0 \leq b \leq 1$, it follows that $\tau(e_n) > \alpha + d$ for some $n > 0$ and all $\tau \in T$. □

The proof of the corollary implies the following:

Corollary 3.7. *Let A be a unital separable C^* -algebra with real rank zero for which $T(A) \neq \emptyset$ and which satisfies (1) in Theorem 3.5. Then $M(B)$ and $M(B)/B$ have property P1, where $B = A \otimes \mathcal{K}$.*

Corollary 3.8. *Let $B = A \otimes \mathcal{K}$, where A is a unital simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Then both $M(B)$ and $M(B)/B$ have property P1.*

Corollary 3.9. *Let $A = C(X)$, where X is a compact Hausdorff space with finite covering dimension d . Then $M(A \otimes \mathcal{K})$ and $M(A \otimes \mathcal{K})/(A \otimes \mathcal{K})$ have property P1.*

Proof. Suppose $e, f \in A \otimes \mathcal{K}$ are projections. We may assume that $e, f \in M_n(C(X))$ for some integer $n > 0$. Suppose that $\tau(e) > \tau(f) + d + 1$ for all $t \in T(A)$. It follows that for each $x \in X$, the rank of $e(x)$ is greater than $d + 1$ + the rank of $f(x)$. It follows from [Husemoller 1966, 8.1.2 and 8.1.6] (see [Blackadar 1998, 6.10.3(d)]) that $f \lesssim e$. So Theorem 3.5(1) holds (for $(d + 1)/2$).

For (2), let $1 \geq b \geq 0$ be an element in $M_k(C(X))$ for which $\tau(b) > \alpha + (d + 1)$. Let f_n be as in (1-1). For some large n , we have $\tau(f_n(b)) > \alpha + (d + 1)$ for all $\tau \in T(A)$. Thus, for each $\xi \in X$, the rank of $f_n(b)(\xi)$ is at least $\alpha + (d + 1)$. By [Blackadar et al. 1991, Lemma C], there is a projection $e \in \overline{bM_k(A)b}$ such that the rank of $e(\xi)$ is greater than α for all $\xi \in X$. It follows that $\tau(e) > \alpha$ for all $\tau \in T(A)$. \square

To discuss property P2, we begin with an easy observation:

Proposition 3.10. *Let B be a unital C^* -algebra having property P2. For any integer $n > 0$, there are $s_{11}, s_{22}, \dots, s_{nn}$ such that $1_B \geq \sum_{i=1}^n s_{ii}$ and there exists an isometry $Z \in B$ such that $ZZ^* = e_{11}$. Moreover:*

- (1) *If for some $n \geq 2$, $1_B = \sum_{i=1}^n s_{ii}$, then there exists a unital embedding from \mathbb{O}_n to B .*
- (2) *There is a unital embedding from \mathbb{O}_∞ to B .*
- (3) *There exists a full embedding $j : \mathbb{O}_2 \rightarrow B$.*

Conversely, if there is a unital embedding of \mathbb{O}_∞ in B , then B has property P2. Furthermore, if B admits a full embedding from \mathbb{O}_2 , then B has property P2.

Proposition 3.11. (1) *Let A be a unital C^* -algebra and $B = A \otimes \mathcal{K}$. Then $M(B)$ and $M(B)/B$ has property P2.*

- (2) *Let A be a nonunital σ -unital simple C^* -algebra which has continuous scale. Then $M(A)/A$ has property P2*
- (3) *Let A be a unital purely infinite simple C^* -algebra and $B = C_0(X, A)$, where X is a locally compact Hausdorff space. Then $M(B)$ and $M(B)/B$ have property P2.*

Proof. For (3), we note there is a unital embedding from \mathbb{O}_∞ to A and the constant maps from X into A are in $C^b(X, A) = M(B)$. \square

Now we turn to property P3. Every unital purely infinite simple C^* -algebra has property P3, by [Lin 2005b, 2.6]. Therefore, if B is a nonunital but σ -unital simple C^* -algebra with continuous scale, then $M(B)/B$ has property P3.

Proposition 3.12. *Let B be a unital C^* -algebra having property P1. Suppose that $0 \leq a, b \leq 1$, where $ab = a$ and a is full. Then there exists $x \in B$ with $\|x\| \leq 1$ and*

$$x^*bx = 1.$$

Note that the proposition includes the case that a is a full projection.

Proof. There is $z \in B$ such that $z^*az = 1$. Then $a^{1/2}zz^*a^{1/2} = p$ must be a projection. Moreover, $p \in \text{Her}(a)$. Therefore $pb = p$. Put $v = a^{1/2}z$. Then $v^*v = 1$ and $vv^* = p$. In particular, $\|v\| = 1$. Now

$$1 \geq \|b\|v^*v \geq v^*bv \geq v^*pv = 1.$$

We conclude that $v^*bv = 1$. \square

Proposition 3.13. *Let A be a unital C^* -algebra and set $B = A \otimes \mathcal{K}$. Then $M(B)/B$ has property P3.*

Proof. Let $\pi : M(B) \rightarrow M(B)/B$ be the quotient map and D a separable C^* -algebra. Let $\{e_{i,j}\}$ be a system of matrix units for \mathcal{K} . Set $E_n = \sum_{i=1}^n e_{i,i}$. By [Pedersen 1979, 3.12.14] and the proof of [Lin 2001, 5.5.3], there is a sequence $\{e_n\} \subset \text{Conv}\{E_k : k = 1, 2, \dots\}$ such that

$$(3-1) \quad e_{n+1}e_n = e_n \quad \text{and} \quad \|e_na - ae_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for all $a \in D$.

Suppose that $e_n = \sum_{i=1}^{k(n)} \alpha_i E_i$, where the α_i are nonnegative scalars such that $\sum_{i=1}^{k(n)} \alpha_i = 1$. There exist $0 \leq \beta_j \leq 1$ such that $e_n = \sum_{i=1}^{k(n)} \beta_j e_{jj}$. Since, for each i ,

$$\|e_m e_{ii} - e_{ii}\| \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

there is $N(n) > 0$ such that, for each $m > N(n)$, $e_m = \sum_{i=1}^{k(m)} \beta_i^{(m)} e_{ii}$ with $\beta_{k(n)+1} > \frac{1}{2}$. It follows that

$$\begin{aligned} (e_m - e_n)e_{k(n)+1, k(n)+1} &= \left(\sum_{i=k(n)+1}^{k(m)} \beta_i^{(m)} e_{ii} + \sum_{i=1}^{k(n)} (\beta_i^{(m)} - \beta_i^{(n)}) e_{ii} \right) e_{k(n)+1, k(n)+1} \\ &= \left(\sum_{i=k(n)+1}^{k(m)} \beta_i^{(m)} e_{ii} \right) e_{k(n)+1, k(n)+1} = \beta_{k(n)+1}^{(m)} e_{k(n)+1, k(n)+1}. \end{aligned}$$

By passing to a subsequence if necessary we may as well assume that

$$(e_{n+1} - e_n)e_{k(n)+1,k(n)+1} = \lambda_n e_{k(n)+1,k(n)+1}$$

for some $\lambda_n > \frac{1}{2}$. Now let $F \subset \mathbb{N}$ be an infinite subset. Then

$$b_F = \sum_{n \in F} (e_{n+1} - e_n) \geq \frac{1}{2} \sum_{n \in F} e_{k(n)+1,k(n)+1}.$$

Thus b_F is a full positive element in $M(B)$. Suppose that $\{F_n\}$ is a sequence of infinite subsets of \mathbb{N} . By Proposition 3.12, the image $\pi(\{\sum_{j \in F_n} e_{k(j)+1,k(j)+1}\})$ is full in $l^\infty(M(B)/B)$. So $\{\pi(b_{F_n})\}$ is full in $l^\infty(M(B)/B)$.

By (3-1), $\pi(b_F)$ commutes with $\pi(d)$ for each $d \in D$. Also by (3-1)

$$(e_{n+1} - e_n)(e_{m+1} - e_m) = 0 \quad \text{if } |n - m| \geq 2.$$

It follows that $b_F b_{F'} = 0$ if $|n - m| \geq 2$ for any $n \in F$ and any $m \in F'$. Note that one may write $b_F = \sum_{n \in S(F)} \lambda_n e_{n,n}$, where each $0 < \lambda_n \leq 1$ is a positive number and $S(F)$ is an infinite subset of \mathbb{N} .

It is easy to find a family of (disjoint) infinite subsets $\{F_{i,j} : i, j = 1, 2, \dots\}$ of \mathbb{N} such that $|n - m| \geq 2$ for any $n \in S_{i,j}$ and any $m \in S_{i',j'}$, if $i \neq i'$ or $j \neq j'$. Define $S_{i,j} = S(F_{i,j})$ as above. We note that $S_{i,j} \cap S_{i',j'} = \emptyset$ if $i \neq i'$ or $j \neq j'$. Write $b_{i,j}$ for $b_{F_{i,j}}$. It follows that $M(B)/B$ has property P3. □

4. Nonstable cases

In [Kirchberg and Rørddam 2000] the notion of purely infinite C^* -algebras was extended to nonsimple C^* -algebras. Let C_1 be a unital C^* -algebra and C_2 be a unital separable purely infinite simple C^* -algebra. Then $C_1 \otimes C_2$ is purely infinite [Kirchberg and Rørddam 2000, 4.5]. Therefore, for any unital C^* -algebra C the C^* -algebra $B = C \otimes \mathbb{O}_\infty$ has properties P1 and P2 as well as P3.

Proof of Proposition 2.2. From the preceding paragraph, we know that B is purely infinite, and so has properties P1 and P2. Let A be a separable C^* -subalgebra of B . There is a separable C^* -subalgebra $C_0 \subset C$ such that $A \subset C_0 \otimes C_1$. It follows from [Kirchberg and Phillips 2000] that $C_1 \otimes \mathbb{O}_\infty \cong C_1$. By [Rørddam 2002, 7.2.6] and [Kirchberg and Phillips 2000, 3.12], there is a sequence of unital monomorphisms $\phi_n : \mathbb{O}_\infty \rightarrow C_0 \otimes C_1$ such that

$$\lim_{n \rightarrow \infty} \|\phi_n(x)a - a\phi_n(x)\| = 0 \quad \text{for all } a \in C_0 \otimes \mathcal{E}_1.$$

Let $\{e_k\}$ be a sequence of nonzero mutually orthogonal projections in \mathbb{O}_∞ . Define $a_n^{(i)} = \phi_n(e_i)$, $n, i = 1, 2, \dots$. One checks that $a_n^{(i)}$ satisfies the requirements for property P3. □

There are σ -unital but *nonstable* separable C^* -algebras B for which the corona C^* -algebra $M(B)/B$ has properties P1 and P2 as well as P3. For example, when B has continuous scale, $M(B)/B$ is a purely infinite simple C^* -algebra [Lin 2004b]. So in those cases $M(B)/B$ has all three properties. There are other nonstable separable C^* -algebras B for which B has properties P1, P2 and P3.

To make a point, we will present a very simple example of a nonstable σ -unital C^* -algebra B for which $M(B)/B$ is not simple but both $M(B)$ and $M(B)/B$ have properties P1, P2 and $M(B)/B$ has P3. Many such examples can be constructed.

Proposition 4.3 is not needed in Example 4.4 but will be used again later.

Lemma 4.1. *Let A be a unital C^* -algebra and let $0 \leq a \leq 1$ be an element in A . Suppose that there is $x \in A$ such that $x^*ax = 1$. Then there is $N > 0$, depending on $\|x\|$ but not on A or a , for which there is $y \in A$ with $\|y\| \leq 1$ such that*

$$y^* f_N(a)y = 1.$$

In particular, $f_N(a)$ is full, where f_N is as defined in (1-1).

Proof. Let $q = a^{1/2}xx^*a^{1/2}$. Then q is a projection. There exists $k > 0$ depending on $\|x\|$ such that

$$\|f_k(t)t^{1/2} - t^{1/2}\| < \frac{1}{16\|x\|^2} \quad \text{for all } t \in [0, 1],$$

where f_k is as in (1-1). Then

$$\|f_k(a)q - q\| = \|(f_k(a)a^{1/2} - a^{1/2})x^*xa^{1/2}\| < \frac{1}{16}.$$

It follows from [Effros 1981, A8] that there is a projection $p \in \overline{f_k(a)Af_k(a)}$ such that

$$\|q - p\| < \frac{1}{2}.$$

Thus there exists $w \in A$ such that $w^*w = 1$ and $ww^* = p$. Choose $N = k + 1$. Then $f_N(a)q = q$, showing that

$$w^* f_N(a)w = 1. \quad \square$$

Lemma 4.2. *Let A be a unital C^* -algebra and let $a \in A$ with $0 \leq a \leq 1$ be a full element. Suppose that there are $x_1, x_2, \dots, x_m \in A$ such that*

$$\sum_{i=1}^m x_i^* a x_i = 1.$$

Set $r = \sum_{i=1}^m \|x_i\|^2$. Suppose also that $1_{M_m(A)} \lesssim 1$. Then there exists an integer $N > 0$, depending on r but not on A or a , such that $f_N(a)$ is full. Moreover, there

are $y_1, y_2, \dots, y_m \in A$ such that $\sum_{i=1}^m \|y_i\|^2 \leq 1$ and

$$\sum_{i=1}^m y_i^* f_N(a) y_i = 1.$$

Proof. Let

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a. \end{pmatrix}$$

Since $1_{M_m(A)} \lesssim 1$, one obtain $Y \in M_m(A)$ with $\|Y\| = 1$ and $Y^* \text{diag}(1, 0, \dots, 0) Y = 1_{M_m(A)}$. Note that $0 \leq b \leq 1$ and $XbX^* = \text{diag}(1, 0, \dots, 0)$. Thus

$$Y^* XbX^* Y = 1_{M_m(A)}.$$

We compute that $\|X^* Y\| \leq r^{1/2}$. It follows from Lemma 4.1 that there exist $N > 0$ and $z \in M_m(A)$ with $\|z\| \leq 1$ such that

$$z^* f_N(b) z = 1_{M_m(A)}.$$

So $Yz^* f_N(b) z Y^* = 1$. An easy computation shows that there are $y_1, y_2, \dots, y_n \in A$ such that $\sum_{i=1}^m \|y_i\|^2 \leq 1$ and

$$\sum_{i=1}^m y_i^* f_N(a) y_i = 1. \quad \square$$

Proposition 4.3. *Let $\{A_n\}$ be a sequence of unital C^* -algebras having property P1. Then $l^\infty(\{A_n\})$ also has property P1.*

Proof. Let $a = \{a_n\}$ be a full element in $l^\infty(\{A_n\})$ such that $0 \leq a \leq 1$. (Note that full elements of $l^\infty(\{A_n\})$ cannot be in $c_0(\{A_n\})$.) By Lemma 4.2, there exists $N > 0$ for which $f_N(a)$ is full. For each n , there exists $x_n \in A_n$ such that $x_n^* f_N(a_n) x_n = 1$. Note that $f_{N+1}(a_n) f_N(a) = f_N(a)$. It follows from Proposition 3.12 that, for each n , there is $y_n \in A$ with $\|y_n\| \leq 1$ such that

$$y_n^* f_{N+1}(a) y_n = 1.$$

Put $y = \{y_n\}$. Then $y \in l^\infty(\{A_n\})$. It is clear that there is $g \in C_0((0, 1])_+$ such that

$$\|g(a) a g(a) - f_{N+1}(a)\| < \frac{1}{4}.$$

Then

$$\|y^* g(a) a g(a) y - 1\| = \|y^* (g(a) a g(a) - f_N(a)) y\| \leq \frac{1}{4}.$$

Hence there is $z \in l^\infty(\{A_n\})$ with $\|z\| < \frac{4}{3}$ such that

$$z^* y^* g(a) a g(a) y z = 1. \quad \square$$

This proposition is not required in the following example. However it will be used in Lemma 6.5.

Example 4.4. Let A be a unital separable amenable purely infinite simple C^* -algebras. Denote by $B = c_0(A)$. Then $M(B) = l^\infty(A)$. Put $q_\infty(A) = l^\infty(A)/c_0(A)$. So $M(B)/B = q_\infty(A)$.

- (1) $M(B)$ and $M(B)/B$ has properties P1 and P2.
- (2) $M(B)/B$ has property P3.

Claim (1) is obvious (it also follows from Proposition 4.3). In fact, if $C = C_0((0, 1), A)$, then $M(C)$ and $M(C)/C$ also have properties P1 and P2. This can be proved rather easily.

To see (2), let D be a separable C^* -subalgebra of $M(B)$. Suppose that $x^{(1)} = \{x_n^{(1)}\}$, $x^{(2)} = \{x_n^{(2)}\}, \dots, x^{(k)} = \{x_n^{(k)}\}, \dots$ is a dense sequence in the unit ball of D . Using the fact that $A \otimes \mathbb{O}_\infty \cong A$ [Kirchberg and Phillips 2000, Theorem 3.15], we obtain a sequence of homomorphisms $\phi_n : \mathbb{O}_\infty \rightarrow A$ such that

$$\lim_{n \rightarrow \infty} \|\phi_n(b)a - a\phi_n(b)\| = 0$$

for all $a \in A$ and $b \in \mathbb{O}_\infty$. Let $e_1 \in \mathbb{O}_\infty$ be a proper projection. There is an integer $n(1) > 0$ such that

$$\|\phi_{n(1)}(e_1)x_1^{(1)} - x_1^{(1)}\phi_{n(1)}(e_1)\| < \frac{1}{2}.$$

There is a projection $e_2 \in \mathbb{O}_\infty$ such that $e_1e_2 = e_2e_1 = 0$ and $1 > e_1 + e_2$. There is $n(2) > 0$ such that

$$\|\phi_{n(2)}(e_j)x_l^{(i)} - x_l^{(i)}\phi_{n(2)}(e_j)\| < \frac{1}{4} \quad \text{for } i, j, l = 1, 2.$$

Continuing in this fashion, we obtain a sequence of mutually orthogonal nonzero projections $\{e_m\} \subset \mathbb{O}_\infty$ and a subsequence $\{n(m)\}$ such that

$$\|\phi_{n(m)}(e_j)x_l^{(i)} - x_l^{(i)}\phi_{n(m)}(e_j)\| < 2^{-m} \quad \text{for } i, j, l = 1, 2, \dots, m.$$

Put $p^{(j)} = \{\phi_{n(m)}(e_j)\} \in l^\infty(A)$, for $j = 1, 2, \dots$. Then $p_m^{(i)}p_m^{(j)} = 0$ if $i \neq j$. Moreover,

$$\|\pi(p^{(j)})\pi(\{x^{(i)}\}) - \pi(\{x^{(i)}\})\pi(p^{(j)})\| = 0.$$

This implies that

$$\pi(p^{(j)})\pi(d) = \pi(d)\pi(p^{(j)}).$$

Put $a_n^{(j)} = p^{(j)}$, $j = 1, 2, \dots$. This shows that $M(B)/B$ has property P3.

It is clear, in fact, that $l^\infty(\{A_n\})/c_0(\{A_n\})$ has property P3 if each A_n is a unital purely infinite simple C^* -algebra.

5. Amenable contractive completely positive linear maps

Lemma 5.1 ([Akemann et al. 1986, 2.3]; see also [Lin 2001, 5.3.2]). *Let A be a separable C^* -algebra and $\psi : A \rightarrow \mathbb{C}$ a pure state. Denote by ψ also the extension of ψ to \tilde{A} and put $L = \{a \in \tilde{A} : \psi(a^*a) = 0\}$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $z_1, z_2, z_3 \in \tilde{A}_+$ such that $\|z_i\| = 1$ and $z_i \notin L$ for $i = 1, 2, 3$, $z_{i+1}z_i = z_i$ for $i = 1, 2$, and*

$$\|z_i(\phi(a) - a)z_i\| < \frac{1}{2}\varepsilon \quad \text{for } i = 1, 2, 3 \text{ and all } a \in \mathcal{F}.$$

Moreover, if $\{e_n\}$ is an approximate identity for A , then, for some large N ,

$$\|e_n z_i e_n (\phi(a) - a) e_n z_i e_n\| < \varepsilon \quad \text{and} \quad e_n z_i e_n \notin L$$

for all $a \in \mathcal{F}$ and all $n \geq N$.

Proof. To simplify notation, we assume that \mathcal{F} is a subset of the unit ball of A . Let

$$N = \{a \in \tilde{A} : \phi(a) = 0\}.$$

Note that L is a closed left ideal. Let C be the hereditary C^* -subalgebra given by $L \cap L^*$. As in the proof of [Lin 2001, 5.3.2], we have $z_1, z_2, z_3 \in \tilde{A}$ with $\|z_i\| = 1$ ($i = 1, 2, 3$) such that $z_i \notin L$, $z_{i+1}y_i = z_i$, $i = 1, 2, 3$, and

$$\|z_i(\psi(a) - a)z_i\| < \frac{1}{2}\varepsilon, \quad i = 1, 2, 3.$$

Let $\{e_n\}$ be an approximate identity for A such that $e_n e_{n+1} = e_n$ for all n . Note that z_i has the form $\lambda_i 1_B + y'_i$, where $y'_i \in A$ and $\lambda_i \in \mathbb{C}$, $i = 1, 2$. Choose a large n such that

$$\|e_k a - a e_k\| < \frac{1}{4}\varepsilon, \quad \|e_k a - a\| < \frac{1}{4}\varepsilon \quad \text{and} \quad \|e_k z_i - z_i e_k\| < \frac{1}{4}\varepsilon$$

for all $a \in \mathcal{F} \cup \{z_1 a z_1, z_2 a z_2, z_3 a z_3 : a \in \mathcal{F}\}$ and for all $k \geq n$. Let $y_i = e_n z_i e_n$. Then, for $n \geq N$,

$$\|y_i(\psi(a) - a)y_i\| < \frac{1}{4}\varepsilon + \|e_n^2 z_i (\psi(a) - a) z_i e_n^2\| < \varepsilon \quad \text{for all } a \in \mathcal{F}. \quad \square$$

The following is folklore.

Lemma 5.2. *Let A be a C^* -subalgebra of B and take $a \in A_+$. Denote by C the hereditary C^* -subalgebra of B generated by a . Then, for any approximate identity $\{e_n\}$ of A , the sequences $\|e_n b - b\|$ and $\|b e_n - b\|$ converge to 0 as $n \rightarrow \infty$, for all $b \in C$.*

Proof. There exists a sequence of positive functions $f_n \in C_0(sp(a))$ with $0 \leq f_n \leq 1$ such that $\{f_n(a)\}$ forms an approximate identity for C . Fix an element $b \in C$. For any $\varepsilon > 0$, there is f_k such that

$$\|f_k(a)b - b\| < \frac{1}{4}\varepsilon \quad \text{and} \quad \|b f_k(a) - b\| < \frac{1}{4}\varepsilon.$$

Choose an integer $N > 0$ such that

$$\|e_n f_k(a) - f_k(a)\| < \frac{\varepsilon}{4(\|b\| + 1)} \quad \text{for all } n \geq N.$$

Then

$$\begin{aligned} \|e_n b - b\| &\leq \|e_n b - e_n f_k(a)b\| + \|e_n f_k(a)b - f_k(a)b\| + \|f_k(a)b - b\| \\ &< \frac{1}{4}\varepsilon + \|b\| \left(\frac{\varepsilon}{4(\|b\| + 1)} \right) + \frac{1}{4}\varepsilon = \frac{3}{4}\varepsilon < \varepsilon. \end{aligned} \quad \square$$

Lemma 5.3. *Let B be a unital C^* -algebra that has the property P1. Let A be a separable C^* -algebra and let I be an ideal of A . Suppose that $j : A \rightarrow B$ is an embedding such that $j(a)$ is a full element of B for all $a \notin I$. Then, for any pure state $\phi : A \rightarrow \mathbb{C}1_B \subset B$ which vanishes on I , any finite subset $\mathcal{F} \subset A$, and any $\varepsilon > 0$, there is a partial isometry $V \in B$ such that*

$$\|\phi(a) - V^* j(a) V\| < \varepsilon \quad \text{for } a \in \mathcal{F}, \quad V^* V = 1_B \quad \text{and} \quad V V^* \in \text{Her}(j(A)).$$

Proof. To simplify notation, we identify A with $j(A)$. Fix $0 < \varepsilon < \frac{1}{2}$. By Lemma 5.1, there are $z_1, z_2, z_3 \in \tilde{A}_+$ such that $\|z_i\| = 1$ and $z_i \notin L$ for $i = 1, 2, 3$, $z_{i+1} z_i = z_i$ for $i = 1, 2$, and

$$\|\phi(a) z_i^2 - z_i j(a) z_i\| < \frac{1}{4}\varepsilon \quad \text{for } i = 1, 2 \text{ and all } a \in \mathcal{F}.$$

Note that $L = \{a \in \tilde{A} : \psi(a^* a) = 0\}$. Therefore $I \subset L \cap L^* \subset L$. Let $\{e_n\}$ be an approximate identity for A such that $e_n e_{n+1} = e_n$, $n = 1, 2, \dots$. Let N be an integer as in Lemma 5.1, such that

$$(5-1) \quad \|\phi(a)(e_n z_i e_n)^2 - e_n z_i e_n j(a) e_n z_i e_n\| < \frac{1}{2}\varepsilon, \quad \text{for } i = 1, 2, 3.$$

Put $y_1 = e_N z_1 e_N$. We may assume that $y_1 \notin L$. By the assumption, y_1 is full. Because B has property P1, there exists $x \in B$ such that $x^* y_1^2 x = 1_B$. Put $v_1 = y_1 x$. Then $v_1^* v_1 = 1_B$ and $v_1 v_1^* = p_1$ is a projection. Note that $p_1 \in \text{Her}(y_1)$. There is a projection in $q_1 \in \text{Her}(z_1^{1/2} e_N z_1^{1/2})$ such that q_1 is equivalent to p_1 . Therefore there is a partial isometry $w_1 \in B$ such that $w_1^* q_1 w_1 = 1_B$ and $w_1 w_1^* = q_1$. Since $z_2^2 z_1 = z_1$, $z_2^2 q_1 = q_1$. By applying Lemma 5.2, one can choose a large integer $k > N$ such that, for all $n \geq k$,

$$(5-2) \quad \|e_n q_1 - q_1\| < \frac{1}{32}\varepsilon \quad \text{and} \quad \|e_n z_i - z_i e_n\| < \frac{1}{32}\varepsilon \quad \text{for } i = 1, 2, 3.$$

Thus

$$\|(e_k z_2 e_k)^2 q_1 - q_1\| = \|e_k z_2 e_k^2 z_2 e_k q_1 - q_1\| < \frac{8}{32}\varepsilon = \frac{1}{4}\varepsilon.$$

Put $y_2 = e_k z_2 e_k$. Then one estimates

$$\|w_1^* y_2^2 w_1 - 1\| = \|w_1^* q_1 y_2^2 q_1 w_1 - w_1^* q_1 w_1\| < \frac{1}{2}\varepsilon.$$

Thus there is $s \in \text{Her}(z_1^{1/2} e_N z_1^{1/2})_+ \subset B_+$ such that $\|s\| \leq \frac{1}{1 - \frac{1}{2}\varepsilon}$ and

$$s^{1/2} w_1^* y_2^2 w_1 s^{1/2} = 1.$$

Note that

$$(5-3) \quad \|w_1 s^{1/2}\| \leq \sqrt{\frac{1}{1 - \frac{1}{2}\varepsilon}} < \sqrt{\frac{2}{2 - \frac{1}{2}}} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}.$$

Define $V = y_2 w_1 s^{1/2}$. Note that

$$(5-4) \quad V^* V = 1_B \quad \text{and} \quad V V^* \in \text{Her}(j(A)).$$

Put $y_3 = e_{k+1} z_3 e_{k+1}$. Then, by (5-2),

$$(5-5) \quad \|y_3 y_2 - y_2\| = \|e_{k+1}(z_3 e_k z_2 - z_2) e_k\| < \frac{1}{32}\varepsilon.$$

Furthermore, by (5-3) and (5-5),

$$(5-6) \quad \|y_3 V - V\| = \|y_3 y_2 w_1 s^{1/2} - y_2 w_1 s^{1/2}\| \leq \left(\frac{1}{32}\varepsilon\right) \frac{2\sqrt{3}}{3} = \frac{\sqrt{3}}{48}\varepsilon.$$

We estimate, by applying (5-6), (5-4) and (5-1), that

$$\begin{aligned} \|\phi(a) - V^* a V\| &= \|\phi(a) V^* V - V^* a V\| \leq \frac{\sqrt{3}}{48}\varepsilon + \|\phi(a) V^* y_2^2 V - V^* y_2 a y_2 V\| \\ &\leq \frac{\sqrt{3}}{48}\varepsilon + \|\phi(a) y_2^2 - y_2 a y_2\| < \frac{\sqrt{3}}{48}\varepsilon + \frac{1}{2}\varepsilon < \varepsilon \quad \text{for all } a \in \mathcal{F}. \quad \square \end{aligned}$$

Remark 5.4. If A has a unit, the proof of Lemma 5.3 is almost identical to that of [Lin 2001, 5.3.2], which has its origin in [Akemann et al. 1986]. When A has no unit, the elements z_1, z_2, z_3 are not in A_+ but in \tilde{A}_+ . By using an approximate identity $\{e_n\}$, one does have $\|y_3 y_2 - y_2\|$ small. However the norm of x could be large and it depends on the choice of z_i as well as N as in the proof above. By introducing q_1 , we are able to control the norm of $w_1 s^{1/2}$.

Lemma 5.5. *Let B be a unital C^* -algebra having property P1 and let A be a separable C^* -algebra. Suppose there exists a sequence of homomorphism $\phi_n : A \rightarrow B$ such that $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set in B for all $a \in A$. Let I be an ideal of A such that $\ker \phi_n \subset I$ and $\phi_n(a)$ is a full element in B for all $a \notin I$ and for all n . Then, for any state $\psi : A/I \rightarrow \mathbb{C}1_B \subset B$, any finite subset $\mathcal{F} \subset A$, and any $\varepsilon > 0$, there is a partial isometry $V \in B$ and an integer n such that $V^* V = 1_B$,*

$$\left\| \psi \circ \pi(a) - V^* \left(\sum_{k=1}^n \phi_k(a) \right) V \right\| < \varepsilon \quad \text{for } a \in \mathcal{F} \quad \text{and} \quad V V^* \in \text{Her} \sum_{i=1}^n \phi_i(A),$$

where $\pi : A \rightarrow A/I$ is the quotient map. If ψ is only assumed to be a nonzero positive linear functional with $\|\psi\| \leq 1$, the conclusion still holds with the difference that V is merely a contraction.

Proof. By the Krein–Milman theorem, we have positive numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ with $\sum_{i=1}^m \alpha_i = 1$ and pure states $\psi_1, \psi_2, \dots, \psi_m$ of A/I such that

$$\left\| \psi \circ \pi(a) - \sum_{i=1}^m \alpha_i \psi_i(a) \right\| < \frac{1}{2} \varepsilon \quad \text{for } a \in \mathcal{F}.$$

Let $\pi_n : A \rightarrow A/\ker \phi_n$ and $\gamma_n : A/\ker \phi_n \rightarrow A/I$ be the quotient maps, for $n = 1, 2, \dots$. Note that $\psi_i \circ \gamma_n$ is a pure state of $A/\ker \phi_n$.

By Lemma 5.5, there are $V_i \in B$ such that $V_i^* V_i = 1_B$, $V_i V_i^* \in \text{Her } \phi_i(A)$ and

$$\left\| \psi_i \circ \gamma_i(\phi_i(a)) - V_i^* \phi_i(a) V_i \right\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

(Note that $\psi_i \circ \gamma_i \circ \phi_i = \psi_i \circ \pi$.)

Set $V = \sum_{i=1}^m \sqrt{\alpha_i} V_i \in B$. Since $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set for each $a \in A$, we compute that

$$V^* V = \sum_{i=1}^m 1_B = 1_B \quad \text{and} \quad V V^* = \sum_{i,j} \sqrt{\alpha_i \alpha_j} V_i V_j^* \in \text{Her } \sum_{i=1}^m \phi_i(A).$$

Moreover

$$\begin{aligned} \left\| \psi \circ \pi(a) - V^* \left(\sum_{i=1}^m \phi_i(a) \right) V \right\| &= \left\| \psi \circ \pi(a) - \sum_{i=1}^m \alpha_i V_i^* \phi_i(a) V_i \right\| \\ &\leq \left\| \psi \circ \pi(a) - \sum_{i=1}^m \alpha_i \psi_i(a) \right\| + \sum_{i=1}^m \alpha_i \left\| \psi_i \circ \pi(a) - V_i^* \phi_i(a) V_i \right\| \\ &< \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon \quad \text{for } a \in \mathcal{F}. \end{aligned}$$

To check the last statement of the lemma, note that there is $0 < \lambda \leq 1$ such that $\psi(a) = \lambda \cdot g(a)$ for some state g and for all $a \in A$. □

Lemma 5.6. *Let A be a separable C^* -algebra and let B be a unital C^* -algebra having properties P1 and P2. Let C be as described in the definition of Property P2, with $n = k$ (see Proposition 3.10). Suppose that $\phi_n : A \rightarrow B$ is a sequence of homomorphisms such that $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set in B . Suppose that I is an ideal of A such that $I \supset \ker \phi_n$ and $\phi_n(a)$ is a full element for all $a \notin I$. Let $\psi : A/I \rightarrow M_k(\mathbb{C}) \subset M_k(C) \subset B$ be a contractive completely positive linear map. Then, for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a contraction $V \in B$*

and an integer $m > 0$ such that

$$\left\| \psi(a) - V^* \left(\sum_{i=1}^m \phi_i(a) \right) V \right\| < \varepsilon \quad \text{for } a \in \mathcal{F} \quad \text{and} \quad VV^* \in \text{Her} \sum_{i=1}^K \phi_i(A),$$

where $\pi : A \rightarrow A/I$ is the quotient map.

Proof. Write $\psi(a) = \sum_{i=1}^k \psi_{ij}(a) \otimes s_{ij}$ for $a \in A$, where $\{s_{ij}\}$ is a system of matrix units for M_n and $\psi_{ij} : A \rightarrow \mathbb{C}$ is linear. Also assume that s_{ii} are as in Property P2 and Proposition 3.10, for $i = 1, 2, \dots, k$. Define $\Phi : M_k(A) \rightarrow \mathbb{C} \subset C$ by $\Phi((a_{ij})_{k \times k}) = \sum_{i,j=1}^k \psi_{ij}(a_{ij})$, where $a_{ij} \in A$. Let Z be as in the definition of Property P2, so that $ZZ^* = s_{11}$. Put $J_n(a) = Z_k \phi_n(a) Z^*$ for all $a \in A$. Then J_n maps A into $C = s_{11} B s_{11}$. Note that $\phi_n \otimes \text{id} : M_k(A) \rightarrow M_k(C)$ is also full. Set $\mathcal{G} = \{(a_{ij}) : a_{ij} \in \mathcal{F} \cup \{0\}\}$. By applying Lemma 5.5, we see there is $W \in M_k(B)$ with $\|W\| \leq 1$ such that

$$\left\| \Phi(b) - W^* \left(\sum_{k=1}^m J_k \otimes \text{id } b \right) W \right\| < \frac{\varepsilon}{2n^2} \quad \text{for } b \in \mathcal{G}.$$

We may also assume that $W^*W \leq \text{diag}(1_C, 0, \dots, 0)$. Choose a positive $d \in A$ such that $0 \leq d \leq 1$ and

$$\|da - a\| < \frac{\varepsilon}{2n^2} \quad \text{for all } a \in \mathcal{F}.$$

Let $v_i = 0, \dots, 0, d, 0, \dots, 0$, where the nonzero entry is in i -th place. Let v'_i be the $n \times n$ matrix whose first row is v_i and whose remaining rows are zero. Put $r_i = \sum_{n=1}^m J_n \otimes \text{id}(v'_i)$. For any $a \in A$,

$$\left\| r_i^* \left(\sum_{n=1}^m J_n(a) \otimes \text{id}(a \otimes s_{11}) \right) r_j - \sum_{n=1}^m J_n \otimes \text{id}(a \otimes s_{ij}) \right\| < \frac{\varepsilon}{2n^2}.$$

Therefore

$$\left\| \psi_{ij}(a) - W^* r_i^* \left(\sum_{k=1}^m J_k \otimes \text{id}(a \otimes s_{11}) \right) r_j W \right\| < \frac{\varepsilon}{n^2} \quad \text{for all } a \in \mathcal{F}.$$

Put $V' = (v'_1 W, v'_2 W, \dots, v'_n W)$. We view V' as an $n \times n$ matrix whose i -th column is the nonzero column $v'_i W$, for $i = 1, 2, \dots, n$. Then

$$\left\| \psi(a) - V'^* \sum_{k=1}^m J_k \otimes \text{id}(a \otimes e_{11}) V' \right\| < \varepsilon \quad \text{for } a \in \mathcal{F}.$$

Define $V = Z^* V'$. We have

$$\left\| \psi(a) - V^* \sum_{n=1}^m \phi_n(a) V \right\| < \varepsilon \quad \text{for } a \in \mathcal{F}.$$

We also note that $VV^* \in \text{Her} \sum_{n=1}^m \phi_n(A)$. □

Lemma 5.7. *Let A be a separable C^* -algebra and let B be a unital C^* -algebra having properties P1 and P2. Suppose that $\phi_n : A \rightarrow B$ is a sequence of homomorphisms such that the embedding $j_n : \phi_n(A) \rightarrow B$ is full, where $\{\phi_n(a) : n = 1, 2, \dots\}$ is an orthogonal set in B . Suppose that $\psi : A \rightarrow B$ is amenable and such that $\ker \psi \supset \ker \phi_n, n = 1, 2, \dots$. For any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a contraction $V \in B$ and an integer $K > 0$ such that*

$$\left\| \psi(a) - V^* \left(\sum_{i=1}^K \phi_i(a) \right) V \right\| < \varepsilon \quad \text{for } a \in \mathcal{F} \quad \text{and} \quad VV^* \in \text{Her} \sum_{i=1}^K \phi_i(A).$$

Proof. Fix a finite subset \mathcal{F} and $\varepsilon > 0$. Since ψ is amenable, we may as well assume that $\psi = \alpha \circ \beta$, where $\beta : A \rightarrow M_n = M_n(\mathbb{C} \cdot 1_C)$ and $\alpha : M_n \rightarrow B$ are contractive completely positive linear maps (however, n depends on \mathcal{F} as well as ε). Write $M_n(\mathbb{C}) \subset B$ as in the definition Property P2 (see also Proposition 3.10). Put $\mathcal{G} = \beta(\mathcal{F})$. It is convenient to assume that \mathcal{F} lies in the unit ball of A so \mathcal{G} lies the unit ball of $M_n(\mathbb{C} \cdot 1_C)$. Note that $\sigma : M_n \rightarrow M_n(\mathbb{C}) \subset B$ is full. There exists an integer $m > 0$ and a contraction $Z \in M_m(B)$ such that

$$\| \alpha(b) - Z^*(b \text{Id}_m)Z \| < \frac{1}{4}\varepsilon \quad \text{for } b \in \mathcal{G},$$

where Id_m is the $m \times m$ identity matrix. It follows from Lemma 5.6 that there is $N(1) > 1$ and a contraction $W_1 \in B$ such that

$$\left\| \beta(a) - W_1^* \sum_{i=1}^{N(1)} \phi_i(a) W_1 \right\| < \frac{\varepsilon}{4m} \quad \text{for } a \in \mathcal{F}$$

as well as integers $N(k+1) > N(k)$ and a contractions $W_k \in B$ such that

$$\left\| \beta(a) - W_{k+1}^* \sum_{i=N(k)+1}^{N(k+1)} \phi_i(a) W_{k+1} \right\| < \frac{\varepsilon}{4m} \quad \text{for } a \in \mathcal{F} \text{ and } k = 1, 2, \dots$$

Note that

$$\| \alpha \circ \beta(a) - Z^*(\beta(a) \text{Id}_m)Z \| < \frac{1}{2}\varepsilon \quad \text{for } a \in \mathcal{F}.$$

It follows that

$$\left\| \psi(a) - Z^* \left(\text{diag} \left(W_1^* \sum_{i=1}^{m(1)} \phi_i(a) W_1, \dots, W_m \sum_{i=N(m-1)+1}^{N(m)} \phi_i(a) W_m \right) \right) Z \right\| < \frac{1}{2}\varepsilon$$

for all $a \in \mathcal{F}$. There exist $d_i \in \text{Her}(\phi_i(A))_+$ with $0 \leq d_i \leq 1$ such that

$$\| d_i \phi_i(a) - \phi_i(a) \| < \frac{\varepsilon}{2m} \quad \text{and} \quad \| d_i \phi_i(a) d_i - \phi_i(a) \| < \frac{\varepsilon}{2m}$$

for all $a \in \mathcal{F}$. Note that $d_i d_j = 0$ if $i \neq j, i, j = 1, 2, \dots, m$. Now let Y be the $n \times n$ matrix whose first row is (d_1, d_2, \dots, d_m) and the rest are zero. Put

$W = \text{diag}(W_1, W_2, \dots, W_m)$ and $V = YWZ$. Then

$$\left\| \text{diag}\left(W_1^* \sum_{i=1}^{m(1)} \phi_i(a) W_1, \dots, W_m \sum_{i=N(m-1)+1}^{N(m)} \phi_i(a) W_m\right) - W^* Y^* \sum_{k=1}^{N(m)} \phi_k(a) Y W \right\|$$

is less than $\frac{1}{2}\varepsilon$ for $a \in \mathcal{F}$. Moreover

$$\left\| \psi(a) - V^* \sum_{k=1}^{N(m)} \phi_k(a) V \right\| < \varepsilon \quad \text{for } a \in \mathcal{F} \quad \text{and} \quad VV^* \in \text{Her} \sum_{k=1}^{N(m)} \phi_k(A). \quad \square$$

6. Commutants in the ultrapower of corona algebras

Definition 6.1. Recall that a family ω of subsets of \mathbb{N} is an *ultrafilter* if

- (i) $X_1, \dots, X_n \in \omega$ implies $\bigcap_{i=1}^n X_i \in \omega$,
- (ii) $\emptyset \notin \omega$,
- (iii) if $X \in \omega$ and $X \subset Y$, then $Y \in \omega$, and
- (iv) if $X \subset \mathbb{N}$ then either X or $\mathbb{N} \setminus X$ is in ω .

An ultrafilter is said to be *free* if $\bigcap_{X \in \omega} X = \emptyset$. The set of free ultrafilters is identified with elements in $\beta\mathbb{N} \setminus \mathbb{N}$, where $\beta\mathbb{N}$ is the Stone–Čech compactification of \mathbb{N} .

A sequence $\{x_n\}$ (in a normed space) is said to converge to x_0 along ω , written $\lim_\omega x_n = x_0$, if for any $\varepsilon > 0$ there exists $X \in \omega$ such that $\|x_n - x_0\| < \varepsilon$ for all $n \in X$.

Let $\{B_n\}$ be a sequence of C^* -algebras. Fix an ultrafilter ω . The ideal of $l^\infty(\{B_n\})$ consisting of those sequences $\{a_n\}$ in $l^\infty(\{B_n\})$ such that $\lim_\omega \|a_n\| = 0$ is denoted by $c_\omega(\{B_n\})$. Define

$$q_\omega(\{A_n\}) = l^\infty(\{B_n\})/c_\omega(\{B_n\}).$$

If $B_n = B, n = 1, 2, \dots$, we write $c_\omega(B)$ for $c_\omega(\{B_n\})$ and $q_\omega(A)$ for $q_\omega(\{A_n\})$.

Lemma 6.2. *Let A be a C^* -algebra, I an ideal of A , and let $a \in A \setminus \{0\}$ be such that $0 \leq a \leq 1$. Suppose that $a \notin I$. Then there is $b \in C^*(a)$ with $0 \leq b \leq 1$ and $\|b\| = 1$ such that if $c \in C^*(b) \setminus J$, then $c \notin I$, where*

$$J = \{f(b) : f \in C_0(sp(b) \setminus \{0\}), f(1) = 0\}.$$

Proof. Let $\pi : A \rightarrow A/I$ be the quotient map. Then $\pi(a) \neq 0$. Suppose that $\xi \in sp(\pi(a)) \setminus \{0\}$. Note $sp(\pi(a)) \subset sp(a)$. Let $f \in C_0(sp(a) \setminus \{0\})$ such that $f(\xi) = 1$ and $0 < f(t) < 1$ for all other $t \in sp(a) \setminus \{0\}$. Set $b = f(a)$. Then, $\pi(b) \neq 0$ and $\|\pi(b)\| = 1$. If $c \notin J, c = g(b)$ for some $g \in C_0(sp(b) \setminus \{0\})$ such that $g(1) \neq 0$. Thus $c = g \circ f(a)$. Note that $g \circ f(\xi) \neq 0$. It follows that $\pi(c) = \pi(g \circ f(a)) \neq 0$. Therefore $c \notin I$. □

Lemma 6.3. *Let B be a unital C^* -algebra and let $a \in B$ be an element with $0 \leq a \leq 1$. Suppose that there is $x \in B$ such that $x^*ax = 1$. Then there exists an element $b \in C^*(a)$ such that c is full for all $c \in C^*(b) \setminus J$, where*

$$J = \{f(b) : f \in C_0(\text{sp}(b) \setminus \{0\}), f(1) = 0\}.$$

Proof. Put $v = a^{1/2}x$. Then $v^*v = 1$ and $vv^* = q$ for some projection $q \in B$. Note that $q \in \text{Her}(a^{1/2}xx^*a^{1/2}) \subset \text{Her}(a)$. For any $0 < \varepsilon < \frac{1}{4}$, there is $N > 0$ such that

$$\|f_n(a)p - p\| < \frac{1}{2}\varepsilon \quad \text{for all } n \geq N,$$

with f_n is as in (1-1). It follows that

$$\|f_n(a)pf_n(a) - p\| < \varepsilon \quad \text{for all } n \geq N.$$

Hence there is a projection $q \in \text{Her}(f_N(a))$ and a partial isometry $w \in B$ such that $w^*qw = 1$ and $wq = q$. Thus $f_{N+1}(a)q = q$. Put $b = f_{N+1}(a)$. For any function $g \in C_0((0, 1])$, if $g(1) \neq 0$, then $g(b)q = q$. It follows that $w^*g(b)w = 1$, so $g(b)$ is full and the lemma follows. \square

Lemma 6.4. *Let A be a unital separable C^* -subalgebra of a unital C^* -algebra B which has properties P1 and P3. Suppose that every nonzero element in A is full in B . Then there exists a sequence of sequences of positive elements $\{a_n^{(i)}\}$, $i = 1, 2, \dots$ with $0 \leq a_n^{(i)} \leq 1$ satisfying the following:*

- (1) $\lim_{n \rightarrow \infty} \|a_n^{(i)}a - aa_n^{(i)}\| = 0$ for all $a \in A$ and $i = 1, 2, \dots$
- (2) $\lim_{n \rightarrow \infty} \|a_n^{(i)}a_n^{(j)}\| = 0$ if $i \neq j$.
- (3) $\Pi(\{a_n^{(i)}\})\Pi \circ J(a)$ is full in $q_\omega(A)$ for any free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$, where $J : B \rightarrow l^\infty(B)$ is defined by $J(b) = (b, b, \dots, b, \dots)$ for $b \in B$ and $\Pi : l^\infty(B) \rightarrow q_\omega(B)$ is the quotient map.

Proof. For each nonzero element $0 \leq a \leq 1$ in A , define

$$r(a) = \inf\{\|x\| : x^*ax = 1\}.$$

Let $b_1, b_2, \dots, b_n, \dots$ be a dense sequence of the unit ball of A . We may assume that $\{b_n\}$ contains a subsequence of positive elements which is dense in the positive part of the unit ball. For each $0 \leq b_k \leq 1$ in the sequence, from the assumption, there is $x_k \in B$ such that $x_k^*b_kx_k = 1$ and $\|x_k\| \leq \frac{4}{3}r(b_k)$. Let D be the separable C^* -subalgebra generated by A and $\{x_k\}$.

We claim that, for each nonzero $a \in A$ with $0 \leq a \leq 1$ there is $x \in D$ such that $x^*ax = 1$. There is $z \in B$ such that $z^*az = 1$ and $\|z\| < \frac{4}{3}r(a)$. There is b_k with $0 \leq b_k \leq 1$ for which

$$\|a - b_k\| < \frac{1}{8(\frac{4}{3}r(a) + 1)^2}.$$

Then

$$\|z^*b_kz - 1\| \leq \|z^*(b_k - a)z\| \leq \frac{1}{8}.$$

We obtain $y \in D$ with $\|y\| < \frac{8}{7}$ such that

$$y^*z^*b_kzy = 1.$$

It follows that $r(b_k) \leq \frac{8}{7}r(a)$. Hence there is $x_k \in D$ with $\|x_k\| \leq \frac{4}{3} \cdot \frac{8}{7}r(a)$ such that $x_k^*b_kx_k = 1$. It follows that

$$\|x_k^*ax_k - 1\| \leq \|x_k^*(a - b_k)x_k\| < \left(\frac{1}{8}\left(\frac{4}{3}r(a) + 1\right)^2\right)\left(\frac{4}{3} \cdot \frac{8}{7}r(a)\right)^2 < \frac{8}{49} < 1.$$

Thus there is $d \in D$ such that

$$d^*x_kax_kd = 1.$$

This proves the claim.

Now since B has property P3 and D is separable, there exists a sequence of sequences of nonzero elements $\{a_n^{(i)}\}$ in B with $0 \leq a_n^{(i)} \leq 1$ such that

- (i) $\lim_{n \rightarrow \infty} \|a_n^{(i)}d - da_n^{(i)}\| = 0$ for all $d \in D$,
- (ii) $\lim_{n \rightarrow \infty} \|a_n^{(i)}a_n^{(j)}\| = 0$ if $i \neq j$, and
- (iii) for each i , $\{a_n^{(i)}\}$ is full in $l^\infty(A)$.

Thus (1) and (2) follow. To see (3), let $a \in A$. From the claim, there is $d \in D$ such that

$$d^*ad = 1.$$

Put $a_i = \{a_n^{(i)}\}$. Then, by Proposition 4.3, there is $z \in l^\infty(A)$ such that $z^*a_iz = 1$. Note that (i) implies that

$$\Pi(a_i)\Pi \circ J(b) = \Pi \circ J(b)\Pi(a_i) \quad \text{for all } b \in D.$$

Put $g = \Pi \circ J(d)\Pi(z)$. Then

$$\begin{aligned} g^*\Pi(a_i)\Pi \circ J(a)g &= \Pi(z^*)\Pi \circ J(d^*)\Pi(a_i)\Pi \circ J(a)\Pi \circ J(d)\Pi(z) \\ &= \Pi(z^*)\Pi(a_i)\Pi \circ J(d^*)\pi \circ J(a)\Pi \circ J(d)\Pi(z) \\ &= \pi(z^*)\Pi(a_i)\Pi(z) = 1. \end{aligned} \quad \square$$

Lemma 6.5. *Let A be a unital separable amenable C^* -algebra and B a unital C^* -algebra having properties P1, P2 and P3. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Suppose that $\tau : A \rightarrow B$ is a full unital embedding. Let $\tau_\infty : A \rightarrow l^\infty(B)$ be defined by $\tau_\infty(a) = (\tau(a), \tau(a), \dots)$ and let $\psi = \Pi \circ \tau_\infty$, where $\Pi : l^\infty(B) \rightarrow q_\omega(B)$. Then there is a unital C^* -subalgebra $C \cong \mathbb{C}_\infty$ in the commutant of $\psi(A)$ in $q_\omega(B)$.*

Proof. Let $\{a_n^{(i)}\}$ be the sequence of sequences of elements given by Lemma 6.4. Put $a_i = \{a_n^{(i)}\}$, $i = 1, 2, \dots$

Applying Lemma 6.3, and introducing and using D as in the proof of Lemma 6.4, we may assume that each a_i has the property that $sp(a_i) \subset [0, 1]$ and $f(\Pi(a_i))$ is full for all $0 \leq f \leq 1$ in $C_0((0, 1])$ for which $f(1) \neq 0$.

Let $X = (0, 1]$ and fix i . Define $\phi'_j, L' : C_0(X) \otimes A \rightarrow q_\omega(B)$ by $\phi'_j(f \otimes a) = f(\Pi(a_i))\psi(a)$ and $L'(f \otimes a) = f(1)\psi(a)$ for $a \in A$. By Lemma 6.4(3), ϕ'_j is full. Let $\{\mathcal{F}_j\}$ be an increasing sequence of finite subsets of A for which $\bigcup_{n=1}^\infty \mathcal{F}_j$ is dense in A and let $\{g_n\}$ be a dense sequence of $C_0((0, 1])$.

Let $\{a_{i(k)}\}_{k=1}^\infty$ be a subsequence of $\{a_i\}$. It follows from Lemma 5.7 that there exists $s_n \in B$ such that

$$\left\| s_n^* \left(\sum_{k=1}^{m(n)} g_j(a_{i(k)})\psi(a) \right) s_n - g_j(1)\psi(a) \right\| < \frac{1}{2^n} \quad \text{for } a \in \mathcal{F}_n \text{ and } j = 1, 2, \dots, n.$$

Moreover, $s_n s_n^* \in \text{Her}(\sum_{k=1}^{m(n)} (a_{i(k)})\psi(A))$. Suppose that $s_n = \Pi((s_{n,1}, s_{n,2}, \dots))$, $n = 1, 2, \dots$. We may assume that

$$\left\| s_{n,k(n)}^* \left(\sum_{k=1}^{m(n)} g_j(a_{k(n)}^{i(k)})\tau(a) \right) s_{n,k(n)} - g_j(1)\tau(a) \right\| < \frac{1}{2^n} \quad \text{for } n = 1, 2, \dots$$

Now put $t_n = s_{n,k(n)}$, $t' = (t_1, t_2, \dots)$ and $t = \Pi(t')$. Define $\Phi : C_0(X) \otimes A \rightarrow l^\infty(B)$ by

$$\Phi(f \otimes a) = \left\{ \sum_{k=1}^{m(n)} f(a_{k(n)}^{i(k)})\tau(a) \right\} \quad \text{for all } f \in C_0(X) \text{ and } a \in A.$$

It follows that

$$t^* \Pi \circ \Phi(f \otimes a) t = f(1)\psi(a) \quad \text{for all } f \in C_0(X) \text{ and } a \in A.$$

Put $b(\{i(k)\}) = \Pi(\{a_{k(n)}^{i(k)}\})$. Note that $0 \leq b(\{i(k)\}) \leq 1$. We have (with $\iota(t) = t$ for all $t \in (0, 1]$)

$$t^* b(\{i(k)\}) t = \iota(1) = 1_{q_\omega(B)}.$$

Put

$$w(\{i(k)\}) = b(\{i(k)\})^{1/2} t \quad \text{and} \quad q = b(\{i(k)\})^{1/2} t^* b(\{i(k)\})^{1/2}.$$

Since $b(\{i(k)\}) \in \psi(A)'$ and $\iota(1) = 1$, we have

$$(6-1) \quad t^* b(\{i(k)\})^{1/2} \psi(a) b(\{i(k)\})^{1/2} t = t^* b(\{i(k)\}) \psi(a) t = \iota(1) \psi(a) = \psi(a) \quad \text{for all } a \in A.$$

It follows from [Rørðam 2002, 6.36] that $w(\{i(k)\}) = b(\{i(k)\})^{1/2} t \in \psi(A)'$. If $\{i(k)\}$ and $\{i(k)'\}$ are disjoint infinite subsets of \mathbb{N} , the corresponding projections q and q' are orthogonal. Thus there is a sequence of isometries $v_k \in \psi(A)'$ such

that $v_k^* v_k = 1_{q_\omega(B)}$ and $1 \geq \sum_{k=1}^n v_k v_k^*$, $n = 1, 2, \dots$. Thus $\psi(A)'$ admits a unital embedding of \mathbb{C}_∞ , □

7. Full extensions

Definition 7.1. Let $\mathbf{Ext}(A, B)$ be the set of *stable* unitary equivalence classes of extensions of the form (2-1). When A is amenable, it is known (by work of Arveson, Choi and Effros) that $\mathbf{Ext}(A, B)$ is a group. Moreover, the group can be identified with $KK^1(A, B)$. Let $\mathcal{T}(A, B)$ be the set of all *stable* unitary equivalence classes of approximately trivial extensions. It is known that $\mathcal{T}(A, B)$ is a subgroup of $KK^1(A, B)$ (see [Lin 2005a]). Following Rørdam, one defines $KL^1(A, B) = KK^1(A, B)/\mathcal{T}(A, B)$.

Let G_1, G_2, G_3 be abelian groups. A group extension $0 \rightarrow G_1 \rightarrow G_3 \rightarrow G_2 \rightarrow 0$ is said to be *pure* if every finitely generated subgroup of G_2 lifts. Denote by $\text{Pext}(G_2, G_1)$ the set of all pure extensions and by $E(G_2, G_1)$ the quotient $\text{ext}_{\mathbb{Z}}(G_2, G_1)/\text{Pext}(G_2, G_1)$.

If A satisfies the Approximate Universal Coefficient Theorem, or AUCT (see [Lin 2005a]), one has the following short exact sequence:

$$(7-1) \quad 0 \rightarrow E(K_i(A), K_i(B)) \rightarrow KL^1(A, B) \rightarrow \text{Hom}(K_i(A), K_{i-1}(B)) \rightarrow 0.$$

So $KL^1(A, B)$ may be computable in theory. It should be noted every separable amenable C^* -algebra which satisfies the Universal Coefficient Theorem (UCT) satisfies the AUCT. Rosenberg and Schochet [1987] have shown that every separable C^* -algebras in the so-called *bootstrap class* satisfies the UCT (therefore the AUCT). We also use the notation $KL(A, B) = KL^1(A, SB)$.

As mentioned in the introduction, two stably unitarily equivalent extensions are in general not unitarily equivalent and trivial extensions are not unitarily equivalent. Furthermore, an essential extension which is zero in $KK^1(A, B)$ may not be trivial (or approximately trivial). We will use $KL^1(A, M(B)/B)$ to give a classification of full essential extensions up to approximately unitary equivalence.

Proposition 7.2. *Let D be a unital C^* -algebra for which there is a unital embedding from \mathbb{C}_2 to D . Let $h_1, h_2 : \mathbb{C}_2 \rightarrow D$ be two full homomorphisms. Suppose that $h_1(1_{\mathbb{C}_2}) \sim h_2(1_{\mathbb{C}_2})$. Then there is a sequence of partial isometries v_n such that*

$$v_n^* v_n = h_2(1_{\mathbb{C}_2}), \quad v_n v_n^* = h_1(1_{\mathbb{C}_2}) \quad \text{and} \quad \lim_{n \rightarrow \infty} \|v_n^* h_1(a) v_n - h_2(a)\| = 0$$

for all $a \in \mathbb{C}_2$.

Proof. This is the combination of Theorem 6.5 and Lemma 7.2 in [Lin 2007]. □

Lemma 7.3. *Let A be a unital separable C^* -algebra and let B and C be unital C^* -algebras such that $B \otimes \mathbb{C}_2$ is a unital C^* -subalgebra of C and C has property P1.*

Suppose that $h_1, h_2 : A \rightarrow B \otimes \mathbb{C} \cdot 1 \subset B \otimes \mathbb{O}_2$ are two unital full monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent in C .

Proof. By [Rørdam 1994] we have $\mathbb{O}_2 \cong \mathbb{O}_2 \otimes \mathbb{O}_2$. Let $p_n = 1_B \otimes q_n \otimes 1_{\mathbb{O}_2}$, where $\{q_n\}$ is a sequence of mutually orthogonal nonzero projections in \mathbb{O}_2 . Note that $p_n \sim 1_{B \otimes \mathbb{O}_2 \otimes \mathbb{O}_2}$, $n = 1, 2, \dots$. Define $\phi_i(a) = p_i h_1(a)$ and $\psi_i(a) = p_i h_2(a)$ for all $a \in A$. Also define $\Phi_n(a) = (1 - \sum_{i=1}^n p_i) h_1(a)$ and $\Psi_n(a) = (1 - \sum_{i=1}^n p_i) h_2(a)$ for all $a \in A$. Then, for each n , $h_1 = \sum_{i=1}^n \phi_i \oplus \Phi_n$ and $h_2 = \sum_{i=1}^n \psi_i \oplus \Psi_n$. Note that ϕ_i, Φ_n, ψ_i and Ψ_n are all full. Now we work in $B \otimes \mathbb{O}_2 \otimes 1$. There are partial isometries $v_{i,j} \in \mathbb{O}_2$ such that $v_{i,j}^* v_{i,j} = p_j$ and $v_{i,j} v_{i,j}^* = p_i$ for $i, j = 1, 2, \dots, n$ and

$$v_{n+1,j}^* v_{n+1,j} = p_j, \quad v_{n+1,j} v_{n+1,j}^* = 1 - \sum_{i=1}^n p_i \quad \text{for } j = 1, 2, \dots, n.$$

Put $w_{i,j} = 1 \otimes v_{i,j} \otimes 1$. Then we also have

$$w_{i,1}^* \phi_1 w_{i,1} = \phi_i \quad \text{for } i = 1, 2, \dots, n \quad \text{and} \quad w_{n+1,1}^* \phi_1 w_{n+1,1} = \Phi_n.$$

Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n, \dots$ be an increasing sequence of finite subsets of A such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A . It follows from [Lin 2001, Lemma 5.4.2] that, for each n , there are isometries $u_n, v_n \in B \otimes \mathbb{O}_2 \otimes 1$ such that

$$\|u_n^* h_1(a) u_n - h_2(a)\| < 1/n \quad \text{and} \quad \|v_n^* h_2(a) v_n - h_1(a)\| < 1/n \quad \text{for } a \in \mathcal{F}_n.$$

Note that the relative commutant of $B \otimes \mathbb{O}_2 \otimes 1$ contains a unital C^* -subalgebra $1_B \otimes 1_{\mathbb{O}_2} \otimes \mathbb{O}_2$ which is isomorphic to \mathbb{O}_2 . It follows from [Kirchberg and Phillips 2000, 1.10] that h_1 and h_2 are approximately unitarily equivalent. \square

Lemma 7.4. *Let A be a unital separable nuclear C^* -algebra, let B_1 and B_2 be two unital C^* -algebra and let C be another unital C^* -algebra. Suppose that $j_i : B_i \otimes \mathbb{O}_2 \rightarrow C$ are two full monomorphisms so that $j_1(1) \sim j_2(1)$ and $h_i : A \rightarrow B_i$ are two full unital monomorphisms. Then there is a sequence of partial isometries $v_n \in C$ such that $v_n^* v_n = j_1(1)$, $v_n v_n^* = j_2(1)$ and*

$$\lim_{n \rightarrow \infty} \|v_n^* (j_2 \circ h_2(a)) v_n - j_1 \circ h_1(a)\| = 0 \quad \text{for all } a \in A.$$

Proof. To simplify notation, we may assume that $j_1(1) = j_2(1)$. Therefore we may assume that both j_1 and j_2 are unital. Define $J_i : B_i \otimes \mathbb{O}_2 \rightarrow l^\infty(C)$ by $J_i(b) = (j_i(b), j_i(b), \dots)$ for $b \in B_i \otimes \mathbb{O}_2$ and $H_i = J_i \circ h_i$, respectively, $i = 1, 2$. Note that these maps are full in $l^\infty(C)$. Since there is a unital \mathbb{O}_2 embedding to $l^\infty(C)$, by Proposition 7.2, we obtain unitaries $u_n \in C$ such that

$$\lim_{n \rightarrow \infty} \|u_n^* J_2(1 \otimes b) u_n - J_1(1 \otimes b)\| = 0 \quad \text{for all } b \in \mathbb{O}_2.$$

Denote $U = \{u_n\}$ in $l^\infty(C)$. Let ω be a free ultrafilter on \mathbb{N} and $\pi : l^\infty(C) \rightarrow q_\omega(C)$ be the quotient map. Let D be the C^* -subalgebra generated by $\pi \circ J_1(B_1 \otimes \mathbb{C} \cdot 1_{\mathbb{O}_2})$

and $\pi \circ \text{ad } U \circ J_2(B_2 \otimes \mathbb{C} \cdot 1_{\mathbb{O}_2})$. It follows that D' , the commutant of D , contains $J_1(1_{B_1} \otimes \mathbb{O}_2)$ which is isomorphic to \mathbb{O}_2 . Therefore we may write $D \subset D \otimes \mathbb{O}_2$. Now $\pi \circ H_1$ and $\pi \circ \text{ad } W \circ H_2$ are two full unital monomorphisms from A into $D \subset D \otimes \mathbb{O}_2$. It follows from Lemma 7.3 that $\pi \circ H_1$ and $\pi \circ \text{ad } W \circ H_2$ are approximately unitarily equivalent. It follows from [Rørdaam 2002, Lemma 6.2.5] that $j_1 \circ h_1$ and $j_2 \circ h_2$ are approximately unitarily equivalent. \square

Theorem 7.5. *Let A be a unital separable amenable C^* -algebra and let B be a unital C^* -algebra which has properties P1, P2 and P3. Let $j_o : A \rightarrow \mathbb{O}_2 \rightarrow B$ be a full embedding of A into B which factors through \mathbb{O}_2 . Suppose that $\tau : A \rightarrow B$ is a full monomorphism. Then there is a sequence of partial isometries $V_n \in M_2(B)$ such that $V_n^*V = 1_B \oplus j_o(1_A)$, $V_nV_n^* = 1_B$ and*

$$\lim_{n \rightarrow \infty} \|V_n(\tau \oplus j_o)(a)V_n^* - \tau(a)\| = 0 \quad \text{for all } a \in A.$$

Proof. Let $J : B \rightarrow l^\infty(B)$ be defined by $J(c) = (c, c, \dots)$ for $c \in B$. Define $\tau_\infty = J \circ \tau$ and $J_o = J \circ j_o$. Let ω be a free ultrafilter on \mathbb{N} and $\pi : l^\infty(B) \rightarrow q_\omega(B)$ be the quotient map. It follows from Lemma 6.5 that $\pi \circ \tau_\infty(A)'$ contains a unital C^* -subalgebra which is isomorphic to \mathbb{O}_∞ . Denote this C^* -subalgebra by \mathbb{O}_∞ . Let $q \in \mathbb{O}_\infty$ be a nonzero projection such that $[q] = 0$ in $K_0(\mathbb{O}_\infty)$. There is a C^* -subalgebra C of \mathbb{O}_∞ for which $1_C = q$ and $C \cong \mathbb{O}_2$. Put $\tau_0(a) = q\pi \circ \tau_\infty(a)$. So we may view τ_0 is a unital full homomorphism from A into $\tau_0(A) \otimes \mathbb{O}_2$. Since $\mathbb{O}_2 \cong \mathbb{O}_2 \otimes \mathbb{O}_2$ by [Rørdaam 1994], it follows from Lemma 7.4 that $\tau_0 \oplus \pi \circ J_o$ and τ_0 are approximately unitarily equivalent. Thus $\pi \circ \tau_\infty$ and $\pi \circ \tau_\infty \oplus \pi \circ J_o$ are approximately unitarily equivalent. It follows from [Rørdaam 2002, 6.2.5] that τ and $\tau \oplus j_o$ are approximately unitarily equivalent. \square

Proof of Theorem 2.6. Since A is separable, there is a unital embedding $j : A \rightarrow \mathbb{O}_2$, by [Kirchberg and Phillips 2000, 2.8]. Since B has property P2, there is a full monomorphism $\sigma : \mathbb{O}_2 \rightarrow B$. Define $\bar{j} = \sigma \circ j$. Note \bar{j} is full. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$ be a finite subset. It follows from [Lin 2005a, Theorem 3.9] that there is an integer n and a unitary $v \in M_{n+1}(B)$ such that

$$\|v^* \text{diag}(h_1(a), \bar{j}(a), \bar{j}(a), \dots, \bar{j}(a))v - \text{diag}(h_2(a), \bar{j}(a), \bar{j}(a), \dots, \bar{j}(a))\| < \frac{1}{4}\varepsilon$$

for all $a \in \mathcal{F}$. On the other hand, by Proposition 7.2, there is an isometry $u \in M_n(\pi \circ \sigma(\mathbb{O}_2))$ with $uu^* = 1_{\pi \circ \sigma(\mathbb{O}_2)}$ such that

$$\|u^* \bar{j}(a)u - \text{diag}(\bar{j}(a), \bar{j}(a), \dots, \bar{j}(a))\| < \frac{1}{4}\varepsilon$$

for $a \in \mathcal{F}$. Thus, we obtain an isometry $w \in M_2(B)$ with $ww^* = 1_B$ such that

$$\|w^* \text{diag}(h_1(a), \bar{j}(a))w - \text{diag}(h_2, \bar{j}(a))\| < \frac{1}{2}\varepsilon \quad \text{for all } a \in \mathcal{F}.$$

By applying Theorem 7.5, we obtain a partial isometry $z \in B$ such that $z^*h_1(1_A)z = h_2(1_A)$, $zh_2(1_A)z^* = h_1(1_A)$ and

$$\|z^*h_1(a)z - h_2(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}. \quad \square$$

Remark 7.6. If both h_1 and h_2 are unital, it is clear that z can be chosen to be unitary. If one of them is unital and the other is not, z can never be unitary. Suppose that both are not unital. Since B has properties P1, P2 and P3, we obtain full \mathbb{C}_2 embeddings into $h_1(1_A)Bh_1(1_A)$ and $h_2(1_A)Bh_2(1_A)$. Therefore there is a projection $e \leq h_1(1_A)$ such that $h_1(1_A)$ is equivalent to $h_1(1_A) - e$ and e is a full projection. So there is a partial isometry $v \in B$ such that $v^*v = h_1(1_A)$ and $vv^* = h_1(1_A) - e$. Thus $1 - \text{ad } v^* \circ h_1(1_A)$ is full. Similarly, there is a partial isometry $w \in B$ with $w^*w = h_2(1_A)$ such that $1 - \text{ad } w^* \circ h_2(1_A)$ is full. Now apply Theorem 2.6 to the case that $A = \mathbb{C}$. we know that $1 - \text{ad } v^* \circ h_1(1_A)$ and $1 - \text{ad } w^* \circ h_2(1_A)$ are equivalent. This implies that we can choose z to be unitary in the proof of Theorem 2.6 in the case that both h_1 and h_2 are not unital.

Corollary 7.7. *Theorem 2.6 also holds for the case that $B = q_\infty(\{C_n\})$, where each C_n is a unital purely infinite simple C^* -algebras.*

Proof. It is clear that B has properties P1 and P2. From the proof of Theorem 2.6 above, we only need an absorbing Theorem 7.5 for this B . Let $\tau : A \rightarrow B$ be a full monomorphism and $j_0 : A \rightarrow \mathbb{C}_2 \rightarrow B$ be a full embedding of A into B which factors through \mathbb{C}_2 . So we may write $j_0 = \Phi \circ j$, where $j : A \rightarrow \mathbb{C}_2$ is a monomorphism and $\Phi : \mathbb{C}_2 \rightarrow B$ is a full homomorphism. Let $L : A \rightarrow l^\infty(\{C_n\})$ be a contractive completely positive linear map for which $\pi \circ L = \tau$, where $\pi : l^\infty(\{C_n\}) \rightarrow q_\infty(\{C_n\})$ is the quotient map. Write $L = \{L_n\}$, where $L_n : A \rightarrow C_n$ is a contractive completely positive linear map. Let $\phi_n : \mathbb{C}_2 \rightarrow C$ such that $\pi \circ \{\psi_n\} = \Phi$. Denote by D_n the separable unital purely infinite simple C^* -algebra containing $L_n(A)$ and $\psi_n(\mathbb{C}_2)$. Then $q_\infty(\{D_n\}) \subset B$ and $\tau : A \rightarrow q_\infty(\{C_n\})$ and $j_0 : A \rightarrow \mathbb{C}_2 \rightarrow q_\infty(\{C_n\})$. Thus one can apply [Lin 2004a, 7.5]. \square

Proof of Proposition 2.9. Let $h_1 : A \rightarrow B \otimes \mathcal{K}$ be a homomorphism. It follows from [Lin 2004a, 4.5] that there is a sequence of asymptotically multiplicative contractive completely positive linear maps $\{\phi_n\}$ from A to $B \otimes \mathcal{K}$ and a sequence of unitaries $u_n \in \widetilde{B \otimes \mathcal{K}}$ such that

$$\lim_{n \rightarrow \infty} \|(h \oplus \phi_n)(a) - \text{ad } u_n \circ j(a)\| = 0 \quad \text{for all } a \in A.$$

Since B has property P2, it is easy to see that we may assume that ϕ_n maps A into B and u_n are unitaries in B . It follows from 6.5 in [Lin 2007] that, for each k , there exists a sequence of unitaries $v_n(k) \in M_2(B)$ such that

$$\lim_{n \rightarrow \infty} \|v_n(k)^*(\phi_n(a) \oplus j_0(a))v_n(k) - (\phi_{n+k}(a) \oplus j_0(a))\| = 0 \quad \text{for all } a \in A.$$

It follows from [Lin 2004a, 4.7] that there exists a homomorphism $h_1 : A \rightarrow M_2(B)$ and a sequence of unitaries $w_n \in M_2(B)$ such that

$$\lim_{n \rightarrow \infty} \|\text{ad } w_n \circ h_1(a) - (\phi_n(a) \oplus j_o(a))\| = 0 \quad \text{for all } a \in A.$$

By applying the fact that B has property P2 and applying Proposition 7.2, we obtain a sequence of isometries $z_n \in M_3(B)$ with $z_n z_n^* = j_o(1_A)$ such that

$$\lim_{n \rightarrow \infty} \|(h \oplus h_1 \oplus j_o)(a) - z_n^* j_o(a) z_n\| = 0 \quad \text{for all } a \in A.$$

Hence $[h_1] = -[h]$ in $H(A, B)$. □

Proof of Corollary 2.10. Combine Proposition 2.9 and Theorem 7.5. □

8. Classification of full extensions

Definition 8.1. Let C_n be a commutative C^* -algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^* -algebra. Put $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$. One has the following six-term exact sequence (see [Schochet 1984]):

$$\begin{array}{ccccc} K_0(A) & \rightarrow & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \rightarrow & K_1(A) \\ & & \uparrow \mathbf{k} & & \downarrow \mathbf{k} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A). \end{array}$$

In [Dadarlat and Loring 1996], $K_i(A, \mathbb{Z}/n\mathbb{Z})$ is identified with $KK^i(\mathbb{l}_n, A)$ for $i = 0, 1$. As in that paper, we use the notation

$$\underline{K}(A) = \bigoplus_{\substack{i=0,1 \\ n \in \mathbb{Z}_+}} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

By $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect to direct sum decomposition and the so-called Bockstein operations; see [Dadarlat and Loring 1996]. It follows from these authors' definition that if $x \in KK(A, B)$, the Kasparov product $KK^i(\mathbb{l}_n, A) \times x$ gives an element in $KK^i(\mathbb{l}_n, B)$, which we identify with $\text{Hom}(K_i(A, \mathbb{Z}/n\mathbb{Z}), K_0(B, \mathbb{Z}/n\mathbb{Z}))$. Thus one obtains a map $\Gamma : KK(A, B) \rightarrow \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$. It was shown in the same paper that if A is in \mathcal{N} then, for any σ -unital C^* -algebra B , the map Γ is surjective and $\ker \Gamma = \text{Pext}(K_*(A), K_*(B))$. In particular,

$$\Gamma : KL(A, B) \rightarrow \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

is an isomorphism. It is shown in [Lin 2005a] that if A satisfies the AUCT, then Γ is also an isomorphism from $KL(A, B)$ onto $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$.

Lemma 8.2. *Let B be a unital C^* -algebra which admits a full \mathbb{C}_2 embedding and let G_i be a countable subgroup of $K_i(B)$ ($i = 0, 1$). There exists a unital separable*

C^* -algebra $B_0 \subset B$ which has a full \mathbb{O}_2 embedding such that $K_i(B_0) \supset G_i$ and $j_{*i} = \text{id}_{K_i(B_0)}$, where $j : B_0 \rightarrow B$ is the embedding.

Proof. Let p_1, \dots, p_n, \dots be projections and $u_1, u_2, \dots, u_n, \dots$ be unitaries in $\bigcup_{k=1}^\infty M_k(B)$ such that $\{p_n\}$ and $\{u_n\}$ generates of G_0 and G_1 , respectively. There is a countable set S such that

$$p_n, u_n \in \bigcup_{n=1}^\infty \{(a_{ij})_{n \times n} \in M_n(B) : a_{ij} \in S\}.$$

Let $j_0 : \mathbb{O}_2 \rightarrow B$ be a full embedding. Let $p = j(1_{\mathbb{O}_2})$ and $x_1, x_2, \dots, x_m \in B$ such that $\sum_{i=1}^m x_i^* p x_i = 1$. Let B_1 be the unital separable C^* -subalgebra generated by $S, \{x_1, x_2, \dots, x_m\}$ and $j(\mathbb{O}_2)$. Then B_1 has a full \mathbb{O}_2 embedding and $p_n, u_n \in \bigcup_{k=1}^\infty M_k(B_1)$ for all n . Note that $K_i(B_1)$ is countable. The embedding $j_1 : B_1 \rightarrow B$ gives homomorphisms $(j_1)_{*i} : K_i(B_1) \rightarrow K_i(B)$. Let $F_{1,i}$ be the subgroup of $K_0(B_1)$ generated by $\{p_n\}$ and $\{u_n\}$, respectively. It is clear that $(j_1)_{*i}$ is injective on $F_{1,i}, i = 0, 1$. In particular, the image of $(j_1)_{*i}$ contains $G_i, i = 0, 1$. Let $N'_{1,i} = \ker(j_1)_{*i}$ and let $N_{1,i}$ be the set of all projections (if $i = 0$), or unitaries (if $i = 1$) in $\bigcup_{k=1}^\infty M_k(B_1)$ which have images in $N'_{1,i}$. Let $\{p_{1,n}\}$ be a dense subset of projections in $\bigcup_{k=1}^\infty M_k(B_1)$. There are countable pairs of projections $\{e_n, e'_n\}$ in $\{p_{1,n}\}$ such that $[e_n] = [e'_n]$ in $K_0(B)$. There are $w_n \in \bigcup_{k=1}^\infty M_k(B)$ such that $w_n^* w_n = e_n \oplus 1_{k(n)}$ and $w_n w_n^* = e'_n \oplus 1_{k(n)}$.

Let $\{u_{1,n}\}$ be a dense subset of unitaries in $\bigcup_{k=1}^\infty M_k(B_1)$. For each $u_{1,n}$, there are unitaries $z_{1,n,k} \in \bigcup_{j=1}^\infty M_j(B), k = 1, 2, \dots, m(n)$ such that

$$\|z_{1,n,1} - 1\| < \frac{1}{2}, \|z_{1,n,m(n)} - u_{1,n}\| < \frac{1}{2} \quad \text{and} \quad \|z_{1,n,k} - z_{1,n,k+1}\| < \frac{1}{2},$$

$k = 1, 2, \dots, m(n), n = 1, 2, \dots$. Let B_2 be a separable unital C^* -algebra containing B_1 such that $\bigcup_{k=1}^\infty M_k(B_2)$ contains all $\{w_{1,n}\}$ and $\{z_{1,n,k}\}$. Note that there is a full embedding of \mathbb{O}_2 to B_2 . Note also that if $p, q \in \bigcup_{k=1}^\infty M_k(B_1)$ are projections such that $[p] - [q] \in N_{1,0}$ then $[p] - [q] = 0$ in $K_0(B_2)$. Similarly, if $u \in B_1$ and $[u] \in N_{1,1}$, then $[u] = 0$ in B_2 . Suppose that B_l has been constructed. Let $j_l : B_l \rightarrow B$ be the embedding. Let $N_{l,i} = \ker(j_l)_{*i}$, for $i = 0, 1$. As before, we obtain a unital separable C^* -algebra $B_{l+1} \supset B_l$ such that every pair of projections $p, q \in \bigcup_{k=1}^\infty M_k(B_l)$ with $[p] - [q] \in N_{l,0}$ has the property that $[p] = [q]$ in $K_0(B_{l+1})$, and every unitary $u \in B_l$ with $[u] \in N_{l,1}$ has the property that $[u] = 0$ in $K_1(B_{l+1})$. Let B_0 be the closure of $\bigcup_{l=1}^\infty B_l$. Note that B_0 admits a full \mathbb{O}_2 embedding, say $j : B_0 \rightarrow B$, and that B_0 is separable.

We claim that j_{*i} is injective. Suppose that $p, q \in M_k(B_0)$ is a pair of projections for which $[p] - [q] \in \ker j_{*0}$ and $[p] - [q] \neq 0$ in B_0 . Without loss of generality, we may assume that $p, q \in M_k(B_l)$ for some large integer l . Then $[p] - [q]$ must be in the $\ker(j_l)_{*0}$. By the construction, $[p] - [q] = 0$ in $K_0(B_{l+1})$. This would imply

that $[p] - [q] = 0$ in $K_0(B_0)$. Thus j_{*0} is injective. An exactly same argument shows that j_{*1} is also injective. The lemma follows. \square

Lemma 8.3. *Let B be a unital C^* -algebra which admits a full \mathbb{O}_2 embedding. Suppose that $G_i \subset K_i(B)$ and $F_i(k) \subset K_i(B, \mathbb{Z}/k\mathbb{Z})$ are countable subgroups such that the image of $F_i(k)$ in $K_{i-1}(B)$ is contained in G_{i-1} , for $i = 0, 1$ and $k = 2, 3, \dots$. Then there exists a separable unital C^* -algebra $C \subset B$ admitting a full \mathbb{O}_2 embedding and such that $K_i(C) \supset G_i$, $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset F_i(k)$ and the embedding $j : C \rightarrow B$ induces an injective map*

$$j_{*i} : K_i(C) \rightarrow K_i(B)$$

and an injective map

$$j_* : K_i(C, \mathbb{Z}/k\mathbb{Z}) \rightarrow K_i(B, \mathbb{Z}/k\mathbb{Z}) \quad \text{for } k = 2, 3, \dots$$

Proof. By Lemma 8.2, there is a separable unital C^* -algebra C_1 admitting a full \mathbb{O}_2 embedding $j : C_1 \rightarrow B$ and such that $K_0(C_1) \supset G_0$, $K_1(C_1) \supset G_1$ and j induces an identity map on $K_0(C_1)$ and $K_1(C_1)$. Fix k , and let $\{x \in K_i(C_1) : kx = 0\} = \{g_1^{(i)}, g_2^{(i)}, \dots, \}$. Suppose that $\{s_1^{(i)}, s_2^{(i)}, \dots, \}$ is a subset of $K_{i-1}(B, \mathbb{Z}/k\mathbb{Z})$ such that the map from $K_{i-1}(B, \mathbb{Z}/k\mathbb{Z})$ to $K_i(B)$ maps $s_j^{(i)}$ to $g_j^{(i)}$. For each $z^{(i)} \in K_{i-1}(C_1, \mathbb{Z}/k\mathbb{Z})$, there is $s_j^{(i)}$ such that

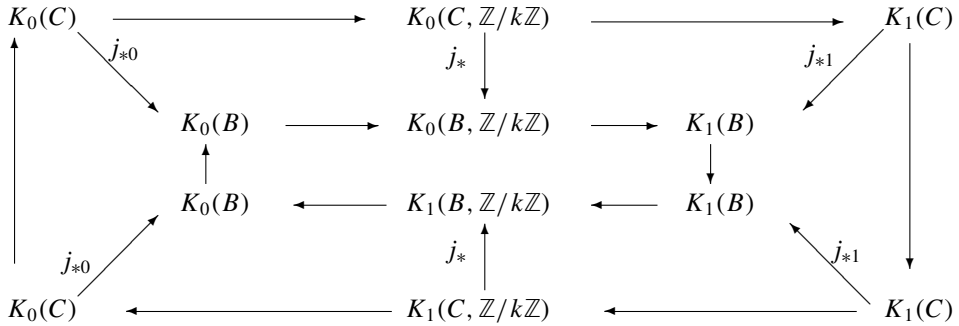
$$z^{(i)} - s_j^{(i)} \in K_i(B)/kK_i(B).$$

Since $K_i(C_1)$ is countable, the set of all possible $z^{(i)} - s_j^{(i)}$ is countable. Thus one obtains a countable subgroup G'_i that contains $K_i(C_1)$ and for which $G'_i/kK_i(B)$ contains the countable set just mentioned, as well as $F_i(k) \cap (K_i(B)/kK_i(B))$ for each k . Since countably many countable sets is still countable, we obtain a countable subgroup $G_i^{(2)} \subset K_i(B)$ such that $G_i^{(2)}$ contains G'_i and $kK_i(B) \cap G_i^{(2)} = kG_i^{(2)}$, $k = 1, 2, \dots$, and $i = 0, 1$. Note also $F_i(k) \cap (K_i(B)/kK_i(B)) \subset G_i^{(2)}/kK_i(B)$. By applying Lemma 8.2, we obtain a separable unital C^* -algebra $C_2 \supset C_1$ such that $K_i(C_2) \supset G_i^{(2)}$ and an embedding from C_2 to B gives an injective map on $K_i(C_2)$, $i = 0, 1$. Repeating what we have done above, we obtain an increasing sequence of countable subgroups $G_i^{(n)} \subset K_i(B)$ such that $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)}$ for all k and $i = 0, 1$ and an increasing sequence of separable C^* -subalgebra $s C_n$ such that $K_i(C_n) \supset G_i^{(n)}$ and embeddings from C_n into B giving injective maps on $K_i(C_n)$, $i = 0, 1$, and $n = 1, 2, \dots$. Moreover $F_i^{(k)} \cap (K_i(B)/kK_i(B)) \subset K_i(C_n)/kK_i(B)$. Let C denote the closure of $\bigcup_n C_n$ and $j : C \rightarrow B$ be the embedding. Then C is a separable unital C^* -algebra and j_{*i} is an injective map, $i = 0, 1$. Since $C \supset C_1$ and C_1 is unital, C admits a full \mathbb{O}_2 embedding. We claim that $K_i(C) \cap kK_i(B) = kK_i(C)$, $k = 1, 2, \dots$, and $i = 0, 1$. Note that $K_i(C) = \bigcup_n G_i^{(n)}$. Since $G_i^{(n)} \cap kK_i(B) = kG_i^{(n)} \subset kK_i(C)$, we see that $K_i(C) \cap kK_i(B) = kK_i(C)$,

$i = 0, 1$. Thus $K_i(C)/kK_i(C) = K_i(C)/kK_i(B)$. Since

$$K_i(C)/kK_i(B) \supset F_i^{(k)} \cap (K_i(B)/kK_0(B)),$$

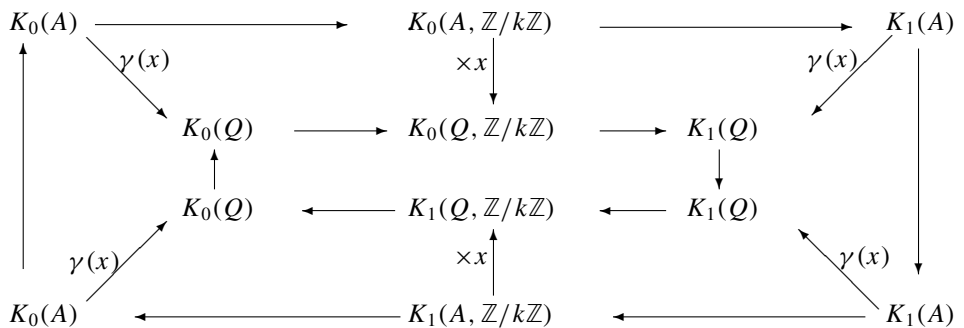
we conclude also that $K_i(C, \mathbb{Z}/k\mathbb{Z})$ contains $F_i(k)$. Since j_{*i} is injective, j induces an injective map from $K_i(C)/kK_i(C)$ into $K_i(B)/kK_i(B)$ for all integers $k \geq 1$. Using this and the fact that $j_{*i} : K_i(C) \rightarrow K_i(B)$ is injective, by chasing around the commutative diagram



one sees that j induces an injective map from $K_i(C, \mathbb{Z}/k\mathbb{Z})$ to $K_i(B, \mathbb{Z}/k\mathbb{Z})$. \square

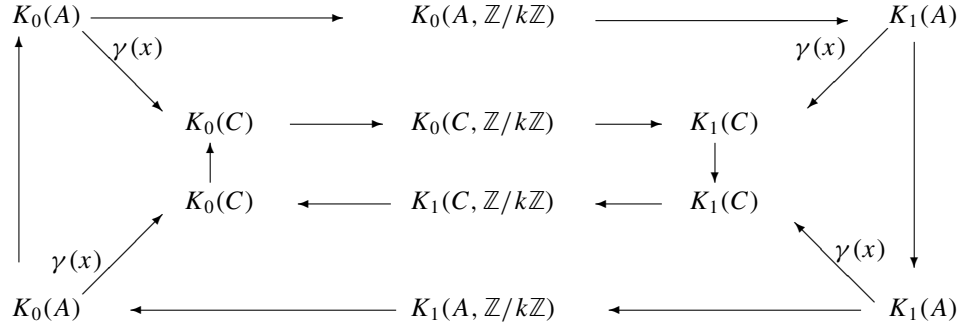
Corollary 8.4. *Without assuming that B has a full \mathbb{O}_2 embedding, both Lemma 8.3 and Lemma 8.2 hold if we do not require that C (or B_0) has a full \mathbb{O}_2 embedding.*

Proof of Theorem 2.13. By Theorem 2.5, it suffices to show that, for each $x \in KL(A, M(B)/B)$, there is a full monomorphism $h : A \rightarrow M(B)/B$ such that $[h] = x$. Put $Q = M(B)/B$. Since A satisfies the AUCT, we may view x as an element in $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q))$. Note that $K_i(A)$ is a countable abelian group ($i = 0, 1$). Let $G_0^{(i)} = \gamma(x)(K_i(A))$, $i = 0, 1$, where $\gamma : \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(Q)) \rightarrow \text{Hom}(K_*(A), K_*(Q))$ is the surjective map. Then $G_0^{(i)}$ is a countable subgroup of $K_i(Q)$, $i = 0, 1$. Consider the commutative diagram

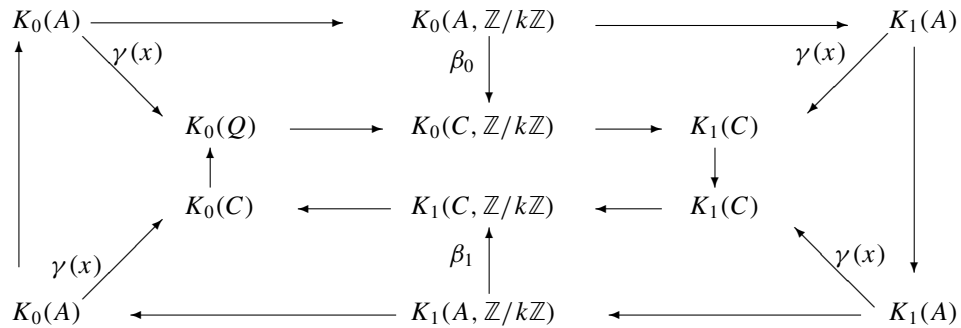


It follows from Lemma 8.3 that there is a unital C^* -algebra $C \subset Q$ which has a full \mathbb{O}_2 embedding such that $K_i(C) \subset G_0^{(i)}$, $K_i(C) \cap kK_i(Q) = kK_i(C)$, $k = 1, 2, \dots$, and $i = 0, 1$, and the embedding $j : C \rightarrow Q$ induces injective maps on $K_i(C)$

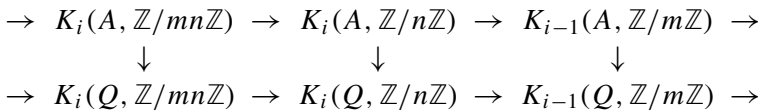
as well as on $K_i(C, \mathbb{Z}/k\mathbb{Z})$ for all k and $i = 0, 1$. Moreover $K_i(C, \mathbb{Z}/k\mathbb{Z}) \supset (\times x)(K_i(A, \mathbb{Z}/k\mathbb{Z}))$ for $k = 1, 2, \dots$ and $i = 0, 1$. We have the commutative diagram



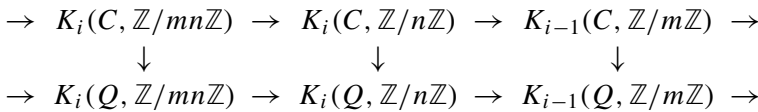
We will add two more maps on the above diagram. From the fact that the image of $K_i(A, \mathbb{Z}/k\mathbb{Z})$ under $\times x$ is contained in $K_i(C, \mathbb{Z}/k\mathbb{Z})$, ($k = 2, 3, \dots, i = 0, 1$), we obtain two maps $\beta_i : K_i(A, \mathbb{Z}/k\mathbb{Z}) \rightarrow K_i(C, \mathbb{Z}/k\mathbb{Z})$, $k = 2, 3, \dots, i = 0, 1$ such that $j_* \circ \beta_i = \times x$ and obtain the commutative diagram



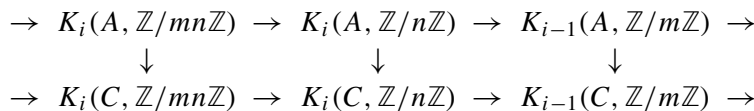
Consider the commutative diagram



Since $j_* \circ \beta_i = \times x$ and all vertical maps in the diagram



are injective, we obtain the commutative diagram



Thus we obtain $y \in KL(A, C)$ such that $y \times [j] = x$. Since A satisfies the AUCT, one checks that $KL(A, C) = KL(A \otimes \mathbb{O}_\infty, C)$. This also follows from the fact that the unital embedding from $A \rightarrow A \otimes \mathbb{O}_\infty$ gives a KK -equivalence; see [Pimsner 1997]. It follows from [Lin 2007, 6.6, 6.7] that there exists a homomorphism $\phi : A \otimes \mathbb{O}_\infty \rightarrow C \otimes \mathcal{K}$ such that $[\phi] = y$. Define $\psi = \phi|_{A \otimes 1}$. By the same result of Pimsner, one obtains that $[\psi] = y$. Since A is unital, we may assume that the image of ψ is in $M_m(C)$ for some integer $m \geq 1$. Since C admits a full \mathbb{O}_2 embedding, C has property P2. Thus 1_m is equivalent to a projection in C . Thus we may further assume that ψ maps A into C . Put $h_1 = j \circ \psi$. To obtain a full monomorphism, we use the embedding $\iota : A \rightarrow \mathbb{O}_2$ given by [Kirchberg and Phillips 2000, Theorem 2.8]. Since $M(B)/B$ has property P2, we obtain a full monomorphism $\psi : \mathbb{O}_2 \rightarrow M(B)/B$. Let $e = \psi(1_{\mathbb{O}_2})$. There is a partial isometry $w \in M_2(M(B)/B)$ such that $w^*w = 1_{M(B)/B}$ and $ww^* = 1 \oplus e$. Define $h = w^*(h_1 \oplus \psi \circ \iota)w$. One checks that $[h] = [h_1] = x$ and h is a full monomorphism. \square

Corollary 8.5. *Let A be a unital separable amenable C^* -algebra satisfying the AUCT. Let B be a unital C^* -algebra which has property P2. Then, for each $x \in KL(A, B)$, there is a full monomorphism $h : A \rightarrow B$ such that $[h] = x$.*

Proof. In the proof above, we may replace $M(B)/B$ by B . \square

Proof of Theorem 2.14. For the first part of the theorem, it suffices to show that every essential and full extension is absorbing. Let τ be a such extension. Following Elliott and Kucerovsky, we will show that τ is purely large. Denote $E = \tau^{-1}(A)$. Choose $c \in E \setminus C$. Then, by Lemma 3.3, c is a full element. Since $M(C)$ has property P1, there exists $x \in M(B)$ such that $x^*cc^*x = 1$. Therefore there exists a projection $p \leq cc^*$ for which there is $v \in M(B)$ such that $v^*v = 1$ and $vv^* = p$. Note $\overline{cBc^*} = \overline{cM(B)c^*} \cap B$. Thus $pBp \subset \overline{cBc^*}$. Now $v^*pBpv = B$, so pBp is stable and pBp is full. Thus τ is purely large, hence absorbing. The last part of the theorem follows from the next corollary. \square

Corollary 8.6. *Let A be a separable unital amenable C^* -algebra and C a unital C^* -algebra, and set $B = C \otimes \mathcal{K}$. Then $\mathbf{Ext}(A, B)$ is the same set as unitary equivalence classes of essential and full extensions of A by B .*

Proof. It suffices to show that given any element $x \in \mathbf{Ext}(A, B)$, there exists an essential and full extension $\tau : A \rightarrow M(B)/B$ so that $[\tau] = x$. There exists an essential extension $\tau_1 : A \rightarrow M(B)/B$ such that $[\tau_1] = x$. Take a monomorphism $j : A \rightarrow \mathbb{O}_2$ (see [Kirchberg and Phillips 2000]). Let $h : \mathbb{O}_2 \rightarrow M(\mathcal{K})$ be a monomorphism (given by a faithful representation of \mathbb{O}_2 on a separable Hilbert space). Let $\phi : M(\mathcal{K}) \rightarrow M(B)$ be the standard unital embedding and $\pi : M(B) \rightarrow M(B)/B$ be the quotient map. Then $\tau_2 = \pi \circ \phi \circ h \circ j$ gives a full essential trivial extension. It follows that $\tau = \tau_1 \oplus \tau_2$ is an full essential extension. Since $[\tau_2] = 0$, $[\tau] = [\tau_1] = x$. \square

Remark 8.7. Let B be a nonstable, nonunital but σ -unital C^* -algebra. Suppose that $M(B)/B$ has properties P1, P2 and P3, and suppose that $\tau : A \rightarrow M(B)/B$ is an essential and full extension. One should not expect that such extension is purely large in general. Let $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$ be an essential and full extension corresponding to τ . Recall that the extension is purely large if cBc^* contains a C^* -subalgebra which is stable and cBc^* is full in B ; see [Elliott and Kucerovsky 2001]. Given any element $c \in E \setminus B$, $\pi(c)$ is full in $M(B)/B$. But, in general, c need not be full in $M(B)$, nor does cBc^* need to be full in B . Examples are easily seen in the case that $B = c_0(C)$, where C is a unital purely infinite simple C^* -algebra. Suppose that $0 \rightarrow c_0(C) \rightarrow E \rightarrow A \rightarrow 0$ is a full extension and $c' \in E \setminus c_0(C)$. Write $c' = \{c'_n\} \in l^\infty(C)$. Define $c_n = c'_n$ if $n \geq N > 1$ and $c_n = 0$ if $n \leq N$. Put $c = \{c_n\}$. Then $c \in E \setminus c_0(C)$. However, it is clear that $cc_0(C)c^*$ is not full in $c_0(C)$. By Theorem 7.5, the full extension τ is approximately absorbing in the sense of Theorem 7.5 but not purely large. It should be also noted that, even if c^*Bc is full for all $c \in E \setminus B$, the full extension may not be purely large. Let B be a nonstable, nonunital but σ -unital simple C^* -algebra with continuous scale (see [Lin 2004b] for more examples). Then B may be stably finite. No hereditary C^* -subalgebra of B contains a stable C^* -subalgebra. So none of the essential extensions of a unital separable amenable C^* -algebra A by B could be possibly purely large in the sense of [Elliott and Kucerovsky 2001]; nevertheless, all of these extensions are approximately absorbing in the sense of Theorem 7.5 (and many of them are actually absorbing: for example, when $A = C(X)$).

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