BOUNDARY-CLUSTERED INTERFACES FOR THE ALLEN–CAHN EQUATION

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We consider the Allen–Cahn equation

$$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$ and $\varepsilon > 0$ is a small parameter. We prove the existence of a radial solution u_ε having N interfaces $\{u_\varepsilon(r) = 0\} = \bigcup_{j=1}^N \{r = r_j^\varepsilon\}$, where $1 > r_1^\varepsilon > r_2^\varepsilon > \cdots > r_N^\varepsilon$ are such that $1 - r_1^\varepsilon \sim \varepsilon \log(1/\varepsilon)$ and $r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log(1/\varepsilon)$ for $j = 2, \ldots, N$. Moreover, the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is exactly N.

1. Introduction

The aim of this paper is to construct a family of *clustered* transitional layered solutions to the Allen–Cahn equation

(1-1) $$\varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator, $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$, $\varepsilon > 0$ is a small parameter, and $\nu(x)$ denotes the unit outer normal at $x \in \partial\Omega$.

Problem (1-1) and its parabolic counterpart have been a subject of extensive research for many years. In order to describe some known results, we define the Allen–Cahn functional (see [Allen and Cahn 1979]),

$$J_\varepsilon[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 - F(u) \right), \quad \text{where } F(u) = -\frac{1}{4}(1 - u^2)^2.$$

The set $\{x \in \Omega \mid u(x) = 0\}$ is called the interface of u. Let $\text{Per}_\Omega(A)$ be the relative perimeter of the set $A \subset \Omega$. Using Γ-convergence techniques (see [Modica 1987]), Kohn and Sternberg [1989] obtained a general result stating that in a neighborhood

of an isolated local minimizer of Per$_\Omega$ there exists a local minimizer to the functional J_ε. They further used this idea to show the existence of a stable solution for (1-1) in two-dimensional nonconvex domains, such as a dumb-bell. Since then, the existence of solutions with a single interface intersecting the boundary has been established and studied by many authors. See [Alikakos et al. 2000; Bronsard and Stoth 1996; Flores et al. 2001; Kowalczyk 2005; Padilla and Tonegawa 1998; Sternberg and Zumbrun 1998] and the references therein. However, the existence of multiple interfaces is only proved, in the one-dimensional case, for the Allen–Cahn equation (with inhomogeneous terms)

$$(1\text{-}2) \qquad \varepsilon^2 u'' + a(x)(u - u^3) = 0, \quad -1 < x < 1, \quad u'(\pm 1) = 0$$

(see [Nakashima 2003; Nakashima and Tanaka 2003]); and, in the higher-dimensional case, for the following nonlinear equation with bistable nonlinearity and inhomogeneous term:

$$(1\text{-}3) \quad \varepsilon^2 \Delta u + u\big(u - a(|x|)\big)(1 - u) = 0 \quad \text{in } B_1(0), \qquad \frac{\partial u}{\partial v} = 0 \quad \text{on } \partial B_1(0)$$

(see [Dancer and Yan 2003]). The result of this last paper states that if $a(r)$ has a critical point $r_0 \in (0, 1)$ such that $a(r_0) = 1/2$, $a'(r_0) = 0$, $a''(r_0) < 0$, then there exists a clustered interior-layer solution to (1-3). All three papers use the properties of the inhomogeneous terms to construct multiple (interior) interfaces. (For the Allen–Cahn equation with inhomogeneity, $\Delta u + a(x)(u - u^3) = 0$ in $\mathbb{R}^2$, see [Rabinowitz and Stredulinsky 2003; 2004].)

Here, we continue our study, initiated in [Malchiodi et al. 2005], of clustered layered solutions for semilinear elliptic equations, and show that the *homogeneous* Allen–Cahn equation *itself* can generate multiple clustered interfaces near the boundary. In that paper we showed that the singularly-perturbed Neumann problem

$$(1\text{-}4) \qquad \begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \\[2mm] u > 0 \ \text{in } \Omega \quad \text{and} \quad \dfrac{\partial u}{\partial v} = 0 \ \text{on } \partial\Omega, \end{cases}$$

has a clustered layered solution near the boundary. (The existence of a one-layer solution to (1-4) near the boundary was first established in [Ambrosetti et al. 2003; 2004].) The purpose of this paper is to show that a similar phenomenon happens to the Allen–Cahn equation. In particular, we establish the existence of clustered interfaces — the so-called "phantom interfaces" — in higher dimensions. Moreover, we show that, for each fixed positive integer N, there exists a solution to (1-1) with Morse index N (in the space of radial functions).

Our main result is this:

Theorem 1.1. *Let N be a fixed positive integer. There exists $\varepsilon_N > 0$ such that, for all $\varepsilon < \varepsilon_N$, problem (1-1) admits a radially symmetric solution u_ε with the following properties:*

(1) the set of interfaces $\{u_\varepsilon(r) = 0\}$ contains N spheres $\{r = r_j^\varepsilon\}$, $j = 1, \ldots, N$, with

(1-5) $$1 - r_1^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \qquad r_{j-1}^\varepsilon - r_j^\varepsilon \sim \varepsilon \log \frac{1}{\varepsilon}, \quad j = 2, \ldots, N.$$

More precisely, we have $u_\varepsilon(r_j^\varepsilon + \varepsilon y) \to (-1)^j H(y)$, where $H(y)$ is the unique heteroclinic solution of

(1-6) $$H'' + H - H^3 = 0, \qquad H(0) = 0, \qquad H(\pm\infty) = \pm 1.$$

(2) The solution u_ε has the energy bound

(1-7) $$J_\varepsilon[u_\varepsilon] = \omega_{n-1} N \varepsilon I[H] + o(\varepsilon),$$

where

$$I[H] = \int_{\mathbb{R}} \left(\tfrac{1}{2}(H')^2 - F(H) \right),$$

and where ω_{n-1} denotes the volume of S^{n-1}.

(3) The Morse index of u_ε in $H_r^1(\Omega)$ is exactly N, where $H_r^1(\Omega)$ denotes the space of radial functions in $H^1(\Omega)$.

Remark 1.2. By a simple transformation, Theorem 1.1 readily extends to (1-3) with $a(r) \equiv 1/2$.

Our approach is similar to that of [Malchiodi et al. 2005], where a finite-dimensional reduction procedure combined with a variational approach is used. Such a method has been used successfully in many other papers, for example, [Ambrosetti et al. 2003; 2004; Dancer and Yan 1999; Gui and Wei 1999; 2000; Gui et al. 2000].

In the rest of section, we introduce some notation to be used later.

By the scaling $x = \varepsilon y$, problem (1-1) is reduced to the ODE

(1-8) $$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + f(u) = 0 & \text{for } r \in (0, \, 1/\varepsilon), \\ u'(0) = u'(1/\varepsilon) = 0, \end{cases}$$

where $f(u) = u - u^3$. From now on, we will work with (1-8).

Let $H(y)$ be the unique solution to (1-6). Set

(1-9) $$\Omega_\varepsilon = (1/\varepsilon) B_1(0) = B_{1/\varepsilon}(0), \qquad \text{and} \qquad I_\varepsilon = (0, \, 1/\varepsilon).$$

For $u \in C^2(\Omega_\varepsilon)$ and $u = u(r)$, we have

(1-10) $$\Delta u = u'' + \frac{n-1}{r} u'.$$

For $k \in \mathbb{N}$, we denote by $H_r^k(\Omega_\varepsilon)$ the space of radial functions in $H^k(\Omega_\varepsilon)$. On $H_r^1(\Omega_\varepsilon)$, we define an inner product as follows:

$$(1\text{-}11) \qquad (u, v)_\varepsilon = \int_0^{1/\varepsilon} (u'v' + 2uv)r^{n-1}dr.$$

Similarly, the inner product on $L_r^2(\Omega_\varepsilon)$ can be defined by

$$(1\text{-}12) \qquad \langle u, v \rangle_\varepsilon = \int_0^{1/\varepsilon} (uv)r^{n-1}dr.$$

We also introduce a new energy functional that, up to a positive multiplicative constant, is equivalent to J_ε:

$$(1\text{-}13) \qquad \mathscr{E}_\varepsilon[u] = \frac{1}{2}\int_0^{1/\varepsilon} |u'|^2 r^{n-1} - \int_0^{1/\varepsilon} F(u)r^{n-1}dr, \quad u \in H_r^1(\Omega_\varepsilon).$$

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants that are independent of ε, for ε sufficiently small. The notation $A_\varepsilon \gg B_\varepsilon$ means that $\lim_{\varepsilon \to 0} |B_\varepsilon|/|A_\varepsilon| = 0$, while $A_\varepsilon \ll B_\varepsilon$ means $(1/A_\varepsilon) \gg (1/B_\varepsilon)$.

2. Some preliminary analysis

In this section we introduce a family of approximate solutions to (1-8) and derive some useful estimates.

Let H be the unique solution of (1-6). It is easy to see that

$$(2\text{-}1) \qquad \begin{cases} H(y) - 1 = -A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}) & \text{for } y > 1, \\ H(y) + 1 = A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}) & \text{for } y < -1, \\ H'(y) = \sqrt{2}A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}) & \text{for } |y| > 1, \end{cases}$$

where $A_0 > 0$ is a fixed constant.

We state the following well-known lemma on H. For a proof, see [Müller 1993, Lemma 4.1].

Lemma 2.1. *For the eigenvalue problem*

$$(2\text{-}2) \qquad \phi'' + f'(H)\phi = \lambda\phi, \quad |\phi| \le 1, \quad in \ \mathbb{R},$$

there holds

$$(2\text{-}3) \qquad \lambda_1 = 0, \qquad \phi_1 = cH', \qquad \lambda_2 < 0.$$

For $u \in H_r^2(\Omega_\varepsilon)$, we define the operator

$$(2\text{-}4) \qquad \mathscr{S}_\varepsilon[u] := u_{rr} + \frac{n-1}{r}u_r + f(u).$$

We introduce the set

$$(2\text{-}5) \quad \Lambda = \left\{ \mathbf{t} = (t_1, \ldots, t_N) \;\middle|\; \begin{array}{l} t_N > 1 - \varepsilon (\log(1/\varepsilon))^2, \quad 1 - t_1 \geq \eta \, \varepsilon \log(1/\varepsilon), \\ t_{j-1} - t_j > \eta \, \varepsilon \log(1/\varepsilon), \, j = 2, \ldots, N \end{array} \right\},$$

where $\eta \in (0, \, 1/8\sqrt{2})$ is a fixed number.

Let $\chi(s)$ be a cut-off function such that $\chi(s) = 1$ for $s \leq 1/4$ and $\chi(s) = 0$ for $s \geq 1/2$. For $t \in (3/4, \, 1)$, we define

$$(2\text{-}6) \qquad \rho_\varepsilon(t) = H'\!\left(\frac{1-t}{\varepsilon}\right); \qquad \beta_\varepsilon(r) = \frac{1}{\sqrt{2}} e^{\sqrt{2}(r - 1/\varepsilon)}, \quad r \in [0, \, 1/\varepsilon],$$

and

$$(2\text{-}7) \qquad \begin{aligned} H_t(r) &= H\!\left(r - \frac{t}{\varepsilon}\right), \\ H_{\varepsilon,t}(r) &= \left(H\!\left(r - \frac{t}{\varepsilon}\right) - \rho_\varepsilon(t) \beta_\varepsilon(r)\right)\!\left(1 - \chi(\varepsilon r)\right) - \chi(\varepsilon r). \end{aligned}$$

It is easy to see that, for $(1-t)/\varepsilon \gg 1$,

$$(2\text{-}8) \qquad \rho_\varepsilon(t) = \sqrt{2} A_0 e^{-\sqrt{2}(1-t)/\varepsilon} + O\!\left(e^{-2\sqrt{2}(1-t)/\varepsilon}\right).$$

We first assume that N is *odd*. For $\mathbf{t} \in \Lambda$, we now define our approximate function:

$$(2\text{-}9) \qquad H_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon, t_j}(r).$$

If N is even, we set

$$(2\text{-}10) \qquad H_{\varepsilon,\mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon,t}(r) - 1 = \sum_{j=1}^{N+1} (-1)^j H_{\varepsilon, t_j}(r)$$

where we use the convention that $H_{\varepsilon, t_{N+1}} = 1$. So, without loss of generality, we can assume that N is *odd*.

Note that, for $r \leq 1/(2\varepsilon)$, there holds

$$(2\text{-}11) \qquad |H_{\varepsilon,t}(r) - (-1)^N| + |H'_{\varepsilon,t}(r)| + |H''_{\varepsilon,t}(r)| \leq e^{-1/(C\varepsilon)}.$$

Observe also that, by construction, $H_{\varepsilon,\mathbf{t}}$ satisfies the Neumann boundary condition, namely $H'_{\varepsilon,\mathbf{t}}(0) = H'_{\varepsilon,\mathbf{t}}(1/\varepsilon) = 0$. Furthermore, $H_{\varepsilon,\mathbf{t}}$ depends smoothly on $\mathbf{t}$ as a map with values in $C^2\big([0, \, 1/\varepsilon]\big)$.

The next lemma shows that $H_{\varepsilon,\mathbf{t}}$ is a good approximate function to (1-8).

Lemma 2.2. *For ε sufficiently small and $\mathbf{t} \in \Lambda$, one has*

$$(2\text{-}12) \quad \big\|\mathscr{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\big\|_{L^\infty} + \varepsilon^{n-1} \int_0^{1/\varepsilon} \big|\mathscr{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\big| r^{n-1} dr$$

$$\leq C\!\left(\varepsilon + \sum_{j=1}^{N} \big(\rho_\varepsilon(t_j)\big)^2 + \sum_{i \neq j} e^{-\sqrt{2}|t_i - t_j|/\varepsilon}\right).$$

Proof. Using (1-6) it is easy to see that

$$(2\text{-}13) \quad \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}] = \frac{n-1}{r} H'_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}})$$
$$- \sum_{l=1}^{N} (-1)^l f(H_{t_l}) - 2 \sum_{l=1}^{N} (-1)^l \rho_\varepsilon(t_l) \beta_\varepsilon(r) + O(e^{-1/(C\varepsilon)}).$$

The first term in the right-hand side of (2-13) can be estimated as

$$\frac{1}{r} H'_{\varepsilon,\mathbf{t}} = \frac{1}{r} \sum_{j=1}^{N} (-1)^j \left(H'_{t_j} - \rho_\varepsilon(t_j) \beta'_\varepsilon(r) \right) + O(e^{-1/(C\varepsilon)}).$$

From the decay of H' and β_ε we deduce that

$$(2\text{-}14) \qquad \left\| \frac{1}{r} H'_{\varepsilon,\mathbf{t}} \right\|_\infty + \varepsilon^{n-1} \int_0^{1/\varepsilon} \frac{1}{r} |H'_{\varepsilon,\mathbf{t}}| r^{n-1} dr \le C\varepsilon.$$

Next, we note that

$$\left| f(H_{\varepsilon,\mathbf{t}}) - \sum_{l=1}^{N} f\left((-1)^l H_{t_l}\right) - 2 \sum_{l=1}^{N} (-1)^l \rho_\varepsilon(t_l) \beta_\varepsilon(r) \right| \le S_1 + S_2,$$

where

$$S_1 = \left| f\left(\sum_{j=1}^{N} (-1)^j H_{t_j} \right) - \sum_{j=1}^{N} f\left((-1)^j H_{t_j} \right) \right|,$$

$$S_2 = \left| f\left(\sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j} \right) - f\left(\sum_{j=1}^{N} (-1)^j H_{t_j} \right) - 2 \sum_{j=1}^{N} (-1)^j \rho_\varepsilon(t_j) \beta_\varepsilon(r) \right|.$$

To estimate S_1 and S_2, we divide the domain $I_\varepsilon = (0, 1/\varepsilon)$ into the N intervals $I_{\varepsilon,1}, \ldots, I_{\varepsilon,N}$ defined by

$$(2\text{-}15) \qquad I_{\varepsilon,1} = \left[\frac{t_1 + t_2}{2\varepsilon}, \frac{1}{\varepsilon} \right), \quad I_{\varepsilon,j} = \left[\frac{t_j + t_{j+1}}{2\varepsilon}, \frac{t_j + t_{j-1}}{2\varepsilon} \right), \ j = 2, \ldots, N-1,$$
$$\text{and} \quad I_{\varepsilon,N} = \left(0, \frac{t_N + t_{N-1}}{2\varepsilon} \right).$$

We choose $t_0 = 2 - t_1$ and $t_{N+1} = -t_N$, so that

$$(2\text{-}16) \qquad I_{\varepsilon,j} = \left[\frac{t_j + t_{j+1}}{2\varepsilon}, \frac{t_j + t_{j-1}}{2\varepsilon} \right), \quad j = 1, \ldots, N, \qquad I_\varepsilon = \bigcup_{j=1}^{N} I_{\varepsilon,j}.$$

For $r \in I_{\varepsilon,j}$ and $j < l$, we note that

$$H_{t_l}(r) = 1 + O\left(e^{-\sqrt{2}|r - t_j/\varepsilon|} \right),$$

while for $j > l$,

$$H_{t_l}(r) = -1 + O\left(e^{-\sqrt{2}|r - t_j/\varepsilon|} \right).$$

Since N is odd, we see that

(2-17) $$\sum_{l \neq j}(-1)^l H_{t_l} = \sum_{l<j}(-1)^l(H_{t_l}+1) + \sum_{l>j}(-1)^l(H_{t_l}-1).$$

Thus, we can rewrite S_1 as:

$$S_1 = f\left(\sum_{l<j}(-1)^l(H_{t_l}+1) + (-1)^j H_{t_j} + \sum_{l>j}(-1)^l(H_{t_l}-1)\right) - (-1)^j f(H_{t_j})$$
$$- \sum_{l \neq j}(-1)^l f(H_{t_l})$$
$$= f'\left((-1)^j H_{t_j}\right)\left(\sum_{l<j}(-1)^l(H_{t_l}+1) + \sum_{l>j}(-1)^l(H_{t_l}-1)\right) - \sum_{l \neq j}(-1)^l f(H_{t_l})$$
$$+ O\left(\sum_{l<j}(H_{t_l}+1)^2 + \sum_{l>j}(H_{t_l}-1)^2\right)$$

This quantity can also be written

$$S_1 = \left(f'\left((-1)^j H_{t_j}\right) - f'(1)\right)\left(\sum_{l<j}(-1)^l(H_{t_l}+1) + \sum_{l>j}(-1)^l(H_{t_l}-1)\right)$$
$$+ O\left(\sum_{l<j}(H_{t_l}+1)^2 + \sum_{l>j}(H_{t_l}-1)^2\right)$$
$$= O\left(\min\{H_{t_j}+1, H_{t_j}-1\}\right)\left(\sum_{l<j}(H_{t_l}+1) + \sum_{l>j}(H_{t_l}-1)\right)$$
$$+ O\left(\sum_{l<j}(H_{t_l}+1)^2 + \sum_{l>j}(H_{t_l}-1)^2\right).$$

Then, with some elementary computations, one finds that

(2-18) $$\|S_1\|_{L^\infty(I_{\varepsilon,j})} + \varepsilon^{n-1}\int_{I_{\varepsilon,j}} |S_1(r)| r^{n-1} dr \leq C \sum_{i \neq j} e^{-\sqrt{2}|t_i - t_j|/\varepsilon}.$$

It remains to estimate S_2. For this, we note that, for $r \in I_{\varepsilon,j}$ and $j \geq 2$, we have

$$\rho_\varepsilon(t_j)\beta_\varepsilon(r) = O\left(e^{-\sqrt{2}(1-t_1)/\varepsilon} e^{\sqrt{2}(r-1/\varepsilon)}\right),$$

from which it follows that, for $j \geq 2$,

$$\|S_2\|_{L^\infty(I_{\varepsilon,j})} + \varepsilon^{n-1}\int_{I_{\varepsilon,j}} |S_2(r)| r^{n-1} dr = O\left(e^{-2\sqrt{2}(1-t_1)/\varepsilon}\right) = O\left(\sum_{j=1}^N (\rho_\varepsilon(t_j))^2\right).$$

Therefore, we just need to consider the case when $r \in I_{\varepsilon,1}$. But, since $f'(\pm 1) = -2$, we have

$$S_2 = f\left(\sum_{l=1}^N (-1)^l H_{t_l} - \sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r)\right)$$
$$- f\left(\sum_{l=1}^N (-1)^l H_{t_l}\right) - f'(-1)\sum_{l=1}^N (-1)^l \rho_\varepsilon(t_l)\beta_\varepsilon(r)$$

$$= \Big(f'\Big(\sum_{l=1}^{N}(-1)^{l}H_{t_{l}}\Big) - f'(-1)\Big)\sum_{l=1}^{N}(-1)^{l}\rho_{\varepsilon}(t_{l})\beta_{\varepsilon}(r) + O\Big(\sum_{l=1}^{N}\rho_{\varepsilon}(t_{l})^{2}\beta_{\varepsilon}(r)^{2}\Big)$$

$$= O\Big(\sum_{l=1}^{N}e^{-\sqrt{2}|r-t_{l}/\varepsilon|}\Big)\Big(\sum_{l=1}^{N}\rho_{\varepsilon}(t_{l})\beta_{\varepsilon}(r)\Big) + O\Big(\sum_{l=1}^{N}\rho_{\varepsilon}(t_{l})^{2}\beta_{\varepsilon}(r)^{2}\Big).$$

Hence, we also get

(2-19) $$\|S_2\|_{L^{\infty}(I_1)} + \varepsilon^{n-1}\!\int_{I_1}|S_2(r)|r^{n-1}dr \le C\rho_{\varepsilon}^2(t_1). \qquad\qquad \square$$

The proof of the next lemma is postponed to the appendix.

Lemma 2.3. *Let* $\mathbf{t} \in \Lambda$. *For* ε *sufficiently small, we have*

(2-20) $$\mathscr{C}_{\varepsilon}\Big[\sum_{j=1}^{N}(-1)^{j}H_{\varepsilon,t_{j}}\Big]$$

$$= I[H]\sum_{i=1}^{N}\Big(\frac{t_{i}}{\varepsilon}\Big)^{n-1} - \Big(\frac{1}{\varepsilon}\Big)^{n-1}\big(\sqrt{2}A_0^2 + o(1)\big)e^{-2\sqrt{2}(1-t_1)/\varepsilon}$$

$$- \sum_{j=2}^{N}\Big(\frac{t_{j}}{\varepsilon}\Big)^{n-1}\big(\sqrt{2}A_0^2 + o(1)\big)e^{-\sqrt{2}|t_{j}-t_{j-1}|/\varepsilon} + O(\varepsilon^{2-n}),$$

where $A_0 > 0$ *is defined in* (2-1).

3. Lyapunov–Schmidt process: finite-dimensional reduction

In this section we outline the so-called Lyapunov–Schmidt reduction process. Since this can be proved along the same ideas of [Malchiodi et al. 2005, Sections 3], we skip some of the details.

Fix $\mathbf{t} \in \Lambda$. Integrating by parts, one can show that orthogonality to $\partial H_{\varepsilon,t_j}/\partial t_j$ in $H^1_r(\Omega_{\varepsilon})$, $j = 1, \ldots, N$, is equivalent to orthogonality in $L^2(\Omega_{\varepsilon})$ to the functions

(3-1) $$Z_{\varepsilon,t_j} = \Delta\Big(\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}\Big) - 2\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}, \qquad j = 1, \ldots, N.$$

By elementary computations, differentiating (1-6) we obtain

(3-2) $$\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} = -\frac{1}{\varepsilon}H'\Big(r - \frac{t_j}{\varepsilon}\Big) + \frac{1}{\varepsilon}H''\Big(\frac{1-t_j}{\varepsilon}\Big)\beta_{\varepsilon}(r) + O\big(e^{-1/(C\varepsilon)}\big),$$

(3-3) $$Z_{\varepsilon,t_j} = \big(f'(H_{t_j}) - f'(\pm 1)\big)\frac{\partial H_{\varepsilon,t_j}}{\partial t_j} + \frac{n-1}{r}\Big(\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}\Big)'$$

$$= -\frac{1}{\varepsilon}H'_{t_j}\big(f'(H_{t_j}) - f'(\pm 1)\big) + o(1/\varepsilon),$$

where $O\big(e^{-1/(C\varepsilon)}\big)$ and $o(1/\varepsilon)$ are intended both in the C^1 and H^1_r sense.

We consider first the following linear problem: Given $h \in L^\infty(\Omega_\varepsilon)$, find a function ϕ satisfying

(3-4)
$$\begin{cases} L_\varepsilon[\phi] := \phi'' + \dfrac{n-1}{r}\phi' + f'(H_{\varepsilon,\mathbf{t}})\phi = h + \displaystyle\sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}; \\ \phi'(0) = \phi'(1/\varepsilon) = 0 \quad\text{and}\quad \langle \phi, Z_{\varepsilon,t_j}\rangle_\varepsilon = 0, \quad j = 1, \dots, N, \end{cases}$$

for some constants c_j, $j = 1, \dots, N$. For this, define the norm

(3-5)
$$\|\phi\|_* = \sup_{r \in (0,\, 1/\varepsilon)} |\phi(r)|.$$

Assuming a solution to (3-4) exists, we have an estimate on ϕ:

Proposition 3.1. *Let ϕ satisfy* (3-4). *For ε sufficiently small, we have*

(3-6)
$$\|\phi\|_* \le C\|h\|_*,$$

where C is a positive constant independent of ε and $\mathbf{t} \in \Lambda$.

Proof. The argument is similar in spirit of that of [Malchiodi et al. 2005, Proposition 3.1]. For the sake of completeness, we include a proof here.

Arguing by contradiction, assume that

(3-7)
$$\|\phi\|_* = 1, \qquad \|h\|_* = o(1).$$

We multiply (3-4) by $\partial H_{\varepsilon,t_j}/\partial t_j$ and integrate over Ω_ε to obtain

(3-8)
$$\sum_{i=1}^{N} c_i \left\langle Z_{\varepsilon,t_i}, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \right\rangle_\varepsilon = -\left\langle h, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \right\rangle_\varepsilon + \left\langle \Delta\phi + f'(H_{\varepsilon,\mathbf{t}})\phi, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \right\rangle_\varepsilon.$$

From the exponential decay of H', one finds

$$\left\langle h, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \right\rangle_\varepsilon = \int_0^{1/\varepsilon} h \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} r^{n-1} dr = O\big(\|h\|_* \varepsilon^{-n}\big).$$

Moreover, integrating by parts and using (3-2) and (3-3), we deduce

$$\left\langle \Delta\phi + f'(H_{\varepsilon,\mathbf{t}})\phi, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \right\rangle_\varepsilon = \left\langle Z_{\varepsilon,t_j} + f'(H_{\varepsilon,\mathbf{t}})\frac{\partial H_{\varepsilon,t_j}}{\partial t_j}, \phi \right\rangle_\varepsilon = o\big(\varepsilon^{-n}\|\phi\|_*\big).$$

From (3-2) and (3-3), we also see that

(3-9)
$$\left\langle Z_{\varepsilon,t_i}, \frac{\partial H_{\varepsilon,t_j}}{\partial t_j} \right\rangle_\varepsilon = -\varepsilon^{-n-1}\left(t_i^{n-1}\delta_{ij} \int_{\mathbb{R}} f'(H)(H')^2 + o(1) \right),$$

where δ_{ij} denotes the Kronecker symbol. Note that, using the equation $H''' + f'(H)H' = 0$, we find

$$\int_{\mathbb{R}} f'(H)(H')^2 = \int_{\mathbb{R}} (H'')^2 > 0.$$

This shows that the left-hand side of the equation (3-8) is diagonally dominant in the indices i and j, and hence, by (3-7), we have

$$(3\text{-}10) \qquad c_i = O\big(\varepsilon \|h\|_*\big) + o\big(\varepsilon \|\phi\|_*\big) = o(\varepsilon), \qquad i = 1, \ldots, N.$$

Also, since we are assuming that $\|h\|_* = o(1)$ and since $\|Z_{\varepsilon, t_j}\|_* = O\,(1/\varepsilon)$, there holds

$$(3\text{-}11) \qquad \Big\| h + \sum_{j=1}^{N} c_j Z_{\varepsilon, t_j} \Big\|_* = o(1).$$

Thus, (3-4) yields

$$(3\text{-}12) \qquad \begin{cases} \phi'' + \dfrac{n-1}{r}\phi' + f'(\pm 1) + \big(f'(H_{\varepsilon, \mathbf{t}}) - f'(\pm 1)\big)\phi = o(1), \\[2mm] \phi'(0) = \phi'(1/\varepsilon) = 0 \quad \text{and} \quad \langle \phi, Z_{\varepsilon, t_j}\rangle_\varepsilon = 0, \quad j = 1, \ldots, N, \end{cases}$$

where $o(1)$ is in the sense of $L^\infty(0, 1/\varepsilon)$.

We show that (3-12) is incompatible with our assumption that $\|\phi\|_* = 1$. First, we claim that

$$(3\text{-}13) \qquad |\phi| \to 0 \quad \text{on} \quad y \in \bigcup_{j=1}^{N}\Big(\frac{t_j}{\varepsilon} - R, \frac{t_j}{\varepsilon} + R\Big), \quad \text{as } \varepsilon \to 0,$$

where R is any fixed positive constant.

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $j \in \{1, \ldots, N\}$, and sequences $\varepsilon_k, \phi_k, y_k \in \big(t_j/\varepsilon - R, \, t_j/\varepsilon + R\big)$ such that ϕ_k satisfies (3-4) and

$$(3\text{-}14) \qquad |\phi_k(y_k)| \geq \delta_0.$$

Let $\tilde{\phi}_k = \phi_k(y - t_j/\varepsilon_k)$. Then, using (3-12) and $\|\phi\|_* = 1$, as $\varepsilon_k \to 0$, $\tilde{\phi}_k$ converges weakly in $H^2_{\mathrm{loc}}(\mathbb{R})$ and strongly in $C^1_{\mathrm{loc}}(\mathbb{R})$ to a bounded function ϕ_0 which satisfies

$$\phi_0'' + f'(H)\phi_0 = 0 \quad \text{in } \mathbb{R}, \quad |\phi_0| \leq C.$$

By Lemma 2.1, we have $\phi_0 = cH'$ for some c. Since $\tilde{\phi}_k \perp Z_{\varepsilon, t_j}$, we conclude that

$$\int_{\mathbb{R}} \phi_0 f'(H)(H')^2(y) = 0,$$

which yields $c = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (3-14), so (3-13) holds true.

Given $\delta > 0$, the decay of $f'(H) - f'(\pm 1)$ together with (3-13) (with R sufficiently large) imply that

$$(3\text{-}15) \qquad \big\|\big(f'(H_{\varepsilon, \mathbf{t}}) - f'(\pm 1)\big)\phi\big\|_* \leq \delta + \tfrac{1}{2}\|\phi\|_*.$$

Using (3-12) and the Maximum Principle, one finds

$$\|\phi\|_* \leq \left\|(f'(H_{\varepsilon,\mathbf{t}}) - f'(\pm 1))\phi\right\|_* + \sum_{j=1}^{N} |c_j| \, \|Z_{\varepsilon,t_j}\|_* + \|h\|_* \leq 2\delta + \tfrac{1}{2}\|\phi\|_*,$$

and hence

$$\|\phi\|_* \leq 4\delta < 1,$$

if we choose $\delta < 1/4$. This contradicts (3-7). $\qquad\square$

Next, we consider the following nonlinear problem: Find a function ϕ such that for some constants c_j, $j = 1, \ldots, N$, the equation

$$(3\text{-}16) \qquad \begin{cases} \Delta(H_{\varepsilon,\mathbf{t}} + \phi) + f(H_{\varepsilon,\mathbf{t}} + \phi) = \displaystyle\sum_{j=1}^{N} c_j Z_{\varepsilon,t_j} & \text{in } \Omega_\varepsilon, \\[2mm] \phi'(0) = \phi'(\tfrac{1}{\varepsilon}) = 0 & \text{and} \qquad \langle \phi, Z_{\varepsilon,t_j} \rangle_\varepsilon = 0, \quad j = 1, \ldots, N. \end{cases}$$

holds true.

The proof of the next result follows the same lines of [Malchiodi et al. 2005, Proposition 4.2].

Proposition 3.2. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, there exists a unique* $\phi = \phi_{\varepsilon,\mathbf{t}}$ *such that* (3-16) *holds. Moreover,* $t \mapsto \phi_{\varepsilon,\mathbf{t}}$ *is of class* C^1 *as a map into* $H_r^1(\Omega_\varepsilon)$, *and we have*

$$(3\text{-}17) \qquad \|\phi_{\varepsilon,\mathbf{t}}\|_* \leq C\left(\varepsilon + \sum_{j=1}^{n} e^{-(3/2)\sqrt{2}(1-t_j)/\varepsilon} + \sum_{i \neq j} e^{-(3/4)\sqrt{2}|t_i - t_j|/\varepsilon}\right).$$

4. Energy computation for reduced energy functional

We expand the quantity

$$(4\text{-}1) \qquad \mathcal{M}_\varepsilon(\mathbf{t}) := \varepsilon^{n-1} \mathcal{E}_\varepsilon\big[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}\big] : \Lambda \to \mathbb{R}$$

in ε and $\mathbf{t}$, where $\phi_{\varepsilon,\mathbf{t}}$ is given by Proposition 3.2. Up to negligible error terms, the same expansion of Lemma 2.3 holds true.

Lemma 4.1. *For* $\mathbf{t} \in \Lambda$ *and* ε *sufficiently small, we have*

$$\mathcal{M}_\varepsilon(\mathbf{t}) = \varepsilon^{n-1} \mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]$$

$$(4\text{-}2) \qquad = I[H] \sum_{j=1}^{N} t_j^{n-1} - \left(\sqrt{2}A_0^2 + o(1)\right)e^{-2\sqrt{2}(1-t_1)/\varepsilon}$$

$$- \left(\sqrt{2}A_0^2 + o(1)\right) \sum_{j=2}^{N} t_j^{n-1} e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon} + O(\varepsilon).$$

Proof. It is sufficient to show that

$$\mathcal{M}_\varepsilon(\mathbf{t}) = \varepsilon^{n-1}\mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] + o\Big(\sum_{j=1}^N e^{-2\sqrt{2}(1-t_j)/\varepsilon} + \sum_{i\neq j} e^{-\sqrt{2}|t_i-t_j|/\varepsilon} \Big) + O(\varepsilon),$$

and to apply Lemma 2.3. In order to do this, we write

$$\varepsilon^{1-n}\mathcal{M}_\varepsilon = \mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] + K_1 + K_2 - K_3,$$

where

$$K_1 = \int_0^{1/\varepsilon} \big(H_{\varepsilon,\mathbf{t}}'\phi_{\varepsilon,\mathbf{t}}' - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big) r^{n-1} dr;$$

$$K_2 = \tfrac{1}{2} \int_0^{1/\varepsilon} \big(|\phi_{\varepsilon,\mathbf{t}}'|^2 - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\big) r^{n-1} dr;$$

$$K_3 = \int_0^{1/\varepsilon} \big(F(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - F(H_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \tfrac{1}{2}f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\big) r^{n-1} dr.$$

Integrating by parts, using Lemma 2.2 and Proposition 3.1, we find

$$(4\text{-}3) \quad |K_1| = \Big| \int_0^{1/\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr \Big| \leq C\|\phi_{\varepsilon,\mathbf{t}}\|_* \int_0^{1/\varepsilon} \big|\mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\big| r^{n-1} dr$$

$$\leq C\varepsilon^{1-n}\Big(\varepsilon^2 + \sum_{j=1}^N \big(\rho_\varepsilon(t_j)\big)^{2+3/2} + \sum_{i\neq j} e^{-(7/4)\sqrt{2}|t_i-t_j|/\varepsilon} \Big).$$

To estimate K_2, we note that $\phi_{\varepsilon,\mathbf{t}}$ satisfies

$$(4\text{-}4) \qquad \Delta\phi_{\varepsilon,\mathbf{t}} + f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) + \mathcal{S}_\varepsilon[w_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^N c_j Z_{\varepsilon,t_j}.$$

Multiplying (4-4) by $\phi_{\varepsilon,\mathbf{t}} r^{n-1}$ and integrating over I_ε, we obtain

$$(4\text{-}5) \quad \int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr = \int_{I_\varepsilon} \big(|\phi_{\varepsilon,\mathbf{t}}'|^2 - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\big) r^{n-1} dr$$

$$+ \int_{I_\varepsilon} \big(f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big)\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr.$$

Hence, we find

$$2K_2 = -\int_{I_\varepsilon} \big(f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big)\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr$$

$$+ \int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}]\phi_{\varepsilon,\mathbf{t}} r^{n-1} dr.$$

From Taylor's formula, we get

$$\big|f(H_{\varepsilon,\mathbf{t}}+\phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) - f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}\big| \leq C|\phi_{\varepsilon,\mathbf{t}}|^2,$$

so we deduce

$$|K_2| \le C \int_{I_\varepsilon} |\phi_{\varepsilon,\mathbf{t}}|^3 r^{n-1} dr + C \|\phi_{\varepsilon,\mathbf{t}}\|_* \int_{I_\varepsilon} \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}}] r^{n-1} dr.$$

From the exponential decay of $H(\pm y) - (\pm 1)$ one finds that $\phi_{\varepsilon,\mathbf{t}}(r)$ satisfies

$$\phi_{\varepsilon,\mathbf{t}}'' + \frac{n-1}{r} \phi_{\varepsilon,\mathbf{t}}' + f(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}}) = O\left(\sum_{j=1}^N e^{-\sqrt{2}|r - t_j/\varepsilon|}\right),$$

$$\phi_{\varepsilon,\mathbf{t}}'(0) = \phi_{\varepsilon,\mathbf{t}}'(1/\varepsilon) = 0.$$

From (4-4) and a comparison principle, we obtain

$$(4\text{-}6) \qquad |\phi_{\varepsilon,\mathbf{t}}(r)| \le C \sum_{j=1}^N e^{-(\sqrt{2}/\tilde{C})|r - t_j/\varepsilon|}$$

for some $\tilde{C} < 1$.

Using Proposition 3.2 and (4-6), we get

$$(4\text{-}7) \qquad |K_2| \le C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^N \big(\rho_\varepsilon(t_j)\big)^3 + \sum_{i \ne j} e^{-2\sqrt{2}|t_i - t_j|/\varepsilon}\right).$$

From the Hölder continuity of f', we deduce

$$\left|F(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - F(H_{\varepsilon,\mathbf{t}}) - f(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}} - \tfrac{1}{2}f'(H_{\varepsilon,\mathbf{t}})\phi_{\varepsilon,\mathbf{t}}^2\right| \le C|\phi_{\varepsilon,\mathbf{t}}|^3,$$

so, again, it follows that

$$(4\text{-}8) \qquad |K_3| \le C\varepsilon^{1-n}\left(\varepsilon^2 + \sum_{j=1}^N \big(\rho_\varepsilon(t_j)\big)^3 + \sum_{i \ne j} e^{-2\sqrt{2}|t_i - t_j|/\varepsilon}\right).$$

Combining with (2-20) of Lemma 2.2, we obtain the conclusion. $\square$

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Fix $\mathbf{t} \in \overline{\Lambda}$ and let $\phi_{\varepsilon,\mathbf{t}}$ be given by Proposition 3.2. Let also $\mathcal{M}_\varepsilon(\mathbf{t})$ denote the reduced energy functional defined by (4-1).

Proposition 5.1. *For ε small, the following maximization problem*

$$(5\text{-}1) \qquad \sup\{\mathcal{M}_\varepsilon(\mathbf{t}) \mid \mathbf{t} \in \Lambda\}$$

has a solution $\mathbf{t}^\varepsilon$ in the interior of Λ.

Proof. Since $\mathcal{M}_\varepsilon(\mathbf{t})$ is continuous in $\mathbf{t}$, it achieves a maximum in $\bar{\Lambda}$. Let $\mathbf{t}^\varepsilon$ be a maximum point. We claim that $\mathbf{t}^\varepsilon \in \Lambda$.

We argue by contradiction and assume that $\mathbf{t}^\varepsilon \in \partial\Lambda$. From the definition of Λ, there are three possibilities: either $1 - t_1 = \eta\varepsilon \log(1/\varepsilon)$, or there exists $j \ge 2$ such that $t_{j-1} - t_j = \eta\varepsilon \log(1/\varepsilon)$, or, finally, $t_N = 1 - \varepsilon\big(\log(1/\varepsilon)\big)^2$.

In the first case, we have

$$I[H]t_1^{n-1} - \left(\sqrt{2}A_0^2 + o(1)\right)e^{-2\sqrt{2}(1-t_1)/\varepsilon}$$

$$= I[H]\left(1 - \eta\varepsilon\log\frac{1}{\varepsilon}\right)^{n-1} - \sqrt{2}A_0^2 e^{-2\eta\sqrt{2}\log(1/\varepsilon)} + o\left(\varepsilon^{2\sqrt{2}\eta}\right)$$

$$\leq I[H] - A_0^2\varepsilon^{2\sqrt{2}\eta}.$$

Since $\eta < 1/8\sqrt{2}$, we obtain

$$(5\text{-}2) \qquad\qquad \mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \leq NI[H] - A_0^2\varepsilon^{2\sqrt{2}\eta}.$$

In the second case, there holds

$$(5\text{-}3) \quad \mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \leq I[H]\sum_{j=1}^N t_j^{n-1} - \left(\sqrt{2}A_0^2 + o(1)\right)\varepsilon^{\sqrt{2}\eta}t_j^{n-1} \leq NI[H] - A_0^2\varepsilon^{\sqrt{2}\eta}.$$

In the latter case, we have $t_N = 1 - \varepsilon\left(\log(1/\varepsilon)\right)^2$, and therefore

$$(5\text{-}4) \quad \mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \leq I[H](N - 1 + t_N^{n-1}) + O(\varepsilon)$$

$$\leq I[H]\left(N - (n-1)\varepsilon(\log(1/\varepsilon))^2\right) + O(\varepsilon).$$

On the other hand, choosing $t_j = 1 - (j/\sqrt{2})\varepsilon\log(1/\varepsilon)$, $j = 1, \ldots, N$, we obtain

$$\sum_{j=1}^N t_j^{n-1} = 1 - \frac{N(N+1)(n-1)}{2\sqrt{2}}\varepsilon\log(1/\varepsilon) + O\left(\varepsilon^2(\log(1/\varepsilon))^2\right);$$

$$(5\text{-}5) \qquad e^{-2\sqrt{2}(1-t_1)/\varepsilon} = \varepsilon^2; \qquad e^{-\sqrt{2}|t_{j-1}-t_j|/\varepsilon} = \varepsilon,$$

and we find

$$\mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) \geq NI[H] - \frac{N(N+1)(n-1)^2}{2\sqrt{2}}\varepsilon\log(1/\varepsilon) + O(\varepsilon),$$

which contradicts either (5-2) or (5-3) or (5-4). This completes the proof of Proposition 5.1. $\qquad\square$

Remark 5.2. The above argument also shows that

$$(5\text{-}6) \qquad\qquad 1 - t_1^\varepsilon \sim \varepsilon\log(1/\varepsilon), \qquad t_{j-1}^\varepsilon - t_j^\varepsilon \sim \varepsilon\log(1/\varepsilon).$$

Finally, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 3.2, there exists ε_N such that, for $\varepsilon < \varepsilon_N$, we have a C^1 map $\mathbf{t} \mapsto \phi_{\varepsilon,\mathbf{t}}$ from $\overline{\Lambda}$ into $C^2(I_\varepsilon)$ such that

$$(5\text{-}7) \qquad\qquad \mathcal{S}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}] = \sum_{j=1}^N c_j Z_{\varepsilon,t_j},$$

for some constants $\{c_j\} \subseteq \mathbb{R}$, which are also of class C^1 in $\mathbf{t}$.

By Proposition 5.1, there exists $\mathbf{t}^\varepsilon \in \Lambda$ that achieves the maximum of $\mathcal{H}_\varepsilon : \mathbf{t} \mapsto \mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}]$. Let

$$u_\varepsilon = \sum_{i=1}^{N} (-1)^i H_{\varepsilon,t_i^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon} = H_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}.$$

Then we have

$$\partial_{t_i}\big|_{\mathbf{t}=\mathbf{t}^\varepsilon} \mathcal{M}_\varepsilon(\mathbf{t}^\varepsilon) = 0, \qquad i = 1, \ldots, N,$$

and hence

$$\int_{I_\varepsilon} \Big(\nabla u_\varepsilon \nabla \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) + u_\varepsilon \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) - f(u_\varepsilon)\partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) \Big)\Big|_{\mathbf{t}=\mathbf{t}^\varepsilon} r^{n-1} dr = 0.$$

Therefore, by (5-7), we find

(5-8) $$\sum_{j=1}^{N} c_j \int_{I_\varepsilon} \big(Z_{\varepsilon,t_j} \partial_{t_i}(H_{\varepsilon,\mathbf{t}} + \phi_{\varepsilon,\mathbf{t}}) \big) r^{n-1} dr = 0.$$

Differentiating the equation $\langle \phi, Z_{\varepsilon,t_j} \rangle_\varepsilon = 0$ with respect to t_j, we get

$$\langle \partial_{t_i}\phi, Z_{\varepsilon,t_j} \rangle_\varepsilon = -\langle \phi, \partial_{t_i} Z_{\varepsilon,t_j} \rangle_\varepsilon = O(\|\phi\|_*)\varepsilon^{-n-1}.$$

Using (3-3), we see that (5-8) is diagonally dominant in the coefficients $\{c_i\}$, which implies that $c_j = 0$ for $j = 1, \ldots, N$. Hence

$$u_\varepsilon = H_{\varepsilon,\mathbf{t}^\varepsilon} + \phi_{\varepsilon,\mathbf{t}^\varepsilon}$$

is a solution of (1-1).

By our construction, one can easily check that $\varepsilon^{n-1}\mathcal{E}_\varepsilon(u_\varepsilon) \to NI[H]$ as $\varepsilon \to 0$, and that u_ε has only N zeroes $s_1^\varepsilon/\varepsilon, \ldots, s_N^\varepsilon/\varepsilon$. By the structure of u_ε, we see that (up to a permutation) $s_i^\varepsilon - t_i^\varepsilon = o(1)$. This proves (1) and (2) of Theorem 1.1.

It remains to prove (3). First we note that u_ε' satisfies

(5-9) $$\Delta u_\varepsilon' + f'(u_\varepsilon)u_\varepsilon' = \frac{n-1}{r^2}u_\varepsilon'.$$

By our construction, at each interval $(s_j^\varepsilon/\varepsilon, s_{j-1}^\varepsilon/\varepsilon)$, for $j = 2, \ldots, N$, there exists a point $\tilde{s}_{j-1}^\varepsilon/\varepsilon \in (s_j/\varepsilon, s_{j-1}/\varepsilon)$ such that $u_\varepsilon'(\tilde{s}_{j-1}^\varepsilon/\varepsilon) = 0$. Now, we set

$$\varphi_1(r) = \begin{cases} u_\varepsilon'(r) & \text{for } r \in \big(\tilde{s}_1^\varepsilon/\varepsilon,\ 1\big), \\ 0 & \text{otherwise}; \end{cases}$$

$$\varphi_j(r) = \begin{cases} u_\varepsilon'(r) & \text{for } r \in \big(\tilde{s}_j^\varepsilon/\varepsilon,\ \tilde{s}_{j-1}^\varepsilon/\varepsilon\big), \\ 0 & \text{otherwise}, \end{cases} \qquad j = 2, \ldots, N-1;$$

$$\varphi_N(r) = \begin{cases} u_\varepsilon'(r), & \text{for } r \in \big(1/(2\varepsilon),\ \tilde{s}_{N-1}^\varepsilon/\varepsilon\big), \\ 2\varepsilon(r - 1/(4\varepsilon))u_\varepsilon'(r), & \text{for } 1/(4\varepsilon) \le r \le 1/(2\varepsilon), \\ 0, & \text{for } r < 1/(4\varepsilon) \text{ or } r \ge \tilde{s}_{N-1}^\varepsilon/\varepsilon. \end{cases}$$

Next, we define a quadratic functional

$$\mathbf{Q}[\phi] = \int_{I_\varepsilon} \left(|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2 \right) r^{n-1} dr. \tag{5-10}$$

It is easy to check that

$$\int_{I_\varepsilon} \varphi_i \varphi_j r^{n-1} dr = 0 \qquad \text{for } i \neq j. \tag{5-11}$$

Using equation (5-9), we obtain

$$\mathbf{Q}[\varphi_i] = -\int_{I_\varepsilon} \varphi_i^2 r^{n-3} dr < 0, \qquad i = 1, \ldots, N-1. \tag{5-12}$$

When $i = N$, we have

$$\mathbf{Q}[\varphi_N] = -\int_{I_\varepsilon} \varphi_N^2 r^{n-3} dr + O\left(e^{-1/(C\varepsilon)}\right) < 0. \tag{5-13}$$

From (5-12) and (5-13), the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at least N.

Finally, we also show that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at most N. In fact, we define

$$z_j^\varepsilon(r) = H'_{\varepsilon, t_j^\varepsilon} \chi \left(\frac{\varepsilon r - t_j^\varepsilon}{\varepsilon \sqrt{|\log(1/\varepsilon)|}} \right), \qquad j = 1, \ldots, N, \tag{5-14}$$

and consider the following minimization problem

$$\mu_j^\varepsilon = \inf_{\substack{\phi \in H^1(I_{\varepsilon,j}) \\ \int_{I_{\varepsilon,j}} \phi z_j^\varepsilon r^{n-1} dr = 0}} \frac{\int_{I_{\varepsilon,j}} \left(|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2 \right) r^{n-1} dr}{\int_{I_{\varepsilon,j}} \phi^2 r^{n-1} dr}. \tag{5-15}$$

Assume that $\mu_j^\varepsilon \leq 0$. By standard regularity theory, μ_j^ε is attained by a function ϕ_j^ε which satisfies

$$\Delta \phi_j^\varepsilon + f'(u_\varepsilon)\phi_j^\varepsilon = -\mu_j^\varepsilon \phi_j^\varepsilon + c_j^\varepsilon z_j^\varepsilon,$$

$$(\phi_j^\varepsilon)'|_{\partial I_{\varepsilon,j}} = 0 \qquad \text{and} \qquad \int_{I_{\varepsilon,j}} \phi_j^\varepsilon z_j r^{n-1} dr = 0, \tag{5-16}$$

where c_j^ε is a constant.

First, we notice that $c_j^\varepsilon = o(\|\phi_j^\varepsilon\|_*)$, which follows by reasoning as for (3-10) of Proposition 3.1. Then, from Lemma 2.1 we deduce that $\mu_j^\varepsilon \to 0$; moreover, the same argument leading to Proposition 3.1 shows that $\phi_j^\varepsilon = 0$.

Thus, $\mu_j^\varepsilon > 0$. Let $\phi = \phi(r)$ be such that $\int_{I_\varepsilon} \phi z_j^\varepsilon r^{n-1} = 0$, $j = 1, \ldots, N$, which is equivalent to $\int_{I_{\varepsilon,j}} \phi z_j^\varepsilon r^{n-1} = 0$. This then implies that

$$(5\text{-}17) \quad \int_{I_{\varepsilon,j}} \left(|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2 \right) r^{n-1} dr \geq \mu_j^\varepsilon \int_{I_{\varepsilon,j}} |\phi|^2 r^{n-1} dr, \qquad j = 1, \ldots, N,$$

and hence

$$(5\text{-}18) \quad \int_{I_\varepsilon} \left(|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2 \right) r^{n-1} dr = \sum_{j=1}^{N} \int_{I_{\varepsilon,j}} \left(|\nabla \phi|^2 - f'(u_\varepsilon)\phi^2 \right) r^{n-1} dr$$

$$\geq \min_{j=1,\ldots,N} \mu_j^\varepsilon \int_{I_\varepsilon} |\phi|^2 r^{n-1} dr.$$

This yields

$$(5\text{-}19) \quad \lambda_{N+1} = \sup_{\substack{v_1,\ldots,v_N \\ j=1,\ldots,N}} \inf_{\substack{\int_{I_\varepsilon} \phi v_j r^{n-1} = 0}} \frac{\int_{I_\varepsilon} \left(|\nabla u|^2 - f'(u_\varepsilon)\phi^2 \right) r^{n-1}}{\int_{I_\varepsilon} \phi^2 r^{n-1}} \geq \min_{j=1,\ldots,N} \mu_j^\varepsilon > 0,$$

and hence the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is at most N.

Combining the upper and lower bound for the Morse index, we see that the Morse index of u_ε in $H_r^1(\Omega_\varepsilon)$ is exactly N. This proves (3) of Theorem 1.1. □

Appendix

In this appendix we expand the quantity $\mathcal{E}_\varepsilon\left[\sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j}\right]$ as a function of ε and $\mathbf{t}$. Several facts will be used repeatedly:

$$H(y) = 1 - A_0 e^{-\sqrt{2}|y|} + O\left(e^{-2\sqrt{2}|y|}\right), \text{ for } y > 1;$$

$$H(y) = -1 + A_0 e^{-\sqrt{2}|y|} + O\left(e^{-2\sqrt{2}|y|}\right), \text{ for } y < -1;$$

$$H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O\left(e^{-2\sqrt{2}|y|}\right), \text{ for } |y| > 1;$$

$$\rho_\varepsilon(t_1) = \sqrt{2}\left(A_0 + o(1)\right) e^{-\sqrt{2}(1-t_1)/\varepsilon};$$

$$\rho_\varepsilon(t_j) = o\left(\rho_\varepsilon(t_1)\right) \text{ for } j \geq 2.$$

From a Taylor expansion we find

$$\mathcal{E}_\varepsilon[H_{\varepsilon,\mathbf{t}}] = I_1 + I_2 + I_3 + O\left(\varepsilon^{1-n} \rho_\varepsilon^3(t_1)\right),$$

where

$$I_1 = \mathscr{E}_\varepsilon\Big[\sum_{j=1}^{N}(-1)^j H_{t_j}\Big],$$

$$I_2 = -\Big(\sum_{l=1}^{K}(-1)^l \rho_\varepsilon(t_l)\Big)\int_{I_\varepsilon}\Big(\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}\Big)'\beta_\varepsilon' - f\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}\Big)\beta_\varepsilon\Big)r^{n-1}dr,$$

$$I_3 = \tfrac{1}{2}\Big(\sum_{l=1}^{N}(-1)^l \rho_\varepsilon(t_l)\Big)^2\int_{I_\varepsilon}\Big(|\beta_\varepsilon'|^2 - f'\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}\Big)\beta_\varepsilon^2\Big)r^{n-1}.$$

Recalling that $f'(\pm 1) = -2$, the term I_3 can be estimated by

$$I_3 = \tfrac{1}{2}\Big(\sum_{j=1}^{N}(-1)^j \rho_\varepsilon(t_j)\Big)^2\int_{I_\varepsilon}\Big(2 - f'\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}\Big)\Big)\beta_\varepsilon^2 r^{n-1}dr + o\big(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)\big)$$

$$= \big(\rho_\varepsilon(t_1)\big)^2\int_{I_\varepsilon}\beta_\varepsilon^2 r^{n-1}dr + o\big(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)\big) = \tfrac{1}{2\sqrt{2}}\varepsilon^{1-n}\big(\rho_\varepsilon(t_1)\big)^2 + o\big(\varepsilon^{1-n}\rho_\varepsilon^2(t_1)\big)$$

$$= \frac{A_0^2 + o(1)}{\sqrt{2}}\varepsilon^{1-n}e^{-2\sqrt{2}(1-t_1)/\varepsilon}.$$

Next we estimate the integral in I_2. We have

$$\int_{I_\varepsilon}\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}'\beta_\varepsilon' - f\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}\Big)\beta_\varepsilon\Big)r^{n-1}dr$$

$$= \int_{I_\varepsilon}\Big(\sqrt{2}\sum_{j=1}^{N}(-1)^j H_{t_j}' - f\Big(\sum_{j=1}^{N}(-1)^j H_{t_j}\Big)\Big)\beta_\varepsilon r^{n-1}dr$$

$$= \int_{I_{\varepsilon,1}}\big(-\sqrt{2}H_{t_1}' - f(-H_{t_1})\big)\beta_\varepsilon r^{n-1}dr + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big)$$

$$= -\tfrac{1}{\sqrt{2}}e^{-\sqrt{2}(1-t_1)/\varepsilon}\int_{\mathbb{R}}\big(\sqrt{2}H' - f(H)\big)e^{\sqrt{2}y}dy\,(t_1/\varepsilon)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big)$$

$$= -A_0 e^{-\sqrt{2}(1-t_1)/\varepsilon}(t_1/\varepsilon)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big),$$

since

$$\int_{\mathbb{R}}\big(\sqrt{2}H' - f(H)\big)e^{\sqrt{2}y}dy = \big(H'e^{\sqrt{2}y}\big)\Big|_{-\infty}^{+\infty} = \sqrt{2}A_0.$$

Thus,

$$I_2 = -\big(\sqrt{2}A_0^2 + o(1)\big)e^{-2\sqrt{2}(1-t_1)/\varepsilon}(t_1/\varepsilon)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big) + O(\varepsilon^{2-n}),$$

which implies that

(5-20) $\quad I_2 + I_3 = -\dfrac{A_0^2 + o(1)}{\sqrt{2}}e^{-2\sqrt{2}(1-t_1)/\varepsilon}(t_1/\varepsilon)^{n-1} + o\big(\varepsilon^{1-n}\rho_\varepsilon(t_1)\big) + O(\varepsilon^{2-n}),$

since $t_1 = 1 + O\big(\varepsilon(\log(1/\varepsilon))^2\big)$.

It remains to consider I_1. For this purpose, we decompose it as

$$I_1 = \sum_{j=1}^{N} E_{\varepsilon,j},$$

where

$$E_{\varepsilon,j} = \int_{I_{\varepsilon,j}} \left(\tfrac{1}{2} \left| \sum_{l=1}^{N} (-1)^l H_{t_l}' \right|^2 - F\left(\sum_{l=1}^{N} (-1)^l H_{t_l} \right) \right) r^{n-1} dr$$

$$= \int_{I_{\varepsilon,j}} \left(\tfrac{1}{2} \left| H_{t_j}' + \sum_{l \neq j} (-1)^{j+l} H_{t_l}' \right|^2 - F\left(H_j + \sum_{l \neq j} (-1)^{j+l} H_{t_l} \right) \right) r^{n-1} dr$$

$$= I_4 + I_5 + I_6 + o\left(\varepsilon^{1-n} \sum_{i \neq j} e^{-\sqrt{2}|t_i - t_j|/\varepsilon} \right),$$

with

$$I_4 = \int_{I_{\varepsilon,j}} \left(\tfrac{1}{2} |H_{t_j}'|^2 - F(H_{t_j}) \right) r^{n-1} dr,$$

$$I_5 = \int_{I_{\varepsilon,j}} \left(H_{t_j}' \sum_{l \neq j} (-1)^{l+j} H_{t_l}' - f(H_{t_j}) \sum_{l \neq j} (-1)^{l+j} H_{t_l} \right) r^{n-1} dr,$$

$$I_6 = \frac{1}{2} \int_{I_{\varepsilon,j}} \left| \sum_{l \neq j} (-1)^{j+l} H_{t_l} \right|^2 (2 - f'((-1)^j H_{t_j})) r^{n-1} dr.$$

Using the fact that $|H'|^2 = 2F(H)$, for I_4 we find

$$I_4 = \int_{I_{\varepsilon,j}} |H_{t_j}'|^2 r^{n-1} dr$$

$$= \int_{R} |H'|^2 dy \, (t_j/\varepsilon)^{n-1} - \frac{A_0^2 + o(1)}{\sqrt{2}} \left(e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon} + e^{-\sqrt{2}|t_j - t_{j+1}|/\varepsilon} \right) (t_j/\varepsilon)^{n-1}$$

$$+ O(\varepsilon^{2-n}).$$

For $j \geq 2$, I_5 can be estimated (by recalling the exponential-decay property of $H(y) \pm 1$) as

$$I_5 = (t_j/\varepsilon)^{n-1} H_{t_j}' \sum_{l \neq j} (-1)^{l+j} H_{t_l} \big|_{\partial I_{\varepsilon,j}} + O(\varepsilon^{2-n})$$

$$= -\left(A_0^2 + o(1) \right) \sqrt{2} \left(e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon} + e^{-\sqrt{2}|t_j - t_{j+1}|/\varepsilon} \right) (t_j/\varepsilon)^{n-1} + O(\varepsilon^{2-n}).$$

For $j = 1$, we have

$$I_5 = (t_1/\varepsilon)^{n-1} H_{t_j}' \sum_{l > 1} (-1)^{l+1} H_{t_l} \big|_{\partial I_{\varepsilon,1}} + O(\varepsilon^{2-n})$$

$$= -\left(A_0^2 + o(1) \right) \sqrt{2} e^{-\sqrt{2}|t_1 - t_2|/\varepsilon} (t_j/\varepsilon)^{n-1} + O(\varepsilon^{2-n}).$$

I_6 can be estimated similarly: for $j \geq 2$, we have

$$
\begin{aligned}
I_6 &= 2 \int_{I_{\varepsilon,j}} \left| \sum_{l \neq j} (-1)^{j+l} H_{t_l} \right|^2 r^{n-1} dr \\
&= \frac{A_0^2 + o(1)}{\sqrt{2}} \left(e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon} + e^{-\sqrt{2}|t_j - t_{j+1}|/\varepsilon} \right)(t_j/\varepsilon)^{n-1} + O(\varepsilon^{2-n}),
\end{aligned}
$$

while for $j = 1$,

$$
I_6 = 2 \int_{I_{\varepsilon,1}} \left| \sum_{l > 1} (-1)^{l+1} H_{t_l} \right|^2 r^{n-1} dr = \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-\sqrt{2}|t_1 - t_2|/\varepsilon}(t_1/\varepsilon)^{n-1} + O(\varepsilon^{2-n}).
$$

Combining the estimates of I_4, I_5, and I_6, we obtain

$$
\begin{aligned}
I_1 &= I[H] \sum_{j=1}^{N} (t_j/\varepsilon)^{n-1} - \sqrt{2} \left(A_0^2 + o(1) \right) \sum_{j=2}^{N} e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon}(t_j/\varepsilon)^{n-1} \\
&\qquad\qquad\qquad\qquad - \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-2\sqrt{2}(1-t_1)/\varepsilon} + O(\varepsilon^{2-n}) \\
&= I[H] \sum_{j=1}^{N} (t_j/\varepsilon)^{n-1} - \sqrt{2} \left(A_0^2 + o(1) \right) \sum_{j=2}^{N} e^{-\sqrt{2}|t_j - t_{j-1}|/\varepsilon}(t_j/\varepsilon)^{n-1} \\
&\qquad\qquad\qquad\qquad - \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-2\sqrt{2}(1-t_1)/\varepsilon}(t_1/\varepsilon)^{n-1} + O(\varepsilon^{2-n}).
\end{aligned}
$$

Adding this to the estimate in (5-20), we obtain the asymptotic expansion (2-20) of $\mathscr{E}_\varepsilon \left[\sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j} \right]$.

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