BOUNDARY-CLUSTERED INTERFACES FOR THE ALLEN–CAHN EQUATION

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We consider the Allen–Cahn equation

\[ \varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \]

where \( \Omega = B_1(0) \) is the unit ball in \( \mathbb{R}^n \) and \( \varepsilon > 0 \) is a small parameter. We prove the existence of a radial solution \( u_\varepsilon \) having \( N \) interfaces \( \{ u_\varepsilon(r) = 0 \} = \bigcup_{j=1}^N \{ r = r^*_j \} \), where \( 1 > r^*_1 > r^*_2 > \cdots > r^*_N \) are such that \( 1 - r^*_1 \sim \varepsilon \log(1/\varepsilon) \) and \( r^*_{j-1} - r^*_j \sim \varepsilon \log(1/\varepsilon) \) for \( j = 2, \ldots, N \). Moreover, the Morse index of \( u_\varepsilon \) in \( H^1_r(\Omega_\varepsilon) \) is exactly \( N \).

1. Introduction

The aim of this paper is to construct a family of clustered transitional layered solutions to the Allen–Cahn equation

\[ \varepsilon^2 \Delta u + u - u^3 = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega, \]

where \( \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \) is the Laplace operator, \( \Omega = B_1(0) \) is the unit ball in \( \mathbb{R}^n \), \( \varepsilon > 0 \) is a small parameter, and \( \nu(x) \) denotes the unit outer normal at \( x \in \partial \Omega \).

Problem (1-1) and its parabolic counterpart have been a subject of extensive research for many years. In order to describe some known results, we define the Allen–Cahn functional (see [Allen and Cahn 1979]),

\[ J_\varepsilon[u] = \int_\Omega \left( \frac{\varepsilon^2}{2} |\nabla u|^2 - F(u) \right), \quad \text{where} \quad F(u) = -\frac{1}{4} (1 - u^2)^2. \]

The set \( \{ x \in \Omega \mid u(x) = 0 \} \) is called the interface of \( u \). Let \( \text{Per}_\Omega(A) \) be the relative perimeter of the set \( A \subset \Omega \). Using \( \Gamma \)-convergence techniques (see [Modica 1987]), Kohn and Sternberg [1989] obtained a general result stating that in a neighborhood of the interface, the solution is concentrated in a small number of interface layers.

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of an isolated local minimizer of \( \text{Per}_2 \) there exists a local minimizer to the functional \( J_{\varepsilon} \). They further used this idea to show the existence of a stable solution for (1-1) in two-dimensional nonconvex domains, such as a dumb-bell. Since then, the existence of solutions with a single interface intersecting the boundary has been established and studied by many authors. See [Alikakos et al. 2000; Bronsard and Stoth 1996; Flores et al. 2001; Kowalczyk 2005; Padilla and Tonegawa 1998; Sternberg and Zumbrun 1998] and the references therein. However, the existence of multiple interfaces is only proved, in the one-dimensional case, for the Allen–Cahn equation (with inhomogeneous terms)

\[
(1-2) \quad \varepsilon^2 u'' + a(x)(u - u^3) = 0, \quad -1 < x < 1, \quad u'(\pm1) = 0
\]

(see [Nakashima 2003; Nakashima and Tanaka 2003]); and, in the higher-dimensional case, for the following nonlinear equation with bistable nonlinearity and inhomogeneous term:

\[
(1-3) \quad \varepsilon^2 \Delta u + u(u - a(|x|))(1 - u) = 0 \text{ in } B_1(0), \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_1(0)
\]

(see [Dancer and Yan 2003]). The result of this last paper states that if \( a(r) \) has a critical point \( r_0 \in (0, 1) \) such that \( a(r_0) = 1/2 \), \( a'(r_0) = 0 \), \( a''(r_0) < 0 \), then there exists a clustered interior-layer solution to (1-3). All three papers use the properties of the inhomogeneous terms to construct multiple (interior) interfaces. (For the Allen–Cahn equation with inhomogeneity, \( \Delta u + a(x)(u - u^3) = 0 \) in \( \mathbb{R}^2 \), see [Rabinowitz and Stredulinsky 2003; 2004].)

Here, we continue our study, initiated in [Malchiodi et al. 2005], of clustered layered solutions for semilinear elliptic equations, and show that the homogeneous Allen–Cahn equation itself can generate multiple clustered interfaces near the boundary. In that paper we showed that the singularly-perturbed Neumann problem

\[
(1-4) \quad \begin{cases} 
\varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

has a clustered layered solution near the boundary. (The existence of a one-layer solution to (1-4) near the boundary was first established in [Ambrosetti et al. 2003; 2004].) The purpose of this paper is to show that a similar phenomenon happens to the Allen–Cahn equation. In particular, we establish the existence of clustered interfaces — the so-called “phantom interfaces” — in higher dimensions. Moreover, we show that, for each fixed positive integer \( N \), there exists a solution to (1-1) with Morse index \( N \) (in the space of radial functions).

Our main result is this:
Theorem 1.1. Let $N$ be a fixed positive integer. There exists $\varepsilon_N > 0$ such that, for all $\varepsilon < \varepsilon_N$, problem (1-1) admits a radially symmetric solution $u_\varepsilon$ with the following properties:

1. The set of interfaces $\{u_\varepsilon(r) = 0\}$ contains $N$ spheres $\{r = r^\varepsilon_j\}$, $j = 1, \ldots, N$, with

   \[
   1 - r^\varepsilon_1 \sim \varepsilon \log \frac{1}{\varepsilon}, \quad r^\varepsilon_{j-1} - r^\varepsilon_j \sim \varepsilon \log \frac{1}{\varepsilon}, \quad j = 2, \ldots, N.
   \]

   More precisely, we have $u_\varepsilon(r^\varepsilon_j + \varepsilon y) \to (-1)^j H(y)$, where $H(y)$ is the unique heteroclinic solution of

   \[
   H'' + H - H^3 = 0, \quad H(0) = 0, \quad H(\pm \infty) = \pm 1.
   \]

2. The solution $u_\varepsilon$ has the energy bound

   \[
   J_\varepsilon[u_\varepsilon] = \omega_{n-1} N \varepsilon I[H] + o(\varepsilon),
   \]

   where

   \[
   I[H] = \int_{\mathbb{R}} \left( \frac{1}{2} (H')^2 - F(H) \right),
   \]

   and where $\omega_{n-1}$ denotes the volume of $S^{n-1}$.

3. The Morse index of $u_\varepsilon$ in $H^1_r(\Omega)$ is exactly $N$, where $H^1_r(\Omega)$ denotes the space of radial functions in $H^1(\Omega)$.

Remark 1.2. By a simple transformation, Theorem 1.1 readily extends to (1-3) with $a(r) \equiv \frac{1}{2}$.

Our approach is similar to that of [Malchiodi et al. 2005], where a finite-dimensional reduction procedure combined with a variational approach is used. Such a method has been used successfully in many other papers, for example, [Ambrosetti et al. 2003; 2004; Dancer and Yan 1999; Gui and Wei 1999; 2000; Gui et al. 2000].

In the rest of section, we introduce some notation to be used later.

By the scaling $x = \varepsilon y$, problem (1-1) is reduced to the ODE

\[
\begin{cases}
u_{rr} + \frac{n-1}{r} u_r + f(u) = 0 \quad &\text{for } r \in (0, 1/\varepsilon),
\end{cases}
\]

where $f(u) = u - u^3$. From now on, we will work with (1-8).

Let $H(y)$ be the unique solution to (1-6). Set

\[
\Omega_\varepsilon = (1/\varepsilon) B_1(0) = B_{1/\varepsilon}(0), \quad \text{and} \quad I_\varepsilon = (0, 1/\varepsilon).
\]

For $u \in C^2(\Omega_\varepsilon)$ and $u = u(r)$, we have

\[
\Delta u = u'' + \frac{n-1}{r} u'.
\]
For $k \in \mathbb{N}$, we denote by $H^k_r(\Omega_\varepsilon)$ the space of radial functions in $H^k(\Omega_\varepsilon)$. On $H^1_r(\Omega_\varepsilon)$, we define an inner product as follows:

$$
(1-11) \quad (u, v)_\varepsilon = \int_0^{1/\varepsilon} (u'v' + 2uv)r^{n-1}dr.
$$

Similarly, the inner product on $L^2_r(\Omega_\varepsilon)$ can be defined by

$$
(1-12) \quad \langle u, v \rangle_\varepsilon = \int_0^{1/\varepsilon} (uv)r^{n-1}dr.
$$

We also introduce a new energy functional that, up to a positive multiplicative constant, is equivalent to $J_\varepsilon$:

$$
(1-13) \quad \mathcal{E}_\varepsilon[u] = \frac{1}{2} \int_0^{1/\varepsilon} |u'|^2 r^{n-1} - \int_0^{1/\varepsilon} F(u)r^{n-1}dr, \quad u \in H^1_r(\Omega_\varepsilon).
$$

Throughout this paper, unless otherwise stated, the letter $C$ will always denote various generic constants that are independent of $\varepsilon$, for $\varepsilon$ sufficiently small. The notation $A_\varepsilon \gg B_\varepsilon$ means that $\lim_{\varepsilon \to 0} |B_\varepsilon|/|A_\varepsilon| = 0$, while $A_\varepsilon \ll B_\varepsilon$ means $(1/A_\varepsilon) \gg (1/B_\varepsilon)$.

2. Some preliminary analysis

In this section we introduce a family of approximate solutions to (1-8) and derive some useful estimates.

Let $H$ be the unique solution of (1-6). It is easy to see that

$$
(2-1) \quad \begin{cases}
H(y) - l = -A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}) \quad \text{for } y > 1, \\
H(y) + l = A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}) \quad \text{for } y < -1, \\
H'(y) = \sqrt{2}A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}) \quad \text{for } |y| > 1,
\end{cases}
$$

where $A_0 > 0$ is a fixed constant.

We state the following well-known lemma on $H$. For a proof, see [Müller 1993, Lemma 4.1].

**Lemma 2.1.** For the eigenvalue problem

$$
(2-2) \quad \phi'' + f'(H)\phi = \lambda\phi, \quad |\phi| \leq 1, \quad \text{in } \mathbb{R},
$$

there holds

$$
(2-3) \quad \lambda_1 = 0, \quad \phi_1 = cH', \quad \lambda_2 < 0.
$$

For $u \in H^2_r(\Omega_\varepsilon)$, we define the operator

$$
(2-4) \quad \mathcal{F}_\varepsilon[u] := u_{rr} + \frac{n-1}{r}u_r + f(u).
$$
We introduce the set
\[
\Lambda = \left\{ \mathbf{t} = (t_1, \ldots, t_N) \mid t_N > 1 - \varepsilon (\log(1/\varepsilon))^2, \quad 1 - t_1 \geq \eta \varepsilon \log(1/\varepsilon), \quad t_{j-1} - t_j > \eta \varepsilon \log(1/\varepsilon), \quad j = 2, \ldots, N \right\},
\]
where \( \eta \in (0, 1/8\sqrt{\pi}) \) is a fixed number.

Let \( \chi(s) \) be a cut-off function such that \( \chi(s) = 1 \) for \( s \leq 1/4 \) and \( \chi(s) = 0 \) for \( s \geq 1/2 \). For \( t \in (3/4, 1) \), we define
\[
\rho_{\varepsilon}(t) = H\left( \frac{1 - t}{\varepsilon} \right); \quad \beta_{\varepsilon}(r) = \frac{1}{\sqrt{2}} e^{\sqrt{2}(r - 1/\varepsilon)}, \quad r \in [0, 1/\varepsilon],
\]
and
\[
H_{\varepsilon}(r) = H\left( r - \frac{t}{\varepsilon} \right),
\]
\[
H_{\varepsilon, i}(r) = \left( H\left( r - \frac{t}{\varepsilon} \right) - \rho_{\varepsilon}(t) \beta_{\varepsilon}(r) \right) \left( 1 - \chi(\varepsilon r) \right) - \chi(\varepsilon r).
\]
It is easy to see that, for \( (1 - t)/\varepsilon \gg 1 \),
\[
\rho_{\varepsilon}(t) = \sqrt{2} A_0 e^{-\sqrt{2}(1 - t)/\varepsilon} + O\left( e^{-2\sqrt{2}(1 - t)/\varepsilon} \right).
\]
We first assume that \( N \) is odd. For \( \mathbf{t} \in \Lambda \), we now define our approximate function:
\[
H_{\varepsilon, \mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon, t_j}(r).
\]
If \( N \) is even, we set
\[
H_{\varepsilon, \mathbf{t}}(r) = \sum_{j=1}^{N} (-1)^j H_{\varepsilon, t_j}(r) - 1 = \sum_{j=1}^{N+1} (-1)^j H_{\varepsilon, t_j}(r)
\]
where we use the convention that \( H_{\varepsilon, t_{N+1}} = 1 \). So, without loss of generality, we can assume that \( N \) is odd.

Note that, for \( r \leq 1/(2\varepsilon) \), there holds
\[
|H_{\varepsilon, \mathbf{t}}(r) - (-1)^N| + |H_{\varepsilon, \mathbf{t}}'(r)| + |H_{\varepsilon, \mathbf{t}}''(r)| \leq e^{-1/(C\varepsilon)}.
\]
Observe also that, by construction, \( H_{\varepsilon, \mathbf{t}} \) satisfies the Neumann boundary condition, namely \( H_{\varepsilon, \mathbf{t}}'(0) = H_{\varepsilon, \mathbf{t}}'(1/\varepsilon) = 0 \). Furthermore, \( H_{\varepsilon, \mathbf{t}} \) depends smoothly on \( \mathbf{t} \) as a map with values in \( C^2([0, 1/\varepsilon]) \).

The next lemma shows that \( H_{\varepsilon, \mathbf{t}} \) is a good approximate function to (1-8).

**Lemma 2.2.** For \( \varepsilon \) sufficiently small and \( \mathbf{t} \in \Lambda \), one has
\[
\| f_\varepsilon[H_{\varepsilon, \mathbf{t}}] \|_{L^\infty} + \varepsilon^{n-1} \int_0^1 \| f_\varepsilon[H_{\varepsilon, \mathbf{t}}] \| r^{n-1} dr \leq C \left( \varepsilon + \sum_{j=1}^{N} \left( \rho_{\varepsilon}(t_j) \right)^2 + \sum_{i \neq j} e^{-\sqrt{2}|t_i - t_j|/\varepsilon} \right).
\]
Proof. Using (1.6) it is easy to see that

\[ \mathcal{G}_\varepsilon[H_{\varepsilon,t}] = \frac{n-1}{r} H'_{\varepsilon,t} + f(H_{\varepsilon,t}) - \sum_{i=1}^{N} (-1)^j f(H_{\varepsilon}) - 2 \sum_{j=1}^{N} (-1)^j \rho_\varepsilon(t_j) \beta_\varepsilon(r) + O(\varepsilon^{-1/2}). \]

The first term in the right-hand side of (2.13) can be estimated as

\[ \frac{1}{r} H'_{\varepsilon,t} = \frac{1}{r} \sum_{j=1}^{N} (-1)^j (H'_{\varepsilon} - \rho_\varepsilon(t_j) \beta_\varepsilon(r)) + O(\varepsilon^{-1/2}). \]

From the decay of \( H' \) and \( \beta_\varepsilon \) we deduce that

\[ \left\| \frac{1}{r} H'_{\varepsilon,t} \right\|_{\infty} + \varepsilon^{n-1} \int_{0}^{1/r} \frac{1}{r} |H'_{\varepsilon,t}| r^{n-1} dr \leq C \varepsilon. \]

Next, we note that

\[ |f(H_{\varepsilon,t}) - \sum_{i=1}^{N} f((-1)^j H_{\varepsilon}) - 2 \sum_{j=1}^{N} (-1)^j \rho_\varepsilon(t_j) \beta_\varepsilon(r)| \leq S_1 + S_2, \]

where

\[ S_1 = \left| f\left( \sum_{j=1}^{N} (-1)^j H_{\varepsilon} \right) - \sum_{j=1}^{N} f((-1)^j H_{\varepsilon}) \right|, \]

\[ S_2 = \left| f\left( \sum_{j=1}^{N} (-1)^j H_{\varepsilon, t_j} \right) - f\left( \sum_{j=1}^{N} (-1)^j H_{\varepsilon} \right) - 2 \sum_{j=1}^{N} (-1)^j \rho_\varepsilon(t_j) \beta_\varepsilon(r) \right|. \]

To estimate \( S_1 \) and \( S_2 \), we divide the domain \( I_{\varepsilon} = (0, 1/\varepsilon) \) into the \( N \) intervals \( I_{\varepsilon, 1}, \ldots, I_{\varepsilon,N} \) defined by

\[ I_{\varepsilon,1} = \left[ \frac{t_1 + t_2}{2 \varepsilon}, \frac{1}{\varepsilon} \right), \quad I_{\varepsilon,j} = \left[ \frac{t_j + t_{j+1}}{2 \varepsilon}, \frac{t_j + t_{j-1}}{2 \varepsilon} \right), \quad j = 2, \ldots, N - 1, \]

and \( I_{\varepsilon,N} = \left( 0, \frac{t_N + t_{N-1}}{2 \varepsilon} \right) \).

We choose \( t_0 = 2 - t_1 \) and \( t_{N+1} = -t_N \), so that

\[ I_{\varepsilon,j} = \left[ \frac{t_j + t_{j+1}}{2 \varepsilon}, \frac{t_j + t_{j-1}}{2 \varepsilon} \right), \quad j = 1, \ldots, N, \]

\[ I_{\varepsilon} = \bigcup_{j=1}^{N} I_{\varepsilon,j}. \]

For \( r \in I_{\varepsilon,j} \) and \( j < l \), we note that

\[ H_\varepsilon(r) = 1 + O\left( \varepsilon^{-\sqrt{2}|r-t_j/\varepsilon}| \right), \]

while for \( j > l \),

\[ H_\varepsilon(r) = -1 + O\left( \varepsilon^{-\sqrt{2}|r-t_j/\varepsilon}| \right). \]
Since $N$ is odd, we see that

\[(2-17) \quad \sum_{l \neq j} (-1)^{l} H_{l} = \sum_{l < j} (-1)^{l} (H_{l} + 1) + \sum_{l > j} (-1)^{l} (H_{l} - 1).\]

Thus, we can rewrite $S_1$ as:

\[
S_1 = f \left( \sum_{l < j} (-1)^{l} (H_{l} + 1) + \sum_{l > j} (-1)^{l} (H_{l} - 1) \right) - (-1)^{j} f (H_{j}) - \sum_{l \neq j} (-1)^{l} f (H_{l})
\]

\[
= f'((-1)^{j} H_{j}) \left( \sum_{l < j} (-1)^{l} (H_{l} + 1) + \sum_{l > j} (-1)^{l} (H_{l} - 1) \right) - \sum_{l \neq j} (-1)^{l} f (H_{l})
\]

\[
+ O \left( \sum_{l < j} (H_{l} + 1)^{2} + \sum_{l > j} (H_{l} - 1)^{2} \right)
\]

This quantity can also be written

\[
S_1 = (f'((-1)^{j} H_{j}) - f'((1))) \left( \sum_{l < j} (-1)^{l} (H_{l} + 1) + \sum_{l > j} (-1)^{l} (H_{l} - 1) \right)
\]

\[
+ O \left( \sum_{l < j} (H_{l} + 1)^{2} + \sum_{l > j} (H_{l} - 1)^{2} \right)
\]

\[
= O \left( \min \{ H_{j} + 1, H_{j} - 1 \} \right) \left( \sum_{l < j} (H_{l} + 1) + \sum_{l > j} (H_{l} - 1) \right)
\]

\[
+ O \left( \sum_{l < j} (H_{l} + 1)^{2} + \sum_{l > j} (H_{l} - 1)^{2} \right).
\]

Then, with some elementary computations, one finds that

\[(2-18) \quad \| S_1 \|_{L^{\infty}(I_{\epsilon,j})} + e^{n-1} \int_{I_{\epsilon,j}} |S_1(r)| r^{n-1} dr \leq C \sum_{i \neq j} e^{-\sqrt{2}(l_{i,j})/\epsilon} e^{\sqrt{2}(r-1/\epsilon)}.
\]

It remains to estimate $S_2$. For this, we note that, for $r \in I_{\epsilon,j}$ and $j \geq 2$, we have

\[\rho_{\epsilon}(t_{j}) \beta_{\epsilon}(r) = O\left( e^{-\sqrt{2}(1-t_{j})/\epsilon} e^{\sqrt{2}(r-1/\epsilon)} \right),\]

from which it follows that, for $j \geq 2$,

\[\| S_2 \|_{L^{\infty}(I_{\epsilon,j})} + e^{n-1} \int_{I_{\epsilon,j}} |S_2(r)| r^{n-1} dr = O\left( e^{-2\sqrt{2}(1-t_{j})/\epsilon} \right) = O\left( \sum_{j=1}^{N} \left( \rho_{\epsilon}(t_{j}) \right)^{2} \right).
\]

Therefore, we just need to consider the case when $r \in I_{\epsilon,1}$. But, since $f'(\pm 1) = -2$, we have

\[
S_2 = f \left( \sum_{l=1}^{N} (-1)^{l} H_{l} - \sum_{l=1}^{N} (-1)^{l} \rho_{\epsilon}(t_{l}) \beta_{\epsilon}(r) \right)
\]

\[
- f \left( \sum_{l=1}^{N} (-1)^{l} H_{l} \right) - f'((-1) \sum_{l=1}^{N} (-1)^{l} \rho_{\epsilon}(t_{l}) \beta_{\epsilon}(r))
\]
\begin{equation}
= \left( f' \left( \sum_{i=1}^{N} (-1)^i H_{t_i} \right) - f'(-1) \right) \sum_{i=1}^{N} (-1)^i \rho_{\epsilon}(t_i) \beta_{\epsilon}(r) + O \left( \sum_{i=1}^{N} \rho_{\epsilon}(t_i)^2 \beta_{\epsilon}(r)^2 \right)
\end{equation}

Hence, we also get
\begin{equation}
\| S_2 \|_{L^\infty(t_1)} + \varepsilon^{n-1} \int_{t_1} S_2(r) |r^{n-1}dr \leq C \rho_{\epsilon}^2(t_1).
\end{equation}

The proof of the next lemma is postponed to the appendix.

**Lemma 2.3.** Let \( t \in \Lambda \). For \( \varepsilon \) sufficiently small, we have
\begin{equation}
\mathcal{E}_{\varepsilon} \left[ \sum_{j=1}^{N} (-1)^j H_{t_j} \right]
= I[H] \sum_{i=1}^{N} \left( \frac{t_i}{\varepsilon} \right)^{n-1} - \left( \frac{1}{\varepsilon} \right)^{n-1} \left( \sqrt{2} A_0^2 + o(1) \right) \varepsilon^{-2\sqrt{2}(1-t_i)/\varepsilon}
\end{equation}

where \( A_0 > 0 \) is defined in (2-1).

### 3. Lyapunov–Schmidt process: finite-dimensional reduction

In this section we outline the so-called Lyapunov–Schmidt reduction process. Since this can be proved along the same ideas of [Malchiodi et al. 2005, Sections 3], we skip some of the details.

Fix \( t \in \Lambda \). Integrating by parts, one can show that orthogonality to \( \partial H_{t_j} / \partial t_j \) in \( H^1(\Omega_{\varepsilon}) \), \( j = 1, \ldots, N \), is equivalent to orthogonality in \( L^2(\Omega_{\varepsilon}) \) to the functions
\begin{equation}
Z_{\varepsilon,t_j} = \Delta \left( \frac{\partial H_{t_j}}{\partial t_j} \right) - 2 \frac{\partial H_{t_j}}{\partial t_j}, \quad j = 1, \ldots, N.
\end{equation}

By elementary computations, differentiating (1-6) we obtain
\begin{equation}
\frac{\partial H_{t_j}}{\partial t_j} = \frac{-1}{\varepsilon} H'(r - \frac{t_j}{\varepsilon}) + \frac{1}{\varepsilon} H'' \left( \frac{1-t_j}{\varepsilon} \right) \beta_{\epsilon}(r) + O(e^{-1/(Ce)})
\end{equation}

\begin{equation}
Z_{\varepsilon,t_j} = \left( f'(H_{t_j}) - f'(\pm 1) \right) \frac{\partial H_{t_j}}{\partial t_j} + \frac{n-1}{r} \left( \frac{\partial H_{t_j}}{\partial t_j} \right)'
\end{equation}

where \( O(e^{-1/(Ce)}) \) and \( o(1/\varepsilon) \) are intended both in the \( C^1 \) and \( H^1 \) sense.
We consider first the following linear problem: Given \( h \in L^\infty(\Omega_\varepsilon) \), find a function \( \phi \) satisfying
\[
\begin{align*}
L_\varepsilon[\phi] &:= \phi'' + \frac{n-1}{\varepsilon} \phi' + f'(H_{\varepsilon,t}) \phi = h + \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j}; \\
\phi'(0) & = \phi'(1/\varepsilon) = 0 \quad \text{and} \quad \langle \phi, Z_{\varepsilon,t_j} \rangle_{\varepsilon} = 0, \quad j = 1, \ldots, N,
\end{align*}
\]
for some constants \( c_j, \quad j = 1, \ldots, N \). For this, define the norm
\[
\|\phi\|_* = \sup_{r \in (0, 1/\varepsilon)} |\phi(r)|.
\]

Assuming a solution to (3-4) exists, we have an estimate on \( \phi \):

**Proposition 3.1.** Let \( \phi \) satisfy (3-4). For \( \varepsilon \) sufficiently small, we have
\[
\|\phi\|_* \leq C \|h\|_*,
\]
where \( C \) is a positive constant independent of \( \varepsilon \) and \( t \in \Lambda \).

**Proof.** The argument is similar in spirit of that of [Malchiodi et al. 2005, Proposition 3.1]. For the sake of completeness, we include a proof here.

Arguing by contradiction, assume that
\[
\|\phi\|_* = 1, \quad \|h\|_* = o(1).
\]

We multiply (3-4) by \( \partial H_{\varepsilon,t_j}/\partial t_j \) and integrate over \( \Omega_\varepsilon \) to obtain
\[
\sum_{i=1}^{N} c_i \langle Z_{\varepsilon,t_i}, \partial H_{\varepsilon,t_j}/\partial t_j \rangle_{\varepsilon} = -\langle h, \partial H_{\varepsilon,t_j}/\partial t_j \rangle_{\varepsilon} + \langle \Delta \phi + f'(H_{\varepsilon,t}) \phi, \partial H_{\varepsilon,t_j}/\partial t_j \rangle_{\varepsilon}.
\]

From the exponential decay of \( H' \), one finds
\[
\langle h, \partial H_{\varepsilon,t_j}/\partial t_j \rangle_{\varepsilon} = \int_0^{1/\varepsilon} h \partial H_{\varepsilon,t_j}/\partial t_j r^{n-1} dr = O(\|h\|_* \varepsilon^{-n}).
\]

Moreover, integrating by parts and using (3-2) and (3-3), we deduce
\[
\langle \Delta \phi + f'(H_{\varepsilon,t}) \phi, \partial H_{\varepsilon,t_j}/\partial t_j \rangle_{\varepsilon} = \langle Z_{\varepsilon,t_j} + f'(H_{\varepsilon,t}) \partial H_{\varepsilon,t_j}/\partial t_j, \phi \rangle_{\varepsilon} = o(\varepsilon^{-n} \|\phi\|_*).
\]

From (3-2) and (3-3), we also see that
\[
\langle Z_{\varepsilon,t_j}, \partial H_{\varepsilon,t_j}/\partial t_j \rangle_{\varepsilon} = -\varepsilon^{-n-1} \int_{\mathbb{R}} (t_j^{n-1} \delta_{ij} \int_{\mathbb{R}} f'(H)(H')^2 + o(1)),
\]
where \( \delta_{ij} \) denotes the Kronecker symbol. Note that, using the equation \( H'' + f'(H) H' = 0 \), we find
\[
\int_{\mathbb{R}} f'(H)(H')^2 = \int_{\mathbb{R}} (H'')^2 > 0.
\]
This shows that the left-hand side of the equation (3-8) is diagonally dominant in the indices $i$ and $j$, and hence, by (3-7), we have

$$c_i = O(\varepsilon \|h\|_a) + o(\varepsilon \|\phi\|_a) = o(\varepsilon), \quad i = 1, \ldots, N. $$

Also, since we are assuming that $\|h\|_a = o(1)$ and since $\|Z_{\varepsilon, t_j}\|_a = O(1/\varepsilon)$, there holds

$$\left\| \sum_{j=1}^N c_j Z_{\varepsilon, t_j} \right\|_a = o(1).$$

Thus, (3-4) yields

$$\phi'' + \frac{n-1}{r} \phi' + f'(\pm 1) + \left( f'(H_{\varepsilon, t}) - f'(\pm 1) \right) \phi = o(1),$$

where $o(1)$ is in the sense of $L^\infty(0, 1/\varepsilon)$.

We show that (3-12) is incompatible with our assumption that $\|\phi\|_a = 1$. First, we claim that

$$|\phi| \to 0 \quad \text{on} \quad y \in \bigcup_{j=1}^N \left( \frac{t_j}{\varepsilon} - R, \frac{t_j}{\varepsilon} + R \right), \quad \text{as} \ \varepsilon \to 0,$$

where $R$ is any fixed positive constant.

Indeed, assuming the contrary, there exist $\delta_0 > 0$, $j \in \{1, \ldots, N\}$, and sequences $\varepsilon_k, \phi_k, y_k \in (t_j - R, t_j + R)$ such that $\phi_k$ satisfies (3-4) and

$$|\phi_k(y_k)| \geq \delta_0.$$

Let $\tilde{\phi}_k = \phi_k(y - t_j/\varepsilon_k)$. Then, using (3-12) and $\|\phi\|_a = 1$, as $\varepsilon_k \to 0$, $\tilde{\phi}_k$ converges weakly in $H^2_{\text{loc}}(\mathbb{R})$ and strongly in $C^1_{\text{loc}}(\mathbb{R})$ to a bounded function $\phi_0$ which satisfies

$$\phi_0'' + f'(H) \phi_0 = 0 \quad \text{in} \ \mathbb{R}, \quad |\phi_0| \leq C.$$

By Lemma 2.1, we have $\phi_0 = c H'$ for some $c$. Since $\tilde{\phi}_k \perp Z_{\varepsilon, t_j}$, we conclude that

$$\int_{\mathbb{R}} \phi_0 f'(H)(H')^2(y) \, dy = 0,$$

which yields $c = 0$. Hence $\phi_0 = 0$ and $\tilde{\phi}_k \to 0$ in $B_{2R}(0)$. This contradicts (3-14), so (3-13) holds true.

Given $\delta > 0$, the decay of $f'(H) - f'(\pm 1)$ together with (3-13) (with $R$ sufficiently large) imply that

$$\left\| \left( f'(H_{\varepsilon, t}) - f'(\pm 1) \right) \phi \right\|_a \leq \delta + \frac{1}{2} \|\phi\|_a.$$
Using (3-12) and the Maximum Principle, one finds
\[
\|\phi\|_\ast \leq \left\| \left( f'(H_\varepsilon, t) - f'(\pm 1) \right) \phi \right\|_\ast + \sum_{j=1}^{N} |c_j| \| Z_{\varepsilon, t_j}\|_\ast + \| h \|_\ast \leq 2\delta + \frac{1}{2} \|\phi\|_\ast ,
\]
and hence
\[
\|\phi\|_\ast \leq 4\delta < 1,
\]
if we choose \(\delta < 1/4\). This contradicts (3-7). \(\square\)

Next, we consider the following nonlinear problem: Find a function \(\phi\) such that for some constants \(c_j\), \(j = 1, \ldots, N\), the equation
\[
(3-16) \left\{ \begin{array}{ll}
\Delta (H_\varepsilon, t + \phi) + f(H_\varepsilon, t + \phi) = \sum_{j=1}^{N} c_j Z_{\varepsilon, t_j} & \text{in } \Omega_\varepsilon, \\
\phi'(0) = \phi'(\frac{1}{2}) = 0 & \text{and} \\
\langle \phi, Z_{\varepsilon, t_j} \rangle_\varepsilon = 0, & j = 1, \ldots, N.
\end{array} \right.
\]
holds true.

The proof of the next result follows the same lines of [Malchiodi et al. 2005, Proposition 4.2].

**Proposition 3.2.** For \(t \in \Lambda\) and \(\varepsilon\) sufficiently small, there exists a unique \(\phi = \phi_{\varepsilon, t}\) such that (3-16) holds. Moreover, \(t \mapsto \phi_{\varepsilon, t}\) is of class \(C^1\) as a map into \(H^1(\Omega_\varepsilon)\), and we have
\[
(3-17) \|\phi_{\varepsilon, t}\|_\ast \leq C \left( \varepsilon + \sum_{j=1}^{n} e^{-(3/2)\sqrt{2}(1-t_j)/\varepsilon} + \sum_{i \neq j} e^{-(3/4)\sqrt{2}|t_i-t_j|/\varepsilon} \right).
\]

### 4. Energy computation for reduced energy functional

We expand the quantity
\[
M_\varepsilon(t) := \varepsilon^{n-1} \varepsilon^t \left[ H_{\varepsilon, t} + \phi_{\varepsilon, t} \right] : \Lambda \rightarrow \mathbb{R}
\]
in \(\varepsilon\) and \(t\), where \(\phi_{\varepsilon, t}\) is given by Proposition 3.2. Up to negligible error terms, the same expansion of Lemma 2.3 holds true.

**Lemma 4.1.** For \(t \in \Lambda\) and \(\varepsilon\) sufficiently small, we have
\[
M_\varepsilon(t) = \varepsilon^{n-1} \varepsilon^t \left[ H_{\varepsilon, t} + \phi_{\varepsilon, t} \right]
\]
\[
= I[H] \sum_{j=1}^{N} t_j^{n-1} - (\sqrt{2} A_0^2 + o(1)) e^{-2\sqrt{2}(1-t_1)/\varepsilon} \sum_{j=2}^{N} t_j^{n-1} e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon} + O(\varepsilon).
\]
Lemma 2.3. In order to do this, we write

\[ \mathcal{M}_\varepsilon(t) = e^{n-1} \varepsilon \phi \left[ H_{\varepsilon,x} \right] + O \left( \sum_{j=1}^{N} e^{-2\sqrt{3} |1-t_j|/\varepsilon} + \sum_{i \neq j} e^{-\sqrt{2} |t_i-t_j|/\varepsilon} \right) + O(\varepsilon), \]

and to apply Lemma 2.3. In order to do this, we write

\[ \varepsilon^{1-n} \mathcal{M}_\varepsilon = \varepsilon \phi \left[ H_{\varepsilon,x} \right] + K_1 + K_2 - K_3, \]

where

\[ K_1 = \int_0^{1/\varepsilon} \left( \varepsilon \phi \left[ H_{\varepsilon,x} \right] - f(H_{\varepsilon,x}) \phi_x \right) r^{n-1} dr; \]
\[ K_2 = \frac{1}{2} \int_0^{1/\varepsilon} \left( |\phi|_e^2 - f'(H_{\varepsilon,x}) \phi_x^2 \right) r^{n-1} dr; \]
\[ K_3 = \int_0^{1/\varepsilon} \left( F(H_{\varepsilon,x} + \phi_{\varepsilon,x}) - F(H_{\varepsilon,x}) - f(H_{\varepsilon,x}) \phi_{\varepsilon,x} - \frac{1}{2} f'(H_{\varepsilon,x}) \phi_x^2 \right) r^{n-1} dr. \]

Integrating by parts, using Lemma 2.2 and Proposition 3.1, we find

(4.3) \[ |K_1| = \left| \int_0^{1/\varepsilon} \left( \varepsilon \phi \left[ H_{\varepsilon,x} \right] - f(H_{\varepsilon,x}) \phi_x \right) r^{n-1} dr \right| \leq C \| \phi \|_\varepsilon \| \phi_x \|_\varepsilon \int_0^{1/\varepsilon} |\phi \left[ H_{\varepsilon,x} \right]| r^{n-1} dr \]
\[ \leq C \varepsilon^{1-n} \left( \varepsilon^2 + \sum_{j=1}^{N} \left( \rho_e (t_j) \right)^{2+3/2} + \sum_{i \neq j} e^{-7/4 \sqrt{2} |t_i-t_j|/\varepsilon} \right). \]

To estimate \( K_2 \), we note that \( \phi_{\varepsilon,x} \) satisfies

(4.4) \[ \Delta \phi_{\varepsilon,x} + f(H_{\varepsilon,x} + \phi_{\varepsilon,x}) - f(H_{\varepsilon,x}) + \phi \left[ w_{\varepsilon,x} \right] = \sum_{j=1}^{N} c_j Z_{\varepsilon,x}. \]

Multiplying (4.4) by \( \phi_{\varepsilon,x} r^{n-1} \) and integrating over \( I_\varepsilon \), we obtain

(4.5) \[ \int_{I_\varepsilon} \phi \left[ H_{\varepsilon,x} \right] \phi_{\varepsilon,x} r^{n-1} dr = \int_{I_\varepsilon} \left( |\phi|_e^2 - f'(H_{\varepsilon,x}) \phi_x^2 \right) r^{n-1} dr \]
\[ + \int_{I_\varepsilon} \left( f(H_{\varepsilon,x} + \phi_{\varepsilon,x}) - f(H_{\varepsilon,x}) - f'(H_{\varepsilon,x}) \phi_{\varepsilon,x} \right) \phi_{\varepsilon,x} r^{n-1} dr. \]

Hence, we find

\[ 2K_2 = -\int_{I_\varepsilon} \left( f(H_{\varepsilon,x} + \phi_{\varepsilon,x}) - f(H_{\varepsilon,x}) - f'(H_{\varepsilon,x}) \phi_{\varepsilon,x} \right) \phi_{\varepsilon,x} r^{n-1} dr \]
\[ + \int_{I_\varepsilon} \phi \left[ H_{\varepsilon,x} \right] \phi_{\varepsilon,x} r^{n-1} dr. \]

From Taylor’s formula, we get

\[ \left| f(H_{\varepsilon,x} + \phi_{\varepsilon,x}) - f(H_{\varepsilon,x}) - f'(H_{\varepsilon,x}) \phi_{\varepsilon,x} \right| \leq C |\phi_{\varepsilon,x}|^2, \]
so we deduce

\[ |K_2| \leq C \int \left| \phi_{\varepsilon,t} \right|^3 r^{n-1} dr + C \| \phi_{\varepsilon,t} \| \epsilon \int \mathcal{G}_\varepsilon[H_{\varepsilon,t}] r^{n-1} dr. \]

From the exponential decay of \( H(\pm y) - (\pm 1) \) one finds that \( \phi_{\varepsilon,t}(r) \) satisfies

\[ \phi_{\varepsilon,t}'' + \frac{n-1}{r} \phi_{\varepsilon,t}' + f(H_{\varepsilon,t} + \phi_{\varepsilon,t}) - f(H_{\varepsilon,t}) = O\left( \sum_{j=1}^{N} e^{-\sqrt{2}|t_j - t'|/\varepsilon} \right), \]

\[ \phi_{\varepsilon,t}'(0) = \phi_{\varepsilon,t}'(1/\varepsilon) = 0. \]

From (4-4) and a comparison principle, we obtain

\[ |\phi_{\varepsilon,t}(r)| \leq C \sum_{j=1}^{N} e^{-(\sqrt{2}/\tilde{C})|t_j - t'|/\varepsilon} \]

for some \( \tilde{C} < 1. \)

Using Proposition 3.2 and (4-6), we get

\[ |K_2| \leq C \varepsilon^{1-n} \left( \varepsilon^2 + \sum_{j=1}^{N} (\rho_\varepsilon(t_j))^3 + \sum_{i \neq j} e^{-2\sqrt{2}|t_j - t_i|/\varepsilon} \right). \]

From the Hölder continuity of \( f' \), we deduce

\[ |F(H_{\varepsilon,t} + \phi_{\varepsilon,t}) - F(H_{\varepsilon,t}) - f(H_{\varepsilon,t})\phi_{\varepsilon,t} - \frac{1}{2} f'(H_{\varepsilon,t})\phi_{\varepsilon,t}^2| \leq C |\phi_{\varepsilon,t}|^3, \]

so, again, it follows that

\[ |K_3| \leq C \varepsilon^{1-n} \left( \varepsilon^2 + \sum_{j=1}^{N} (\rho_\varepsilon(t_j))^3 + \sum_{i \neq j} e^{-2\sqrt{2}|t_j - t_i|/\varepsilon} \right). \]

Combining with (2-20) of Lemma 2.2, we obtain the conclusion. \( \square \)

5. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Fix \( t \in \Lambda \) and let \( \phi_{\varepsilon,t} \) be given by Proposition 3.2. Let also \( M_\varepsilon(t) \) denote the reduced energy functional defined by (4-1).

**Proposition 5.1.** For \( \varepsilon \) small, the following maximization problem

\[ \sup \{ M_\varepsilon(t) \mid t \in \Lambda \} \]

has a solution \( t^\varepsilon \) in the interior of \( \Lambda \).

**Proof.** Since \( M_\varepsilon(t) \) is continuous in \( t \), it achieves a maximum in \( \bar{\Lambda} \). Let \( t^\varepsilon \) be a maximum point. We claim that \( t^\varepsilon \in \Lambda \).

We argue by contradiction and assume that \( t^\varepsilon \in \partial \Lambda \). From the definition of \( \Lambda \), there are three possibilities: either \( 1 - t_1 = \eta \varepsilon \log(1/\varepsilon) \), or there exists \( j \geq 2 \) such that \( t_{j-1} - t_j = \eta \varepsilon \log(1/\varepsilon) \), or, finally, \( t_N = 1 - \varepsilon \left( \log(1/\varepsilon) \right)^2 \).
In the first case, we have
\[
I[H]t_1^{n-1} - (\sqrt{2}A_0^2 + o(1))e^{-2\sqrt{2}(1-t_1)/\varepsilon} \\
= I[H]\left(1 - \eta \varepsilon \log \frac{1}{\varepsilon}\right)^{n-1} - \sqrt{2}A_0^2 e^{-2\eta \sqrt{2} \log(1/\varepsilon)} + o(\varepsilon \sqrt{2} \eta) \\
\leq I[H] - A_0^2 e^{2\sqrt{2} \eta}.
\]

Since \(\eta < 1/\sqrt{2}\), we obtain
\[
(5-2) \quad M_\varepsilon(t^\varepsilon) \leq NI[H] - A_0^2 e^{2\sqrt{2} \eta}.
\]

In the second case, there holds
\[
(5-3) \quad M_\varepsilon(t^\varepsilon) \leq I[H] \sum_{j=1}^{N} t_j^{n-1} - (\sqrt{2}A_0^2 + o(1))e^{\sqrt{2} \eta_j^{n-1}} \leq NI[H] - A_0^2 e^{2\sqrt{2} \eta}.
\]

In the latter case, we have \(t_N = 1 - \varepsilon (\log(1/\varepsilon))^2\), and therefore
\[
(5-4) \quad M_\varepsilon(t^\varepsilon) \leq I[H](N - 1 + t_N^{n-1}) + O(\varepsilon) \\
\leq I[H](N - (n - 1) \varepsilon (\log(1/\varepsilon))^2) + O(\varepsilon).
\]

On the other hand, choosing \(t_j = 1 - (j/\sqrt{2})\varepsilon \log(1/\varepsilon), j = 1, \ldots, N\), we obtain
\[
\sum_{j=1}^{N} t_j^{n-1} = 1 - \frac{N(N + 1)(n - 1)}{2\sqrt{2}} \varepsilon \log(1/\varepsilon) + O(\varepsilon^2 (\log(1/\varepsilon))^2);
\]
(5-5) \quad \varepsilon^{-2\sqrt{2}(1-t_1)/\varepsilon} = \varepsilon^2; \quad \varepsilon^{-\sqrt{2}|\eta_j - 1|/\varepsilon} = \varepsilon,

and we find
\[
M_\varepsilon(t^\varepsilon) \geq NI[H] - \frac{N(N + 1)(n - 1)^2}{2\sqrt{2}} \varepsilon \log(1/\varepsilon) + O(\varepsilon),
\]
which contradicts either (5-2) or (5-3) or (5-4). This completes the proof of Proposition 5.1.

□

**Remark 5.2.** The above argument also shows that
\[
(5-6) \quad 1 - t_1^\varepsilon \sim \varepsilon \log(1/\varepsilon), \quad t_j^{\varepsilon} - t_{j-1}^{\varepsilon} \sim \varepsilon \log(1/\varepsilon).
\]

Finally, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 3.2, there exists \(\varepsilon_N\) such that, for \(\varepsilon < \varepsilon_N\), we have a \(C^1\) map \(t \mapsto \phi_{\varepsilon,t}\) from \(\overline{\Lambda}\) into \(C^2(I_\varepsilon)\) such that
\[
(5-7) \quad \mathcal{G}_\varepsilon[H_{\varepsilon,t} + \phi_{\varepsilon,t}] = \sum_{j=1}^{N} c_j Z_{\varepsilon,t_j},
\]
for some constants \(\{c_j\} \subseteq \mathbb{R}\), which are also of class \(C^1\) in \(t\).
By Proposition 5.1, there exists \( t^\varepsilon \in \Lambda \) that achieves the maximum of \( \mathcal{K}_\varepsilon : t \mapsto \mathcal{K}_\varepsilon [H_{\varepsilon, t} + \phi_{\varepsilon, t}] \). Let
\[
  u_\varepsilon = \sum_{i=1}^{N} (-1)^i H_{\varepsilon, t^\varepsilon} + \phi_{\varepsilon, t^\varepsilon} = H_{\varepsilon, v} + \phi_{\varepsilon, v}.
\]
Then we have
\[
  \partial_i \bigg|_{t=0} \mathcal{M}_\varepsilon (t^\varepsilon) = 0, \quad i = 1, \ldots, N,
\]
and hence
\[
  \int_{I_\varepsilon} \left( \nabla u_\varepsilon \nabla \partial_i (H_{\varepsilon, t} + \phi_{\varepsilon, t}) + u_\varepsilon \partial_i (H_{\varepsilon, t} + \phi_{\varepsilon, t}) - f(u_\varepsilon) \partial_i (H_{\varepsilon, t} + \phi_{\varepsilon, t}) \right) \bigg|_{t=t^\varepsilon} r^{n-1} dr = 0.
\]
Therefore, by (5-7), we find
\[
\sum_{j=1}^{N} c_j \int_{I_\varepsilon} (Z_{\varepsilon, t_j} \partial_i (H_{\varepsilon, t} + \phi_{\varepsilon, t})) r^{n-1} dr = 0. \tag{5-8}
\]
Differentiating the equation \( \langle \partial_i \phi, Z_{\varepsilon, t_j} \rangle_\varepsilon = 0 \) with respect to \( t_j \), we get
\[
\langle \partial_i \phi, Z_{\varepsilon, t_j} \rangle_\varepsilon = -\langle \phi, \partial_i Z_{\varepsilon, t_j} \rangle_\varepsilon = O(\|\phi\|_1)\varepsilon^{-n-1}.
\]
Using (3-3), we see that (5-8) is diagonally dominant in the coefficients \( \{c_i\} \), which implies that \( c_j = 0 \) for \( j = 1, \ldots, N \). Hence
\[
  u_\varepsilon = H_{\varepsilon, v} + \phi_{\varepsilon, v}
\]
is a solution of (1-1).

By our construction, one can easily check that \( \varepsilon^{n-1} \mathcal{K}_\varepsilon (u_\varepsilon) \to N I [H] \) as \( \varepsilon \to 0 \), and that \( u_\varepsilon \) has only \( N \) zeroes \( s_1^\varepsilon / \varepsilon, \ldots, s_N^\varepsilon / \varepsilon \). By the structure of \( u_\varepsilon \), we see that (up to a permutation) \( s_j^\varepsilon - t_j^\varepsilon = o(1) \). This proves (1) and (2) of Theorem 1.1.

It remains to prove (3). First we note that \( u'_\varepsilon \) satisfies
\[
\Delta u'_\varepsilon + f'(u_\varepsilon) u'_\varepsilon = \frac{n-1}{r^2} u'_\varepsilon. \tag{5-9}
\]
By our construction, at each interval \( (s_j^\varepsilon / \varepsilon, s_{j+1}^\varepsilon / \varepsilon) \), for \( j = 2, \ldots, N \), there exists a point \( \tilde{s}_j^\varepsilon / \varepsilon \in (s_j / \varepsilon, s_{j+1} / \varepsilon) \) such that \( u'_\varepsilon (\tilde{s}_j^\varepsilon / \varepsilon) = 0 \). Now, we set
\[
\varphi_1(r) = \begin{cases} u'_\varepsilon (r) & \text{for } r \in \left( \tilde{s}_1^\varepsilon / \varepsilon, 1 \right), \\ 0 & \text{otherwise}; \end{cases}
\]
\[
\varphi_j(r) = \begin{cases} u'_\varepsilon (r) & \text{for } r \in \left( \tilde{s}_j^\varepsilon / \varepsilon, \tilde{s}_{j-1}^\varepsilon / \varepsilon \right), \\ 0 & \text{otherwise}, \end{cases} \quad j = 2, \ldots, N - 1;
\]
\[
\varphi_N(r) = \begin{cases} u'_\varepsilon (r), & \text{for } r \in \left( 1/(2\varepsilon), \tilde{s}_{N-1}^\varepsilon / \varepsilon \right), \\ 2\varepsilon (r - 1/(4\varepsilon)) u'_\varepsilon (r), & \text{for } 1/(4\varepsilon) \leq r \leq 1/(2\varepsilon), \\ 0, & \text{for } r < 1/(4\varepsilon) \text{ or } r \geq \tilde{s}_{N-1}^\varepsilon / \varepsilon. \end{cases}
\]
Next, we define a quadratic functional

\begin{equation}
Q[\phi] = \int_{I_\varepsilon} \left( |\nabla \phi|^2 - f'(u_\varepsilon) \phi^2 \right) r^{n-1} dr.
\end{equation}

It is easy to check that

\begin{equation}
\int_{I_\varepsilon} \varphi_i \varphi_j r^{n-1} dr = 0 \quad \text{for } i \neq j.
\end{equation}

Using equation (5-9), we obtain

\begin{equation}
Q[\varphi_i] = -\int_{I_\varepsilon} \varphi_i^2 r^{n-3} dr < 0, \quad i = 1, \ldots, N - 1.
\end{equation}

When \( i = N \), we have

\begin{equation}
Q[\varphi_N] = -\int_{I_\varepsilon} \varphi_N^2 r^{n-3} dr + O(e^{-1/(C\varepsilon)}) < 0.
\end{equation}

From (5-12) and (5-13), the Morse index of \( u_\varepsilon \) in \( H^1_r(\Omega_\varepsilon) \) is at least \( N \).

Finally, we also show that the Morse index of \( u_\varepsilon \) in \( H^1_r(\Omega_\varepsilon) \) is at most \( N \). In fact, we define

\begin{equation}
z_j^\varepsilon(r) = H_{r,j}^\varepsilon \chi \left( \frac{\varepsilon r - t_j^\varepsilon}{\varepsilon \sqrt{\log(1/\varepsilon)}} \right), \quad j = 1, \ldots, N,
\end{equation}

and consider the following minimization problem

\begin{equation}
\mu_j^\varepsilon = \inf_{\phi \in H^1_r(I_{r,j})} \frac{\int_{I_{r,j}} \left( |\nabla \phi|^2 - f'(u_\varepsilon) \phi^2 \right) r^{n-1} dr}{\int_{I_{r,j}} \phi^2 r^{n-1} dr}.
\end{equation}

Assume that \( \mu_j^\varepsilon \leq 0 \). By standard regularity theory, \( \mu_j^\varepsilon \) is attained by a function \( \phi_j^\varepsilon \) which satisfies

\begin{equation}
\Delta \phi_j^\varepsilon + f'(u_\varepsilon) \phi_j^\varepsilon = -\mu_j^\varepsilon \phi_j^\varepsilon + c_j^\varepsilon z_j^\varepsilon,
\end{equation}

\begin{equation}
(\phi_j^\varepsilon)' \big|_{0_{I_{r,j}}} = 0 \quad \text{and} \quad \int_{I_{r,j}} \phi_j^\varepsilon z_j r^{n-1} dr = 0,
\end{equation}

where \( c_j^\varepsilon \) is a constant.

First, we notice that \( c_j^\varepsilon = o(\|\phi_j^\varepsilon\|_s) \), which follows by reasoning as for (3-10) of Proposition 3.1. Then, from Lemma 2.1 we deduce that \( \mu_j^\varepsilon \to 0 \); moreover, the same argument leading to Proposition 3.1 shows that \( \phi_j^\varepsilon = 0 \).
Thus, \( \mu_j^\varepsilon > 0 \). Let \( \phi = \phi(r) \) be such that \( \int_{I_j^\varepsilon} \phi z_j^\varepsilon r^{n-1} = 0 \), \( j = 1, \ldots, N \), which is equivalent to \( \int_{I_j^\varepsilon} \phi z_j^\varepsilon r^{n-1} = 0 \). This then implies that

\[
(5-17) \quad \int_{I_j^\varepsilon} (|\nabla \phi|^2 - f'(u_\varepsilon) \phi^2) r^{n-1} dr \geq \mu_j^\varepsilon \int_{I_j^\varepsilon} |\phi|^2 r^{n-1} dr, \quad j = 1, \ldots, N,
\]

and hence

\[
(5-18) \quad \int_{I_j^\varepsilon} (|\nabla \phi|^2 - f'(u_\varepsilon) \phi^2) r^{n-1} dr = \sum_{j=1}^{N} \int_{I_j^\varepsilon} (|\nabla \phi|^2 - f'(u_\varepsilon) \phi^2) r^{n-1} dr \geq \min_{j=1,\ldots,N} \mu_j^\varepsilon \int_{I_j^\varepsilon} |\phi|^2 r^{n-1} dr.
\]

This yields

\[
(5-19) \quad \lambda_{N+1} = \sup_{v_1, \ldots, v_N} \inf_{\phi \in \ker I_j^\varepsilon \rVert r^{n-1} = 0} \frac{\int_{I_j^\varepsilon} (|\nabla u|^2 - f'(u_\varepsilon) \phi^2) r^{n-1}}{\int_{I_j^\varepsilon} |\phi|^2 r^{n-1}} \geq \min_{j=1,\ldots,N} \mu_j^\varepsilon > 0,
\]

and hence the Morse index of \( u_\varepsilon \) in \( H^1_\varepsilon(\Omega_\varepsilon) \) is at most \( N \).

Combining the upper and lower bound for the Morse index, we see that the Morse index of \( u_\varepsilon \) in \( H^1_\varepsilon(\Omega_\varepsilon) \) is exactly \( N \). This proves (3) of Theorem 1.1. \( \square \)

**Appendix**

In this appendix we expand the quantity \( \mathcal{E}_\varepsilon[\sum_{j=1}^{N} (-1)^j H_{\varepsilon,t_j}] \) as a function of \( \varepsilon \) and \( t \). Several facts will be used repeatedly:

\[
H(y) = 1 - A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}), \quad \text{for } y > 1;
\]
\[
H(y) = -1 + A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}), \quad \text{for } y < -1;
\]
\[
H'(y) = \sqrt{2} A_0 e^{-\sqrt{2}|y|} + O(e^{-2\sqrt{2}|y|}), \quad \text{for } |y| > 1;
\]
\[
\rho_\varepsilon(t_1) = \sqrt{2}(A_0 + o(1)) e^{-\sqrt{2}(1-t_1)/\varepsilon};
\]
\[
\rho_\varepsilon(t_j) = o(\rho_\varepsilon(t_1)) \quad \text{for } j \geq 2.
\]

From a Taylor expansion we find

\[
\mathcal{E}_\varepsilon[H_{\varepsilon,t}] = I_1 + I_2 + I_3 + O(\varepsilon^{1-n} \rho_\varepsilon^3(t_1)),
\]
Thus, where

\[ I_1 = \varepsilon \left[ \sum_{j=1}^{N} (-1)^j H_{t_j} \right], \]

\[ I_2 = -\left( \sum_{j=1}^{K} (-1)^j \rho_{\varepsilon}(t_j) \right) \int_{I_0} \left( \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \right) \beta_{\varepsilon}' - f \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \right) \beta_{\varepsilon} \right) r^{n-1} dr, \]

\[ I_3 = \frac{1}{2} \left( \sum_{j=1}^{N} (-1)^j \rho_{\varepsilon}(t_j) \right)^2 \int_{I_0} \left( \beta_{\varepsilon}'^2 - f' \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \right) \beta_{\varepsilon}^2 \right) r^{n-1}. \]

Recalling that \( f'(\pm 1) = -2 \), the term \( I_3 \) can be estimated by

\[ I_3 = \frac{1}{2} \left( \sum_{j=1}^{N} (-1)^j \rho_{\varepsilon}(t_j) \right)^2 \int_{I_0} \left( 2 - f' \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \right) \right) \beta_{\varepsilon}^2 r^{n-1} dr + o(\varepsilon^{1-n} \rho_{\varepsilon}^2(t_1)) \]

\[ = \left( \rho_{\varepsilon}(t_1) \right)^2 \int_{I_0} \beta_{\varepsilon}^2 r^{n-1} dr + o(\varepsilon^{1-n} \rho_{\varepsilon}^2(t_1)) = \frac{1}{2\sqrt{2}} \varepsilon^{1-n} \left( \rho_{\varepsilon}(t_1) \right)^2 + o(\varepsilon^{1-n} \rho_{\varepsilon}^2(t_1)) \]

\[ = \frac{A_0^2 + o(1)}{\sqrt{2}} \varepsilon^{1-n} e^{-2\sqrt{2}(1-t_1)/\varepsilon}. \]

Next we estimate the integral in \( I_2 \). We have

\[ \int_{I_0} \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \beta_{\varepsilon}' - f \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \right) \beta_{\varepsilon} \right) r^{n-1} dr \]

\[ = \int_{I_0} \left( \sqrt{2} \sum_{j=1}^{N} (-1)^j H_{t_j}' - f \left( \sum_{j=1}^{N} (-1)^j H_{t_j} \right) \right) \beta_{\varepsilon} r^{n-1} dr \]

\[ = \int_{I_0} \left( -\sqrt{2} H_{t_1}' - f \left( (-H_{t_1}) \right) \right) \beta_{\varepsilon} r^{n-1} dr + o(\varepsilon^{1-n} \rho_{\varepsilon}(t_1)) \]

\[ = -\frac{1}{\sqrt{2}} \varepsilon^{-\sqrt{2}(1-t_1)/\varepsilon} \int_{\mathbb{R}} \left( \sqrt{2} H' - f(H) \right) e^{\sqrt{2}y} dy (t_1/\varepsilon)^{n-1} + o(\varepsilon^{1-n} \rho_{\varepsilon}(t_1)) \]

\[ = -A_0 e^{-\sqrt{2}(1-t_1)/\varepsilon} (t_1/\varepsilon)^{n-1} + o(\varepsilon^{1-n} \rho_{\varepsilon}(t_1)), \]

since

\[ \int_{\mathbb{R}} \left( \sqrt{2} H' - f(H) \right) e^{\sqrt{2}y} dy = \left( H' e^{\sqrt{2}y} \right) \bigg|_{-\infty}^{+\infty} = \sqrt{2} A_0. \]

Thus,

\[ I_2 = -\left( \sqrt{2} A_0^2 + o(1) \right) e^{-2\sqrt{2}(1-t_1)/\varepsilon} (t_1/\varepsilon)^{n-1} + o(\varepsilon^{1-n} \rho_{\varepsilon}(t_1)) + O(\varepsilon^{2-n}), \]

which implies that

\[ (5.20) \quad I_2 + I_3 = -\frac{A_0^2 + o(1)}{\sqrt{2}} e^{-2\sqrt{2}(1-t_1)/\varepsilon} (t_1/\varepsilon)^{n-1} + o(\varepsilon^{1-n} \rho_{\varepsilon}(t_1)) + O(\varepsilon^{2-n}), \]

since \( t_1 = 1 + O(\varepsilon (\log(1/\varepsilon))^2) \).
It remains to consider $I_1$. For this purpose, we decompose it as

$$I_1 = \sum_{j=1}^{N} E_{\epsilon,j},$$

where

$$E_{\epsilon,j} = \int_{I_{\epsilon,j}} \left( \frac{1}{2} \left| \sum_{l=1}^{j} (-1)^l H_{\epsilon}^l \right|^2 - F \left( \sum_{l=1}^{j} (-1)^l H_{\epsilon}^l \right) \right) r^{n-1} dr$$

$$= \int_{I_{\epsilon,j}} \left( \frac{1}{2} \left| H_{\epsilon}^j + \sum_{l \neq j} (-1)^{j+l} H_{\epsilon}^l \right|^2 - F \left( H_{\epsilon}^j + \sum_{l \neq j} (-1)^{j+l} H_{\epsilon}^l \right) \right) r^{n-1} dr$$

$$= I_4 + I_5 + I_6 + o(e^{1-n} \sum_{l \neq j} e^{-\sqrt{2}|t_l-t_j|/\epsilon}),$$

with

$$I_4 = \int_{I_{\epsilon,j}} \left( \frac{1}{2} |H_{\epsilon}^j|^2 - F (H_{\epsilon}^j) \right) r^{n-1} dr,$$

$$I_5 = \int_{I_{\epsilon,j}} \left( H_{\epsilon}^j \sum_{l \neq j} (-1)^{j+l} H_{\epsilon}^l - f (H_{\epsilon}^j) \sum_{l \neq j} (-1)^{j+l} H_{\epsilon}^l \right) r^{n-1} dr,$$

$$I_6 = \frac{1}{2} \int_{I_{\epsilon,j}} \left| \sum_{l \neq j} (-1)^{j+l} H_{\epsilon}^l \right|^2 (2 - f'((-1)^j H_{\epsilon}^j)) r^{n-1} dr.$$

Using the fact that $|H'|^2 = 2F(H)$, for $I_4$ we find

$$I_4 = \int_{I_{\epsilon,j}} |H_{\epsilon}^j|^2 r^{n-1} dr$$

$$= \int_R |H'|^2 dy (t_j/\epsilon)^{n-1} - \frac{A_0^2 + o(1)}{\sqrt{2}} \left( e^{-\sqrt{2}|t_j-t_{j-1}|/\epsilon} + e^{-\sqrt{2}|t_j-t_{j+1}|/\epsilon} \right) (t_j/\epsilon)^{n-1}$$

$$+ O(\epsilon^{2-n}).$$

For $j \geq 2$, $I_5$ can be estimated (by recalling the exponential-decay property of $H(y) \pm 1$) as

$$I_5 = (t_j/\epsilon)^{n-1} H_{\epsilon}^j \sum_{l \neq j} (-1)^{j+l} H_{\epsilon}^l|_{\partial I_{\epsilon,j}} + O(\epsilon^{2-n})$$

$$= -(A_0^2 + o(1)) \sqrt{2} \left( e^{-\sqrt{2}|t_j-t_{j-1}|/\epsilon} + e^{-\sqrt{2}|t_j-t_{j+1}|/\epsilon} \right) (t_j/\epsilon)^{n-1} + O(\epsilon^{2-n}).$$

For $j = 1$, we have

$$I_5 = (t_1/\epsilon)^{n-1} H_{\epsilon}^j \sum_{l \geq 1} (-1)^{l+1} H_{\epsilon}^l |_{\partial I_{\epsilon,1}} + O(\epsilon^{2-n})$$

$$= -(A_0^2 + o(1)) \sqrt{2} e^{-\sqrt{2}|t_1-t_2|/\epsilon} (t_1/\epsilon)^{n-1} + O(\epsilon^{2-n}).$$
\[ I_6 \text{ can be estimated similarly: for } j \geq 2, \text{ we have} \]
\[ I_6 = 2 \int_{I_{t,j}} \left| \sum_{i \neq j} (-1)^{i+j} H_{t_i} \right|^2 r^{n-1} dr \]
\[ = \frac{A_0^2 + o(1)}{\sqrt{2}} \left( e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon} + e^{-\sqrt{2}|t_j-t_{j+1}|/\varepsilon} \right) (t_j/\varepsilon)^{n-1} + O(\varepsilon^{2-n}), \]
while for \( j = 1 \),
\[ I_6 = 2 \int_{I_{t,1}} \left| \sum_{l=1}^N (-1)^{l+1} H_{t_l} \right|^2 r^{n-1} dr = \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-\sqrt{2}|t_1-t_2|/\varepsilon} (t_1/\varepsilon)^{n-1} + O(\varepsilon^{2-n}). \]
Combining the estimates of \( I_4, I_5, \) and \( I_6 \), we obtain
\[ I_1 = I[H] \sum_{j=1}^N (t_j/\varepsilon)^{n-1} - \sqrt{2} (A_0^2 + o(1)) \sum_{j=2}^N e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon} (t_j/\varepsilon)^{n-1} \]
\[ - \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-2\sqrt{2}(1-t_1)/\varepsilon} + O(\varepsilon^{2-n}) \]
\[ = I[H] \sum_{j=1}^N (t_j/\varepsilon)^{n-1} - \sqrt{2} (A_0^2 + o(1)) \sum_{j=2}^N e^{-\sqrt{2}|t_j-t_{j-1}|/\varepsilon} (t_j/\varepsilon)^{n-1} \]
\[ - \frac{A_0^2 + o(1)}{\sqrt{2}} e^{-2\sqrt{2}(1-t_1)/\varepsilon} (t_1/\varepsilon)^{n-1} + O(\varepsilon^{2-n}). \]
Adding this to the estimate in (5-20), we obtain the asymptotic expansion (2-20) of \( \mathcal{E}_\varepsilon \left[ \sum_{j=1}^N (-1)^j H_{t,j} \right] \).

References


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