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ON THE UNIT GROUP OF SOME MULTIQUADRATIC NUMBER FIELDS

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We study the index of the group of units in the genus field of an imaginary quadratic number field modulo the subgroup generated by the units of the quadratic subfields (over $\mathbb{Q}$) of the genus field.

1. Introduction

One major problem in algebraic number theory is the computation of the class number $h(K)$ for a number field K. In the case of quadratic fields, this problem is easily solved by elementary methods. Once the field degree is larger than 2, the problem becomes more challenging. Historically, the oldest case after the quadratic fields seems to be when K runs through a particular family of quartic bicyclic fields over $\mathbb{Q}$, meaning that $\mathrm{Gal}(K/\mathbb{Q}) \simeq (2, 2)$ (here $(a_1, \ldots, a_r)$ denotes the direct sum of cyclic groups of order a_i, for $i = 1, \ldots, r$). Dirichlet [1842] in essence computed the class number $h(K)$ for the family of quartic fields $K = \mathbb{Q}(\sqrt{-1}, \sqrt{m})$, m a positive nonsquare integer. Namely, let $k_1 = \mathbb{Q}(\sqrt{-1})$, $k_2 = \mathbb{Q}(\sqrt{m})$, and $k_3 = \mathbb{Q}(\sqrt{-m})$, and denote by E_F the group of units of a number field F. Then Dirichlet discovered the class number formula

$$h(K) = \tfrac{1}{2}\, q(K/\mathbb{Q}) h(k_2) h(k_3),$$

where $q = q(K/\mathbb{Q}) = (E_K : E_{k_1} E_{k_2} E_{k_3})$. Dirichlet went on to show that the unit index q could be determined and was equal to 1 or 2.

Over time, Dirichlet's formula has been generalized in several directions; see in particular [Herglotz 1922; Kubota 1953; 1956; Kuroda 1950; Lemmermeyer 1994b; Wada 1966], and references therein. One particularly striking formula is usually attributed to Kuroda [1950], but in fact goes back to Herglotz [1922] in an equivalent, if less convenient, form for q. Let $L = \prod_i k_i$ be the multiquadratic field generated as the composite of all its quadratic subfields k_i, and suppose further that $[L : \mathbb{Q}] = 2^m$. Then

$$h(L) = \frac{1}{2^{\nu}}\, q(L/\mathbb{Q}) \prod_i h(k_i),$$

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where $q = q(L/\mathbb{Q}) = (E_L : \prod_i E_{k_i})$ and

$$\nu = \begin{cases} m(2^{m-1} - 1) & \text{if } L \text{ is real,} \\ (m-1)(2^{m-2} - 1) + 2^{m-1} - 1 & \text{if } L \text{ is complex.} \end{cases}$$

Hence $h(L)$ can be computed easily provided that the unit index $q(L/\mathbb{Q})$ can be computed. Herein lies the obstruction to an easy determination of the class number of multiquadratic number fields. For quartic bicyclic fields, Kubota [1956] gave a method for finding a system of fundamental units and thus for computing q. Wada [1966], generalized Kubota's method, creating an algorithm for computing fundamental units in any given multiquadratic field. However, in general there seems to be no explicit formula for q, even when L is of degree 4 over $\mathbb{Q}$.

This brings us to the purpose of this article. We try to glean some understanding of the difficulties in computing the unit index by giving explicit computations of q for special families of multiquadratic fields L. We consider the special case of the genus field $L = k_{\text{gen}}$ of a complex quadratic field k for which the 2-rank of the class group $\mathrm{Cl}(k)$ of k is ≤ 3. (Recall that the 2^n-rank of a finite abelian group G is the minimal number of generators of the factor group $G^{2^{n-1}}/G^{2^n}$.) If the 2-rank of $\mathrm{Cl}(k)$ is 1, then $[L : \mathbb{Q}] = 4$, by genus theory, and in this case it is known that $q = 1$ (see [Lemmermeyer 1995], for instance; the proof is easy — see the next section).

Next, if the 2-rank is 2, then $[L : \mathbb{Q}] = 8$ by genus theory. In this case, we reduce the problem to that of computing $q(K/\mathbb{Q})$ where K is the maximal real subfield of L. But then K is a totally real bicyclic field and we may apply the results of [Kubota 1956] to compute $q(K/\mathbb{Q})$. We find that $q(L/\mathbb{Q}) = 8$ or 2 according as the 2-class field tower of k is of length 1 or > 1. (Here k^1 is the Hilbert 2-class field of k and $k^{n+1} = (k^n)^1$; the length of the 2-class field tower of k is the cardinality of the set of k^n.)

For the case where the 2-rank of $\mathrm{Cl}(k)$ is 3, we seem to be in new territory. We restrict to the case of elementary 2-class group. Specifically, we assume $\mathrm{Cl}_2(k) \simeq (2, 2, 2)$, so $L = k_{\text{gen}} = k^1$. If the rank of $\mathrm{Cl}_2(k^1)$ is 2 as a module over the integral group ring $\Lambda = \mathbb{Z}[\mathrm{Gal}(k^1/k)]$, then $q(L/\mathbb{Q}) = 2^7$. This condition on the Λ-rank is, by the way, a natural one; see [Benjamin et al. 2003]. We then obtain less complete information about q for the other case where $\mathrm{Cl}_2(k^1)$ is of Λ-rank 3. In the particular fields we consider, $q = 2^4$ or 2^5.

2. The Main Results

Let k be an imaginary quadratic field for which the 2-rank of $\mathrm{Cl}(k)$ is $t - 1$. Hence, by genus theory, $k = \mathbb{Q}(\sqrt{d_1 \cdots d_t})$, where $\operatorname{disc} k = d_1 \cdots d_t$ is a factorization of the discriminant of k into distinct prime discriminants d_i divisible by the rational prime p_i for $i = 1, \ldots, t$. Then $L = k_{\text{gen}} = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_t})$ and hence multiquadratic of degree 2^t over $\mathbb{Q}$. Hence for $t \geq 2$ Kuroda's class number formula above yields

$$h(L) = \frac{1}{2^\nu}\, q(L/\mathbb{Q}) \prod_i h(k_i),$$

where k_i range over the $2^t - 1$ quadratic subfields of L, $q(L/\mathbb{Q}) = (E_L : \prod_i E_{k_i})$, and $\nu = (t-1)(2^{t-2}-1) + 2^{t-1} - 1$ since L is complex.

We start our computations of $q = q(L/\mathbb{Q})$ by first considering $t = 2$ (for the sake of completeness).

Theorem 1. *Let k be a complex quadratic number field and $L = k_{\text{gen}}$. If the 2-rank of* $\mathrm{Cl}(k)$ *equals* 1, *then* $q(L/\mathbb{Q}) = 1$.

Proof. Since the 2-rank of $\mathrm{Cl}(k)$ is 1, $t = 2$ so $k = \mathbb{Q}(\sqrt{d_1 d_2}\,)$ for prime discriminants d_1, d_2. Then $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}\,)$. Now, by Kuroda's class number formula (where $t = 2$ implies $\nu = 1$),

$$h(L) = \tfrac{1}{2} q(L/\mathbb{Q}) h(d_1 d_2) h(d_1) h(d_2),$$

where $h(n) = h(\mathbb{Q}(\sqrt{n}\,))$. Now, it is well known that $h(d)$ is odd for any prime discriminant d. Moreover, by the Artin map, $\mathrm{Gal}(k^1/k) \simeq \mathrm{Cl}_2(k)$ where $\mathrm{Cl}_2(k)$ is the 2-class group of k, the Sylow 2-subgroup of the class group, the order of which is $h_2(k)$, the 2-class number of k. Now consider $G = \mathrm{Gal}(k^2/k)$. Since the commutator subgroup $G' = \mathrm{Gal}(k^2/k^1)$, we see $G/G' \simeq \mathrm{Gal}(k^1/k) \simeq \mathrm{Cl}_2(k)$. But in the present case, $\mathrm{Cl}_2(k)$ is cyclic, whence $G' = \langle 1 \rangle$, and thus $k^2 = k^1$. But then since in general $k^1 \subseteq L^1 \subseteq k^2$, we have $L^1 = k^1$. Therefore, $h_2(L) = [L^1 : L] = [k^1 : L] = [k^1 : k]/2 = h_2(k)/2$. Now by restricting to 2-class numbers and using the fact that q is a power of two, (see [Wada 1966], for instance) the Kuroda class number formula becomes

$$h_2(L) = \tfrac{1}{2} q(L/\mathbb{Q}) h_2(k) h_2(d_1) h_2(d_2).$$

From the preceding discussion we get $\frac{1}{2} h_2(k) = \frac{1}{2} q\, h_2(k)$, as needed. $\square$

Next, we consider the case where the 2-rank of $\mathrm{Cl}(k)$ is 2, i.e. $t = 3$. Hence $k = \mathbb{Q}(\sqrt{d_1 d_2 d_3}\,)$, with prime discriminants d_i. Moreover, since k is complex, $\operatorname{disc} k < 0$ so either all the d_i are negative or exactly two are positive, say $d_1, d_2 > 0$, $d_3 < 0$. Notice that we have $L = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}\,)$. Let $K = L^+$ be the maximal real subfield of L, (so $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}\,)$ if say $d_1, d_2 > 0, d_3 < 0$, and $K = \mathbb{Q}(\sqrt{d_1 d_2}, \sqrt{d_2 d_3}\,)$ if $d_i < 0$, for $i = 1, 2, 3$). But then it follows that $q(L/\mathbb{Q}) = Q(L/K) q(K/\mathbb{Q})$, where $Q(L/K) = (E_L : W_L E_K)$ with W_L the group of roots of unity of L. To see this apply for example [Benjamin et al. 2003, Proposition 1], where we notice that any primitive eighth root ζ_8 of unity is not contained in L since any ramification index of a prime in $L/\mathbb{Q}$ must divide 2, whereas 2 is totally ramified in $\mathbb{Q}(\zeta_8)$. Now suppose d_1, $d_2 > 0$, $d_3 < 0$. By [Lemmermeyer

1995, Theorem 1], $Q(L/K) = 1$, since $L = K(\sqrt{d_3}\,)$ implies that L/K is essentially ramified if $d_3 \neq -4$ and that $2\mathcal{O}_K$ is not an ideal square if $d_3 = -4$, see [Lemmermeyer 1995] again for the details. Thus we have $q(L/\mathbb{Q}) = q(K/\mathbb{Q})$. If however all the $d_i < 0$, then for any i, $L = K(\sqrt{d_i}\,)$. In this case, it can be shown that $Q(L/K) = 2$ by [Lemmermeyer 1995], but we shall see that this is the case by another method. In either case, it is well known that $Q(L/K) = 1, 2$ (see [Hasse 1985, Satz 14]), and moreover, by Kubota [Kubota 1956], $q(K/\mathbb{Q})$ divides 4. Thus $q(L/\mathbb{Q})$ must divide 8.

Theorem 2. *Let k be a complex quadratic field with* $2\text{-rank}\,\mathrm{Cl}(k) = 2$. *Then for* $L = k_{\text{gen}}$, $q(L/\mathbb{Q}) = 8$ *or* 2 *according as the* 2*-class field tower of k is of length* 1 *or* > 1.

Proof. By assumption, $k = \mathbb{Q}(\sqrt{d_1 d_2 d_3}\,)$ for prime discriminants d_i. Now notice that $K_i = k(\sqrt{d_i}\,)$ for $i = 1, 2, 3$ are the three unramified quadratic extensions of k in L. These fields are quartic bicyclic extensions of $\mathbb{Q}$ and so Kuroda's class number yields

$$h(K_1) = \tfrac{1}{2}\, q(K_1/\mathbb{Q}) h(k) h(d_2 d_3) h(d_1),$$

since $m = 2$, so $\nu = 1$, (analogously for K_2 and K_3). Now since the K_i are unramified quadratic extensions of a complex quadratic field k, it is known that $q(K_i/\mathbb{Q}) = 1$; see for example [Lemmermeyer 1995]. Hence by considering 2-class numbers so that we may use $h_2(d_i) = 1$, we have

$$h_2(L) = \tfrac{1}{32}\, q(L/\mathbb{Q}) h_2(k) h_2(d_1 d_2) h_2(d_1 d_3) h_2(d_2 d_3), \qquad h_2(K_1) = \tfrac{1}{2} h_2(k) h_2(d_2 d_3).$$

Now we rewrite the formula for $h_2(L)$ in terms of $h_2(K_i)$. From above, notice that for example $h_2(d_2 d_3) = 2h_2(K_1)/h_2(k)$, etc. and so by class field theory,

$$h_2(d_2 d_3) = \frac{2[K_1^1 : K_1]}{[k^1 : k]} = \frac{2[K_1^1 : k^1][k^1 : K_1]}{[k^1 : K_1][K_1 : k]} = [K_1^1 : k^1].$$

Substituting into the above formula yields

$$[L^1 : L] = \tfrac{1}{32} q(L/\mathbb{Q}) [k^1 : k][K_1^1 : k^1][K_2^1 : k^1][K_3^1 : k^1]$$

and since $[L : k] = 4$, we have

$$[L^1 : k^1] = \tfrac{1}{8} q(L/\mathbb{Q}) [K_1^1 : k^1][K_2^1 : k^1][K_3^1 : k^1].$$

Notice, in particular, that if the 2-class field tower of k is of length 1, then all the field degrees in the above formula equal 1, and therefore $q = 8$. Now, the length of the 2-class field tower of k is 1 precisely when $d_i < 0$ for $i = 1, 2, 3$; see for example [Benjamin et al. 1997]. From this we have $8 = q(L/\mathbb{Q}) = Q(L/K) q(K/\mathbb{Q})$ from which it follows (by the comments before the proposition) that $Q(L/K) = 2$ and $q(K/\mathbb{Q}) = 4$.

Now suppose that $d_1, d_2 > 0, d_3 < 0$. In this case we have $q(L/\mathbb{Q}) = q(K/\mathbb{Q})$, where $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$. Kuroda's class number formula implies

$$h_2(K) = \tfrac{1}{4} q(K/\mathbb{Q}) h_2(F),$$

where $F = \mathbb{Q}(\sqrt{d_1 d_2})$. Then notice that $\mathrm{Cl}_2(F)$ is cyclic and thus $F^1 = F^2$. Thus since K/F is unramified so $K \subseteq F^1$, $h_2(K) = h_2(F)/2$. Plugging this into the formula above yields $q(K/\mathbb{Q}) = 2$. Thus $q(L/\mathbb{Q}) = 2$. □

Proposition 3. *Let k be a complex quadratic field with* 2-rank $\mathrm{Cl}(k) \leq 2$ *and with* 4-rank $\mathrm{Cl}(k) \leq 1$. *Then*

$$\prod_i K_i^1 = \Big(\prod_i K_i\Big)^1,$$

where K_i range over all the unramified quadratic extensions of k.

Proof. If $k^1 = k^2$, then the proposition is trivially true, since both fields are k^1. Thus, assume $k^1 \neq k^2$. Hence we know $k = \mathbb{Q}(\sqrt{d_1 d_2 d_3})$ where $d_1, d_2 > 0, d_3 < 0$. From the proof of Theorem 2,

$$[L^1 : k^1] = \tfrac{1}{4}[K_1^1 : k^1][K_2^1 : k^1][K_3^1 : k^1],$$

where $L = K_1 K_2 K_3$ with $K_i = k(\sqrt{d_i})$. But notice that

$$[K_1^1 K_2^1 K_3^1 : k^1] = \frac{[K_1^1 : k^1]}{[K_1^1 \cap K_2^1 K_3^1 : k^1]} \frac{[K_2^1 : k^1]}{[K_2^1 \cap K_3^1 : k^1]} [K_3^1 : k^1].$$

(Also notice this equation is true for any permutation of the indices.) Now since

$$[L^1 : k^1] = [L^1 : K_1^1 K_2^1 K_3^1][K_1^1 K_2^1 K_3^1 : k^1],$$

we see by putting these equations together that

$$[L^1 : K_1^1 K_2^1 K_3^1] = \tfrac{1}{4}[K_1^1 \cap K_2^1 K_3^1 : k^1][K_2^1 \cap K_3^1 : k^1].$$

To finish the proof, it suffices to show that

$$[K_1^1 \cap K_2^1 K_3^1 : k^1] = [K_2^1 \cap K_3^1 : k^1] = 2.$$

Here is where some group theory comes in. Let $G = \mathrm{Gal}(k^2/k)$, and further let H_1, H_2, H_3 be the three maximal subgroups of G such that $\mathrm{Gal}(k^2/K_i) = H_i$. Then we need to show that

$$(G' : H_2' H_3') = (G' : H_1'(H_2' \cap H_3')) = 2.$$

Here is a sketch of the proof. If G' is cyclic, say $G' = \langle c \rangle$, by the table of possible groups and their presentations at the end of [Benjamin et al. 1997], we have (without loss of generality) $H_3' = \langle c^2 \rangle$ and $H_1' H_2' = \langle c^2 \rangle$, from which our result follows.

Now suppose G' is not cyclic. Then by our assumption on the class group of k, G must be nonmetacyclic with $G/G' \simeq (2, 2^n)$ for some $n > 1$. Now we assume the notation before [Benjamin et al. 2001, Lemma 1]. Hence let $G = \langle a, b\rangle$ where $a^2 \equiv b^{2^n} \equiv 1 \bmod G'$. Let $[a, b] = c$ and define inductively, $c_2 = c$ and $c_{j+1} = [b, c_j]$. We have $G' = \langle c_2, c_3, \dots\rangle$, and $G_3 = \langle c_2^2, c_3, \dots\rangle$, and $G_4 = \langle c_2^4, c_3^2, c_4, \dots\rangle$; see [Benjamin et al. 1997, Lemma 2]. Now if $H_3 = \langle a, b^2, G'\rangle$, then it is easy to see that $H_3'G_4 = G_3$. Thus $H_3' = G_3$ by [Hall 1933, Theorem 2.49ii]. Hence $(G' : H_3') = (G' : G_3) = 2$. Similarly, if $H_1 = \langle b, G'\rangle$ and $H_2 = \langle ab, G'\rangle$, then $H_1'H_2'G_4 = G_3$ so once again $H_1'H_2' = G_3$. This shows the result and finishes the proof of the proposition. □

Now we consider $\mathrm{Cl}_2(k) \simeq (2, 2, 2)$, and thus in particular $\operatorname{disc} k = d_1d_2d_3d_4$ for distinct prime discriminants d_i. If we assume the Λ-rank of $\mathrm{Cl}_2(k^1/k)$ is 2, then by [Benjamin et al. 2003, Theorem 2], exactly three of the d_i's must be negative, say $d_1, d_2, d_3 < 0$, $d_4 > 0$.

Theorem 4. *Let k be a complex quadratic field with $\mathrm{Cl}_2(k) \simeq (2, 2, 2)$. If the Λ-rank of $\mathrm{Cl}_2(k^1)$ equals 2, the unit index $q(k^1/\mathbb{Q})$ equals 2^7.*

Proof. If $\mathrm{Cl}_2(k) \simeq (2, 2, 2)$, then $\mathrm{Cl}_2(k^1/k)$ has Λ-rank 2 if and only if $G/G' \simeq (2, 2, 2)$ and $G'/G_3 \simeq (2, 2)$, where $G = \mathrm{Gal}(k^2/k)$. Thus $(G : G_3) = 32$ and $G'/G_3 \simeq (2, 2)$, and by [Hall and Senior 1964], G/G_3 must be one of the seven groups 32.033, 32.035, 32.036, 32.037, 32.038, 32.040, 32.041, in the notation of that same reference.

Let $L = k^1 = k_{\mathrm{gen}}$. Kuroda's class number formula (with $t = 4$, so $\nu = 16$) gives

$$h_2(L) = \frac{1}{2^{16}} q(L/\mathbb{Q}) h_2(k) \prod_i h_2(k_i), \tag{1}$$

where the k_i are the quadratic subfields of L excluding k.

The following table lists the 2-class numbers $h_2(k_i)$ and $h_2(L)$:

G/G_3	$h_2(k_i)$	$h_2(L)$
32.041, 32.040	1 (7), 2 (6), 4	4
32.035, 32.037, 32.038	1 (7), 2 (5), 4, 2^n	2^{n+1}
32.036	1 (7), 2 (5), 2^{m+1}, 2^n	2^{m+n}
32.033	1 (7), 2 (3), 4, 2^l, 2^m, 2^n	$2^{l+m+n-1}$

Here "1 (7)" means that 7 quadratic subfields have 2-class number equal to 1. Plugging these data into (1) we immediately find the values of the unit index $q(L/\mathbb{Q})$ in each of the cases.

The 2-class numbers of the quadratic subfields k_i of L are easily determined using genus theory (see [Kaplan 1976], for instance). The 2-class numbers of L

were computed in [Benjamin et al. 2003], except for the group $G/G_3 = 32.033$ (for the first five groups, we have given the structure of G' explicitly; in the case $G/G_3 = 32.036$, we computed the 2-class number and actually showed that $q(L/\mathbb{Q}) = 2^7$).

We will now study the case $G/G_3 \simeq 32.033$ in detail. By Proposition 16 in the same reference, we have $k = \mathbb{Q}(\sqrt{d_1d_2d_3d_4}\,)$ with $d_i < 0$, $(i = 1, 2, 3)$, $d_4 > 0$ and such that

$$\left(\frac{d_1}{p_2}\right) = \left(\frac{d_2}{p_3}\right) = \left(\frac{d_3}{p_1}\right) = \left(\frac{d_1}{p_4}\right) = -1, \quad \left(\frac{d_4}{p_2}\right) = \left(\frac{d_4}{p_3}\right) = +1.$$

Here is a list of the 2-class numbers of the quadratic subfields of $L = k^1$. Along with $h_2(k) = 8$, we have

$$\begin{aligned} h_2(d_j) = h_2(d_1d_2) = h_2(d_2d_3) = h_2(d_1d_3) = 1, &\quad (j = 1, 2, 3, 4)\\ h_2(d_1d_4) = h_2(d_1d_2d_4) = h_2(d_1d_3d_4) = 2, &\qquad h_2(d_1d_2d_3) = 4,\\ h_2(d_3d_4) = 2^l,\ \ h_2(d_2d_3d_4) = 2^m,\ \ h_2(d_2d_4) = 2^n &\qquad (l, m, n \geq 2). \end{aligned}$$

Let $K = L^+ = \mathbb{Q}(\sqrt{d_1d_2}, \sqrt{d_1d_3}, \sqrt{d_4}\,)$, the maximal real subfield of L. Then $q(L/\mathbb{Q}) = Q(L/K)q(K/\mathbb{Q})$ by [Benjamin et al. 2003, Proposition 1]. But by [Lemmermeyer 1995], $Q(L/K) = 2$. In fact, if $w_L (= \# W_L) \equiv 2 \bmod 4$, then $L = K(\sqrt{d_1}\,)$ and $(p_1) = (\pi)^2$ in $Q(\sqrt{d_1d_2}\,)$ since p_1 ramifies and the field has odd class number. But then $d_1\mathcal{O}_K = (\pi\mathcal{O}_K)^2$, and part (i)2(a) of [Lemmermeyer 1995, Theorem 1] implies $Q(L/K) = 2$. If instead $w_L \equiv 4 \bmod 8$, then $2\mathcal{O}_K = (1+i)^2\mathcal{O}_K$, whence part (ii)2(a) of the same theorem shows again that $Q(L/K) = 2$.

Now we compute $q(K/\mathbb{Q})$. To this end, consider the quadratic number field $k_0 = \mathbb{Q}(\sqrt{d_2d_3d_4}\,)$ with 2-class group $\mathrm{Cl}_2(k_0) = (2^m)$ and fundamental unit ε_{234}. Then K/k_0 is a V_4-extension with the quadratic subextensions $K_1 = k_0(\sqrt{d_1d_2}\,)$, $K_2 = k_0(\sqrt{d_1d_3}\,)$, $K_3 = k_0(\sqrt{d_2d_3}\,)$. Let ε_{ij} denote the fundamental unit of $\mathbb{Q}(\sqrt{d_id_j}\,)$ for $1 \leq i < j \leq 3$. We shall determine $\mathrm{Cl}_2(K_1)$ and $q(K_1/\mathbb{Q})$. Since k_0 has cyclic 2-class group of type (2^m) and since K_1/k_0 is ramified, its class group contains (2^m) as a subgroup. If we can show that $h_2(K_1) = 2^m$, then $\mathrm{Cl}_2(K_1) \simeq (2^m)$; since K/K_1 is unramified, it would then follow that $\mathrm{Cl}_2(K) \simeq (2^{m-1})$. Applying Kuroda's class number formula to $K/\mathbb{Q}$ would then give $q(K/\mathbb{Q}) = 2^6$, and this in turn implies $q(L/\mathbb{Q}) = 2^7$ and $h_2(L) = 2^{l+m+n-1}$.

For computing the 2-class number of K_1 we use Kuroda's formula

$$h_2(K_1) = \tfrac{1}{4}q(K_1/\mathbb{Q})h_2(d_1d_2)h_2(d_1d_3d_4)h_2(d_2d_3d_4) = q(K_1/\mathbb{Q})2^{m-1}.$$

It suffices to show that $q(K_1/\mathbb{Q}) \leq 2$ (which implies $q(K_1/\mathbb{Q}) = 2$ by the argument above).

We consider two cases: $d_k := \operatorname{disc} k \not\equiv 4 \bmod 8$ and $d_k \equiv 4 \bmod 8$. Assume $d_k \not\equiv 4 \bmod 8$. The prime ideal above d_1 in $\mathbb{Q}(\sqrt{d_1d_2}\,)$ is principal; hence $X^2 - d_1d_2y^2 =$

$$\begin{tikzcd}
 & & K \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
 & K_1 \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & K_2 \arrow[d, no head] & K_3 \arrow[dl, no head] \\
\mathbb{Q}(\sqrt{d_1d_2}\,) \arrow[dr, no head] & \mathbb{Q}(\sqrt{d_1d_3d_4}\,) \arrow[d, no head] & k_0 \arrow[dl, no head] & \\
 & \mathbb{Q} & &
\end{tikzcd}$$

Figure 1. Some subfields of $K/\mathbb{Q}$.

$\pm 4d_1$ is solvable, and so is $d_1x^2 - d_2y^2 = -4$ (the minus sign must occur since $(d_1/p_2) = -1$). Then $\eta = \frac{1}{2}(x\sqrt{d_1} + y\sqrt{d_2}\,)$ is a unit in $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}\,)$; note that $\eta^2 < 0$ in $\mathbb{Q}(\sqrt{d_1d_2}\,)$ since otherwise $\eta \in \mathbb{R} \cap F = \mathbb{Q}(\sqrt{d_1d_2}\,)$. Therefore $\eta^2 = -\varepsilon_{12}^u$; notice that u is odd since otherwise $\sqrt{-1} \in \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}\,)$, a contradiction. Thus $-d_1\varepsilon_{12} = (\sqrt{d_1}\eta\,\varepsilon_{12}^{(1-u)/2})^2$ is a square in $\mathbb{Q}(\sqrt{d_1d_2}\,)$.

Next consider $\mathbb{Q}(\sqrt{d_1d_3d_4}\,)$, along with the diophantine equations

$$d_1x^2 - d_3d_4y^2 = \pm 4, \qquad d_3x^2 - d_1d_4y^2 = \pm 4, \qquad d_4x^2 - d_1d_3y^2 = \pm 4,$$

which are solvable if the prime above d_1, d_3, d_4, respectively, is principal. The first implies $(d_1/p_4) = +1$, which contradicts the assumptions. The last implies $(d_4/p_1) = (d_4/p_3)$, which also leads to a contradiction. Thus the second equation must have a solution, and reduction mod p_3 shows that we must have $d_3x^2 - d_1d_4y^2 = -4$. Thus $-d_3\varepsilon_{134}$ is a square in $\mathbb{Q}(\sqrt{d_1d_3d_4}\,)$. Hence none of ε_{12}, ε_{134}, $\varepsilon_{12}\varepsilon_{134}$ can be squares in K_1. Therefore $q(K_1/\mathbb{Q}) \le 2$, as desired.

Now suppose $d_k \equiv 4 \bmod 8$. Then our assumptions imply that $d_3 = -4$ or $d_2 = -4$. First assume that $d_3 = -4$. Then the argument above shows that $\varepsilon_{12} = p_1\kappa^2$ for some $\kappa \in \mathbb{Q}(\sqrt{d_1d_2}\,)$. Now consider $\mathbb{Q}(\sqrt{d_1d_3d_4}\,) = \mathbb{Q}(\sqrt{p_1p_4}\,)$. Then by genus theory ([Lemmermeyer 2000, page 76]) there is a principal ideal (α) in $\mathbb{Q}(\sqrt{p_1p_4}\,)$ different from (1) and $(\sqrt{p_1p_4}\,)$ which is a product of distinct ramified prime ideals. We now consider the possibilities. First notice that the prime ideals above p_1 and p_4 are not principal, since otherwise $p_1x^2 - p_4y^2 = \pm 1$ is solvable which cannot happen. Now assume that the prime ideal above 2 is principal, equal to say (π) with $\pi = x + y\sqrt{p_1p_4}$, for some $x, y \in \mathbb{N}$. Then $\pi^2/2 = \mu$ a positive unit in $\mathbb{Q}(\sqrt{p_1p_4}\,)$. Clearly μ is not a square in $\mathbb{Q}(\sqrt{p_1p_4}\,)$ since otherwise $\sqrt{2} \in \mathbb{Q}(\sqrt{p_1p_4}\,)$, a contradiction. Hence $\varepsilon_{134} = 2\kappa^2$, for some $\kappa \in \mathbb{Q}(\sqrt{p_1p_4}\,)$. Similarly ε_{134} could be of the form $2p_1\kappa^2$ or $2p_4\kappa^2$. But in all of these cases we see that none of ε_{12}, ε_{134}, $\varepsilon_{12}\varepsilon_{134}$ can be squares in K_1. Once again we have $q(K_1/\mathbb{Q}) \le 2$.

Finally, suppose $d_2 = -4$. The argument above shows $\varepsilon_{134} = p_3\kappa^2$, for some $\kappa \in \mathbb{Q}(\sqrt{d_1d_3d_4})$. Now consider $\mathbb{Q}(\sqrt{d_1d_2}) = \mathbb{Q}(\sqrt{p_1})$. Then arguing as above we see $\varepsilon_{12} = 2\kappa^2$ or $\varepsilon_{12} = 2p_1\kappa^2$ for some $\kappa \in \mathbb{Q}(\sqrt{p_1})$. But again this implies $q(K_1/\mathbb{Q}) \le 2$; whence the result is established. □

Now we come to the case where $\mathrm{Cl}_2(k) \simeq (2,2,2)$ but with $\operatorname{disc} k$ divisible by three positive prime discriminants, say $\operatorname{disc} k = d_1d_2d_3d_4$ with $d_i > 0$ for $i = 1, 2, 3$ and $d_4 < 0$. Our results in this case will be far less complete since our knowledge of $\mathrm{Gal}(k^2/k)$ is much more spotty. But we now simplify things somewhat by reducing to the maximal real subfield of k^1. To this end, from now on, let $L = k^1$ and $K = L^+$ the maximal real subfield of L. But then

$$q(L/\mathbb{Q}) = q(K/\mathbb{Q}),$$

because $q(L/\mathbb{Q}) = Q(L/K)q(K/\mathbb{Q})$ (by [Benjamin et al. 2003, Proposition 1], for example). By [Lemmermeyer 1995, Theorem 1] we get $Q(L/K) = 1$ since $L = K(\sqrt{d_4})$ is essentially ramified if $d_4 \ne -4$ and $2\mathcal{O}_K$ is not an ideal square when $d_4 = -4$.

Now we need only consider $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ where p_i are the rational primes dividing d_i. We set up the following notation. Let $k_0 = \mathbb{Q}(\sqrt{p_1p_2p_3})$. Let $K_i = k_0(\sqrt{p_i})$ for $i = 1, 2, 3$ and let k_i be the quadratic subfield of K_i not equal to k_0 and $\mathbb{Q}(\sqrt{p_i})$. (Notice that $k_i = \mathbb{Q}(\sqrt{\operatorname{disc} k_0/p_i})$.) We now let ε_i for $i = 0, 1, 2, 3$ be the fundamental unit > 1 in k_i and $N\varepsilon_i$ the norm from k_i to $\mathbb{Q}$; also let ε_{p_i} be the fundamental unit in $\mathbb{Q}(\sqrt{p_i})$. Finally let $H_i = \mathrm{Gal}(k_0^2/K_i)$.

Now we assume that k_0 is a particular type of field. Namely, assume that $\mathrm{Cl}_2(k_0) \simeq (2,2)$. This assumption implies that $G = \mathrm{Gal}(k_0^2/k_0)$ is one of the following types: abelian, quaternion, dihedral, semidihedral. Moreover notice that in this case $k_0^1 = K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ since K/k_0 is unramified and $h_2(k_0) = 4$. Without loss of generality we now pick K_1 above so that H_1 is cyclic. We are in a position to state and prove the following (rather technical) theorem.

Theorem 5. *Let k be a complex quadratic field with $\mathrm{Cl}_2(k) \simeq (2,2,2)$ and with $\operatorname{disc} k = d_1d_2d_3d_4$ where d_i are distinct prime discriminants divisible by primes p_i and d_1, d_2, d_3 are positive. With the notation above, assume that $\mathrm{Cl}_2(k_0) \simeq (2,2)$. Then $q = q(k^1/\mathbb{Q})$ takes on the two values 2^4 and 2^5 as follows:*

- *If G is abelian, then $q = 2^4$.*
- *If G is nonabelian, then $N\varepsilon_0 = +1$ implies $q = \left\{\begin{matrix} 2^4 & \text{if } N\varepsilon_1 = -1, \\ 2^5 & \text{otherwise,}\end{matrix}\right\}$, while*

$$N\varepsilon_0 = -1 \ \text{ implies } \ q = \begin{cases} 2^4 & \text{if } \big(N\varepsilon_1 = 1 \text{ or } (N\varepsilon_1 = -1 \text{ and } \sqrt{\varepsilon_{p_1}\varepsilon_0\varepsilon_1} \notin K_1)\big) \\ & \qquad \text{and } \left(\dfrac{p_1}{p_2}\right) = \left(\dfrac{p_1}{p_3}\right) = -\left(\dfrac{p_2}{p_3}\right) = -1, \\ 2^5 & \text{otherwise.} \end{cases}$$

Proof. Since H_1 is cyclic (and so in particular abelian), we have $k_0^2 = K_1^1$. Thus $h_2(K_1) = [k_0^2 : k_0^1][k_0^1 : K_1] = 2h_2(k_0^1)$, and hence

$$(*) \qquad h_2(k_0^1) = \tfrac{1}{2}\, h_2(K_1).$$

Next notice by Kuroda's class number formula that

$$(**) \qquad h_2(k_0^1) = \frac{1}{2^7}\, q\, h_2(k_1)h_2(k_2)h_2(k_3),$$

where we have used $\nu = 9$ and $h_2(k_0) = 4$. Now Kuroda's class number formula for K_i yields

$$(***) \qquad h_2(K_i) = \tfrac{1}{4}\, q_i\, h_2(p_i)h_2(k_i)h_2(k_0) = q_i\, h_2(k_i).$$

But then $(*)$, $(**)$, $(***)$ imply

$$\frac{1}{2}\, q_1 h_2(k_1) = \frac{1}{2^7}\, q\, h_2(k_1)\frac{h_2(K_2)}{q_2}\,\frac{h_2(K_3)}{q_3},$$

and therefore

$$q = \frac{2^6 q_1 q_2 q_3}{h_2(K_2)h_2(K_3)}.$$

Suppose first of all that G is abelian. Then $k_0^2 = k_0^1$ and so $h_2(K_i) = 2$. Thus $q = 2^4 q_1 q_2 q_3$. But $h_2(k_i) \equiv 0 \bmod 2$, since the $d_j > 0$ for $j = 1, 2, 3$. So $(***)$ implies that $h_2(K_i) = 2 = q_i h_2(k_i)$ and this in turn yields $h_2(k_i) = 2$ and $q_i = 1$, for $i = 1, 2, 3$. Thus when G is abelian, $q = 2^4$.

Now assume that G is not abelian. Then for $G \simeq H_8$, the quaternion group of order 8, or $G \simeq D_4$, the dihedral group of order 8, H_i has order 4 for $i = 1, 2, 3$ and so in particular $h_2(K_i) = 4$. If $G \not\simeq H_8$ or D_4, then H_2, H_3 are either dihedral, semidihedral, or quaternion, whence in particular the abelianizations $H_2^{ab} \simeq H_3^{ab} \simeq (2, 2)$ and thus $h_2(K_2) = h_2(K_3) = 4$. Then

$$q = 2^2 q_1 q_2 q_3.$$

Case 1. Assume $N\varepsilon_0 = 1$. We now compute the q_i's. First consider q_2. By [Couture and Derhem 1992, Theorem 1], $(p_1/p_3) = -1$, whence $h_2(k_2)(= h_2(p_1p_3)) = 2$. But $4 = h_2(K_2) = q_2 h_2(k_2) = 2q_2$, so thus

$$q_2 = 2.$$

Next consider q_3. Again by [Couture and Derhem 1992, Theorem 1], $(p_2/p_1) = 1$ and $(p_1/p_2)_4 = -(p_2/p_1)_4$. Since $4 = h_2(K_3) = q_3 h_2(p_1p_2)$, then either $\big(h_2(p_1p_2) = 2\ \ \&\ \ q_3 = 2\big)$ or $\big(h_2(p_1p_2) = 4\ \ \&\ \ q_3 = 1\big)$. We claim the latter does not hold. For, first by (α) on page 318 of [Kaplan 1976], $\mathrm{Cl}_2^+(k_3) \simeq (4)$.

Hence if the latter holds, then $N\varepsilon_3 = -1$, which is not possible by [Kaplan 1976, Corollary 1]. Hence

$$q_3 = 2.$$

Finally consider q_1. First assume $N\varepsilon_1 = -1$. Then by [Kubota 1956] the only possible square root of a nonsquare unit in K_1 would be $\sqrt{\varepsilon_0}$ since the others have negative norm. Now applying [Benjamin et al. 1998, Proposition 3], for example, we see $k_0(\sqrt{\varepsilon_0}\,) = k_0(\sqrt{\delta}\,)$, where by genus theory $\delta | p_1 p_2 p_3$ (but $\neq$) and $\chi_j(\delta) = 1$ for all genus characters of k_0. But then since $(p_1/p_2) = (p_3/p_2) = 1$, p_2 is trivial for all the genus characters and no other p_i has this property. Thus we may assume $\delta = p_2$ which is not a square in K_1. Thus

$$N\varepsilon_1 = -1 \quad \text{implies} \quad q_1 = 1.$$

Now assume $N\varepsilon_1 = +1$. Then $\delta_{k_0} = p_2$ again and $\delta_{k_1} = p_2$ so that $\varepsilon_0\varepsilon_1$ is a square in K_1 this time; again see [Kubota 1956]. Hence

$$N\varepsilon_1 = +1 \quad \text{implies} \quad q_1 = 2.$$

Therefore, for $N\varepsilon_0 = 1$, $q = 2^4$ if $N\varepsilon_1 = -1$ and $q = 2^5$ if $N\varepsilon_1 = +1$.

Case 2. Assume $N\varepsilon_0 = -1$. Since $\mathrm{Cl}_2(k_0) = \mathrm{Cl}_2^+(k_0)$ is elementary, the Rédei–Reichardt conditions [1933] imply that

$$a) \quad \left(\frac{p_i}{p_j}\right) = -1, \quad \text{for all } i \neq j, \quad \text{or} \quad b) \quad \left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{p_3}\right) = -\left(\frac{p_2}{p_3}\right) = -1.$$

First consider a). Then $(p_i p_j / p_\ell) = 1$ for all distinct $i, j, \ell = 1, 2, 3$. By [Couture and Derhem 1992, Theorem 2], $G \simeq (2, 2)$ or H_8. Hence in the present situation $G \simeq H_8$. Thus as noted above the order of H_i is 4 whence $h_2(K_i) = 4$ so that $4 = h_2(K_i) = q_i h_2(k_i)$, for $i = 1, 2, 3$. But $(p_i/p_j) = -1$ implies $h_2(k_i) = 2$. Therefore, $q_i = 2$ for $i = 1, 2, 3$ so $q = 2^5$.

Next consider b). As immediately above, $q_2 = q_3 = 2$. Now consider q_1. If $N\varepsilon_1 = +1$, then arguing as above shows $q_1 = 1$. If $N\varepsilon_1 = -1$, so that the norms of $\varepsilon_{p_1}, \varepsilon_1, \varepsilon_0$ are negative, then $q_1 = 1$ if $\sqrt{\varepsilon_{p_1}\varepsilon_1\varepsilon_0} \notin K_1$, and $q_1 = 2$ otherwise.

This establishes the theorem. □

As a corollary to this theorem, we see that the structure $G = \mathrm{Gal}(k_0^2/k_0)$ determines $q(k^1/\mathbb{Q})$:

Corollary 6. *Let k_0 satisfy all the conditions in Theorem 5. For $G = \mathrm{Gal}(k_0^2/k_0)$,*

$$q(k^1/\mathbb{Q}) = \begin{cases} 2^4 & \text{if } G \text{ is abelian or dihedral,} \\ 2^5 & \text{if } G \text{ is semidihedral or quaternion.} \end{cases}$$

Proof. This follows immediately by Theorem 1 of [Couture and Derhem 1992] and a stronger form of part of Theorem 2 of the same paper, as found in [Lemmermeyer 1994a]. The main change in Theorem 2 is the following: with the notation above Theorem 5 suppose $N\varepsilon_0 = N\varepsilon_1 = -1$. If $\sqrt{\varepsilon_{p_1}\varepsilon_1\varepsilon_0} \in K_1$, then G is quaternion (of order 8 or larger). If $\sqrt{\varepsilon_{p_1}\varepsilon_1\varepsilon_0} \notin K_1$, then G is dihedral. □

The previous theorem is a special case of the following proposition:

Proposition 7. *Let k be a complex quadratic field with $\mathrm{Cl}_2(k) \simeq (2,2,2)$ and with $\operatorname{disc} k = d_1d_2d_3d_4$ where d_i are distinct prime discriminants divisible by primes p_i and d_1, d_2, d_3 are positive. With the notation above, assume that $k_0^1 = k_0^2$. Then $q = q(k^1/\mathbb{Q}) = 2^4$.*

Proof. Recall that $k_0 = \mathbb{Q}(\sqrt{p_1p_2p_3})$, $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$, and that $q = q(K/\mathbb{Q})$. Then Kuroda's class number formula yields

$$h_2(K) = \frac{1}{2^9}\, q\, h_2(k_0)h_2(k_1)h_2(k_2)h_2(k_3);$$

(again, refer to the notation before the previous theorem). Now since $k_0 \subseteq K \subseteq k_0^1$, we see $k_0^1 \subseteq K^1 \subseteq k_0^2$; but since by assumption $k_0^1 = k_0^2$, we have $K^1 = k_0^1$. Hence

$$h_2(K) = [K^1 : K] = [k_0^1 : K] = \tfrac14 [k_0^1 : k_0] = \tfrac14\, h_2(k_0).$$

Similarly $K_i^1 = k_0^1$, for $i = 1, 2, 3$, whence

$$h_2(K_i) = \tfrac12\, h_2(k_0).$$

On the other hand Kuroda's class number formula again yields

$$h_2(K_i) = \tfrac14\, q_i\, h_2(k_0)h_2(k_i).$$

All this implies

$$\tfrac12 h_2(k_0) = \tfrac14\, q_i\, h_2(k_0)h_2(k_i)$$

so that $2 = q_i\, h_2(k_i)$. But then since $2 \mid h_2(k_i)$, we must have $h_2(k_i) = 2$. But then from above we have

$$\frac14 h_2(k_0) = h_2(K) = \frac{1}{2^9}\, q\, h_2(k_0)h_2(k_1)h_2(k_2)h_2(k_3) = \frac{1}{2^6}\, q\, h_2(k_0).$$

Therefore by solving for q, we obtain

$$q = 2^4.$$

□

3. Examples

We now give numerical examples illustrating Theorem 5 with $q = 2^4$ and $q = 2^5$.

Example 1. Let $k_0 = \mathbb{Q}(\sqrt{2405}\,) = \mathbb{Q}(\sqrt{5\cdot 13\cdot 37}\,)$ and $K = \mathbb{Q}(\sqrt{5}, \sqrt{13}, \sqrt{37}\,)$. By [Rédei and Reichardt 1933] or [Kaplan 1976] we see that $\mathrm{Cl}_2(k_0) \simeq (2, 2)$. Moreover, we have $N\varepsilon_0 = -1$ and $(13/5) = (37/5) = (37/13) = -1$. Thus by [Couture and Derhem 1992, Theorem 2], $\mathrm{Gal}(k_0^2/k_0) \simeq H_8$ or $(2, 2)$; but [Benjamin et al. 1998, Theorem 1] then shows that $\mathrm{Gal}(k_0^2/k_0) \simeq H_8$. Finally Theorem 4 above shows $q = 2^5$.

Example 2. Consider $k_0 = \mathbb{Q}(\sqrt{290}\,) = \mathbb{Q}(\sqrt{2\cdot 5\cdot 29}\,)$; see the examples in [Couture and Derhem 1992]. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{29}\,)$. By [Rédei and Reichardt 1933] or [Kaplan 1976] or even [Couture and Derhem 1992], we see that $\mathrm{Cl}_2(k_0) \simeq (2, 2)$. Moreover, we have $N\varepsilon_0 = -1$, where $\varepsilon_0 = 17 + \sqrt{290}$ is the fundamental unit of k_0; and $(2/5) = (2/29) = -(29/5) = -1$. Now by genus theory $K_1 = \mathbb{Q}(\sqrt{5\cdot 29}, \sqrt{2}\,)$ (notation as in above). Also $N\varepsilon_1 = -1$ where $\varepsilon_1 = 12+\sqrt{145}$ is the fundamental unit of $\mathbb{Q}(\sqrt{5\cdot 29}\,)$. Finally, $\varepsilon_2 = 1 + \sqrt{2}$. By the techniques described in [Kubota 1956] we see that $\varepsilon_0\varepsilon_1\varepsilon_2$ is not a square in K_1. Theorem 5 above then shows $q = 2^4$. Furthermore, [Couture and Derhem 1992, Theorem 2] and PARI show $\mathrm{Gal}(k_0^2/k_0) \simeq D_4$.

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