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CHARACTERIZATION OF A GENERALIZED SHANKS SEQUENCE

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We consider generalizations of Shanks' sequence of quadratic fields $\mathbb{Q}(\sqrt{S_n})$ where $S_n = (2^n + 1)^2 + 2^{n+2}$. Quadratic fields of this type are of interest because it is possible to explicitly determine the fundamental unit. If a sequence of quadratic fields given by $D_n = A^2x^{2n} + Bx^n + C^2$ satisfies certain conditions (notably that the regulator is of order $\Theta(n^2)$), then we determine the exact form such a sequence must take.

1. History of creepers

We will be interested in simple continued fractions, which we denote by $\alpha = [\, a_0 \,, \, a_1 \,, \, a_1 \,, \, a_2 \,, \, \dots \,]$; the a_i are called the *partial quotients* of α. It is well-known that a continued fraction expansion is periodic if and only if it is the expansion of a real quadratic irrational. We denote the period length of a real quadratic irrational α by $lp(\alpha)$.

For real quadratic fields, it is expected that the class number will usually be small; see [Cohen and Lenstra 1984]. By the correspondence between ideals and continued fractions this is equivalent to the continued fraction expansion of $\sqrt{D}$ being long, generally of length about $\sqrt{D}$. Thus, examples of short expansions of $\sqrt{D}$ should be considered as unusual and worthy of interest.

It is easy to find sequences of integers D_i such that $\sqrt{D_i}$ has a bounded period length. Many results have been determined for such families and we refer the reader to [Perron 1950; van der Poorten and Williams 1999; Schinzel 1960; 1961].

Shanks [1969] examined the class numbers of quadratic fields with discriminants given by $n^2 - 2^{2k+1}$. He noticed that for the family $S_n = (2^n + 3)^2 - 8$, the class number of S_n grows infinitely large. This sequence of fields is known as *Shanks' sequence*. It happens that Shanks' sequence is just a special case of an earlier

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example given in [Nyberg 1949]:

$$D_n = \big(x^n + (x \pm 1)/2\big)^2 \mp x^n. \tag{1-1}$$

The ring of algebraic integers of a quadratic number field $K = \mathbb{Q}(\sqrt{D})$, denoted by $\mathbb{O}_K$, is equal to $\mathbb{Z}[\omega']$, where d_K is the squarefree kernel of D and

$$\omega' = \begin{cases} \sqrt{d_K} & \text{if } d_K \not\equiv 1 \pmod 4, \\ (1+\sqrt{d_K})/2 & \text{if } d_K \equiv 1 \pmod 4. \end{cases}$$

The *discriminant*, D_K, of $\mathbb{O}_K$ is equal to d_K if d_K is congruent to 1 modulo 4 and $4d_K$ otherwise. An *order* $\mathbb{O}$ of K is defined to be a subring of K, containing 1, such that the quotient $\mathbb{O}_K/\mathbb{O}$ is finite, such an order must be of the form $\mathbb{O} = \mathbb{Z}[f\omega']$. The number f is called the *conductor* of $\mathbb{O}$. The *discriminant* of an order $\mathbb{O} \subset \mathbb{O}_K$ is equal to $D = f^2 D_k$. Thus, the discriminant of an order is always congruent to 0 or 1 modulo 4. The discriminant of the maximal order is called a *fundamental* discriminant.

For any discriminant, $D \equiv t \pmod 4$, the element $\omega = (t + \sqrt{D})/2$ is an algebraic integer since $t \equiv 0, 1 \pmod 4$. If we know that D is fundamental then we usually write ω' instead of ω. With this notation, the expansion of $\omega_n = (1+\sqrt{S_n})/2$ corresponding to Shanks' sequence has a period length of $2n+1$,

$$\omega_n = [\, 2^{n-1}+1\,,\, \overline{1\,,\, 2^{n-1}\,,\, 2\,,\, 2^{n-2}\,,\, 2^2\,,\, \ldots\,,\, 2^{n-1}\,,\, 1\,,\, 2^n+1}\,].$$

The fundamental unit of the order with discriminant D_n is given by

$$\varepsilon_n = \big((2^n+1+\sqrt{D_n})/2\big)\big((2^n+3+\sqrt{D_n})/4\big)^n.$$

The regulator of the order $\mathbb{O}$, denoted by $R(\mathbb{O})$ or by $R(D)$ if $\mathbb{O}$ has discriminant D, is defined as the logarithm of the fundamental unit. Thus, sequences of discriminants D_n, where ω_n has a bounded period length have regulators of order $O(\log D_n)$. Examples like Shanks' sequence have regulators of order $O((\log D_n)^2)$.

Several people have since generalized Shanks' sequence. They include Hendy [1974], Bernstein [1976a; 1976b; 1976c], Azuhata [1984; 1987], and Levesque and Rhin [1986]. A more synthetic account was given in [Williams 1985]. The most general form was presented in [Williams 1995] as

$$D_n = \big(qrx^n + \mu(x^k - \lambda)/q\big)^2 + 4\lambda r x^n,$$

with $\mu, \lambda \in \{-1, 1\}$ and $rq \,\big|\, x^k - \lambda$. The automata of Raney were used in [van der Poorten 1994] to provide an alternate way of constructing ω_n and ε_n.

Kaplansky [1998] coined the terms "sleepers", a sequence of discriminants whose period lengths are bounded, "creepers", a sequence whose lengths gently[1] go to infinity, and "leapers", the generic discriminants whose period lengths increase exponentially.

By selecting a sequence of discriminants from families of sleepers appropriately, one can form a sequence of discriminants with linear period length. These are known as "beepers" and can be found in [Mollin and Cheng 2002; van der Poorten 1999; Williams and Buck 1994]. Since these discriminants are selected from sleepers they have a regulator of order $O(\log D_n)$.

These two ideas were used simultaneously by Madden [2001], who explicitly constructed a sequence of discriminants whose continued fraction expansions possessed slowly growing period lengths. These examples were distinct from the known creepers since they were not polynomially parametrized. However, they can be viewed as selecting specific discriminants from various families we will construct here, much as beepers are specially selected sleepers.

We define a *creeper* to be an infinite family of discriminants D_n, such that $f(X, n) \in \mathbb{Q}[X, X^n]$ and for a fixed $x \in \mathbb{Z}$ we have $D_n = f(x, n)$ satisfying

$$lp(\omega_n) = an + b \quad \text{with} \quad a,\ b \in \mathbb{Q} \qquad \text{and} \qquad R(D_n) = \Theta(n^2).$$

Kaplansky [1998] made several conjectures about creepers which are quadratic in x^n. He suggested that every such creeper could be written as $D_n = A^2x^{2n} + Bx^n + C^2$ with $A, B, C \in \mathbb{Q}$. Each of the examples upon which these conjectures were based has a principal ideal whose norm is a fixed power of x. Consequently, we define a *kreeper* to be an infinite sequence of discriminants D_n such that

(1) $D_n = A^2x^{2n} + Bx^n + C^2$, where $A, B, C \in \mathbb{Q}$, and $x \in \mathbb{Z}^+$.

(2) $lp(\omega_n) = an + b$, where a and b are rational numbers.

(3) In the principal cycle there exists an element whose norm is x^g for some g fixed independently of n.

Note that the existence of some $Q_h = x^g$ implies $R(D_n) = \Theta(n^2)$. In other words, every kreeper is also a creeper. A proof of this is given in [Patterson 2003, Theorem 17]. Indeed many details are excluded here, and can be found in the same reference.

The main results here are the following.

Theorem 1.1. *Any kreeper D_n can be written as*

$$d^2D_n = c^2\left(\left(qrx^n + (mz^2x^k - ly^2)/q\right)^2 + 4ly^2rx^n\right), \tag{1-2}$$

[1]By "gently" he meant that the periods could be written in an arithmetic progression involving a parameter n used in the presentation of the family.

where each term in the above equation is an element of $\mathbb{Z}$, *the terms* r, l, m *are squarefree*, r, x *are positive*, *and the following conditions hold*:

$$(qrx, mlzy) = 1\,, \quad (qr, x) = 1\,, \quad (mz, ly) = 1\,, \qquad \text{(1-3)}$$
$$q \,\big|\, mz^2x^k - ly^2\,, \quad c^2rly^2mz^2 \,\big|\, d^2D_n.$$

Theorem 1.2. *Any sequence of discriminants given by* (1-2) *and satisfying the conditions* (1-3) *as above must in fact be a kreeper.*

As a final introductory remark, we mention that Shanks' sequence has also been generalised to certain cubic fields with unit rank one, see [Adam 1995; 1998; Levesque and Rhin 1991].

2. Preliminary results

Removing nonpositive partial quotients from a continued fraction is not difficult, however removing a fractional partial quotient is, in general, quite difficult. The following few results do provide some assistance. Multiplication can be accomplished via

$$x[\,a\,,\,b\,,\,c\,,\,d\,,\,\ldots\,] = [\,ax\,,\,b/x\,,\,xc\,,\,d/x\,,\,\ldots\,],$$

which leads to:

Lemma 2.1 (Folding Lemma [Mendès France 1973]). *Let* $x/y = [a_0, a_1, \ldots, a_h]$ *with* $(x, y) = 1$, *and denote the sequence* $a_1, \ldots, a_h$ *by* $\overrightarrow{w}$ (*where* $\overleftarrow{w}$ *corresponds to the sequence* $a_h, \ldots, a_1$). *Then*

$$\frac{x}{y} + \frac{(-1)^h}{cy^2} = \big[\,a_0\,,\,\overrightarrow{w}\,,\,c - y'/y\,\big] = [\,a_0\,,\,\overrightarrow{w}\,,\,c\,,\,-\overleftarrow{w}\,],$$

where $y/y' = [\,a_h\,,\,\ldots\,,\,a_1\,]$, $(y, y') = 1$, $y' > 0$.

This result is more than just a novelty. Besides our use of it here, in [van der Poorten 2002] it is used to rediscover the symmetry formulas.

A result which will be pivotal to our expansions later on is the following simple lemma.

Lemma 2.2. *If* $x/y = [\,a_0\,,\,\overrightarrow{w}\,]$, *where* $\overrightarrow{w}$ *is defined as above then*

$$x/y + \gamma = \left[\,a_0\,,\,\overrightarrow{w}\,,\,\frac{(-1)^h}{\gamma y^2} - \frac{b}{y}\,\right],$$

where b *is equal to* $(-1)^{h+1}/x$ *modulo* y.

Proof. Using the Folding Lemma we obtain,

$$\frac{x}{y} + \gamma = \frac{x}{y} + \frac{(-1)^h}{y^2}(-1)^h\gamma y^2 = \left[\,a_0\,,\,\overrightarrow{w}\,,\,\frac{(-1)^h}{\gamma y^2} - \frac{b}{y}\,\right],$$

where b satisfies $xb - cy = (-1)^{h+1}$, which implies $b \equiv (-1)^{h+1}/x \pmod y$.

□

If the minimal polynomial of α is $x^2 - tx + n$ then a typical line in the continued fraction expansion of α appears as

$$\frac{\alpha + P_h}{Q_h} = a_h - \frac{(\overline{\alpha} + P_{h+1})}{Q_h},$$

where $P_0 = 0$, $Q_0 = 1$ and $\overline{\alpha}$ represents the nontrivial automorphism of the quadratic number field[2]. We then have

$$P_{h+1} = a_h Q_h - P_h - t \quad \text{and} \quad Q_{h+1} = -\frac{n + P_h(P_h + t)}{Q_h}$$

A quadratic irrational α is called *reduced* if $\alpha > 1$ and $-1 < \overline{\alpha} < 0$. An integral ideal $\mathfrak{a}$ is called *primitive* if it is not divisible by any element of $\mathbb{Z}$. An integral ideal $\mathfrak{a}$ is *reduced* if there does not exist any nonzero $\alpha \in \mathfrak{a}$ satisfying

$$|\alpha| < N(\mathfrak{a}) \quad \text{and} \quad |\overline{\alpha}| < N(\mathfrak{a}).$$

If $\mathfrak{a}$ is a primitive ideal such that $N(\mathfrak{a}) < \sqrt{D}/2$ then $\mathfrak{a}$ is reduced. From now on, ideal will mean "integral ideal".

One of the uses of the continued fraction expansion of a quadratic irrational is the determination of the fundamental unit. Rather than keeping track of the convergents, this can be done via the following result.

Proposition 2.3. *If* $\alpha = [\, a_0\, , a_1\, , \ldots , a_i\, , \alpha_{i+1}\,]$ *and* $x_j/y_j = [\, a_0\, , \ldots , a_j\,]$, *where* $(x_j, y_j) = 1$, *then*

$$\alpha_1 \alpha_2 \ldots \alpha_{h+1} = (-1)^{h+1}(x_h - y_h \alpha)^{-1}.$$

Corollary 2.4. *If* $\mathbb{O}$ *is an order of* $\mathbb{Q}(\sqrt{D})$ *and* α_i, $1 \le i \le h+1$ *represents a system of reduced elements in any cycle of quadratic irrationals in* $\mathbb{O}$, *then* $\varepsilon = \prod_{i=1}^{h+1} \alpha_i$ *is the fundamental unit of* $\mathbb{O}$.

Such a cycle of quadratic irrationals is produced by the continued fraction expansion. To be more precise, if $\mathfrak{a}_0 = Q_0\mathbb{Z} + (P_0 + \omega)\mathbb{Z}$ is an ideal of $\mathbb{O}$ then the continued fraction expansion of $(\omega + P_0)/Q_0$ produces a sequence of complete quotients $(\omega + P_i)/Q_i$ such that the ideals associated to each complete quotient, that is $\mathfrak{a}_i = Q_i\mathbb{Z} + (\omega + P_i)\mathbb{Z}$, are all equivalent to $\mathfrak{a}_0$.

Later we will need to transfer results from one order to another, where the following proposition will be useful.

[2]The P_h, Q_h appearing here are not, in general, the same as those used in [Perron 1950]

Proposition 2.5. *Let $\mathcal{O}_1$, $\mathcal{O}_2$ be two orders of a real quadratic field given by $\mathcal{O}_1 = \mathbb{Z}[f\omega']$, $\mathcal{O}_2 = \mathbb{Z}[g\omega']$ then*

$$R(\mathcal{O}_1) > \frac{(g, f)}{2g} R(\mathcal{O}_2).$$

Let $\mathfrak{a}$ be any ideal of the order $\mathcal{O}$ having conductor f in K, and suppose that $(N(\mathfrak{a}), f) = 1$. Then $\mathfrak{a} = (t)\mathfrak{r}\mathfrak{s}$, where $t \in \mathbb{Z}$ and any prime ideal divisor of $\mathfrak{r}$ lies over a prime which ramifies and any prime ideal divisor of $\mathfrak{s}$ lies over a prime which splits in $\mathcal{O}$. Furthermore, $\mathfrak{r}$ and $\mathfrak{s}$ are primitive. We will denote by $s(\mathfrak{a})$ the ideal $\mathfrak{s}$. Note that if $t = 1$ (in which case $\mathfrak{a}$ is primitive) then $\mathfrak{a}^2 = (r)\mathfrak{s}^2$, where r is squarefree and $r \mid D$. Also note that $N(s(\mathfrak{a})) = N(s(\bar{\mathfrak{a}}))$.

We now introduce a generalisation of a result of Yamamoto [1971].

Definition 1. Let $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ be invertible ideals of an order $\mathcal{O}$. We say that $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ are *independent* in $\mathcal{O}$ if whenever there exist nonzero integers u, v and $\alpha_i, \beta_i \in \mathbb{Z}^+$ $(i = 1, \dots, n)$ such that

$$(u) \prod_{i=1}^{n} \mathfrak{b}_i^{\alpha_i} = (v) \prod_{i=1}^{n} \mathfrak{b}_i^{\beta_i}$$

then $\alpha_i = \beta_i$ $(i = 1, \dots, n)$.

A sufficient condition for the independence of two ideals $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathcal{O}$ is given in Theorem 2.7, which needs the following Lemma.

Lemma 2.6. *If $\mathfrak{b}$ and $\mathfrak{c}$ are dependent in $\mathcal{O}$ then there exist nonzero integers u, v and nonnegative integers m, n, with $m + n > 0$, such that*

$$(u)\mathfrak{b}^m = (v)\mathfrak{d}^n,$$

where $\mathfrak{d}$ is equal to $\mathfrak{c}$ or $\bar{\mathfrak{c}}$.

Theorem 2.7. *If $\mathfrak{b}$ and $\mathfrak{c}$ are dependent in $\mathcal{O}$ and $(N(\mathfrak{b})N(\mathfrak{c}), f) = 1$ then for some nonnegative integers m, n, with $m + n > 0$ we have*

$$N(s(\mathfrak{b}))^m = N(s(\mathfrak{c}))^n.$$

Proof. By the preceding Lemma, we know there must exist integers m, n, u, v such that $(u)\mathfrak{b}^m = (v)\mathfrak{d}^n$, where m, n are nonnegative and at least one of m, n is positive. Then we can write $(ut_1^m)\tilde{\mathfrak{b}}^m = (vt_2^n)\tilde{\mathfrak{d}}^n$, where $\tilde{\mathfrak{b}}$ and $\tilde{\mathfrak{d}}$ are primitive.

The condition $(f, N(\mathfrak{b})N(\mathfrak{c})) = 1$ and primitivity allow us to write

$$(ut_1^m)\tilde{\mathfrak{b}}^m = (ut_1^m)r(\mathfrak{b})^m s(\mathfrak{b})^m \quad \text{and} \quad (vt_2^n)\tilde{\mathfrak{d}}^n = (vt_2^n)r(\mathfrak{d})^n s(\mathfrak{d})^n.$$

Squaring these yields

$$(u^2 r_1^m t_1^{2m}) s(\mathfrak{b})^{2m} = (v^2 r_2^n t_2^{2n}) s(\mathfrak{d})^{2n},$$

where r_1 and r_2 divide D_n. Dividing out any common factors of $u^2 r_1^m t_1^{2m}$ and $v^2 r_2^n t_2^{2n}$ provides coprime integers u_1, v_1 such that

$$(u_1)\,(s(\mathfrak{b}))^{2m} = (v_1)\,(s(\mathfrak{d}))^{2n}\,.$$

Let $\mathfrak{p}^{\alpha} \,\|\, s(\mathfrak{b})$ and $\mathfrak{p}^{\beta} \,\|\, s(\mathfrak{d})$. If $\mathfrak{p}^{2m\alpha} \nmid s(\mathfrak{d})$ then $N(\mathfrak{p}) \mid v_1$ implies that $\bar{\mathfrak{p}} \mid (v_1)$. The coprimality of u_1 and v_1 means $\bar{\mathfrak{p}} \mid (s(\mathfrak{b}))^{2m}$, which gives $N(\mathfrak{p}) \mid s(\mathfrak{b})$, which is impossible because $s(\mathfrak{b})$ is primitive.

Hence, $\mathfrak{p}^{2m\alpha} \mid s(\mathfrak{d})$, which means that $2m\alpha \le 2n\beta$. By symmetry, $2n\beta \le 2m\alpha$ and so $m\alpha = n\beta$. Thus, $s(\mathfrak{b})^m = s(\mathfrak{d})^n$ and we find $N(s(\mathfrak{b}))^m = N(s(\mathfrak{d}))^n$. □

Suppose that $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ are independent ideals of $\mathbb{O}$ of discriminant D and let

$$S = \left\{ \prod_{i=1}^{n} \mathfrak{b}_i^{\alpha_i} : \alpha_i \ge 0 \;\; i = 1, \dots, n \;\text{ and }\; \prod_{i=1}^{n} N(\mathfrak{b}_i)^{\alpha_i} < \sqrt{D}/2 \right\}$$

If $\mathfrak{a}_i = Q_{i-1}\mathbb{Z} + (\omega + P_{i-1})\mathbb{Z}$ is a reduced ideal then

$$\frac{\omega + P_{i-1}}{Q_{i-1}} > \frac{\sqrt{D}/2}{Q_{i-1}} = \frac{\sqrt{D}/2}{|N(\mathfrak{a}_i)|}.$$

Now suppose $(v)\mathfrak{a} \in S$, where $\mathfrak{a}$ is reduced and $\mathfrak{a} = \langle Q_{i-1}\,,\ \omega + P_{i-1}\rangle$. We have

$$\frac{\omega + P_{i-1}}{Q_{i-1}} > \frac{v^2\sqrt{D}/2}{\prod_{i=1}^{n} N(\mathfrak{b}_i)^{\alpha_i}} \ge \frac{\sqrt{D}/2}{\prod_{i=1}^{n} N(\mathfrak{b}_i)^{\alpha_i}}.$$

Theorem 2.8. *Let $\mathbb{O}_1, \mathbb{O}_2, \dots,$ be a sequence of orders, each of discriminant D_i, where $D_i < D_{i+1}$. Suppose further that in each $\mathbb{O}_i$ there exists an independent set of principal ideals $\{\mathfrak{b}_{i,j} : (j = 1, \dots, n)\}$ such that $N(\mathfrak{b}_{i,j})$ is fixed for each value of i. Then*

$$R(D_i) \gg (\log D_i)^{n+1}.$$

See [Patterson 2003] for a proof.

3. Basic observations on kreepers

Given discriminants of the form $D_n = U^2x^{2n} + Vx^n + W^2$, where $U, V, W \in \mathbb{Q}$, there is no loss in generality in supposing that x is not a power. We may write our discriminants D_n as

$$D_n = \frac{c^2}{d^2}\big[(Ax^n + C)^2 + 4Gx^n\big], \qquad \text{where} \quad G = (B - 2AC)/4, \tag{3-1}$$

for $A, B, C, G, c, d \in \mathbb{Z}$ and $(c, d) = 1$.

Any common factors of C and x can be moved into the square divisors. By considering

$$W := \max_{i\in\mathbb{N}}(x^i, C)\,, \quad m := \min\{i \in \mathbb{N} : (x^i, C) = W\}\,, \quad v := n-2m,$$

so that $(x, C/W) = 1$, we have

$$D_n = \frac{c^2W^2}{d^2}\left(\left(AW\frac{x^{2m}}{W^2}x^v + \frac{C}{W}\right)^2 + 4G\frac{x^{2m}}{W^2}x^v\right) = \frac{c^2W^2}{d^2}\big((\bar{A}x^v+\bar{C})^2+4\bar{G}x^v\big),$$

where $(x, \bar{C}) = 1$ and $\bar{A}, \bar{C}, \bar{G} \in \mathbb{Z}$. Any square factors of (A^2, C^2, G) can also be removed, so that without loss of generality we may suppose that

$$D_n = \left(\frac{c}{d}\right)^2\big((Ax^n+C)^2+4Gx^n\big)$$

with $A, C, G \in \mathbb{Z}$, $(x, C) = 1$ and (A^2, C^2, G) is squarefree.

The next few results don't use any of the properties of kreepers, we are merely interested in determining an explicit formula for $(Ax^n+C)^2+4Gx^n$. Consequently, we define

$$E_n := (Ax^n+C)^2+4Gx^n, \tag{3-2}$$

where $(x, C) = 1$ and (A^2, C^2, G) is squarefree. The first result is a representation of A, C, G.

Theorem 3.1. *Given E_n as in* (3-2) *and the conditions on A, C, G, x stated above, we have that*

$$E_n = \big(qrNx^n + P(M-L)/q\big)^2 + 4rLNPx^n, \tag{3-3}$$

where $r, q, P, N \in \mathbb{Z}^+$, $M, L \in \mathbb{Z}$, and the following conditions are satisfied:

(1) r *is squarefree*, (2) $(P, rqN) = 1$, (3) $rq \mid M-L$,
(4) $(M, L) = 1$, (5) $(rq, ML) = 1$, (6) $(r, LNP) = 1$.

Proof. The selections we make are

$$r := (A, C, G)\,, \quad N := \left(\frac{A}{r}, \frac{G}{r}\right)\,, \quad q := \frac{A}{rN}\,,$$
$$P := \left(\frac{C}{r}, \frac{G}{Nr}\right)\,, \quad L := \frac{G}{rNP}\,, \quad M := \frac{AC+G}{rNP}.$$

These selections make (3-2) and (3-3) equivalent, so it only remains to show that the conditions indicated hold. This is not difficult; see [Patterson 2003]. □

Our next objective is to determine the terms that divide E_n and those that are coprime to E_n.

Lemma 3.2. $(E_n, xNP) = 1$.

Proof. See [Patterson 2003]. □

Theorem 3.3. *For the family of discriminants given by* (3-3) *with* $n \in I$ (*an infinite subset of* $\mathbb{N}$) *there exists an infinite set* $I' \subseteq I$ *such that for every* $n \in I'$ *we have* $M = v'mz^2Z$, $L = vly^2Y$ *with* Z, Y, z, y *positive and*

$$ml \mid E'_n, \quad vv'zy \mid F_n, \quad vv' \mid 2, \quad (ZY, E'_n) = 1, \quad \left(\frac{F_n}{zy}, mlZY\right) = 1$$

and $E_n = F_n^2 E'_n$, *where* E'_n *is squarefree. Here* m, l, z, y, Z, Y *are the same for all* $n \in I'$.

Proof. Define $y_n^2 v_n := (F_n^2, L)$, where v_n is squarefree. Next, write

$$\tilde{F}_n = \frac{F_n}{y_n v_n} \quad \text{so that} \quad \left(v_n \tilde{F}_n{}^2, \frac{L}{v_n y_n^2}\right) = 1,$$

and

$$E_n = y_n^2 v_n^2 \tilde{F}_n{}^2 E'_n = S_n^2 + 4r v_n y_n^2 \frac{L}{v_n y_n^2} NPx^n, \tag{3-4}$$

where

$$S_n = qrNx^n + \frac{P(M-L)}{q}.$$

It is not difficult to show that $v_n \mid 2$; see [Patterson 2003].

We now investigate the factors of $L/v_n y_n^2$. (Since $(F_n/(v_n y_n), L/(v_n y_n^2)) = 1$ it follows that for any prime factor, p, of $L/v_n y_n^2$, we have $p \mid S_n/(v_n y_n)$ if and only if $p \mid E'_n$). We define l_n to be the product of all prime powers p^α of $L/v_n y_n^2$ that satisfy $p^\alpha \,\|\, L/v_n y_n^2$ and $p \mid E'_n$, and Y_n to be the product of all prime powers p^β that satisfy $p^\beta \,\|\, L/v_n y_n^2$ and $(E'_n, p) = 1$. We also absorb the sign of L into l_n. Clearly, $L/v_n y_n^2 = l_n Y_n$. It is straightforward to show that l_n is squarefree and so we find $L = v_n l_n y_n^2 Y_n$, $(Y_n, E'_n) = 1$, $v_n y_n \mid F_n$, $l_n \mid E'_n$, $v_n \mid 2$.

Equation (3-4) can also be written as

$$E_n = z_n^2 (v'_n)^2 \tilde{F}_n{}^2 E'_n = (S'_n)^2 + 4r v'_n z_n^2 \frac{M}{v'_n z_n^2} NPx^n,$$

where v'_n is squarefree and

$$S'_n = qrNx^n - P(M-L)/q \quad \text{and} \quad z_n^2 v'_n = (F_n^2, M).$$

By similar reasoning we get $M = m_n v'_n z_n^2 Z_n$, $(Z_n, E'_n) = 1$, $v'_n z_n \mid F_n$, $m_n \mid E'_n$, $v'_n \mid 2$, and $4 \nmid v_n v'_n$ because $(M, L) = 1$.

As there are only a finite number of choices for m_n, l_n, z_n, y_n, v_n, v'_n, Z_n, Y_n for fixed values of L and M there must exist some infinite set $I' \subseteq I$ for which

$$M = v'mz^2Z\,,\quad L = vly^2Y\,,\quad ml \,\big|\, E'_n\,,\quad zyvv' \,\big|\, F_n\,,\quad (ZY, E'_n) = 1\,,\quad vv' \,\big|\, 2.$$

The coprime conditions all follow easily. □

This completes our investigation of E_n; we have determined the factors coprime to E_n and those which divide each part of it.

4. Independent ideals in kreepers

Completing the square on A^2E_n gives,

$$A^2E_n = \big(A^2x^n + B/2\big)^2 - 4GH$$

and $B/2$ is an integer because $4 \,\big|\, (B - 2AC)$. In terms of the constants found in the previous section, this equation becomes

$$q^2F_n^2E'_n = \big(q^2rNx^n + P(L+M)\big)^2 - 4P^2LM. \tag{4-1}$$

We define $\omega'_n := (\sigma'_n - 1 + \sqrt{E'_n})/\sigma'_n$, where E'_n is squarefree and $\sigma'_n = 2$ if $E'_n \equiv 1 \pmod 4$ and $\sigma'_n = 1$ otherwise. We further define $\mathbb{O}_n = \mathbb{Z} + \omega_n\mathbb{Z}$, where

$$\omega_n := \frac{t_n + \sqrt{D_n}}{2} \qquad \text{and} \qquad t_n = \begin{cases} 0 & \text{if } D_n \equiv 0 \pmod 4, \\ 1 & \text{if } D_n \equiv 1 \pmod 4. \end{cases}$$

Proposition 4.1. $\mathbb{O}_n = \mathbb{Z} + \dfrac{cF_n\sigma'_n}{2d}\omega'_n\mathbb{Z}$.

Our objective now is to find an ideal arising from (4-1) which has a norm coprime to the conductor of some order.

Proposition 4.2. *For each n, there exists an element $\alpha_n \in \mathbb{Z} + f_1\omega'_n\mathbb{Z}$, where $f_1 = F_n/(\lambda zy)$, and $\lambda = 1$ or 2, and $(N(\alpha_n), f_1) = 1$.*

Proof. We take α_n as

$$\alpha_n := \begin{cases} s_n/2 - \dfrac{qF_n}{2zy} + \dfrac{qF_n}{zy}\omega'_n & \text{if } 2 \nmid q \text{ and } 2 \nmid F_n/zy, \\[2ex] s_n/2 + \dfrac{qF_n}{2zy}\sqrt{E'_n} & \text{otherwise.} \end{cases}$$

The remaining details are in [Patterson 2003]. □

Bounded norms in kreepers. By the definition of a kreeper, the continued fraction expansion of ω_n has some $Q_i = x^g$ with g fixed independently of n. In other words, there exists some $\mu_n \in \mathbb{O}_n$ such that $N(\mu_n) = x^g$. Also recall Proposition 4.1, which states that $\mathbb{O}_n = \mathbb{Z} + f_2\omega'_n\mathbb{Z}$, where $f_2 = cF_n\sigma'_n/2d$. Let $f_3 = (f_1, f_2)$, so that α_n, μ_n are both contained in $\mathbb{O}^*_n := \mathbb{Z} + f_3\omega'_n\mathbb{Z}$. We know that $(x, E_n) = 1$, so then $(x, F_n/zy) = 1$. Hence,

$$(N(\alpha_n), f_3) = (N(\mu_n), f_3) = 1.$$

Thus in the order $\mathbb{O}^*_n$ we have the two principal ideals $\mathfrak{a}_n = (\alpha_n)$ and $\mathfrak{b}_n = (\mu_n)$ (whose norms are fixed for infinitely many $n \in I$). Both ideals have norms coprime to the conductor. Hence, we may apply Theorems 2.7 and 2.8.

Proposition 4.3. *If $\mathfrak{a}_n$ and $\mathfrak{b}_n$ are independent ideals in $\mathbb{O}^*_n$ then*

$$R(\mathbb{O}_n) \gg (\log D_n)^3.$$

Proof. By Theorem 2.8,

$$R(\mathbb{O}^*_n) \gg \big(\log \Delta(\mathbb{O}^*_n)\big)^3 = \big(\log(f_3^2\Delta(\mathbb{O}'_n))\big)^3,$$

where $\mathbb{O}'_n := \mathbb{Z} + \omega'_n\mathbb{Z}$ and $\Delta(\mathbb{O})$ denotes the discriminant of the order $\mathbb{O}$. By Proposition 2.5,

$$R(\mathbb{O}_n) > \frac{(f_3, f_2)}{2f_3}R(\mathbb{O}^*_n) \text{ implies } R(\mathbb{O}_n) > \tfrac{1}{2}R(\mathbb{O}^*_n) \gg \big(\log(f_3^2\Delta(\mathbb{O}'_n))\big)^3. \tag{4-2}$$

Since $f_3 = F_n(2d, c\sigma'_n zy\lambda)/2\lambda zyd$ we find

$$\big(\log f_3^2\Delta(\mathbb{O}'_n)\big)^3 \gg \big(\log F_n^2\Delta(\mathbb{O}'_n)\big)^3 \gg \big(\log F_n^2E'_n\big)^3 \gg (\log D_n)^3.$$

Hence, from (4-2), we have that $R(\mathbb{O}_n) \gg (\log D_n)^3$. □

Consequently, in order for the sequence of discriminants $\{D_n\}$ to be a kreeper, $\mathfrak{a}_n$ and $\mathfrak{b}_n$ must be dependent ideals in $\mathbb{O}^*_n$. By Theorem 2.7 there must exist nonnegative integers e and f, with e and f not both 0 such that $N(s(\mathfrak{a}_n))^e = N(s(\mathfrak{b}_n))^f$.

With not too much effort (see [Patterson 2003, Chapter V, §15]) one shows that $N(s(\mathfrak{a}_n)) = P^2ZY$ and $N(s(\mathfrak{b}_n)) = x^g$. If $e = 0$ then $x^{gf} = 1$ and since in this case we must have $f > 0$ we find $x = 1$, which is impossible. On the other hand, if $f = 0$ then Y, Z and P are all ± 1, in which case

$$\frac{q^2E_n}{(zy)^2} = s_n^2 \pm 4mlvv'$$

But $mlvv'$ divides $E_n/(zy)^2$, which by a result of Schinzel [1961] implies that the period length of $q^2E_n/(zy)^2$ is bounded for all n. In the terminology of [Kaplansky 1998], this says that $q^2E_n/(zy)^2$ is a sleeper. It is shown in [Patterson

2003, Chapter III] that any rational multiple of a sleeper is again a sleeper. Hence, $c^2E_n/d^2 = D_n$ must also be a sleeper; in other words, D_n is not a kreeper. It follows that if D_n is to be a kreeper we must have $N(s(\mathfrak{a}_n))^e = N(s(\mathfrak{b}_n))^f$ with e, f each being positive. Since $(x, P) = 1$ we must have $P = \pm 1$. Hence, we may replace PM by M and PL by L and all the previous conditions hold. Taking $d := (gf, e)$, $k := gf/d$, $h := e/d$ gives,

$$(ZY)^h = x^k\,, \qquad (h, k) = 1,$$

which implies that $x = R^h$, $ZY = R^k$. Hence, $h = 1$ and $ZY = x^k$ because x is not a power. From Theorem 3.1 we have $(Z, Y) = 1$; hence

$$Z = U^k\,, \quad Y = T^k\,, \quad \text{where } (U, T) = 1.$$

The objective of the next 2 sections will be to show that $T = 1$ and $U = x$.

5. Part of the continued fraction expansion of ω_n

There is no longer any need to distinguish between factors of E'_n and F_n, which means that we may absorb the terms v and v' into l and m respectively. The form of a kreeper is now given by

$$D_n = \Big(\frac{c}{d}\Big)^2 \Big(\big(qrN(UT)^n + (z^2mU^k - y^2lT^k)/q\big)^2 + 4rNly^2T^{k+n}U^n\Big)\,, \tag{5-1}$$

where m, l, r are squarefree and

$$(qr, UT) = 1\,,\ \ (qrNUT, yzml) = 1\,,\ \ (Tyl, mzU) = 1\,,\ \ qr \,\big|\, z^2mU^k - y^2lT^k$$

and for every $n \in I$,

$$yl \,\big|\, q^2rNx^n + z^2mU^k \quad \text{and} \quad mz \,\big|\, q^2rNx^n + y^2lT^k, \tag{5-2}$$

and $N > 0$, $x = UT$. Let μ be the least positive difference of any two integers in I. Then ν, $\nu + \mu \in I$ for some ν, and

$$yl \,\big|\, q^2rNx^\nu + z^2mU^k \quad \text{and} \quad yl \,\big|\, q^2rNx^{\nu+\mu} + z^2mU^k,$$

which means $yl \,\big|\, q^2rNx^\nu(x^\mu - 1)$, so $yl \,\big|\, x^\mu - 1$ because $(yl, qrNx) = 1$. By symmetry, $zm \,\big|\, x^\mu - 1$. Thus, $ylmz \,\big|\, x^\mu - 1$. Hence the conditions (5-2) become

$$yl \,\big|\, q^2rN(UT)^\nu + z^2mU^k\,,\ mz \,\big|\, q^2rN(UT)^\nu + y^2lT^k\,,\ (TU)^\mu \equiv 1 \pmod{ylmz}$$

for any $n \in I$, such that $n \equiv \nu \pmod{\mu}$. Since the signs of m and l have not yet been specified, there is no loss of generality in supposing that q, U, T are all positive.

Since (5-1) can also be represented as

$$(5\text{-}3)\quad D_n = \left(\frac{c}{d}\right)^2 \left(\left(qrN(UT)^n - (z^2mU^k - y^2lT^k)/q\right)^2 + 4rNmz^2T^nU^{k+n}\right)$$

we may assume without loss of generality that $U > T$.

Some notation. We now rewrite equations (5-1) and (5-3) as

$$D_n = \left(\frac{s_1}{d}\right)^2 + 4\left(\frac{c}{d}\right)^2 rNly^2T^{k+n}U^n = \left(\frac{s_2}{d}\right)^2 + 4\left(\frac{c}{d}\right)^2 rNmz^2T^nU^{k+n},$$

where

$$s_1 = cqrNx^n + c(z^2mU^k - y^2lT^k)/q,\;\; s_2 = cqrNx^n - c(z^2mU^k - y^2lT^k)/q.$$

We choose an infinite subset of I such that $t \equiv D_n \pmod 4$ is fixed. Then we take $S_1 := (s_1 - td)/2$ and $S_2 := (s_2 - td)/2$. We also write

$$\alpha = \omega_n + S_1/d\ ,\quad \overline{\alpha} = \overline{\omega}_n + S_1/d \quad \text{and} \quad \beta = \omega_n + S_2/d\ ,\quad \bar{\beta} = \overline{\omega}_n + S_2/d.$$

Then

$$\alpha\overline{\alpha} = -c^2rNly^2T^kx^n/d^2 \quad \text{and} \quad \beta\bar{\beta} = -c^2rNmz^2U^kx^n/d^2.$$

Further, we have

$$q^2d^2D_n = (qs_3)^2 - 4c^2ml(zy)^2(UT)^k,$$

where

$$qs_3 = cq^2rNx^n + mz^2U^k + ly^2T^k.$$

Also of relevance will be $S_1 + S_2 + td = Ax^n = cqrNx^n$.

We now detail an initial segment of the continued fraction expansion of ω_n. In the case of $T > 1$, this segment will have length $O(n^{1+\varepsilon})$, hence the entire expansion could not satisfy $lp(\omega_n) = an + b$, as required by kreepers.

Before commencing we need to determine the common factors between some of the terms. First, we define $g := (s_1, s_2, d)$. It is not to difficult to show that $(z, s_1/c)$ and $(y, s_2/c)$ each divides 2. We also define $d_y := (S_1, d)$, $d'_y := (s_1/g, d/g)$ and $\tau_y := d_y/d'_y$. It follows easily that τ_y is an integer. Similarly, we define $d_z := (S_2, d)$, $d'_z := (s_2/g, d/g)$ and $\tau_z := d_z/d'_z$. Next, we write $d = \bar{d}d_zd_y$. Here are some simple results:

- $g \mid 2$, moreover, $g = 2$ if and only if $2 \mid d$.
- $\tau_z \mid g$ and $\tau_y \mid g$.
- $\tau_y d'_y \mid y$ and $\tau_z d'_z \mid z$; in other words $d_y \mid y$ and $d_z \mid z$.

Expansion of ω_n. The continued fraction expansion of ω_n begins as

$$\omega_n = \frac{(S_1+td)/d_y}{d/d_y} - (\overline{\omega}_n + S_1/d)$$

and $\big((S_1+td)/d_y, d/d_y\big) = 1$ by the definition of d_y. Hence we may apply Lemma 2.2, and find that after the expansion of $(S_1+td)/d$, of length h_0, a new complete quotient in the continued fraction expansion is

$$\frac{-(-1)^{h_0+1}}{(\overline{\omega}_n + S_1/d)(d/d_y)^2} - \frac{c_0}{d/d_y}, \tag{5-4}$$

where

$$c_0 \equiv \frac{(-1)^{h_0}}{(S_1+td)/d_y} \pmod{d/d_y}.$$

By choosing $(-1)^{h_0+1} = \operatorname{sign}(l)$ the element in (5-4) then becomes

(5-5)

$$\frac{\omega_n + S_1/d}{c^2rN\,|l|\,y^2T^kx^n/(d_y)^2} - \frac{c_0}{d/d_y} = \frac{\omega_n + S_1/d - c_0c^2rN\,|l|\,d_y(y/d_y)^2T^kx^n/d}{c^2rN\,|l|\,(y/d_y)^2T^kx^n}.$$

Now define: $s := \max_{i\in\mathbb{N}}\{(x^i, c)\}$ and $u := c/s$. Hence $(u, x) = 1$. Recall, $(x, qrzyml) = 1$ so then $(s, qrzyml) = 1$. We will denote the element in (5-5) as θ_{h_0}. From it, we can write

$$\theta_{h_0} = \frac{\omega_n + P_{h_0}}{Q_{h_0}} = \frac{A_0}{B_0} - \frac{\overline{\beta}}{c^2rN\,|l|\,(y/d_y)^2T^kx^n}, \tag{5-6}$$

where

$$A_0 = cqrNx^n - c_0c^2rN\,|l|\,d_y(y/d_y)^2T^kx^n, \quad B_0 = c^2drN\,|l|\,(y/d_y)^2T^kx^n.$$

Next, we define $\Delta_0 := (A_0, B_0)$. We need to determine Δ_0 before we can apply Lemma 2.2 again. One finds,

$$\Delta_0 = crNx^n\delta d_z \qquad \text{and} \qquad B_0/\Delta_0 = c/\delta d/d_z\,|l|\,(y/d_y)^2T^k.$$

From (5-6), by applying Lemma 2.2, we find the next partial quotients are those of the continued fraction expansion of A_0/B_0 of length p_0. By choosing $(-1)^{p_0+1} = \operatorname{sign}(m)$, the next element in the continued fraction expansion is θ_{h_1}, where $h_1 := h_0 + p_0$,

$$\theta_{h_1} = \frac{c^2rN\,|l|\,(y/d_y)^2T^kx^n}{-\overline{\beta}\operatorname{sign}(m)\,(B_0/\Delta_0)^2} - \frac{c_1}{B_0/\Delta_0}$$

and $c_1 \equiv -\operatorname{sign}(m)\,\Delta_0/A_0 \pmod{B_0/\Delta_0}$. We can write

$$\theta_{h_1} = \frac{\omega_n + S_2/d - c_1(c/\delta)|m|z(z/d_z)U^k/d}{(c/\delta)^2|ml|(y/d_y)^2(z/d_z)^2(UT)^k}. \tag{5-7}$$

Finding other complete quotients. The set of conditions

$$u_{2i-1}\,\big|\,u\,,\;\; z_{2i-1}\,\big|\,z/d_z\,,\;\; z'_{2i-1}\,\big|\,z\,,\;\; (c_{2i-1}, sT) = 1\,,\;\; (z_{2i-1}y_{2i-1}, u/u_{2i-1}) = 1\,,$$
$$m_{2i-1} \in \{1, |m|\}\,,\;\; l_{2i-1} \in \{1, |l|\}\,,\;\; r_{2i-1} \in \{1, r\}\,,\;\; y_{2i-1}\,\big|\,y/d_y\,,\;\; u'_{2i-1}\,\big|\,u.$$

will be denoted by C_{2i-1}. The set of conditions C_{2i} are the same as C_{2j-1} (with $2j-1$ replaced by $2i$) except that instead of requiring $z'_{2j-1}\,\big|\,z$, we need $y'_{2i}\,\big|\,y$.

Theorem 5.1. *Suppose there exists a complete quotient $(\omega_n + P_{h_{2i-1}})/Q_{h_{2i-1}}$ satisfying*

$$P_{h_{2i-1}} = S_2/d - sm_{2i-1}r_{2i-1}u_{2i-1}u'_{2i-1}z_{2i-1}z'_{2i-1}c_{2i-1}U^{ki}/d \tag{5-8a}$$

$$Q_{h_{2i-1}} = r_{2i-1}l_{2i-1}m_{2i-1}(su_{2i-1}y_{2i-1}z_{2i-1})^2(UT)^{ki}, \tag{5-8b}$$

where $n > ki$ and the set of conditions C_{2i-1} are satisfied. Then there is a complete quotient $(\omega_n + P_{h_{2i}})/Q_{h_{2i}}$, where

$$P_{h_{2i}} = S_1/d - Nsr_{2i}l_{2i}u_{2i}u'_{2i}y_{2i}y'_{2i}c_{2i}U^{n-ki}T^{n+k}/d \tag{5-9a}$$

$$Q_{h_{2i}} = r_{2i}l_{2i}m_{2i}(su_{2i}y_{2i}z_{2i})^2NT^{n+k(i+1)}U^{n-ki} \tag{5-9b}$$

and the conditions C_{2i} are satisfied.

Observe that with appropriate selections, θ_{h_1} is a complete quotient satisfying (5-8) and the conditions C_1.

Proof. From (5-8), we find that the current line in the continued fraction expansion is

$$\frac{\omega_n + P_{h_{2i-1}}}{Q_{h_{2i-1}}} = \frac{cqrNx^n - sr_{2i-1}m_{2i-1}u_{2i-1}u'_{2i-1}z_{2i-1}z'_{2i-1}c_{2i-1}U^{ki}}{dr_{2i-1}m_{2i-1}l_{2i-1}(su_{2i-1}z_{2i-1}y_{2i-1})^2(UT)^{ki}} - \frac{(\overline{\omega}_n + S_1/d)}{r_{2i-1}m_{2i-1}l_{2i-1}(su_{2i-1}z_{2i-1}y_{2i-1})^2(UT)^{ki}}.$$

Now, define

$$A_{2i-1} := cqrNx^n - sr_{2i-1}m_{2i-1}u_{2i-1}u'_{2i-1}z_{2i-1}z'_{2i-1}c_{2i-1}U^{ki},$$
$$B_{2i-1} := dr_{2i-1}m_{2i-1}l_{2i-1}(su_{2i-1}z_{2i-1}y_{2i-1})^2(UT)^{ki},$$
$$\Delta_{2i-1} := (A_{2i-1}, B_{2i-1}).$$

The next few results aid in determining common factors.

Lemma 5.2. *If $A_{2i-1} = dP_{h_{2i-1}} + (s_1 + td)/2$ then $d_y u_{2i-1}l_{2i-1}y_{2i-1}\,\big|\,A_{2i-1}$.*

Lemma 5.3. *For any rational integers a, b, d, f such that $d \mid ab$ and $(f, d) = 1$ there exist rational integers x, y such that*

$$dxy = ab$$

and $x \mid a$, $y \mid b$, $(fx, b/y) = 1$.

Proof. Take $x = a/(a, d)$ and $y = b(a, d)/d$. □

Returning to the expansion of ω_n, we write

$$w_{2i-1} := (m_{2i-1}, u/u_{2i-1}) \quad \text{and} \quad e_{2i-1} := sd_y u_{2i-1} l_{2i-1} y_{2i-1} r_{2i-1} w_{2i-1}.$$

Note that if $n \ge ki$ then $w_{2i-1}u_{2i-1}sU^{ki}r_{2i-1} \mid A_{2i-1}$ and by Lemma 5.2 we get $d_y u_{2i-1} l_{2i-1} y_{2i-1} \mid A_{2i-1}$. Now we define $G_{2i-1} := A_{2i-1}/(U^{ki}e_{2i-1})$. It follows easily (see [Patterson 2003, Chapter 16]) that $(G_{2i-1}, sTz_{2i-1}m_{2i-1}/w_{2i-1}) = 1$. In summary,

$$\Delta_{2i-1} = U^{ki}e_{2i-1}(G_{2i-1}, \bar{d}d_z y_{2i-1}u_{2i-1}).$$

Since $A_{2i-1} = dP_{h_{2i-1}} + S_1 + td$, we have $(d, A_{2i-1}) = d_y$. Thus $(\bar{d}d_z, G_{2i-1}) = 1$. Hence, $\Delta_{2i-1} = U^{ki}e_{2i-1}\overline{\Delta}_{2i-1}$, where $\overline{\Delta}_{2i-1} := (y_{2i-1}u_{2i-1}, G_{2i-1})$.

From the complete quotient

$$\theta_{h_{2i-1}} = \frac{A_{2i-1}}{B_{2i-1}} - \frac{\overline{\alpha}}{r_{2i-1}l_{2i-1}m_{2i-1}(su_{2i-1}y_{2i-1}z_{2i-1})^2(UT)^{ki}}$$

we apply Lemma 2.2, so the next partial quotients are those of the expansion of A_{2i-1}/B_{2i-1} of length p_{2i-1}. The parity of p_{2i-1} is determined by $(-1)^{p_{2i-1}+1} = \operatorname{sign}(l)$. Following this, the next complete quotient is $\theta_{h_{2i}}$, where $h_{2i} := h_{2i-1} + p_{2i-1}$. By Lemma 2.2, $\theta_{h_{2i}}$ is equal to

$$\text{(5-10)} \quad \frac{r_{2i-1}l_{2i-1}m_{2i-1}(su_{2i-1}y_{2i-1}z_{2i-1})^2(UT)^{ki}}{-\overline{\alpha}(B_{2i-1}/\Delta_{2i-1})^2} - \frac{c_{2i}}{B_{2i-1}/\Delta_{2i-1}} = \frac{r_{2i-1}l_{2i-1}(\omega_n + S_1/d)(w_{2i-1}d_y\overline{\Delta}_{2i-1})^2}{s^2u^2rN\,|l|\,y^2m_{2i-1}z_{2i-1}^2U^{n-ki}T^{n+k(i+1)}} - \frac{c_{2i}}{B_{2i-1}/\Delta_{2i-1}},$$

where

$$c_{2i} \equiv -\operatorname{sign}(l)\,\Delta_{2i-1}/A_{2i-1} \pmod{B_{2i-1}/\Delta_{2i-1}}.$$

Also note that $sT \mid B_{2i-1}/\Delta_{2i-1}$ and $(A_{2i-1}/\Delta_{2i-1}, B_{2i-1}/\Delta_{2i-1}) = 1$ imply that $(c_{2i}, sT) = 1$.

According to Lemma 5.3 there exists y_{2i}, u_{2i} such that

$$y_{2i} \mid y/d_y\,, \qquad u_{2i} \mid u/w_{2i-1}\,, \qquad y_{2i}u_{2i}\overline{\Delta}_{2i-1} = \frac{y}{d_y}\frac{u}{w_{2i-1}}$$

and $(z_{2i-1}y_{2i}, u/(u_{2i}w_{2i-1})) = 1$. By taking $l_{2i} := |l|/l_{2i-1}$, $r_{2i} := r/r_{2i-1}$, $m_{2i} := m_{2i-1}$, $z_{2i} := z_{2i-1}$, $y'_{2i} := y/y_{2i-1}$, and $u'_{2i} := u/u_{2i-1}$, one finds that

$(y_{2i}z_{2i}, u/u_{2i}) = 1$. Moreover, the complete quotient $\theta_{h_{2i}}$ now satisfies (5-9) and the set of conditions C_{2i}. □

There is also an analogous result for $\theta_{h_{2i}}$.

Theorem 5.4. *Suppose there is a complete quotient $\theta_{h_{2i}} = (\omega_n + P_{h_{2i}})/Q_{h_{2i}}$ satisfying* (5-9) *and the set of conditions C_{2i}. Then there is a complete quotient $\theta_{h_{2i+1}}$, where*

$$P_{h_{2i+1}} = S_2/d - sm_{2i+1}r_{2i+1}u_{2i+1}u'_{2i+1}z_{2i+1}z'_{2i+1}c_{2i+1}U^{k(i+1)}/d$$
$$Q_{h_{2i+1}} = r_{2i+1}l_{2i+1}m_{2i+1}(su_{2i+1}z_{2i+1}y_{2i+1})^2(UT)^{k(i+1)}$$

and the set of conditions C_{2i+1} are satisfied.

Thus, from the complete quotient $\theta_{h_{2i-1}}$ we find another complete quotient $\theta_{h_{2i+1}}$ satisfying exactly the same requirements as $\theta_{h_{2i-1}}$. Moreover, only a bounded number of the these complete quotients are not reduced. More precisely,

$$\frac{A_{2i}}{B_{2i}} = \frac{k_1x^n - k_2U^{n-ki}T^{n+k}}{k_3T^{n+k(i+1)}U^{n-ki}} = \frac{k_1U^{ki}}{k_3T^{k(i+1)}} - \frac{k_2}{k_3T^{ki}} > 1$$

for some $i \geq W$, where W only depends on the parameters m, l, y, z, r, d, c, N, U, T.

Similarly,

$$\frac{A_{2i+1}}{B_{2i+1}} = \frac{k_1x^n - k_2U^{k(i+1)}}{k_3(UT)^{k(i+1)}} = \frac{k_1x^n - k'_2U^{ki}}{k'_3(UT)^{ki}} = \frac{k_1x^{n-ki}}{k'_3} - \frac{k'_2}{k'_3T^{ki}} > 1$$

for some V such that $n - ki \geq V$. Again, V depends only on the parameters m, l, y, z, r, d, c, N, U, T.

Since the pairs (P_{h_i}, Q_{h_i}) are all distinct for $i = 1, 2, \ldots, 2(n-V)/k + 1$, we see that

$$lp(\omega_n) > \sum_{j=W}^{(n-V)/k} p_{2j}.$$

Our interest now falls on the length of the expansion of A_{2i}/B_{2i}. Basically, since we have $\Theta(n)$ of these expansions, if the lengths are unbounded then, from above, the period length can not possibly be linear in n. In the next section we show that in order to have the length of A_{2i}/B_{2i} bounded for all i we must have $T = 1$.

6. The length of the continued fraction of A_{2i}/B_{2i}

Let i be fixed with $W \leq i \leq (n-V)/k$, so that $\theta_{h_{2i}}$ is reduced. We have

$$P_{h_{2i}} = S_1/d - Nsl_{2i}r_{2i}u_{2i}u'_{2i}y_{2i}y'_{2i}c_{2i}U^{n-ki}T^{n+k}/d = S_1/d - J/d.$$

As usual, we have

$$d^2(t-D_n)/4+d^2(tP_{h_{2i}}+P_{h_{2i}}^2)\equiv 0 \pmod{d^2Q_{h_{2i}}},$$

where in the above case

$$Q_{h_{2i}}=r_{2i}m_{2i}l_{2i}N(su_{2i}z_{2i}y_{2i})^2U^{n-ki}T^{n+k(i+1)}.$$

We can write this as

$$-Gx^n-tdJ-2S_1J+J^2\equiv 0 \pmod{d^2Q_{h_{2i}}}.$$

Modulo $T^{k(i+1)}$ we have

$$GU^{ki}\equiv -CNsl_{2i}r_{2i}u_{2i}u'_{2i}y_{2i}y'_{2i}c_{2i}T^k \pmod{T^{k(i+1)}}. \tag{6-1}$$

Returning now to A_{2i}/B_{2i}, we previously found that

$$\frac{A_{2i}}{B_{2i}}=\frac{AU^{ki}-Nsr_{2i}l_{2i}u_{2i}u'_{2i}y_{2i}y'_{2i}c_{2i}T^k}{dNl_{2i}r_{2i}m_{2i}(su_{2i}y_{2i}z_{2i})^2T^{k(i+1)}}.$$

Writing (6-1) as

$$CNsl_{2i}r_{2i}u_{2i}u'_{2i}y_{2i}y'_{2i}c_{2i}T^k=-GU^{ki}-f_{2i}T^{k(i+1)},$$

where $f_{2i}\in\mathbb{Z}$, we find,

$$\frac{A_{2i}}{B_{2i}}=\frac{c^2rNmz^2U^{k(i+1)}+f_{2i}T^{k(i+1)}}{CNdl_{2i}r_{2i}m_{2i}(su_{2i}y_{2i}z_{2i})^2T^{k(i+1)}}=\frac{1}{F_{2i}}\left(E\xi^{i+1}+f_{2i}\right),$$

where F_{2i} and E are bounded integers and $\xi=U^k/T^k$.

Depth of a sequence of rationals. We now provide an aside regarding the depth of $(a/b)^h$ as $h\to\infty$ for coprime integers a and b. The depth of the regular continued fraction expansion of $\alpha\in\mathbb{Q}$ is denoted by $\delta(\alpha)$. This is defined as the number of partial quotients in the even length continued fraction expansion of α.

Theorem 6.1. *If a and b are two coprime integers with $1<b<a$ then*

$$\lim_{i\to\infty}\delta\left((a/b)^i\right)=\infty$$

Whether this is so was asked by Mendès France and proved by Pourchet in a letter to him. A summary of Pourchet's response is given in [van der Poorten 1984].[3]

[3] Van der Poorten provides the following correction to the given argument: Consider $p_{n+1}<p_na^{h\varepsilon_n}$ so that $a^h=p_{\psi(h)}<a^{h(\varepsilon_1+\cdots+\varepsilon_{\psi(h)})}$ and then consider $\varepsilon_1+\ldots\varepsilon_{\psi(h)}=\psi(h)\varepsilon$.

It is clear that $\delta(1/\alpha) \geq \delta(\alpha) - 2$, $\delta(-\alpha) \geq \delta(\alpha) - 2$ and $\delta(\alpha + n) = \delta(\alpha)$ for any $n \in \mathbb{Z}$. Furthermore, Mendès France [1973] has shown that for any $\alpha \in \mathbb{Q}$, $n \in \mathbb{Z}^+$,

$$\delta(n\alpha) \geq \frac{\delta(\alpha) - 1}{\kappa(n) + 2} - 1,$$

where $\kappa(n)$ is a positive valued function whose values depend only on n. Consequently, for any sequence $\alpha_i \in \mathbb{Q}$ satisfying $\lim_{i\to\infty} \delta(\alpha_i) = \infty$ and any $n \in \mathbb{Z}^+$ we have $\lim_{i\to\infty} \delta(\alpha_i/n) = \infty$. It follows that, if $T > 1$, then

$$\lim_{i\to\infty} \delta\left(\frac{A_{2i}}{B_{2i}}\right) = \lim_{i\to\infty} \delta\left(\frac{1}{F_{2i}}(E\xi^{i+1} + f_{2i})\right) = \infty. \tag{6-2}$$

Period length of ω_n. By our criteria for a kreeper we must have for some $a, b \in \mathbb{Q}$,

$$an + b = lp(\omega_n) > \sum_{j=W}^{(n-V)/k} p_{2j}$$

for all $n \in I$. Hence we must have

$$\sum_{i=W}^{(n-V)/k} \delta\left(\frac{A_{2i}}{B_{2i}}\right) < an + b. \tag{6-3}$$

By (6-2) there exists a $\gamma > W$ such that $\delta\left(A_{2i}/B_{2i}\right) > k(a+1)$ for all $i \geq \gamma$. Then,

$$\sum_{i=W}^{(n-V)/k} \delta\left(\frac{A_{2i}}{B_{2i}}\right) > k(a+1)\big[(n-V)/k - \gamma\big]. \tag{6-4}$$

When $n > b + (a+1)\big(V + k\gamma\big)$ we have (6-4) is greater than $an + b$. And since all the terms on the right side of this inequality are bounded, there must exist an infinitude of values of $n \in I$ such that $lp(\omega_n) > an + b$ for any fixed a, b. In conclusion D_n can not be a kreeper if $T > 1$. In other words, we must have $T = 1$ and $U = x$.

Our only remaining objective is to show that necessarily $N = 1$. Clearly, there is no loss in generality in supposing that N is not a power of x. In Section 5 we established the existence of the following complete quotient in all kreepers,

$$\theta_{h_{2i}} = \frac{\omega_n + S_1/d - Nsr_{2i}l_{2i}u_{2i}u'_{2i}y_{2i}y'_{2i}c_{2i}x^{n-ki}/d}{r_{2i}m_{2i}l_{2i}(su_{2i}y_{2i}z_{2i})^2 N x^{n-ki}}.$$

By taking $i = \lfloor n/k \rfloor$ we find an element, $\eta \in \mathbb{O}_n$, whose norm can be written as RNx^ν, where $R \,\big|\, D_n$, $(N, E_n) = 1$ and $0 \leq \nu < k$. Hence the norm of η is bounded independently of n, and coprime to the conductor of the order $\mathbb{O}_n^*$. First, recall the ideal $\mathfrak{b}_n \in \mathbb{O}_n^*$ from page 195, which has norm x^g, with g fixed independently of

n. Suppose that $N > 1$ and that the ideals (η), $\mathfrak{b}_n$ are dependent in $\mathbb{O}_n^*$. Then by Theorem 2.7 we must have $N^e x^{\nu e} = x^{gf}$ for some nonnegative integers e, f, not both zero. Since N is not a power of x, we must have $e = 0$, $f > 0$. But this would imply $x = 1$, which is impossible. Hence (η) and $\mathfrak{b}_n$ are independent ideals in $\mathbb{O}_n^*$. Then by Proposition 4.3, $R(\mathbb{O}_n) \gg (\log D_n)^3$; that is the sequence of discriminants $\{D_n\}$ is not a kreeper. Thus we must have $N = 1$.

This completes the proof of Theorem 1.1.

7. Function field kreepers

Almost everything done here is also valid in function fields. Instead of considering quadratic fields over $\mathbb{Q}$, we can consider the so-called congruence quadratic function fields $\mathbb{F}_q(X, \sqrt{f})$, where $f \in \mathbb{F}_q[X]$. It is well-known that the results in Section 2 have similar analogues for expansions over $\mathbb{F}_q[X]$. The main exception for our interests here, is that the continued fraction expansion of a rational function $f(X) \in \mathbb{F}_q[X]$ has a fixed length. This was used in Section 5 to ensure each Q_{h_j} was positive. In the function field case the term $(-1)^{p_{2i-1}+1} l$ is equal to ul, where $u \in \mathbb{F}_q^*$. By interpreting $|l|$ to be equal to ul for some $u \in \mathbb{F}_q^*$, the results carry over.

Other minor details: P lies in $\mathbb{F}_q^*$ rather than in $\{\pm 1\}$, which is easily handled by just renaming M; further, L, g should be defined to be 2 and τ_y, τ_z can be ignored.

The main problem is that Theorem 6.1 does not hold for coprime functions in $\mathbb{F}_q[X]$. Consequently, we have no direct proof that $T = 1$ and $U = x$. However, if we suppose only the weaker condition: that x is a monomial in X (then necessarily $x = X$ by assumption about powers of x) then trivially, $T \in \mathbb{F}_q$ and $U = x/T$. Then by renaming l and m we have:

Theorem 7.1. *A function field kreeper, that is a sequence of polynomials $f_n(X)$ such that*

1. $f_n(X) = A(X)^2 X^{2n} + B(X) X^n + C^2$, *where* $A, B, C \in \mathbb{F}_q[X]$
2. $lp(\sqrt{f_n(X)}) = an + b$ *for some* $a, b \in \mathbb{Q}$
3. *In the principal cycle there exists an element whose norm is X^g for some g fixed independently of n.*

must satisfy

$$d^2 f_n(X) = c^2 \left((qrX^n + (mz^2X^k - ly^2)/q)^2 + 4rly^2X^n \right),$$

where $q, r, l, m \in \mathbb{F}_q[X]$ and

$$(qrX, mlzy) = 1\,, \quad (ml, zy) = 1\,, \quad (qr, X) = 1\,,$$
$$c^2rly^2mz^2 \,\big|\, d^2D_n\,, \quad q \,\big|\, mz^2X^k - ly^2.$$

8. Some more notations

Define $s := \max_{i\in\mathbb{N}} \{(x^i, c)\}$, $u := c/s$ so that $(u, x) = 1$. In order to make things easier for ourselves when we come to the expansion of ω_n, we would like to have

$$x^{\mu} \equiv 1 \pmod{u^2mz^2ly^2} \quad \text{and} \quad x^{\mu} \equiv 0 \pmod{s^2}.$$

Clearly, such a μ must exist. We shall want to consider the congruence class, $I_\nu := \{n \in \mathbb{N} : n \equiv \nu \pmod{\mu}, n > \mu\}$. Our proof will show that D_n is a kreeper for $n \in I_\nu$, and since every n lies in some I_ν, we shall not be losing any generality in this restriction. Moreover, the value of $x^n \pmod{u^2s^2mz^2ly^2}$ is the same for all $n \in I_\nu$.

If θ_{i+1} represents the $(i+1)$-th complete quotient of ω_n, that is, if

$$\omega_n = [\, a_0\,, a_1\,, \ldots, a_i\,, \theta_{i+1}\,],$$

we define

$$\Psi_{i+1} = \theta_1 \ldots \theta_{i-1}\theta_i Q_i \in \mathbb{Z}[\omega_n].$$

Then, we have $N(\Psi_{i+1}) = (-1)^i Q_i$. If we write the complete quotient $\theta_{h_i} = (\omega_n + P_{h_i})/Q_{h_i}$ as $\theta_{h_i} = A_i/B_i - \gamma$, where we take

$$B_i = dQ_{h_i} \quad \text{and} \quad A_i = \begin{cases} dP_{h_i} + cqrx^n - S_m & \varepsilon_i = 0 \text{ or } i \equiv 0 \pmod{2}, \\ dP_{h_i} + s_1 - S_2 & \varepsilon_{i-1} = 1 \text{ and } i \equiv 1 \pmod{2}, \end{cases}$$

where $P_{h_i} = S_m/d - J$, S_m being one of S_1 and S_2 and J is some function of r, l, m, z, y, c, x. Applying Lemma 2.2 (where $\Delta_i = (A_i, B_i)$), we find

$$\theta_{h_{i+1}} = \frac{(-1)^{p_i}\overline{\gamma}}{\gamma\overline{\gamma}(B_i/\Delta_i)^2} - \frac{c_i}{B_i/\Delta_i},$$

where $c_i \equiv (-1)^{p_i+1}\Delta_i/A_i \pmod{B_i/\Delta_i}$. Thus,

$$Q_{h_{i+1}} = (-1)^{p_i}\gamma\overline{\gamma}(B_i/\Delta_i)^2 Q_{h_i}.$$

Hence,

$$\frac{(-1)^{p_i} Q_{h_{i+1}}}{Q_{h_i}} = \frac{(A_i - B_i\theta_{h_i})(A_i - B_i\overline{\theta}_{h_i})}{\Delta_i^2}.$$

By Corollary 2.4,

$$\theta_{h_i+1}\theta_{h_i+2} \ldots \theta_{h_{i+1}} = \frac{(A_i - \overline{\theta}_{h_i} B_i) Q_{h_i}}{\Delta_i Q_{h_{i+1}}}.$$

Hence,

$$\Psi_{h_{i+1}+1} = \left(\frac{A_i - B_i\overline{\theta}_{h_i}}{\Delta_i}\right)\Psi_{h_i+1}. \tag{8-1}$$

9. Determination of some specific elements in the expansion

In Section 5 we determined the existence of the following complete quotient in the expansion of ω_n,

$$\theta_{h_1} = \frac{\omega_n + S_2/d - c_1(c/\delta)z(z/d_z)x^k/d}{(c/\delta)^2|ml|(y/d_y)^2(z/d_z)^2x^k} , \tag{9-1}$$

Moreover, for sufficiently large n, θ_{h_1} is reduced. Furthermore,

$$\Psi_{h_1+1} = \alpha\beta/(d\Delta_0). \tag{9-2}$$

As shown earlier, the development of the expansion of ω_n depends upon whether or not a power of x^k can be factored out from Q_{h_i} or not. In order to accommodate this we define

$$\varepsilon_i = \begin{cases} 1 & \text{if } \lambda_i \geq n-k, \\ 0 & \text{if } \lambda_i < n-k, \end{cases}$$

where λ_i is defined recursively as

$$\lambda_1 := k, \qquad \lambda_{i+2} := \begin{cases} \lambda_i + k - n\varepsilon_i & \text{if } i \equiv 1 \pmod 2, \\ 0 & \text{if } i \equiv 0 \pmod 2. \end{cases}$$

Note that $\lambda_{2i-1} \equiv ki \pmod n$.

Theorem 9.1. *Suppose there exists a complete quotient $(\omega_n + P_{h_i})/Q_{h_i}$ satisfying*

$$P_{h_i} = S_2/d - sm_ir_iu_iu_i'z_iz_i'c_ix^{\lambda_i}/d , \quad Q_{h_i} = r_il_im_i(su_iy_iz_i)^2x^{\lambda_i} , \tag{9-3}$$

and the set of conditions C_i are satisfied. Then there exists a complete quotient $(\omega_n + P_{h_{i+2}})/Q_{h_{i+2}}$ satisfying (9-3) *and C_{i+2}. Furthermore,*

$$\Psi_{h_{i+2}+1} = \left(\frac{\alpha^{\varepsilon_i+1}\beta}{d^2\Delta_i\Delta_{i+1}}\right)\Psi_{h_i+1}$$

Note: It is clear that with the appropriate selections, the complete quotient θ_{h_1} in (9-1) satisfies the conditions of the theorem.

Proof. In Section 5 we determined the existence of a complete quotient satisfying the conditions C'_{i+1}. Furthermore,

$$\Psi_{h_{i+1}+1} = \left(\frac{A_i - B_i\overline{\theta}_{h_i}}{\Delta_i}\right)\Psi_{h_i+1} = \left(\frac{\alpha}{d\Delta_i}\right)\Psi_{h_i+1}. \tag{9-4}$$

Moreover we have $h_{i+1} = h_i + p_i$, where p_i is the length of the appropriately selected continued fraction expansion of A_i/B_i.

Suppose $\varepsilon_i = 0$. If $\lambda_i < n-k$ then $\varepsilon_i = 0$ and $\lambda_{i+2} = \lambda_i + k$. This now follows as in Section 5.

Suppose $\varepsilon_i = 1$. When $\lambda_i \geq n-k$ we have $\varepsilon_i = 1$ and this situation has not yet been considered. In this situation, the previous choice of A_{i+1} is not appropriate. Instead, we need to consider

$$\theta_{h_{i+1}} = \frac{s_1/d - sr_{i+1}l_{i+1}u_{i+1}u'_{i+1}y_{i+1}y'_{i+1}c_{i+1}x^{n-\lambda_i}/d}{r_{i+1}l_{i+1}m_{i+1}(su_{i+1}y_{i+1}z_{i+1})^2x^{n-\lambda_i}} - \frac{\overline{\alpha}}{r_{i+1}l_{i+1}m_{i+1}(su_{i+1}y_{i+1}z_{i+1})^2x^{n-\lambda_i}}.$$

There is a slight problem in notation because there is going to be an extra intermediate complete quotient. Consequently, we will use overlines to represent the terms involved. This time we take

$$\begin{aligned}\overline{A}_{i+1} &:= s_1 - sr_{i+1}l_{i+1}u_{i+1}u'_{i+1}y_{i+1}y'_{i+1}c_{i+1}x^{n-\lambda_i},\\ \overline{B}_{i+1} &:= dr_{i+1}l_{i+1}m_{i+1}(su_{i+1}y_{i+1}z_{i+1})^2x^{n-\lambda_i},\\ \overline{\Delta}_{i+1} &:= (\overline{A}_{i+1}, \overline{B}_{i+1}).\end{aligned}$$

Then, $\overline{A}_{i+1} = dP_{h_{i+1}} + (s_1 + td)/2$, which leads to $d_y u_{i+1}l_{i+1}y_{i+1} \,\big|\, \overline{A}_{i+1}$ after a short calculation. We also define

$$\bar{w}_{i+1} := (m_{i+1}, u/u_{i+1}), \quad \bar{e}_{i+1} := sd_y u_{i+1}l_{i+1}y_{i+1}r_{i+1}\bar{w}_{i+1}.$$

By writing $\Delta'_{i+1} = (\overline{G}_{i+1}, u_{i+1}y_{i+1})$, a little calculation gives

$$\overline{\Delta}_{i+1} = sd_y u_{i+1}l_{i+1}y_{i+1}r_{i+1}\bar{w}_{i+1}\Delta'_{i+1}.$$

Note that

$$\frac{\overline{B}_{i+1}}{\overline{\Delta}_{i+1}} = \frac{sdm_{i+1}u_{i+1}y_{i+1}z_{i+1}^2x^{n-\lambda_i}}{d_y\bar{w}_{i+1}\Delta'_{i+1}}.$$

According to Lemma 5.3 there exists two numbers $\bar{y}_{i+2}$ and $\bar{u}_{i+2}$, such that

$$\bar{y}_{i+2} \,\big|\, y/d_y, \quad \bar{u}_{i+2} \,\big|\, u/\bar{w}_{i+1}, \quad \bar{y}_{i+2}\bar{u}_{i+2}\Delta'_{i+1} = \frac{y}{d_y}\frac{u}{\bar{w}_{i+1}},$$

and

$$\left(z_{i+1}\bar{y}_{i+2}, \frac{u}{\bar{u}_{i+2}\bar{w}_{i+1}}\right) = 1.$$

From

$$\theta_{h_{i+1}} = \frac{\overline{A}_{i+1}}{\overline{B}_{i+1}} - \frac{\overline{\alpha}}{r_{i+1}l_{i+1}m_{i+1}(su_{i+1}y_{i+1}z_{i+1})^2x^{n-\lambda_i}},$$

we apply Lemma 2.2 and find that the next partial quotients are those of the continued fraction expansion of $\overline{A}_{i+1}/\overline{B}_{i+1}$ of length $\bar{p}_{i+1}$, where the parity of $\bar{p}_{i+1}$

is determined by $(-1)^{\bar{p}_{i+1}+1} = \operatorname{sign}(l)$. The next complete quotient is then $\theta_{j_{i+2}}$, where $j_{i+2} = h_{i+1} + \bar{p}_{i+1}$ and

$$\text{(9-5)} \qquad \theta_{j_{i+2}} = \frac{r_{i+1}l_{i+1}m_{i+1}(su_{i+1}y_{i+1}z_{i+1})^2x^{n-\lambda_i}}{-\operatorname{sign}(l)\,\overline{\alpha}\left(\dfrac{sdm_{i+1}u_{i+1}y_{i+1}z_{i+1}^2x^{n-\lambda_i}}{d_y\bar{w}_{i+1}\Delta'_{i+1}}\right)^2} - \frac{\bar{c}_{i+2}}{\overline{B}_{i+1}/\overline{\Delta}_{i+1}},$$

where $\bar{c}_{i+2} \equiv -\operatorname{sign}(l)\,\overline{\Delta}_{i+1}/\overline{A}_{i+1} \pmod{\overline{B}_{i+1}/\overline{\Delta}_{i+1}}$. The standard choices yield

$$\text{(9-6)} \qquad \theta_{j_{i+2}} = \frac{\omega_n + S_1/d - s\bar{r}_{i+2}\bar{l}_{i+2}\bar{u}_{i+2}\bar{u}'_{i+2}\bar{y}_{i+2}\bar{y}'_{i+2}\bar{c}_{i+2}x^{2n-\lambda_i}/d}{\bar{r}_{i+2}\bar{l}_{i+2}\bar{m}_{i+2}(s\bar{u}_{i+2}\bar{y}_{i+2}\bar{z}_{i+2})^2x^{2n-\lambda_i}}$$

Since $s \,\big|\, \overline{B}_{i+1}/\overline{\Delta}_{i+1}$ we get $(\bar{c}_{i+2}, s) = 1$, also $(\bar{z}_{i+2}\bar{y}_{i+2}, u/\bar{u}_{i+2}) = 1$. Hence, the conditions $\overline{C}'_{i+2}$ are satisfied. Furthermore,

$$\Psi_{j_{i+2}+1} = \left(\frac{\alpha}{d\overline{\Delta}_{i+1}}\right)\Psi_{h_{i+1}+1}.$$

The complete quotient in (9-6) satisfies the conditions of Theorem 5.4 Hence, there exists a complete quotient

$$\text{(9-7)} \qquad \theta_{j_{i+3}} = \frac{\omega_n + S_2/d - s\bar{m}_{i+3}\bar{r}_{i+3}\bar{u}_{i+3}\bar{u}'_{i+3}\bar{z}_{i+3}\bar{z}'_{i+3}\bar{c}_{i+3}x^{\lambda_i+k-n}/d}{\bar{r}_{i+3}\bar{l}_{i+3}\bar{m}_{i+3}(s\bar{u}_{i+3}\bar{y}_{i+3}\bar{z}_{i+3})^2x^{\lambda_i+k-n}}$$

with the conditions $\overline{C}_{i+3}$ satisfied, and

$$\Psi_{j_{i+3}+1} = \left(\frac{\beta}{d\overline{\Delta}_{i+2}}\right)\Psi_{j_{i+2}+1} = \left(\frac{\alpha\beta}{d^2\overline{\Delta}_{i+1}\overline{\Delta}_{i+2}}\right)\Psi_{h_{i+1}+1}.$$

Since $\varepsilon_i = 1$ we have $\lambda_{i+2} = \lambda_i + k - n$. With appropriate renaming, the complete quotient $\theta_{j_{i+3}}$ in (9-7) becomes

$$\theta_{h_{i+2}} = \frac{\omega_n + S_2/d - sr_{i+2}m_{i+2}u_{i+2}u'_{i+2}z_{i+2}z'_{i+2}c_{i+2}x^{\lambda_{i+2}}/d}{r_{i+2}l_{i+2}m_{i+2}(su_{i+2}y_{i+2}z_{i+2})^2x^{\lambda_{i+2}}}$$

with the conditions C_{i+2} being satisfied. Combining both the $\varepsilon_i = 0$ and $\varepsilon_i = 1$ cases, we have

$$\Psi_{h_{i+2}+1} = \left(\frac{\alpha^{\varepsilon_i}\beta}{d\Delta_{i+1}}\right)\Psi_{h_{i+1}+1} = \left(\frac{\alpha^{\varepsilon_i+1}\beta}{d^2\Delta_i\Delta_{i+1}}\right)\Psi_{h_i+1},$$

and $h_{i+2} = h_i + p_i + p_{i+1}$. □

Our next step will be to investigate the number of partial quotients in the period of the continued fraction expansion of ω_n.

10. Determining the number of partial quotients

By defining, $f_0 := h_0$ and $f_{i+1} := p_i$ for $i \geq 1$ we find

$$h_i = \sum_{k=0}^{i} f_k. \tag{10-1}$$

The value of f_i is dependent only upon the set

$$Z_i := \{l_i, m_i, r_i, s_i, s_i', z_i, z_i', u_i, u_i', y_i, y_i', c_i, L_i, x\},$$

where $L_i := \lambda_i \pmod{\mu}$. Moreover, if $Z_i = Z_j$ then $f_i = f_j$. There are only finitely many distinct sets Z_i; we denote the total number of distinct sets by Z. For a fixed t, there are precisely tk values of $i \in \{1, \ldots, 2tn-1\}$, where $\varepsilon_i = 1$. Let $i_1, \ldots, i_{tk}$ represent these points. Then

$$i_h = \begin{cases} \dfrac{2nh}{k} - 3 & \text{if } k \mid 2nh, \\ \left\lfloor \dfrac{2nh}{k} \right\rfloor - 2 & \text{if } k \nmid 2nh. \end{cases}$$

From (10-1),

$$h_{2nt-1} = \sum_{i=0}^{i_1-1} f_i + \sum_{h=1}^{tk-1} f_{i_h} + \sum_{h=1}^{tk-1} \sum_{i=i_h+1}^{i_{h+1}-1} f_i.$$

The number of summands in $\sum_{i_h+1}^{i_{h+1}-1} f_i$ is $i_{h+1} - 1 - (i_h + 1) + 1 \geq \lfloor 2n/k \rfloor - 2$. Hence the distance between i_h and i_{h+1} can be made arbitrarily large. But Z is independent of n, which by the box principle means that for large enough n, there exists ρ_h and τ_h such that

$$Z_{i_h+\tau_h} = Z_{i_h+\tau_h+\rho_h} \qquad i_h + \tau_h + \rho_h \leq i_{h+1} - 1 \quad 1 \leq \tau_h, \rho_h \leq Z.$$

Now we examine $Z_{\kappa+i_h+\tau_h+j\rho_h}$, where $\kappa+i_h+\tau_h+j\rho_h \leq i_{h+1}-1$ and $0 \leq \kappa < \rho_h$. Since $\varepsilon_{i_h+\tau_h} = 0$ and $\varepsilon_{i_h+\tau_h+\rho_h} = 0$ we have

$$Z_{i_h+\tau_h+1} = Z_{i_h+\tau_h+\rho_h+1} \qquad \text{provided } i_h + \tau_h + \rho_h + 1 \leq i_{h+1} - 1.$$

By induction,

$$Z_{i_h+\tau_h+\kappa} = Z_{i_h+\tau_h+\kappa+j\rho_h} \qquad \text{provided } i_h + \tau_h + \kappa + j\rho_h \leq i_{h+1} - 1,$$

which implies $f_{i_h+\tau_h+\kappa} = f_{i_h+\tau_h+\kappa+j\rho_h}$. Define

$$\upsilon := \left\lfloor \frac{\lfloor 2n/k \rfloor - \tau_h - \rho_h}{\rho_h} \right\rfloor - 1.$$

It is straightforward that $i_h + \tau_h + \kappa + \upsilon\rho_h < i_{h+1}$. Then

$$\sum_{i=i_h+1}^{i_{h+1}-1} f_i = \sum_{j=1}^{\tau_h-1} f_{i_h+j} + \sum_{\kappa=0}^{\rho_h-1}\Bigl(\sum_{j=0}^{\upsilon} f_{i_h+\tau_h+\kappa+j\rho_h}\Bigr) + \sum_{j=i_h+\tau_h+\rho_h+\upsilon\rho_h}^{i_{h+1}-1} f_j.$$

The number of terms in $\sum_{j=i_h+\tau_h+\rho_h+\upsilon\rho_h}^{i_{h+1}-1} f_j$ is $\le \rho_h + 2$. Hence,

$$\sum_{i=i_h+1}^{i_{h+1}-1} f_i = \left\lfloor \frac{2n - k\tau_h - k\rho_h}{k\rho_h} \right\rfloor \zeta_h + \xi_h, \tag{10-2}$$

where ζ_h, ξ_h are independent of n.

We now take $\rho = \prod_{i=1}^{tk} \rho_h$ and $w = \operatorname{lcm}[k, \mu, \rho]$ both of which are independent of n. We write $n = w\gamma + \phi$, where $0 \le \phi < w$. From the original set I_ν, we now wish to consider the following subset, $I_{\nu,\phi} = \{n \in I_\nu : n \equiv \phi \pmod{w}\}$. Without loss of generality, we may suppose that $n \in I_{\nu,\phi}$. Consequently,

$$\left\lfloor \frac{2n - k\tau_h - k\rho_h}{k\rho_h} \right\rfloor = 2\gamma \frac{w}{k\rho_h} + \left\lfloor \frac{2\phi - k\tau_h - k\rho_h}{k\rho_h} \right\rfloor.$$

Thus, the sum (10-2) becomes,

$$\sum_{i=i_h+1}^{i_{h+1}-1} f_i = 2\zeta_h\gamma \frac{w}{k\rho_h} + \zeta_h \left\lfloor \frac{2\phi - k\tau_h - k\rho_h}{k\rho_h} \right\rfloor + \xi_h,$$

which means we can now write h_{2nt-1} as

$$h_{2nt-1} = \sum_{i=0}^{i_1-1} f_i + \sum_{h=1}^{tk-1} f_{i_h} + 2\gamma \sum_{h=1}^{tk-1} \zeta_h \frac{w}{k\rho_h} + \sum_{h=1}^{tk-1}\left(\zeta_h \left\lfloor \frac{2\phi - k\tau_h - k\rho_h}{k\rho_h} \right\rfloor + \xi_h\right).$$

Now, let

$$x_t = 2\sum_{h=1}^{tk-1} \zeta_h \frac{w}{k\rho_h}$$

$$y_t = \sum_{h=1}^{tk-1}\left(\zeta_h \left\lfloor \frac{2\phi - k\tau_h - k\rho_h}{k\rho_h} \right\rfloor + \xi_h\right) + \sum_{i=0}^{i_1-1} f_i + \sum_{h=1}^{tk-1} f_{i_h},$$

both of which are integers and independent of γ. Then

$$h_{2nt-1} = \gamma x_t + y_t = \bigl((n-\phi)/w\bigr)x_t + y_t = a_t n + b_t,$$

where a_t, b_t are both rational numbers independent of n for all $n \in I_{\nu,\phi}$.

This shows that the length of the expansion up to h_{2nt-1} is linear in n. It remains to show that there exists some h_{2nt-1}, where $Q_{h_{2nt-1}} = 1$, and if $Q_j = 1$ then $j = h_{2nt-1}$ for some t independent of n.

11. Finding an element of norm 1

We now examine the product of the elements in the expansion. By Theorem 9.1 and (9-2),

$$\Psi_{h_{2i-1}+1} = \frac{(\alpha\beta)^i \alpha^{\sum_{j=1}^{i-1} \varepsilon_{2j-1}}}{d^{2i-1} \prod_{j=0}^{2i-2} \Delta_j}.$$

So,

$$\Psi_{h_{2nt-1}+1} = \frac{(\alpha\beta)^{nt} \alpha^{kt}}{d^{2nt-1} \prod_{j=0}^{2nt-2} \Delta_j} \quad \text{and} \quad |N(\Psi_{h_{2nt-1}+1})| = Q_{h_{2nt-1}}.$$

Hence,

$$Q_{h_{2nt-1}} \left(d^{2nt-1} \prod_{j=0}^{2nt-2} \Delta_j \right)^2 = \left| N\left((\alpha\beta)^{nt}\right) N\left(\alpha^{kt}\right) \right| = \left| N\big((\alpha\beta)^n\big) N(\alpha^k) \right|^t$$

as well as

$$Q_{h_{2n-1}} \left(d^{2n-1} \prod_{j=0}^{2n-2} \Delta_j \right)^2 = \left| N\big((\alpha\beta)^n\big) N(\alpha^k) \right|.$$

Thus, $\sqrt{Q_{h_{2nt-1}} / \left(Q_{h_{2n-1}}\right)^t} \in \mathbb{Q}$. Since $\lambda_{2j-1} \equiv jk \pmod{n}$ we have that $\lambda_{2ni-1} = 0$ for positive i. Hence, $\lambda_{2n-1} = \lambda_{2nt-1} = 0$ and so

$$\begin{aligned} Q_{h_{2n-1}} &= l_{2n-1} m_{2n-1} r_{2n-1} \left(s u_{2n-1} z_{2n-1} y_{2n-1} \right)^2, \\ Q_{h_{2nt-1}} &= l_{2nt-1} m_{2nt-1} r_{2nt-1} \left(s u_{2nt-1} z_{2nt-1} y_{2nt-1} \right)^2. \end{aligned}$$

Thus,

$$\sqrt{\frac{l_{2nt-1} m_{2nt-1} r_{2nt-1}}{(l_{2n-1} m_{2n-1} r_{2n-1})^t}} \in \mathbb{Q}.$$

Since l_{2n-1}, m_{2n-1}, r_{2n-1} are each squarefree and relatively prime, if $2 \mid t$ then $\sqrt{l_{2nt-1} m_{2nt-1} r_{2nt-1}} \in \mathbb{Q}$, which implies $l_{2nt-1} m_{2nt-1} r_{2nt-1} = 1$. Conversely, if $2 \nmid t$, then $l_{2nt-1} m_{2nt-1} r_{2nt-1} = l_{2n-1} m_{2n-1} r_{2n-1}$.

Now, we construct an element of norm 1. Put

$$\varepsilon = \begin{cases} 1 & \text{if } l_{2n-1} m_{2n-1} r_{2n-1} \neq 1, \\ 0 & \text{if } l_{2n-1} m_{2n-1} r_{2n-1} = 1, \end{cases}$$

and

$$\Gamma = \frac{\Psi_{h_{2n-1}+1}^{1+\varepsilon}}{(l_{2n-1}m_{2n-1}r_{2n-1})^{\varepsilon}(su_{2n-1}y_{2n-1}z_{2n-1})^{1+\varepsilon}}.$$

Then it is easily shown that $N(\Gamma) = 1$.

Lemma 11.1. *If $D_n = F_n^2 D_n'$, where D_n' is squarefree, then $su_i y_i z_i \mid F_n$.*

The lemma and above results imply that we have V_1, $Y_1 \in \mathbb{Z}$ such that

$$\Gamma = V_1 + Y_1 \frac{F_n}{su_{2n-1}y_{2n-1}z_{2n-1}} \omega_n',$$

where $\mathbb{Z}[\omega_n']$ is the maximal order of $\mathbb{Q}(\sqrt{D_n'})$ and D_n' is the squarefree kernel of D_n, that is $D_n = F_n^2 D_n'$. If we now define V_j and Y_j by

$$\left(V_j + Y_j \frac{F_n}{su_{2n-1}y_{2n-1}z_{2n-1}} \omega_n' \right) = \Gamma^j.$$

Then Y_j/Y_1 is the Lucas function, $(\Gamma^j - \overline{\Gamma}^j)/(\Gamma - \overline{\Gamma})$. Since $\Gamma\overline{\Gamma} = 1$, there must exist some minimal positive p such that $su_{2n-1}y_{2n-1}z_{2n-1} \mid Y_p$. Putting $t = g := (1+\varepsilon)p$ we get

$$\Psi_{h_{2ng-1}+1} = \Gamma^p \sqrt{l_{2ng-1}m_{2ng-1}r_{2ng-1}}\, su_{2ng-1}y_{2ng-1}z_{2ng-1}.$$

If $\varepsilon = 1$ then $2 \mid g$ implies that $l_{2ng-1}m_{2ng-1}r_{2ng-1} = 1$. If $\varepsilon = 0$ then

$$l_{2n-1}m_{2n-1}r_{2n-1} = 1,$$

so $l_{2ng-1}m_{2ng-1}r_{2ng-1} = 1$. Since $\Gamma^p \in \mathbb{Z}[\omega_n]$, this implies $su_{2ng-1}y_{2ng-1}z_{2ng-1} = 1$. Hence $|N(\Psi_{h_{2ng-1}+1})| = 1$, which means that $Q_{h_{2ng-1}} = 1$. The values of p depend only on $su_{2n-1}y_{2n-1}z_{2n-1}$, which divides $suyz$. Thus, there can only be a finite number of possible values for p.

Conversely, one can also show (see [Patterson 2003, Chapter 25]) that such a solution is either fundamental or the square of the fundamental solution, although this is superfluous in showing that D_n is a kreeper.

12. Returning to the regular continued fraction expansion

Up to now we have determined

$$\omega_n = [\, a_0 \,, \ldots, a_{h_0-1} \,, b_1 \,, a_{h_0+1} \,, \ldots, a_{h_1-1} \,, b_2 \,, \ldots, a_{h_{2nt-1}-1} \,, 2a_0 - t \,],$$

but in this evaluation we never insisted that $b_i \geq 1$. In other words this expansion might not correspond to the regular continued fraction expansion of ω_n. This is equivalent to saying that the expansions of A_i/B_i might have an initial nonpositive partial quotient.

Proposition 12.1. *In the expansion of* ω_n *given by the earlier procedure, the number of nonpositive partial quotients is bounded independently of* n.

Proof. See [Patterson 2003]. □

This proposition means that there are only finitely many partial quotients that need to be altered in order to find the regular continued fraction expansion of ω_n.

The removal of nonpositive partial quotients is covered in [Dirichlet 1999]. There it is shown that any negative partial quotient can be moved to the left in the continued fraction expansion. In our situation, we discover that either the negative partial quotient disappears easily or we are left with

$$[\, b_{i-1}\, ,\, -1\, ,\, 1\, ,\, u\, ,\, v\, ,\, \dots\,] = [\, b_{i-1} - 2 - u\, ,\, 1\, ,\, v - 1\, ,\, \dots\,],$$

where $u > 0$ and is bounded independently of n, and $b_{i-1} = \lfloor A_{i-1}/B_{i-1} \rfloor$. When

$$\theta_{h_{i-1}} = \frac{\omega_n + S_2/d - E_{i-1}x^{\lambda_i}/d}{E'_{i-1}x^{\lambda_i}} = \frac{A_{i-1}}{B_{i-1}} + e$$

we have $\lambda_i \ll 1$, implies $\lfloor A_{i-1}/B_{i-1} \rfloor > x^{n-J_1}$, where J_1 is bounded independently of n. Consequently, $b_{i-1} - 2 - u > 0$ for all sufficiently large n. The other case follows similarly.

Finally, we note that a_{h_i+j} with $j > 0$ can not be the end of the period since $Q_{h_i+j} > 1$. For sufficiently large n, each b_j which is not some $b_{h_{2nt-1}}$ satisfies $b_j < x^n$ since $B_i \geq x$.

In conclusion, we have shown that,

$$lp(\omega_n) = h_{2ng-1} + c_n = a_g n + b_g + c_n = a_g n + b'_g,$$

where $c_n \in \mathbb{Z}$ can be bounded independently of n. Then a_g, b'_g are rational numbers bounded independently of n.

Hence there must exist an infinitude of values of $n \in I$ such that

$$lp(\omega_n) = an + b,$$

where a, $b \in \mathbb{Q}$ and are fixed independently of n. This completes the proof of Theorem 1.2.

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