A RESULT ABOUT $C^3$-RECTIFIABILITY OF LIPSCHITZ CURVES

Silvano Delladio
A RESULT ABOUT $C^3$-RECTIFIABILITY
OF LIPSCHITZ CURVES

SILVANO DELLAIO

Let $\gamma_0 : [a, b] \to \mathbb{R}^{1+k}$ be Lipschitz. Our main result provides a sufficient condition, expressed in terms of further accessory Lipschitz maps, for the $C^3$-rectifiability of $\gamma_0([a, b])$.

1. Introduction

A set in $\mathbb{R}^n$ is $C^3$-rectifiable if $\mathcal{H}^1$-almost all of it can be covered by countably many curves of class $C^3$ embedded in $\mathbb{R}^n$. The main goal of this paper is to prove the following result.

Theorem 1.1. Let there be given Lipschitz maps

$$\gamma_0, \gamma_1 : [a, b] \to \mathbb{R}^{1+k} \quad \text{and} \quad \gamma_2 = (\gamma_2^\top, \gamma_2^\perp) : [a, b] \to \mathbb{R}^{1+k} \times \mathbb{R}^{1+k}$$

and a function $\omega : [a, b] \to \{\pm 1\}$ such that

$$(1-1) \quad \gamma_0'(t) = \omega(t) \|\gamma_0'(t)\| \gamma_1(t),$$

$$(1-2) \quad (\gamma_0'(t), \gamma_1'(t)) = \omega(t) \|\gamma_0'(t), \gamma_1'(t)\| \gamma_2(t)$$

for almost every $t \in [a, b]$. Then $\gamma_0([a, b])$ is a $C^3$-rectifiable set.

Remark. In the special case when $\omega := 1$ while $\gamma_0$ is regular and at least of class $C^2$, the conditions (1-1) and (1-2) say that $\gamma_1(t)$ and $\gamma_2(t)$ are, respectively, the unit tangent vector of $\gamma_0$ at $t$ and the unit tangent vector of $(\gamma_0, \gamma_1)$ at $t$. This remark is at the root of the applications to geometric variational problems mentioned below.

Theorem 1.1 should be considered a step forward in a project, stated in [Delladio 2005], aimed at providing sufficient conditions for the $C^H$-rectifiability of a $n$-dimensional rectifiable set. Results concerning the case $H = 2$ were first obtained in [Anzellotti and Serapioni 1994; Delladio 2003; Fu 1998], but subtle mistakes seriously invalidating their proofs were discovered later [Delladio 2004; Fu 2004]. Then the paper [Delladio 2005], cleaning-up the simplest case $n = 1$ and $H = 2$, followed. Our future efforts will be aimed at extending the theory to any value of $n$.


Keywords: rectifiable sets, geometric measure theory, Whitney extension theorem.
and $H$. Joint work with Joseph Fu on $C^2$-rectifiability in all dimensions (invoking slicing in order to reduce to dimension one) is in progress.

Further work is in progress to apply these results to geometric variational problems via geometric measure theory and more precisely through the notion, first introduced in [Anzellotti et al. 1990], of a generalized Gauss graph. Former achievements in this direction include [Delladio 2001] (a somehow surprising application to differential geometry context), [Anzellotti and Delladio 1995] (an application to Willmore problem) and [Delladio 1997] (an application to a problem introduced in [Bellettini et al. 1993]). These last two papers followed the idea by De Giorgi of relaxing the functional with respect to $L^1$-convergence of the domains of integration. Now we expect that our results can be applied to handle functionals with integrands involving curvatures with their derivatives and, in particular, to get explicit representation formulas after relaxation.

The proof of Theorem 1.1 starts from the $C^2$-rectifiability of $\gamma_0([a, b])$, which is guaranteed by condition (1-1), as shown in [Delladio 2005]. The problem is reduced in Section 2 to proving that $\gamma_0([a, b])$ intersects the graph of any $C^2$ map

$$f : \mathbb{R} \rightarrow (\mathbb{R}u)^\perp \quad (u \in \mathbb{R}^{1+k}, \|u\| = 1)$$

in a $C^3$-rectifiable set. From the first and second derivatives of $f$ expressed in terms of the $\gamma_i$, we obtain in Section 3 a second order Taylor-type formula for $f$ with the remainder in terms of the $\gamma_i$. Theorem 1.1 then follows by the Whitney Extension Theorem, also involving a Lusin-type argument (Section 4). Finally, the absolute curvature for a one-dimensional $C^2$-rectifiable set $P$ is defined and proved to be approximately differentiable almost everywhere whenever $P$ is $C^3$-rectifiable (Section 5).

2. Reduction to graphs

By virtue of the main result stated in [Delladio 2005], the equality (1-1) implies that $\gamma_0([a, b])$ is $C^2$-rectifiable. As a consequence, there must be countably many unit vectors

$$u_j \in \mathbb{R}^{1+k}$$

and maps of class $C^2$

$$f_j : \mathbb{R} \rightarrow (\mathbb{R}u_j)^\perp$$

such that

$$\mathcal{H}^1(\gamma_0([a, b]) \setminus \bigcup_j G_{f_j}) = 0$$

where

$$G_{f_j} := \{xu_j + f_j(x) \mid x \in \mathbb{R} \}.$$
Hence we need only show that the sets $\gamma_0([a, b]) \cap G_f$ are $C^3$-rectifiable. In other words, Theorem 1.1 becomes an immediate corollary of the following result.

**Theorem 2.1.** Let $\gamma_0, \gamma_1, \gamma_2$ be as in Theorem 1.1. Consider a map 

$$f : \mathbb{R} \to (\mathbb{R}u)'^\perp \quad (u \in \mathbb{R}^{1+k}, \|u\| = 1)$$

of class $C^2$ and define 

$$G_f := \{ xu + f(x) \mid x \in \mathbb{R} \}.$$ 

Then the set $G_f \cap \gamma_0([a, b])$ is $C^3$-rectifiable.

In this section we take the first step toward the proof of Theorem 2.1, which will be concluded later in Section 4. Define 

$$L := \gamma_0^{-1}(G_f) \cap \{ t \in [a, b] \mid \gamma_0'(t), \gamma_1'(t) \text{ exist}, \gamma_0'(t) \neq 0, (1-1) \text{ and (1-2) hold} \}.$$ 

By Lusin’s Theorem, for any given real number $\varepsilon > 0$, there exists a closed subset $L_\varepsilon$ of $L$ such that 

$$(2-1) \quad \gamma_0^' | L_\varepsilon \text{ and } \omega | L_\varepsilon \text{ are continuous and } \mathcal{L}^1(L \setminus L_\varepsilon) \leq \varepsilon.$$ 

If $L_\varepsilon^*$ denotes the set of the density points of $L_\varepsilon$, then 

$$(2-2) \quad L_\varepsilon^* \subset L_\varepsilon$$

since $L_\varepsilon$ is closed. The equality 

$$(2-3) \quad \mathcal{L}^1(L_\varepsilon \setminus L_\varepsilon^*) = 0$$

also holds by a celebrated result of Lebesgue. In the special case that $L$ has measure zero, we take $L_\varepsilon := \emptyset$, hence $L_\varepsilon^* := \emptyset$.

Now observe that 

$$G_f \cap \gamma_0([a, b]) \setminus \gamma_0(L_\varepsilon^*) \subset \gamma_0(\gamma_0^{-1}(G_f) \cap [a, b] \setminus L_\varepsilon^*)$$

hence 

$$\mathcal{H}^1(G_f \cap \gamma_0([a, b]) \setminus \gamma_0(L_\varepsilon^*)) \leq \mathcal{H}^1(\gamma_0(\gamma_0^{-1}(G_f) \cap [a, b] \setminus L_\varepsilon^*))$$

$$\leq \int_{\gamma_0^{-1}(G_f) \cap [a, b] \setminus L_\varepsilon^*} \| \gamma_0^' \| \, dL_\varepsilon^* \leq \int_{L_\varepsilon^*} \| \gamma_0^' \| \leq \varepsilon \text{ Lip} \gamma_0,$$

which implies 

$$\mathcal{H}^1(G_f \cap \gamma_0([a, b]) \setminus \bigcup_{j=1}^\infty \gamma_0(L_{1/j}^*)) = 0.$$ 

Hence, to prove Theorem 2.1, it will be enough to verify that 

$$(2-4) \quad \gamma_0(L_\varepsilon^*) \text{ is } C^3\text{-rectifiable}$$

for all $\varepsilon > 0.$
3. Second order Taylor formula and estimates

Proposition 3.1 below gives formulas for the first and second derivatives of \( f \) in terms of the \( \gamma_i \). This yields a suitable second order Taylor formula in Theorem 3.1.

Throughout this section we shall assume \( L^1(L) > 0 \). Notice that
\[
(3-1) \quad \gamma_{2+}(s) \neq 0 \quad \text{for all } s \in L
\]
by (1-2), so the map
\[
\mu : \{ t \in [a, b] \mid \gamma_{2+}(t) \neq 0 \} \to \mathbb{R}^{1+k}, \quad \mu(t) := \frac{\gamma_{2+}(t)}{\| \gamma_{2+}(t) \|}
\]
is well-defined in \( L \).

Lemma 3.1. Let \( A, B, u \in \mathbb{R}^{1+k} \), with \( \| u \| = 1 \). Then
\[
(A \wedge B) \circ u = (A \cdot u)B - (B \cdot u)A.
\]

Proof. Let \( \{ e_j \} \) be an orthonormal basis of \( \mathbb{R}^{1+k} \) such that \( e_1 = u \). One has
\[
((A \wedge B) \circ u) \cdot e_i = (A \wedge B, u \wedge e_i) = \sum_{j<l} (A_j B_l - A_l B_j) \langle e_j \wedge e_i, e_1 \wedge e_l \rangle
\]
\[
= A_1 B_i - A_i B_1 = ((A \cdot u)B - (B \cdot u)A) \cdot e_i
\]
for all \( i = 1, 2, \ldots, 1+k \). \( \Box \)

Proposition 3.1. Set
\[
x(t) := \gamma_0(t) \cdot u, \quad t \in \mathbb{R}.
\]
Then, for all \( s \in L^*_\epsilon \), one has
\[
(3-2) \quad x'(s) = \gamma_0'(s) \cdot u \neq 0 \quad \text{(that is, } \gamma_1(s) \cdot u \neq 0 \text{)}
\]
and
\[
(3-3) \quad f'(x(s)) = \frac{\gamma_1(s)}{\gamma_1(s) \cdot u} - u.
\]
Moreover
\[
(3-4) \quad f''(x(s)) = \frac{\left( \gamma_1(s) \wedge \mu(s) \right) \circ u}{(\gamma_1(s) \cdot u)^3}.
\]

Proof. Observe that
\[
f(x(t)) = \gamma_0(t) - (\gamma_0(t) \cdot u)u = \gamma_0(t) - x(t)u
\]
for all \( t \in \gamma_0^{-1}(G_f) \). The sides of this equality are both differentiable in \( L^*_\epsilon \) and since each point in \( L^*_\epsilon \subset \gamma_0^{-1}(G_f) \) is a limit point of \( L_\epsilon \subset \gamma_0^{-1}(G_f) \), the derivatives
have to coincide in $L_e^*$. Thus

$$x'(s)f''(x(s)) = \gamma_0'(s) - (\gamma_0'(s) \cdot u)u = \gamma_0'(s) - x'(s)u \tag{3-5}$$

for all $s \in L_e^*$. We obtain (3-2) by recalling that $\gamma_0'(s) \neq 0$ at all $s \in L_e^*$.

Formula (3-3) follows at once from (3-5) and (1-1).

By virtue of (3-2), the sides of (3-3) are both differentiable in $L_e^*$. The derivatives must coincide in $L_e^*$, since each point of $L_e^*$ is a limit point of $L_e^*$. In view of Lemma 3.1, we then get

$$x'(s)f''(x(s)) = \left(\gamma_1(s) \cdot u\right)\gamma_1'(s) - \left(\gamma_1'(s) \cdot u\right)\gamma_1(s) \tag{3-4}$$

for all $s \in L_e^*$. Formula (3-4) finally follows from (3-2), (1-1) and (1-2).

Now set

$$\Delta_s(t) := \gamma_0(t) - \gamma_0(s), \quad s, t \in [a, b].$$

The map

$$\Sigma_s(t) := \Delta_s(t) - (\Delta_s(t) \cdot \gamma_1(s)) \gamma_1(s) - \frac{(\Delta_s(t) \cdot u)^2}{2(\gamma_1(s) \cdot u)^2} \mu(s), \quad t \in [a, b],$$

is well-defined for any given $s \in L_e^*$, by Proposition 3.1.

If $s \in L_e^*$, hence $s \in (a, b)$ and (3-1) holds, one has

$$\|\gamma_2 T(s)\| \geq \frac{1}{2}\|\gamma_2 T(s)\| > 0 \quad \text{for all } \sigma \in I_s,$$

where $I_s$ denotes a certain nontrivial open interval centered at $s$ and included in $[a, b]$, existing by the continuity of $\gamma_2 T$. In particular, this inequality shows that $\mu|I_s$ is Lipschitz, so the map given, for $\sigma \in I_s$, by

$$\Psi_s(\sigma) := \mu(\sigma) - (\mu(\sigma) \cdot \gamma_1(s)) \gamma_1(s) - \frac{\mu(s)}{(\gamma_1(s) \cdot u)^2}\left((\gamma_1(\sigma) \cdot u)^2 + (\Delta_s(\sigma) \cdot u)(\mu(\sigma) \cdot u)\right)$$

is well-defined and Lipschitz, provided $s \in L_e^*$. Moreover

$$\Psi_s(s) = 0,$$

as follows at once from (1-2) and from the following simple result.

**Proposition 3.2.** If $s \in L_e^*$ then $\gamma_1(s) \cdot \gamma_1'(s) = 0$.

**Proof.** Let $\{s_j\}$ be a sequence in $L_e$ converging to $s$, with $s_j \neq s$ for all $j$. Since

$$\|\gamma_1(s_j)\| = \|\gamma_1(s)\| = 1 \quad \text{for all } j,$$

by (1-1) and (2-2), we have

$$0 = \frac{\|\gamma_1(s_j)\|^2 - \|\gamma_1(s)\|^2}{s_j - s} = \frac{\gamma_1(s_j) - \gamma_1(s)}{s_j - s} \cdot (\gamma_1(s_j) + \gamma_1(s)).$$
The conclusion follows by letting $j \to \infty$. \hfill \square

**Theorem 3.1.** Let $s \in L^1_{r*}$.

(1) For all $t \in \gamma_0^{-1}(G_f)$,

\begin{equation}
(3-6) \quad f(x(t)) - f(x(s)) - f'(x(s))(x(t) - x(s)) - \frac{1}{2} f''(x(s))(x(t) - x(s))^2 = \frac{1}{\gamma_1(s) \cdot u} (\gamma_1(s) \wedge \Sigma_s(t)) \sqsubseteq u.
\end{equation}

(2) For all $t \in I_s$,

$$
\Sigma_s(t) = \int_s^t \omega(\rho) \|\gamma_0(\rho)\| \left( \int_s^\rho \omega(\sigma) \|\gamma_0(\sigma)\| \Phi_s(\sigma) d\sigma \right) d\rho.
$$

**Proof.** (1) By recalling Proposition 3.1 and Lemma 3.1, we get

$$
\begin{align*}
&\quad f(x(t)) - f(x(s)) - f'(x(s))(x(t) - x(s)) - \frac{1}{2} f''(x(s))(x(t) - x(s))^2 \\
&= \gamma_0(t) - x(t)u - (\gamma_0(s) - x(s)u) - \left( \frac{\gamma_1(s)}{\gamma_1(s) \cdot u} - u \right) (x(t) - x(s)) \\
&\quad - \frac{(\gamma_1(s) \wedge \gamma_2(t)) \sqsubseteq u}{2 \|\gamma_2(t)\| (\gamma_1(s) \cdot u)^2} (x(t) - x(s))^2 \\
&= \Delta_s(t) - \frac{\gamma_1(s)}{\gamma_1(s) \cdot u} (\Delta_s(t) \cdot u) - \frac{(\gamma_1(s) \wedge \gamma_2(t)) \sqsubseteq u}{2 \|\gamma_2(t)\| (\gamma_1(s) \cdot u)^2} (\Delta_s(t) \cdot u)^2 \\
&= \frac{1}{\gamma_1(s) \cdot u} \left( (\gamma_1(s) \wedge \Delta_s(t)) \sqsubseteq u - \frac{(\gamma_1(s) \wedge \gamma_2(t)) \sqsubseteq u}{2 \|\gamma_2(t)\| (\gamma_1(s) \cdot u)^2} (\Delta_s(t) \cdot u)^2 \right).
\end{align*}
$$

This is just (3-6), in view of the definition of $\Sigma_s(t)$.

(2) Since $\Delta_s$ is Lipschitz and $\Delta_s(s) = 0$, one has

$$
\begin{align*}
\Sigma_s(t) &= \int_s^t \gamma_0(\rho) - (\gamma_0'(\rho) \cdot \gamma_1(s)) \gamma_1(s) - \frac{\mu(s)}{2(\gamma_1(s) \cdot u)^2} \frac{d}{d\rho} \left( (\Delta_s(\rho) \cdot u)^2 \right) d\rho \\
&= \int_s^t \gamma_0(\rho) - (\gamma_0'(\rho) \cdot \gamma_1(s)) \gamma_1(s) - \frac{\mu(s)}{(\gamma_1(s) \cdot u)^2} (\Delta_s(\rho) \cdot u)(\gamma_0'(\rho) \cdot u) d\rho
\end{align*}
$$

namely

\begin{equation}
(3-7) \quad \Sigma_s(t) = \int_s^t \omega(\rho) \|\gamma_0(\rho)\| \Phi_s(\rho) d\rho
\end{equation}

by (1-1), where $\Phi_s$ is the Lipschitz map defined by

$$
\Phi_s(\rho) := \gamma_1(\rho) - (\gamma_1(\rho) \cdot \gamma_1(s)) \gamma_1(s) - \frac{\mu(s)}{(\gamma_1(s) \cdot u)^2} (\Delta_s(\rho) \cdot u)(\gamma_1(\rho) \cdot u)
$$
for $\rho \in [a, b]$. Observe that
\[
\|\gamma_0'(\sigma)\|_{\gamma_2\bot} = \|\gamma_0'(\sigma)\|_{\gamma_2\bot} = \omega(\sigma)\|\gamma_0'(\sigma)\|_{\gamma_2\bot}
\]
for a.e. $\sigma \in [a, b]$, by (1-2). Hence
\[
y_1'(\sigma) = \omega(\sigma)\|\gamma_0'(\sigma)\|_{\gamma_2\bot}
\]
for a.e. $\sigma \in [a, b]$ such that $\gamma_2\bot(\sigma) \neq 0$ — in particular, for a.e. $\sigma \in I_s$. By recalling the definition of $\Psi_1$, it follows at once that
\[
(3-8) \quad \Phi_1'(\sigma) = \omega(\sigma)\|\gamma_0'(\sigma)\|_{\Psi_1}(\sigma)
\]
for a.e. $\sigma \in I_s$. We conclude using (3-7), (3-8) and noting that $\Phi_1(s) = 0$. □

As a consequence, we get the following integral representation of $\Sigma_1'$ and the related first order Taylor formula for $f'$.

**Corollary 3.1.** Let $s \in L^*_{\varepsilon}$ and $t \in L^*_{\varepsilon} \cap I_s$. Then

(1) The map $\Sigma_1'$ is differentiable at $t$ and
\[
\Sigma_1'(t) = \omega(t)\|\gamma_0'(t)\|_{\Psi_1}(t) \int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|_{\Psi_1}(\sigma) d\sigma.
\]

(2) One has
\[
f'(x(t)) - f'(x(s)) - f''(x(s))(x(t) - x(s)) = \\
\quad \frac{1}{(\gamma_1(t) \cdot u)(\gamma_1(s) \cdot u)} \left( \gamma_1(s) \wedge \int_s^t \omega(\sigma)\|\gamma_0'(\sigma)\|_{\Psi_1}(\sigma) d\sigma \right) \cdot u.
\]

**Proof.** (1) Observe that $t + h \in I_s \subset (a, b)$ provided $|h|$ is small enough. By Theorem 3.1(2), then,
\[
\frac{\Sigma_1(t + h) - \Sigma_1(t)}{h} = \frac{1}{h} \int_t^{t+h} \omega(\rho)\|\gamma_0'(\rho)\| \left( \int_s^\rho \omega(\sigma)\|\gamma_0'(\sigma)\|_{\Psi_1}(\sigma) d\sigma \right) d\rho
\]
\[
= I_1(h) + I_2(h)
\]
for all small enough values of $|h|$, where we have set — with a harmless abuse of notation and recalling that $\omega|L^*_{\varepsilon}$ is continuous, by (2-1) and (2-2) —
\[
I_1(h) := \frac{\omega(t)}{h} \int_{[t, t + h] \cap L^*_{\varepsilon}} \omega(\rho)\|\gamma_0'(\rho)\|_{\Psi_1}(\rho) d\rho,
\]
\[
I_2(h) := \frac{1}{h} \int_{[t, t + h] \cap L^*_{\varepsilon}} \omega(\rho)\|\gamma_0'(\rho)\|_{\Psi_1}(\rho) d\rho.
\]
We have

\[ I_1(h) = \frac{\omega(t)}{h} \int_{[s,t+h] \cap L_1^\epsilon} \left( \| \gamma_0'(\rho) \| - \| \gamma_0'(t) \| \right) \left( \int_s^t \omega(\sigma) \| \gamma_0'(\sigma) \| \Psi_s(\sigma) \, d\sigma \right) \, d\rho \]

\[ + \frac{\omega(t) \| \gamma_0'(t) \|}{h} \]

\[ \times \int_{[s,t+h] \cap L_1^\epsilon} \left( \int_s^t \omega(\sigma) \| \gamma_0'(\sigma) \| \Psi_s(\sigma) \, d\sigma + \int_t^\rho \omega(\sigma) \| \gamma_0'(\sigma) \| \Psi_s(\sigma) \, d\sigma \right) \, d\rho. \]

Recalling that

(i) \( \gamma_0' \big| L_\epsilon^* \) is continuous, by (2-1) and (2-2),

(ii) \( \gamma_0 \) is Lipschitz and \( \Psi_s \) is bounded (in fact it is Lipschitz!), and

(iii) \( t \) is a density point of \( L_\epsilon \) (hence of \( L_\epsilon^* \), by (2-3)),

we see that

\[ \lim_{h \to 0} I_1(h) = \omega(t) \| \gamma_0'(t) \| \int_s^t \omega(\sigma) \| \gamma_0'(\sigma) \| \Psi_s(\sigma) \, d\sigma. \]

The conclusion follows now by observing that, as an easy consequence of (ii) and (iii), one also has

\[ \lim_{h \to 0} I_2(h) = 0. \]

(2) The two members of (3-6) are differentiable at \( t \), by (1). Since \( t \) is a limit point of \( L_\epsilon \subset \gamma_0^{-1}(G_f) \) the derivatives have to coincide, by Theorem 3.1(1), namely

\[ \left( f'(x(t)) - f'(x(s)) - f''(x(s))(x(t) - x(s)) \right)x'(t) = \frac{1}{\gamma_1(s) \cdot u} \left( \gamma_1(s) \cap \Sigma_f(t) \right) u. \]

We conclude by recalling Proposition 3.1, part (1) of the corollary and (1-1). □

4. Conclusion of the proof of Theorem 1.1

To complete the proof of Theorem 2.1, hence of Theorem 1.1, we have to verify (2-4). For \( i = 1, 2, \ldots \), define \( \Gamma^{(i)} \) as the set of the points \( s \in L_\epsilon^* \) satisfying, for all \( t \in L_\epsilon^* \) such that \( |t - s| \leq (b - a)/i \), the estimates

\[ \| f(x(t)) - f(x(s)) - f'(x(s))(x(t) - x(s)) - \frac{1}{2} f''(x(s))(x(t) - x(s))^2 \| \leq i \| x(t) - x(s) \|^3, \]

\[ \| f'(x(t)) - f'(x(s)) - f''(x(s))(x(t) - x(s)) \| \leq i \| x(t) - x(s) \|^2, \]

\[ \| f''(x(t)) - f''(x(s)) \| \leq i \| x(t) - x(s) \|. \]

Obviously,

\[ \Gamma^{(i)} \subset \Gamma^{(i+1)} \subset L_\epsilon^* \]
for all \(i\), and it is easy to verify that

(4-1) \[ \bigcup_i \Gamma^{(i)} = L^s_c; \]

indeed, for \(s \in L^s_c\), Theorem 3.1 and the equality \(\Psi_s(s) = 0\) for the Lipschitz function \(\Psi_s\) imply that

\[
\|f(x(t)) - f(x(s)) - f'(x(s))(x(t) - x(s)) - \frac{1}{2} f''(x(s))(x(t) - x(s))^2\| \\
\leq \frac{\|\Sigma_s(t)\|}{|\gamma_1(s) \cdot u|} \leq \frac{\text{Lip}(\gamma_0) \text{Lip}(\Psi_s)}{|\gamma_1(s) \cdot u|} \int_s^{t'} \left( \int_s^\rho |\sigma - s| d\sigma \right) d\rho \\
= A(s)|t-s|^3
\]

for all \(t \in L^s_c \cap I_s\), where

\[
A(s) := \frac{\text{Lip}(\gamma_0)^2 \text{Lip}(\Psi_s)}{6 |\gamma_1(s) \cdot u|}.
\]

Since

\[
\frac{x(t) - x(s)}{t-s} \to x'(s) \quad \text{(as } t \to s)\]

and \(x'(s) \neq 0\) by Proposition 3.1, it follows that

(4-2) \[ \left| \frac{x(t) - x(s)}{t-s} \right| \geq \frac{|x'(s)|}{2} > 0 \]

provided \(|t-s|\) is small enough. Then

(4-3) \[
\|f(x(t)) - f(x(s)) - f'(x(s))(x(t) - x(s)) - \frac{1}{2} f''(x(s))(x(t) - x(s))^2\| \\
\leq \frac{8A(s)}{|x'(s)|^3} |x(t) - x(s)|^3
\]

whenever \(t\) lies in \(L^s_c\) and \(|t-s|\) is small enough.

Analogously, from Corollary 3.1(2) we get

\[
\|f''(x(t)) - f''(x(s)) - f''(x(s))(x(t) - x(s))\| \\
\leq \frac{1}{|\gamma_1(t) \cdot u| |\gamma_1(s) \cdot u|} \left\| \int_s^{t'} \omega(\sigma) \|\gamma_0'(\sigma)\| \Psi_s(\sigma) d\sigma \right\| \\
\leq \frac{\text{Lip}(\gamma_0) \text{Lip}(\Psi_s)}{|\gamma_1(t) \cdot u| |\gamma_1(s) \cdot u|} \left( \int_s^{t'} |\sigma - s| d\sigma \right) = \frac{B(s)}{|\gamma_1(t) \cdot u|} |t-s|^2
\]

for all \(t \in L^s_c \cap I_s\), where

\[
B(s) := \frac{\text{Lip}(\gamma_0) \text{Lip}(\Psi_s)}{2 |\gamma_1(s) \cdot u|}.
\]
Since $\gamma_1(t) \to \gamma_1(s)$ as $t \to s$ and since $\gamma_1(s) \cdot u \neq 0$ by Proposition 3.1, one also has

\[(4-4)\quad |\gamma_1(t) \cdot u| \geq \frac{|\gamma_1(s) \cdot u|}{2} > 0\]

provided $|t - s|$ is small enough. Recalling (4-2), we obtain

\[(4-5)\quad \|f''(x(t)) - f''(x(s))(x(t) - x(s))\| \leq 8B(s)\frac{|\gamma_1(s) \cdot u|}{\|x'(s)\|^2} \|x(t) - x(s)\|^2\]

on condition that $t \in L^*_e$ and $|t - s|$ is small enough.

Since $\mu|[s]$ is Lipschitz and by (4-4), it follows that the map

\[t \mapsto \gamma_1(t) \wedge \mu(t) \bmod u \quad (\gamma_1(t) \cdot u)^3\]

is Lipschitz in a neighborhood of $s$. Then, by also recalling Proposition 3.1, a number $C(s)$ has to exist such that

\[\|f''(x(t)) - f''(x(s))\| \leq C(s) |t - s|\]

provided $t \in L^*_e$ and $|t - s|$ is small enough. By (4-2) one has

\[(4-6)\quad \|f''(x(t)) - f''(x(s))\| \leq 2C(s) \frac{\|x'(s)\|}{\|x'(s)\|} \|x(t) - x(s)\|\]

whenever $t \in L^*_e$ and $|t - s|$ is small enough.

Now (4-3), (4-5) and (4-6) imply that $s \in \Gamma^{(i)}$, for $i$ big enough. Hence (4-1) follows.

As a consequence of (4-1), we are reduced to verifying that

\[(4-7)\quad \gamma_0(\Gamma^{(i)}) \text{ is } C^3\text{-rectifiable for all } i.\]

To prove this, we first set

\[a^{(i)}_j := a + \frac{(b - a)j}{i} \quad (j = 0, \ldots, i),\]

\[\Gamma^{(i)}_j := \Gamma^{(i)} \cap (a^{(i)}_j, a^{(i)}_{j+1}) \quad (j = 0, \ldots, i - 1),\]

\[F^{(i)}_j := x(\Gamma^{(i)}_j) \quad (j = 0, \ldots, i - 1).\]

For any pair $\xi, \eta \in F^{(i)}_j$, there are two sequences $\{s_h\}, \{t_h\} \subset \Gamma^{(i)}_j$ such that

\[\lim_{h \to \infty} x(s_h) = \xi, \quad \lim_{h \to \infty} x(t_h) = \eta.\]
Since the three estimates at the beginning of this section (page 264) hold with $s = s_h, t = t_h$, we obtain, by letting $h \to \infty$,

$$
\| f(\eta) - f(\xi) - f'(\xi)(\eta - \xi) \| \leq i |\eta - \xi|^{3},
$$

$$
\| f'(\eta) - f'(\xi) - f''(\xi)(\eta - \xi) \| \leq i |\eta - \xi|^{2},
$$

$$
\| f''(\eta) - f''(\xi) \| \leq i |\eta - \xi|.
$$

Therefore $f|F_{i}^{j}$ can be extended to a map of class $C^2.1$

$$
f_{i}^{j} : \mathbb{R} \to (\mathbb{R}u)^{1}
$$

by invoking the Whitney extension Theorem [Stein 1970, Chapter VI, §2.3].

Finally, a Lusin-type result [Federer 1969, §3.1.15] implies that $\gamma_0(F_{i}^{j})$ has to be $C^3$-rectifiable (compare [Anzellotti and Serapioni 1994, Proposition 3.2]). Hence (4-7) follows.

5. Approximately differentiable absolute curvature of a one-dimensional $C^3$-rectifiable set

We now extend the notion of absolute curvature to arbitrary one-dimensional $C^2$-rectifiable subsets $P$ of $\mathbb{R}^{1+k}$. Consider a “$C^2$-covering of $P$”, that is, a countable family

$$
\mathcal{A} = \{C_i\},
$$

where the $C_i$ are compact curves of class $C^2$, embedded in the base space and such that

$$
\mathcal{H}^1(P \setminus \bigcup_i C_i) = 0.
$$

Part (1) of the next proposition and the remark following it provide the argument proving the well-posedness of Definition 5.1 below.

**Proposition 5.1.** Let $\varphi, \psi : \mathbb{R} \to \mathbb{R}^{1+k}$ be maps of class $C^2$ and let $x_0$ be a density point of

$$
F := \{x \in \mathbb{R} \mid \varphi(x) = \psi(x)\}.
$$

(1) $\varphi'(x_0) = \psi'(x_0)$ and $\varphi''(x_0) = \psi''(x_0)$.

(2) $\varphi'''(x_0) = \psi'''(x_0)$ if $\varphi$ and $\psi$ are of class $C^3$.

**Proof.** The set $F^*$ of density points of $F$ satisfies $F^* \subset F$ and $\mathcal{H}^1(F \setminus F^*) = 0$; hence every point in $F^*$ is a limit point of $F^*$. The proposition follows. \qed

**Remark.** The following facts follow easily from Proposition 5.1(1).
(a) If $x$ is a density point of both $P \cap C_i$ and $P \cap C_j$, then the absolute curvatures of $C_i$ and $C_j$ coincide at $x$. Hence, denoting by $(P \cap C_i)^*$ the set of density points of $P \cap C_i$, the function
\[ \alpha_P^i : \bigcup_i (P \cap C_i)^* \to \mathbb{R}, \quad x \mapsto \text{the absolute curvature of } C_{i(x)} \text{ at } x \]
where $i(x)$ is any index such that $x \in (P \cap C_{i(x)})^*$, is well-defined. Moreover,
\[ \mathcal{H}^1 \left( P \setminus \bigcup_i (P \cap C_i)^* \right) = \mathcal{H}^1 (P \setminus j_1 (P \cap C_i)) = \mathcal{H}^1 (P \setminus j_1 C_i) = 0, \]
by a well-known result of Lebesgue.

(b) If $\mathcal{B}$ is another $C^2$-covering of $P$, then $\alpha_P^A$ and $\alpha_P^B$ are representatives of the same measurable function, with domain $P$.

**Definition 5.1.** The measurable real-valued function with domain $P$ and having $\alpha_P^A$ as a representative (see preceding remark) is said to be the absolute curvature of $P$ and is denoted by $\alpha_P$.

**Proposition 5.2.** If $P$ is $C^3$-rectifiable, then $\alpha_P$ is approximately differentiable; that is:

1. For any given $C^3$-covering $\mathcal{A} = \{C_i\}$ of $P$, the function $\alpha_P^A$ is approximately differentiable at every point in $(P \cap C_i)^*$, for all $i$.
2. If $\mathcal{A}$ and $\mathcal{B}$ are $C^3$-coverings of $P$, then one has
\[ apD \alpha_P^A = apD \alpha_P^B, \quad a.e. \text{ in } P. \]

**Proof.** (1) Consider any point $a \in (P \cap C_{i_0})^*$. Without loss of generality, we can assume that $C_{i_0}$ is the graph of a function of class $C^3$, namely
\[ C_{i_0} = \{ tu + h(t) \mid t \in I \} \]
where $u$ is a unit vector in $\mathbb{R}^{1+k}$, $I$ is a closed interval centered at 0 and
\[ h \in C^3 (I, (\mathbb{R}u)^{\perp}), \quad h(0) = a. \]
Set $U := I^o \times \mathbb{R}^k$ and let $g : U \to \mathbb{R}$ be defined as the function mapping $(t, v) \in U$ to the absolute curvature of $C_{i_0}$ at $tu + h(t)$, that is,
\[ g(t, v) = \frac{\| (u + h'(t)) \wedge h''(t) \|}{(1 + \| h'(t) \|)^{3/2}}, \quad (t, v) \in U \]
by (8.4.13.1) in [Berger and Gostiaux 1988].

Obviously, the function $g$ is differentiable at $a$. Moreover, since
\[ (P \cap C_{i_0})^* \subset E := \{ x \in \bigcup_i (P \cap C_i)^* \mid \alpha_P^A(x) = g(x) \} \]
by the definition of $\alpha^{\delta}_P$, the set $E$ has density 1 at $a$. According to [Federer 1969, §3.2.16], the function $\alpha^{\delta}_P$ is approximately differentiable at $a$ and one has

\[(5-2) \quad apD\alpha^{\delta}_P(a) = Dg(a)\|\mathbb{R}\tau, \text{ with } \tau := (1, h'(0)).\]

(2) This follows easily from (5-1) and (5-2), by recalling Proposition 5.1. □

References


Received August 18, 2005.

SILVANO DELLADIO
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI TRENTO
VIA SOMMARIVE 14, POVO
38050 TRENTO
ITALY
silvano.delladio@unitn.it