Pacific Journal of Mathematics

THE TWO-PARAMETER QUANTUM GROUP OF EXCEPTIONAL TYPE G_2 AND LUSZTIG'S SYMMETRIES

NAIHONG HU AND QIAN SHI

Volume 230 No. 2 April 2007

PACIFIC JOURNAL OF MATHEMATICS
Vol. 230, No. 2, 2007

THE TWO-PARAMETER QUANTUM GROUP OF EXCEPTIONAL TYPE G_2 AND LUSZTIG'S SYMMETRIES

NAIHONG HU AND QIAN SHI

We give the defining structure of the two-parameter quantum group of type G_2 defined over a field $\mathbb{Q}(r, s)$ (with $r \neq s$), and prove the Drinfel'd double structure as its upper and lower triangular parts, extending a result of Benkart and Witherspoon in type A and a recent result of Bergeron, Gao, and Hu in types B, C, D. We further discuss Lusztig's $\mathbb{Q}$-isomorphisms from $U_{r,s}(G_2)$ to its associated object $U_{s^{-1},r^{-1}}(G_2)$, which give rise to the usual Lusztig symmetries defined not only on $U_q(G_2)$ but also on the centralized quantum group $U_q^c(G_2)$ only when $r = s^{-1} = q$. (This also reflects the distinguishing difference between our newly defined two-parameter object and the standard Drinfel'd–Jimbo quantum groups.) Some interesting (r, s)-identities holding in $U_{r,s}(G_2)$ are derived from this discussion.

1. The two-parameter quantum group $U_{r,s}(G_2)$

Let $\mathbb{K} = \mathbb{Q}(r, s)$ be a field of rational functions with two indeterminates r, s.

Let Φ be a finite root system of G_2 with Π a base of simple roots, which is a subset of a Euclidean space $E = \mathbb{R}^3$ with an inner product $(\, ,\,)$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ denote an orthonormal basis of E. Then $\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 + \epsilon_3 - 2\epsilon_1\}$ and $\Phi = \pm\{\alpha_1, \alpha_2, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \alpha_2 + 3\alpha_1, 2\alpha_2 + 3\alpha_1\}$. In this case, we set $r_1 = r^{(\alpha_1,\alpha_1)/2} = r,\ r_2 = r^{(\alpha_2,\alpha_2)/2} = r^3$ and $s_1 = s^{(\alpha_1,\alpha_1)/2} = s,\ s_2 = s^{(\alpha_2,\alpha_2)/2} = s^3$.

We begin by defining the two-parameter quantum group of type G_2, which is new.

Definition 1.1. Let $U = U_{r,s}(G_2)$ be the associative algebra over $\mathbb{Q}(r, s)$ generated by the symbols $e_i,\ f_i,\ \omega_i^{\pm 1}$ and $\omega_i'^{\pm 1}$ $(1 \le i \le 2)$, subject to the relations

(G1) $[\,\omega_i^{\pm 1}, \omega_j^{\pm 1}\,] = [\,\omega_i^{\pm 1}, \omega_j'^{\pm 1}\,] = [\,\omega_i'^{\pm 1}, \omega_j'^{\pm 1}\,] = 0, \quad \omega_i\omega_i^{-1} = 1 = \omega_j'\omega_j'^{-1};$

MSC2000: primary 17B37, 81R50; secondary 17B35.

Keywords: 2-parameter quantum group, Hopf skew-dual pairing, Hopf dual pairing, Drinfel'd quantum double, Lusztig symmetries.

Hu is supported in part by the NNSF (Grant 10431040), the PCSIRT, the TRAPOYT and the FUDP from the MOE of China, the SRSTP from the STCSM, and the Shanghai Priority Academic Discipline from the SMEC.

(G2)
$$\begin{aligned} &\omega_1\, e_1\, \omega_1^{-1} = (rs^{-1})\, e_1, && \omega_1 f_1\, \omega_1^{-1} = (r^{-1}s)\, f_1,\\ &\omega_1\, e_2\, \omega_1^{-1} = s^3\, e_2, && \omega_1 f_2\, \omega_1^{-1} = s^{-3} f_2,\\ &\omega_2\, e_1\, \omega_2^{-1} = r^{-3}\, e_1, && \omega_2 f_1\, \omega_2^{-1} = r^3 f_1,\\ &\omega_2\, e_2\, \omega_2^{-1} = (r^3s^{-3})\, e_2, && \omega_2 f_2\, \omega_2^{-1} = (r^{-3}s^3)\, f_2; \end{aligned}$$

(G3)
$$\begin{aligned} &\omega_1'\, e_1\, \omega_1'^{-1} = (r^{-1}s)\, e_1, && \omega_1' f_1\, \omega_1'^{-1} = (rs^{-1})\, f_1,\\ &\omega_1'\, e_2\, \omega_1'^{-1} = r^3\, e_2, && \omega_1' f_2\, \omega_1'^{-1} = r^{-3} f_2,\\ &\omega_2'\, e_1\, \omega_2'^{-1} = s^{-3}\, e_1, && \omega_2' f_1\, \omega_2'^{-1} = s^3 f_1,\\ &\omega_2'\, e_2\, \omega_2'^{-1} = (r^{-3}s^3)\, e_2, && \omega_2' f_2\, \omega_2'^{-1} = (r^3s^{-3})\, f_2; \end{aligned}$$

(G4)
$$[\, e_i, f_j\,] = \delta_{ij}\frac{\omega_i - \omega_i'}{r_i - s_i} \quad \text{for } 1 \le i,\ j \le 2;$$

(G5) ((r, s)-Serre relations in positive part)

$$e_2^2e_1 - (r^{-3}+s^{-3})\, e_2e_1e_2 + (rs)^{-3}\, e_1e_2^2 = 0,$$
$$e_1^4e_2 - (r+s)(r^2+s^2)\, e_1^3e_2e_1 + rs(r^2+s^2)(r^2+rs+s^2)\, e_1^2e_2e_1^2 \\ -(rs)^3(r+s)(r^2+s^2)\, e_1e_2e_1^3 + (rs)^6e_2e_1^4 = 0;$$

(G6) ((r, s)-Serre relations in negative part)

$$f_1f_2^2 - (r^{-3}+s^{-3}) f_2f_1f_2 + (rs)^{-3} f_2^2f_1 = 0,$$
$$f_2f_1^4 - (r+s)(r^2+s^2) f_1f_2f_1^3 + rs(r^2+s^2)(r^2+rs+s^2) f_1^2f_2f_1^2 \\ -(rs)^3(r+s)(r^2+s^2) f_1^3f_2f_1 + (rs)^6 f_1^4f_2 = 0.$$

Proposition 1.2. *The algebra $U_{r,s}(G_2)$ is a Hopf algebra with comultiplication, counit and antipode given by*

$$\begin{aligned} &\Delta(\omega_i^{\pm1}) = \omega_i^{\pm1}\otimes\omega_i^{\pm1}, && \Delta(e_i) = e_i\otimes 1 + \omega_i\otimes e_i,\\ &\Delta(\omega_i'^{\pm1}) = \omega_i'^{\pm1}\otimes\omega_i'^{\pm1}, && \Delta(f_i) = 1\otimes f_i + f_i\otimes\omega_i',\\ &\varepsilon(\omega_i^{\pm}) = \varepsilon(\omega_i'^{\pm1}) = 1, && S(\omega_i^{\pm1}) = \omega_i^{\mp1},\ \ S(e_i) = -\omega_i^{-1}e_i,\\ &\varepsilon(e_i) = \varepsilon(f_i) = 0, && S(\omega_i'^{\pm1}) = \omega_i'^{\mp1},\ S(f_i) = -f_i\,\omega_i'^{-1}. \end{aligned}$$

Remark 1.3. (I) When $r = q = s^{-1}$, the quotient Hopf algebra of $U_{r,s}(G_2)$ modulo the Hopf ideal generated by elements $\omega_i' - \omega_i^{-1}$ $(1 \le i \le 2)$ is just the standard quantum group $U_q(G_2)$ of Drinfel'd–Jimbo type; the quotient modulo the Hopf ideal generated by elements $\omega_i' - z_i\omega_i^{-1}$ $(1 \le i \le 2)$, where z_i runs over the center, is the *centralized quantum group $U_q^c(G_2)$*.

(II) In any Hopf algebra $\mathcal{H}$, there exist left-adjoint and right-adjoint actions defined by the Hopf algebra structure:

$$\mathrm{ad}_l\, a\,(b) = \sum_{(a)} a_{(1)}\, b\, S(a_{(2)}), \qquad \mathrm{ad}_r\, a\,(b) = \sum_{(a)} S(a_{(1)})\, b\, a_{(2)},$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \in \mathcal{H} \otimes \mathcal{H}$, for any $a, b \in \mathcal{H}$.

From the viewpoint of adjoint actions, the (r, s)-Serre relations (G5) and (G6) take on the simpler forms

$$(\mathrm{ad}_l\, e_i)^{1-a_{ij}}\,(e_j) = 0 \qquad \textit{for any } i \neq j,$$
$$(\mathrm{ad}_r\, f_i)^{1-a_{ij}}\,(f_j) = 0 \qquad \textit{for any } i \neq j.$$

2. The Drinfel'd quantum double

Definition 2.1. A (Hopf) dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ (see [Bergeron et al. 2006] or [Klimyk and Schmüdgen 1997]) is a bilinear form $\langle\,,\rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{K}$ such that

(1) $$\langle f,\, 1_{\mathcal{A}}\rangle = \varepsilon_{\mathcal{U}}(f), \quad \langle 1_{\mathcal{U}},\, a\rangle = \varepsilon_{\mathcal{A}}(a),$$

(2) $$\langle f,\, a_1 a_2\rangle = \langle \Delta_{\mathcal{U}}(f),\, a_1 \otimes a_2\rangle, \quad \langle f_1 f_2,\, a\rangle = \langle f_1 \otimes f_2,\, \Delta_{\mathcal{A}}(a)\rangle,$$

for all $f,\, f_1,\, f_2 \in \mathcal{U}$, and $a,\, a_1,\, a_2 \in \mathcal{A}$, where $\varepsilon_{\mathcal{U}},\, \varepsilon_{\mathcal{A}}$ denote the counits of $\mathcal{U}$, $\mathcal{A}$ and $\Delta_{\mathcal{U}},\, \Delta_{\mathcal{A}}$ the comultiplications.

A direct consequence of the defining properties above is that

$$\langle S_{\mathcal{U}}(f),\, a\rangle = \langle f,\, S_{\mathcal{A}}(a)\rangle, \quad f \in \mathcal{U},\, a \in \mathcal{A},$$

where $S_{\mathcal{U}}$, $S_{\mathcal{A}}$ denote the respective antipodes of $\mathcal{U}$ and $\mathcal{A}$.

Definition 2.2. A bilinear form $\langle\,,\rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{K}$ is called a skew-dual pairing of two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ (see [Bergeron et al. 2006]) if $\langle\,,\rangle : \mathcal{U}^{\mathrm{cop}} \times \mathcal{A} \to \mathbb{K}$ is a Hopf dual pairing of $\mathcal{A}$ and $\mathcal{U}^{\mathrm{cop}}$, where $\mathcal{U}^{\mathrm{cop}}$ is the Hopf algebra having the opposite comultiplication to $\mathcal{U}$, and $S_{\mathcal{U}^{\mathrm{cop}}} = S_{\mathcal{U}}^{-1}$ is invertible.

Denote by $\mathcal{B} = B(G_2)$ the Hopf subalgebra of $U_{r,s}(G_2)$ generated by e_j, $\omega_j^{\pm 1}$, and by $\mathcal{B}' = B'(G_2)$ the one generated by f_j, $\omega_j'^{\pm 1}$, where $j = 1, 2$.

Proposition 2.3. *There exists a unique skew-dual pairing* $\langle\,,\rangle : \mathcal{B}' \times \mathcal{B} \to \mathbb{Q}(r, s)$ *of the Hopf subalgebras* $\mathcal{B}$ *and* $\mathcal{B}'$ *such that*

(3) $$\langle f_i,\, e_j\rangle = \delta_{ij}\,\frac{1}{s_i - r_i} \qquad (1 \le i,\, j \le 2),$$

(4) $$\langle \omega_1',\, \omega_1\rangle = rs^{-1}, \quad \langle \omega_1',\, \omega_2\rangle = r^{-3},$$
$$\langle \omega_2',\, \omega_1\rangle = s^3, \qquad \langle \omega_2',\, \omega_2\rangle = r^3 s^{-3},$$

(5) $$\langle \omega_i'^{\pm 1},\, \omega_j^{-1}\rangle = \langle \omega_i'^{\pm 1},\, \omega_j\rangle^{-1} = \langle \omega_i',\, \omega_j\rangle^{\mp 1} \quad (1 \le i,\, j \le 2),$$

and all other pairs of generators yield 0*. Furthermore*, $\langle S(a),\, S(b)\rangle = \langle a,\, b\rangle$ *for* $a \in \mathcal{B}'$, $b \in \mathcal{B}$.

Proof. Since any skew-dual pairing of bialgebras is determined by its values on generators, uniqueness is clear. We proceed to prove the existence of the pairing.

We begin by defining a bilinear form $\langle\,,\rangle : \mathcal{B}'^{\mathrm{cop}} \times \mathcal{B} \to \mathbb{Q}(r,\, s)$ first on the generators satisfying (3), (4), and (5). Then we extend it to a bilinear form on $\mathcal{B}'^{\mathrm{cop}} \times \mathcal{B}$ by requiring that (1) and (2) hold for $\triangle_{\mathcal{B}'^{\mathrm{cop}}} = \triangle^{\mathrm{op}}_{\mathcal{B}'}$. We will verify that the relations in $\mathcal{B}$ and $\mathcal{B}'$ are preserved, ensuring that the form is well-defined and so is a dual pairing of $\mathcal{B}$ and $\mathcal{B}'^{\mathrm{cop}}$ by definition.

It is direct to check that the bilinear form preserves all the relations among the $\omega_i^{\pm 1}$ in $\mathcal{B}$ and the $\omega_i'^{\pm 1}$ in $\mathcal{B}'$. Next, the structure constants (4) ensure the compatibility of the form defined above with those relations of (G2) and (G3) in $\mathcal{B}$ or $\mathcal{B}'$, respectively. We are left to verify that the form preserves the $(r,\, s)$-Serre relations in $\mathcal{B}$ and $\mathcal{B}'$. It suffices to show that the form on $\mathcal{B}'^{\mathrm{cop}} \times \mathcal{B}$ preserves the $(r,\, s)$-Serre relations in $\mathcal{B}$; the verification for $\mathcal{B}'^{\mathrm{cop}}$ is similar.

First, let us show that the form preserves the $(r,\, s)$-Serre relation of degree 2 in $\mathcal{B}$, that is,

$$\langle X,\, e_2^2 e_1 - (r^{-3}+s^{-3})\, e_2 e_1 e_2 + r^{-3}s^{-3}\, e_1 e_2^2 \,\rangle = 0,$$

where X is any word in the generators of $\mathcal{B}'$. It suffices to consider three monomials: $X = f_2^2 f_1,\ f_2 f_1 f_2,\ f_1 f_2^2$. However, in the degree 2's situation for type G_2, its proof is the same as that of type C_2 (see [Bergeron et al. 2006, (7C) and thereafter]).

Next, we verify that the $(r,\, s)$-Serre relation of degree 4 in $\mathcal{B}$ is preserved by the form; that is, we show that

$$\Big\langle X,\ e_1^4 e_2 - (r+s)(r^2+s^2)\, e_1^3 e_2 e_1 + rs(r^2+s^2)(r^2+rs+s^2)\, e_1^2 e_2 e_1^2 \\ -(rs)^3(r+s)(r^2+s^2)\, e_1 e_2 e_1^3 + (rs)^6\, e_2 e_1^4\Big\rangle$$

vanishes, where X is any word in the generators of $\mathcal{B}'$. By definition, this expression equals

$$\begin{aligned}(6)\quad \Big\langle \triangle^{(4)}(X),\ & e_1\otimes e_1\otimes e_1\otimes e_1\otimes e_2 - (r+s)(r^2+s^2)\, e_1\otimes e_1\otimes e_1\otimes e_2\otimes e_1 \\ & + rs(r^2+s^2)(r^2+rs+s^2)\, e_1\otimes e_1\otimes e_2\otimes e_1\otimes e_1 \\ & -(rs)^3(r+s)(r^2+s^2)\, e_1\otimes e_2\otimes e_1\otimes e_1\otimes e_1 + (rs)^6\, e_2\otimes e_1\otimes e_1\otimes e_1\otimes e_1 \Big\rangle,\end{aligned}$$

where $\triangle$ in the left-hand side of the pairing $\langle\,,\rangle$ indicates $\triangle^{\mathrm{op}}_{\mathcal{B}'}$. In order for any one of these terms to be nonzero, X must involve exactly four f_1 factors, one f_2 factor, and arbitrarily many $\omega_j'^{\pm 1}$ factors $(j = 1, 2)$.

It suffices to consider five key cases:

(i) If $X = f_1^4 f_2$, we have

$$
\begin{aligned}
\triangle^{(4)}(X) = \big(&\omega_1' \otimes \omega_1' \otimes \omega_1' \otimes \omega_1' \otimes f_1 + \omega_1' \otimes \omega_1' \otimes \omega_1' \otimes f_1 \otimes 1 \\
&+ \omega_1' \otimes \omega_1' \otimes f_1 \otimes 1 \otimes 1 + \omega_1' \otimes f_1 \otimes 1 \otimes 1 \otimes 1 + f_1 \otimes 1 \otimes 1 \otimes 1 \otimes 1\big)^4 \\
&\cdot \big(\omega_2' \otimes \omega_2' \otimes \omega_2' \otimes \omega_2' \otimes f_2 + \omega_2' \otimes \omega_2' \otimes \omega_2' \otimes f_2 \otimes 1 \\
&+ \omega_2' \otimes \omega_2' \otimes f_2 \otimes 1 \otimes 1 + \omega_2' \otimes f_2 \otimes 1 \otimes 1 \otimes 1 + f_2 \otimes 1 \otimes 1 \otimes 1 \otimes 1\big).
\end{aligned}
$$

Expanding $\triangle^{(4)}(X)$, we get 120 relevant terms having a nonzero contribution to (6). They are listed in Table 1, together with their pairing values, where we have

Table 1. Terms of $\Delta^{(4)}(f_1^4 f_2)$ in (6) and their pairing values. We write β instead of ω' and $\cdot$ instead of $\otimes$ to save space. We have also set $a = \langle f_1, e_1\rangle^4 \langle f_2, e_2\rangle$, $x = \langle \omega_1', \omega_1\rangle$, $\bar{x} = \langle \omega_1', \omega_2\rangle$.

Summand in (6)	**1**	Summand in (6)	**2**
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	a	$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}a$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	xa	$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}xa$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	xa	$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^5a	$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot f_1\beta_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^5a$
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	xa	$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^2a	$f_1\beta_1^3\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^5a	$\beta_1^3 f_1\beta_2\cdot \beta_1 f_1\beta_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^5a$
$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^3a	$f_1\beta_1^3\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^3a$
$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^4a	$\beta_1 f_1\beta_1^2\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^4a$
$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^5a	$\beta_1^2 f_1\beta_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^5a$
$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot f_1\beta_2\cdot f_2$	x^6a	$\beta_1^3 f_1\beta_2\cdot \beta_1^2 f_1\beta_2\cdot \beta_1 f_1\beta_2\cdot \beta_1 f_2\cdot f_1$	$\bar{x}x^6a$

Table 1. (Continued)

Summand in (6)	**3**	Summand in (6)	**4**
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2xa$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^2a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^4a$
$f_1\beta_1^3\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^2a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^4a$
$\beta_1^3 f_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^5a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^5a$
$f_1\beta_1^3\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^3a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^4a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^5a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^5a$
$\beta_1^3 f_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^2x^6a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^3x^6a$
$f_1\beta_1^3\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3a$
$\beta_1 f_1\beta_1^2\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2xa$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3xa$
$\beta_1^2 f_1\beta_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^2a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1^3 f_1\beta_2\cdot f_1\beta_1^2\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^3a$
$f_1\beta_1^3\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2xa$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3xa$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^2a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^3 f_1\beta_2\cdot\beta_1 f_1\beta_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^4a$
$f_1\beta_1^3\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^2a$	$f_1\beta_1^3\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^2a$
$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^3a$	$\beta_1 f_1\beta_1^2\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^3a$
$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^4a$	$\beta_1^2 f_1\beta_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^4a$
$\beta_1^3 f_1\beta_2\cdot\beta_1^2 f_1\beta_2\cdot\beta_1^2 f_2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^2x^5a$	$\beta_1^3 f_1\beta_2\cdot\beta_1^3 f_2\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^3x^5a$

Summand in (6)	**5**	Summand in (6)	**5**
$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4xa$	$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4a$
$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^2a$	$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4xa$
$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^3a$	$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^2a$
$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot f_1\beta_1^2\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^4a$	$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot f_1\beta_1^2\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^3a$
$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^2a$	$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4xa$
$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^3a$	$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^2a$
$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^4a$	$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^3a$
$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1 f_1\beta_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^5a$	$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1 f_1\beta_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^4a$
$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^3a$	$\beta_1^4 f_2\cdot f_1\beta_1^3\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^2a$
$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^4a$	$\beta_1^4 f_2\cdot\beta_1 f_1\beta_1^2\cdot\beta_1^2 f_1\cdot' f_1\beta_1\cdot f_1$	$\bar{x}^4x^3a$
$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^5a$	$\beta_1^4 f_2\cdot\beta_1^2 f_1\beta_1\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^4a$
$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1^2 f_1\cdot\beta_1 f_1\cdot f_1$	$\bar{x}^4x^6a$	$\beta_1^4 f_2\cdot\beta_1^3 f_1\cdot\beta_1^2 f_1\cdot f_1\beta_1\cdot f_1$	$\bar{x}^4x^5a$

introduced

$$a = \langle f_1, e_1\rangle^4 \langle f_2, e_2\rangle, \quad x = \langle \omega_1', \omega_1\rangle, \quad \bar{x} = \langle \omega_1', \omega_2\rangle.$$

The expression in (6) equals

$$\begin{aligned}&(\text{sum of expressions in part 1 of Table 1})\\ &\qquad - (\text{sum of expressions in part 2}) \cdot (r+s)(r^2+s^2)\\ &\qquad + (\text{sum of expressions in part 3}) \cdot rs(r^2+s^2)(r^2+rs+s^2)\\ &\qquad - (\text{sum of expressions in part 4}) \cdot (rs)^3(r+s)(r^2+s^2)\\ &\qquad + (\text{sum of expressions in part 5}) \cdot (rs)^6.\end{aligned}$$

Thus, if we sum up all the pairing values listed in each part of Table 1 and multiply by the appropriate factor, we obtain the pairing value of (6):

$$\begin{aligned}&a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)\cdot\big(1-(r+s)(r^2+s^2)\bar{x}\\ &\quad + rs(r^2+s^2)\cdot(r^2+rs+s^2)\bar{x}^2 - (rs)^3(r+s)(r^2+s^2)\bar{x}^3 + (rs)^6\bar{x}^4\big)\\ &= a(1+3x+5x^2+6x^3+5x^4+3x^5+x^6)(1-r^3\bar{x})(1-r^2s\bar{x})(1-rs^2\bar{x})(1-s^3\bar{x})\\ &= 0 \qquad\qquad (\text{because } \bar{x} = \langle \omega_1', \omega_2\rangle = r^{-3}).\end{aligned}$$

(ii) $X = f_2 f_1^4$.

(iii) $X = f_1^2 f_2 f_1^2$.

(iv) $X = f_1^3 f_2 f_1$.

(v) $X = f_1 f_2 f_1^3$.

These four other cases are handled similarly, using the formulas given in the Appendix of [Hu and Shi 2006]. This takes care of $\Delta^{(4)}(X)$. The proof is completed by checking that the relations in B'^{cop} are preserved for G_2. □

Definition 2.4. For any two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ connected by a skew-dual pairing $\langle\,,\rangle$, one may form the Drinfel'd quantum double $\mathcal{D}(\mathcal{A}, \mathcal{U})$ as in [Klimyk and Schmüdgen 1997, 3.2], which is a Hopf algebra whose underlying coalgebra is $\mathcal{A} \otimes \mathcal{U}$ with the tensor product coalgebra structure, whose algebra structure is defined by

$$(a\otimes f)(a'\otimes f') = \sum \langle \mathcal{S}_{\mathcal{U}}(f_{(1)}), a'_{(1)}\rangle \langle (f_{(3)}), a'_{(3)}\rangle a a'_{(2)} \otimes f_{(2)} f' \tag{7}$$

for $a, a' \in \mathcal{A}$ and $f, f' \in \mathcal{U}$, and whose antipode S is given by

$$S(a\otimes f) = (1\otimes \mathcal{S}_{\mathcal{U}}(f))(\mathcal{S}_{\mathcal{A}}(a)\otimes 1). \tag{8}$$

Clearly, both mappings $\mathcal{A} \ni a \mapsto a\otimes 1 \in \mathcal{D}(\mathcal{A}, \mathcal{U})$ and $\mathcal{U} \ni f \mapsto 1\otimes f \in \mathcal{D}(\mathcal{A}, \mathcal{U})$ are injective Hopf algebra homomorphisms. Denote the image $a\otimes 1$ of a in $\mathcal{D}(\mathcal{A}, \mathcal{U})$ by $\hat{a}$, and the image $1\otimes f$ of f by $\hat{f}$. By (7), we have the following cross relations between elements $\hat{a}$ (for $a \in \mathcal{A}$) and $\hat{f}$ (for $f \in \mathcal{U}$) in the algebra $\mathcal{D}(\mathcal{A}, \mathcal{U})$:

$$\hat{f}\hat{a} = \sum \langle \mathcal{S}_{\mathcal{U}}(f_{(1)}), a_{(1)}\rangle \langle (f_{(3)}), a_{(3)}\rangle \hat{a}_{(2)} \hat{f}_{(2)}, \tag{9}$$

$$\sum \langle f_{(1)}, a_{(1)}\rangle \hat{f}_{(2)}\hat{a}_{(2)} = \sum \hat{a}_{(1)}\hat{f}_{(1)}\langle f_{(2)}, a_{(2)}\rangle. \tag{10}$$

In fact, as an algebra the double $\mathcal{D}(\mathcal{A}, \mathcal{U})$ is the universal algebra generated by the algebras $\mathcal{A}$ and $\mathcal{U}$ with cross relations (9) or, equivalently, (10).

Theorem 2.5. *The two-parameter quantum group $U_{r,s}(G_2)$ is isomorphic to the Drinfel'd quantum double $\mathcal{D}(\mathcal{B}, \mathcal{B}')$.*

The proof is the same as that of [Bergeron et al. 2006, Theorem 2.5].

Remark 2.6. The proofs of Proposition 2.3 and Theorem 2.5 show the compatibility of the defining relations of $U_{r,s}(G_2)$. The proof of Theorem 2.5 indicates that the cross relations between $\mathcal{B}$ and $\mathcal{B}'$ are precisely half the ones appearing in (G1)–(G4), and the proof of Proposition 2.3 then shows the compatibility of the remaining relations appearing in $\mathcal{B}$ and $\mathcal{B}'$, including the other half of (G1)–(G4) and the (r, s)-Serre relations (G5)–(G6).

3. Lusztig's symmetries from $U_{r,s}(G_2)$ to $U_{s^{-1},r^{-1}}(G_2)$

As we did in [Bergeron et al. 2006] for the classical types A, B, C, D, we call $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$ the quantum group associated to $(U_{r,s}(G_2), \langle\,,\,\rangle)$, where the pairing $\langle \omega_i' \,|\, \omega_j\rangle$ is defined by replacing (r, s) with (s^{-1}, r^{-1}) in the defining formula for $\langle \omega_i', \omega_j\rangle$. Obviously,

$$\langle \omega_i' \,|\, \omega_j\rangle = \langle \omega_j', \omega_i\rangle.$$

We now study Lusztig's symmetry property between $(U_{r,s}(G_2), \langle\,,\,\rangle)$ and its associated object $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$, which indeed indicates the difference in structures between the two-parameter quantum group introduced above and the usual one-parameter quantum group of Drinfel'd–Jimbo type.

To define the Lusztig symmetries, we introduce the notation of divided-power elements (in $(U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$). For any nonnegative integer $k \in \mathbb{N}$, set

$$\langle k\rangle_i = \frac{s_i^{-k} - r_i^{-k}}{s_i^{-1} - r_i^{-1}}, \qquad \langle k\rangle_i! = \langle 1\rangle_i\langle 2\rangle_i \cdots \langle k\rangle_i,$$

and for any element $e_i, f_i \in (U_{s^{-1},r^{-1}}(G_2), \langle\,|\,\rangle)$, define the divided-power elements

$$e_i^{(k)} = e_i^k/\langle k\rangle_i!\,, \qquad f_i^{(k)} = f_i^k/\langle k\rangle_i!\,.$$

Definition 3.1. To every i $(i=1,2)$, there corresponds a $\mathbb{Q}$-linear mapping $\mathcal{T}_i$: $(U_{r,s}(G_2),\,\langle\,,\,\rangle)\to(U_{s^{-1},r^{-1}}(G_2),\,\langle\,|\,\rangle)$ such that $\mathcal{T}_i(r)=s^{-1}$, $\mathcal{T}_i(s)=r^{-1}$, which acts on the generators $\omega_j,\,\omega_j',\,e_j,\,f_j$ $(1\le j\le 2)$ as

$$\mathcal{T}_i(\omega_j)=\omega_j\,\omega_i^{-a_{ij}},\qquad \mathcal{T}_i(\omega_j')=\omega_j'\,\omega_i'^{-a_{ij}},$$
$$\mathcal{T}_i(e_i)=-\,\omega_i'^{-1}f_i,\qquad \mathcal{T}_i(f_i)=-(r_is_i)\,e_i\,\omega_i^{-1},$$

and for $i\neq j$,

$$\mathcal{T}_i(e_j)=\sum_{\nu=0}^{-a_{ij}}(-1)^{\nu}(rs)^{\frac{\nu}{2}(-a_{ij}-\nu)}\langle\omega_j',\omega_i\rangle^{-\nu}\langle\omega_i',\omega_i\rangle^{\frac{\nu}{2}(1+a_{ij})}e_i^{(\nu)}e_j\,e_i^{(-a_{ij}-\nu)},$$

$$\mathcal{T}_i(f_j)=(r_js_j)^{\delta_{ij}^+}\sum_{\nu=0}^{-a_{ij}}(-1)^{\nu}(rs)^{\frac{\nu}{2}(-a_{ij}-\nu)}\langle\omega_i',\omega_j\rangle^{\nu}\langle\omega_i',\omega_i\rangle^{-\frac{\nu}{2}(1+a_{ij})}f_i^{(-a_{ij}-\nu)}f_jf_i^{(\nu)},$$

where (a_{ij}) is the Cartan matrix of the simple Lie algebra $\mathfrak{g}$ of type G_2, and for any $i\neq j$,

$$\delta_{ij}^+=\begin{cases}2, & \text{if } i<j \text{ and } a_{ij}\neq 0,\\ 1, & \text{otherwise}.\end{cases}$$

Lemma 3.2. $\mathcal{T}_i$ $(i=1,2)$ *preserves the defining relations* (G1)–(G3) *of* $U_{r,s}(G_2)$ *into its associated object* $U_{s^{-1},r^{-1}}(G_2)$.

Proof. For G_2, we have

$$\langle\omega_1',\omega_1\rangle=rs^{-1}=\langle\omega_1'|\,\omega_1\rangle,\qquad \langle\omega_1',\omega_2\rangle=r^{-3}=\langle\omega_2'|\,\omega_1\rangle,$$
$$\langle\omega_2',\omega_1\rangle=s^{3}=\langle\omega_1'|\,\omega_2\rangle,\qquad \langle\omega_2',\omega_2\rangle=r^3s^{-3}=\langle\omega_2'|\,\omega_2\rangle.$$

We show that $\mathcal{T}_1,\mathcal{T}_2$ preserve the defining relations (G1)–(G3). (G1) are automatically satisfied. To check (G2) and (G3): first of all, by direct calculation, we have $\mathcal{T}_k(\langle\omega_i',\omega_j\rangle)=\langle\mathcal{T}_k(\omega_i'),\mathcal{T}_k(\omega_j)\rangle=\langle\omega_j',\omega_i\rangle=\langle\omega_i'|\,\omega_j\rangle$, for $i,j,k\in\{1,2\}$. This fact ensures that $\mathcal{T}_k(k=1,2)$ preserve (G2) and (G3), that is,

$$\mathcal{T}_k(\omega_j)\mathcal{T}_k(e_i)\mathcal{T}_k(\omega_j)^{-1}=\langle\omega_i'|\,\omega_j\rangle\mathcal{T}_k(e_i),$$
$$\mathcal{T}_k(\omega_j)\mathcal{T}_k(f_i)\mathcal{T}_k(\omega_j)^{-1}=\langle\omega_i'|\,\omega_j\rangle^{-1}\mathcal{T}_k(f_i),$$
$$\mathcal{T}_k(\omega_j')\mathcal{T}_k(e_i)\mathcal{T}_k(\omega_j')^{-1}=\langle\omega_j'|\,\omega_i\rangle^{-1}\mathcal{T}_k(e_i),$$
$$\mathcal{T}_k(\omega_j')\mathcal{T}_k(f_i)\mathcal{T}_k(\omega_j')^{-1}=\langle\omega_j'|\,\omega_i\rangle\mathcal{T}_k(f_i),$$

where checking the other three identities is equivalent to checking the first one. □

Lemma 3.3. $\mathcal{T}_i$ $(i=1,2)$ *preserves the defining relations* (G4) *into its associated object* $U_{s^{-1},r^{-1}}(G_2)$.

Proof. Put $\Delta = r^2 + rs + s^2$. To check (G4): for $i = 1, 2$, we have

$$
\begin{aligned}
[\mathcal{T}_i(e_i),\, \mathcal{T}_i(f_i)] &= (r_i s_i)\omega_i'^{-1}(f_i e_i - e_i f_i)\omega_i^{-1} = \mathcal{T}_i([e_i, f_i]),\\
[\mathcal{T}_2(e_1), \mathcal{T}_2(f_1)] &= [e_1e_2 - r^3 e_2 e_1,\, rs(f_2 f_1 - s^3 f_1 f_2)]\\
&= rs\big(f_2[e_1, f_1]e_2 + e_1[e_2, f_2]f_1 - r^3([e_2, f_2]f_1e_1 + e_2 f_2[e_1, f_1])\\
&\qquad - s^3([e_1, f_1]f_2e_2 + e_1 f_1[e_2, f_2]) + (rs)^3(e_2[e_1, f_1]f_2 + f_1[e_2,\, f_2]e_1)\big)\\
&= \frac{\omega_2\omega_1 - \omega_2'\omega_1'}{s^{-1} - r^{-1}} = \frac{\mathcal{T}_2(\omega_1) - \mathcal{T}_2(\omega_1')}{s^{-1} - r^{-1}} = \mathcal{T}_2([e_1, f_1]),
\end{aligned}
$$

and as for

$$
\begin{aligned}
[\mathcal{T}_1(e_2), \mathcal{T}_1(f_2)] = \frac{r^3 s^3}{(r+s)^2\Delta^2}\big[&(rs^2)^3 e_2 e_1^3 - rs^3\Delta e_1 e_2 e_1^2 + s\Delta e_1^2 e_2 e_1 - e_1^3 e_2,\\
&(r^2 s)^3 f_1^3 f_2 - sr^3\Delta f_1^2 f_2 f_1 + r\Delta f_1 f_2 f_1^2 - f_2 f_1^3\big],
\end{aligned}
$$

we have to show that the bracket on the right-hand side is equal to

$$
\Delta(r+s)^2\,\frac{\omega_2\omega_1^3 - \omega_2'\omega_1'^3}{r-s}.
$$

To do so, we introduce the notations of “quantum root vectors” in terms of adjoint actions, as follows:

$$
\begin{aligned}
E_{12} &= (\mathrm{ad}_l e_1)(e_2) = e_1 e_2 - s^3 e_2 e_1,\\
F_{12} &= (\mathrm{ad}_r f_1)(f_2) = f_2 f_1 - r^3 f_1 f_2,\\
E_{112} &= (\mathrm{ad}_l e_1)^2(e_2) = e_1 E_{12} - rs^2 E_{12} e_1,\\
F_{112} &= (\mathrm{ad}_r f_1)^2(f_2) = F_{12} f_1 - r^2 s f_1 F_{12},\\
E_{1112} &= (\mathrm{ad}_l e_1)^3(e_2) = e_1^3 e_2 - s\Delta e_1^2 e_2 e_1 + rs^3\Delta e_1 e_2 e_1^2 - (rs^2)^3 e_2 e_1^3,\\
F_{1112} &= (\mathrm{ad}_r f_1)^3(f_2) = f_2 f_1^3 - r\Delta f_1 f_2 f_1^2 + sr^3\Delta f_1^2 f_2 f_1 - (r^2 s)^3 f_1^3 f_2.
\end{aligned}
$$

That is, we need to verify that

$$
[\, E_{1112},\, F_{1112}\,] = \Delta(r+s)^2\,\frac{\omega_2\omega_1^3 - \omega_2'\omega_1'^3}{r-s}.
$$

By direct calculation using the Leibniz rule, we have

$$
\begin{aligned}
&[e_1,\, F_{12}] = -\Delta\omega_1 f_2, \qquad [e_2,\, F_{12}] = f_1\omega_2',\\
&[E_{12}, f_1] = -\Delta e_2\omega_1', \qquad [E_{12}, f_2] = \omega_2 e_1,\\
&[E_{12},\, F_{12}] = \frac{\omega_1\omega_2 - \omega_1'\omega_2'}{r-s},
\end{aligned}
$$

$$[e_1, F_{112}] = -(r+s)^2\omega_1 F_{12}, \quad [e_2, F_{112}] = s(s^2-r^2) f_1^2\omega_2',$$
$$[E_{112}, f_1] = -(r+s)^2 E_{12}\omega_1', \quad [E_{112}, f_2] = r(r^2-s^2)\omega_2 e_1^2,$$
$$[E_{112}, F_{12}] = (r+s)^2\omega_1\omega_2 e_1, \ [E_{12}, F_{112}] = (r+s)^2 f_1\omega_1'\omega_2',$$
$$[E_{112}, F_{112}] = (r+s)^2\,\frac{\omega_1^2\omega_2 - \omega_1'^2\omega_2'}{r-s},$$

as well as

$$[e_1, F_{1112}] = [e_1, F_{112} f_1 - rs^2 f_1 F_{112}] = -\Delta\omega_1 F_{112},$$
$$\begin{aligned}[E_{112}, F_{1112}] &= [E_{112}, F_{112} f_1 - rs^2 f_1 F_{112}]\\ &= [E_{112}, F_{112}] f_1 - rs^2 f_1 [E_{112}, F_{112}] + F_{112}[E_{112}, f_1]\\ &\qquad - rs^2[E_{112}, f_1]F_{112}\\ &= \Delta(r+s)^2 f_1\omega_1'^2\omega_2',\end{aligned}$$
$$\begin{aligned}[E_{1112}, F_{1112}] &= [e_1E_{112} - r^2sE_{112}e_1, F_{1112}]\\ &= [e_1, F_{1112}]E_{112} - r^2sE_{112}[e_1, F_{1112}] + e_1[E_{112}, F_{1112}]\\ &\qquad - r^2s[E_{112}, F_{1112}]e_1\\ &= \Delta\omega_1[E_{112}, F_{112}] + \Delta(r+s)^2[e_1, f_1]\omega_1'^2\omega_2'\\ &= \Delta(r+s)^2\,\frac{\omega_2\omega_1^3 - \omega_2'\omega_1'^3}{r-s}.\end{aligned}$$

Thus, we arrive at $[\mathcal{T}_1(e_2), \mathcal{T}_1(f_2)] = \mathcal{T}_1([e_2,\ f_2]) \in U_{s^{-1},r^{-1}}(G_2)$. □

Lemma 3.4. *$\mathcal{T}_2$ preserves the (r, s)-Serre relations* $(G5)_1$, $(G6)_1$ *into its associated object $U_{s^{-1},r^{-1}}(G_2)$:*

(11) $\mathcal{T}_2(e_2)^2\mathcal{T}_2(e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + (rs)^3\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)^2 = 0,$

(12) $\mathcal{T}_2(f_1)\mathcal{T}_2(f_2)^2 - (r^3+s^3)\mathcal{T}_2(f_2)\mathcal{T}_2(f_1)\mathcal{T}_2(f_2) + (rs)^3\mathcal{T}_2(f_1)\mathcal{T}_2(f_2)^2 = 0.$

Proof. For the degree 2 (r, s)-Serre relation $(G5)_1$

$$e_2^2 e_1 - (r^{-3}+s^{-3})e_2e_1e_2 + r^{-3}s^{-3}e_1e_2^2 = 0,$$

observe that

(13) $\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) = r^{-3}\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) - r^{-3}e_1, \quad \mathcal{T}_2(e_2)e_1 = s^3e_1\mathcal{T}_2(e_2).$

Making $\mathcal{T}_2$ act algebraically on the left-hand side of $(G5)_1$, we have

$$\begin{aligned}&\mathcal{T}_2(e_2)^2\mathcal{T}_2(e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + (rs)^3\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)^2\\ &\quad = \mathcal{T}_2(e_2)r^3(\mathcal{T}_2(e_1)\mathcal{T}_2(e_2) + r^{-3}e_1) - (r^3+s^3)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)\\ &\qquad + (rs)^3(r^{-3}\mathcal{T}_2(e_2)\mathcal{T}_2(e_1) - r^{-3}e_1)\mathcal{T}_2(e_2)\\ &\quad = 0,\end{aligned}$$

proving (11). The proof of (12) is similar. □

To prove that $\mathscr{T}_1$ preserves the Serre relations, we need three auxiliary lemmas.

Lemma 3.5. *In the notation in Lemma 3.3, we have*

$$[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_2] = 0.$$

Proof. Since $e_1 E_{1112} - r^3 E_{1112} e_1 = \mathrm{ad}_l(e_1)^4(e_2) = 0$ (Serre relation), and

$$\begin{aligned}[][E_{1112},\ f_2] &= [e_1 E_{112} - r^2 s E_{112} e_1,\ f_2] = e_1 [E_{112},\ f_2] - r^2 s [E_{112},\ f_2] e_1 \\ &= r^3(r-s)(r^2-s^2)\,\omega_2 e_1^3,\end{aligned}$$

we obtain

$$\begin{aligned}&[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_2] \\ &\quad = E_{1112}[E_{112},\ f_2] + [E_{1112},\ f_2]E_{112} - r^3\big(E_{112}[E_{1112},\ f_2] + [E_{112},\ f_2]E_{1112}\big) \\ &\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1^3 E_{112} - r\Delta e_1 E_{1112} e_1 - (r^2 s)^3 E_{112} e_1^3\big) \\ &\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1^3 E_{112} - r\Delta e_1^2 E_{112} e_1 + r^3 s \Delta e_1 E_{112} e_1^2 - (r^2 s)^3 E_{112} e_1^3\big) \\ &\quad = r^3(r-s)(r^2-s^2)\,\omega_2\,\big(e_1\cdot(\mathscr{SR}) - rs^2(\mathscr{SR})\cdot e_1\big) \\ &\quad = 0,\end{aligned}$$

where $(\mathscr{SR})$ denotes the left-hand-side presentation of the (r,s)-Serre relation $(\mathrm{G5})_2$

$$e_1^2 E_{112} - r^2(r+s) e_1 E_{112} e_1 + r^5 s E_{112} e_1^2 = 0,$$

and we used the replacement $E_{1112} = e_1 E_{112} - r^2 s E_{112} e_1$ in the third equality. □

Lemma 3.6. *In the notation of Lemma 3.3, we have*

$$[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_1] = 0.$$

Proof. It is easy to check that $[E_{1112}, f_1] = -\Delta E_{112}\omega_1'$. Thus

$$\begin{aligned}&[E_{1112}E_{112} - r^3 E_{112}E_{1112},\ f_1] \\ &\quad = E_{1112}[E_{112},\ f_1] + [E_{1112},\ f_1]E_{112} - r^3\big(E_{112}[E_{1112},\ f_1] + [E_{112},\ f_1]E_{1112}\big) \\ &\quad = (r+s)\,\big((r+s)((rs)^3 E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2\big)\omega_1'.\end{aligned}$$

It suffices to show that

$$E_{1112}E_{12} = (rs)^3 E_{12}E_{1112} + r(r-s)(r+s)^{-1}\Delta E_{112}^2. \tag{14}$$

At first, we note that the (r,s)-Serre relation $(\mathrm{G5})_1$ is equivalent to

$$E_{12}e_2 = r^3 e_2 E_{12}.$$

Since $e_1e_2 = E_{12} + s^3e_2e_1$, we get

$$\begin{aligned} E_{112}e_2 &= (e_1E_{12} - rs^2E_{12}e_1)e_2 = r^3e_1e_2E_{12} - rs^2E_{12}e_1e_2 \\ &= r^3(E_{12} + s^3e_2e_1)E_{12} - rs^2E_{12}(E_{12} + s^3e_2e_1) \\ &= r(r^2-s^2)E_{12}^2 + (rs)^3e_2(e_1E_{12} - rs^2E_{12}e_1) \\ &= r(r^2-s^2)E_{12}^2 + (rs)^3e_2E_{112}. \end{aligned}$$

Next, we claim

$$E_{1112}e_2 = (rs^2)^3e_2E_{1112} - r(rs-r^2+s^2)E_{112}E_{12} + (rs)^2(r^2+rs-s^2)E_{12}E_{112}.$$

Indeed, since $E_{1112} = e_1E_{112} - r^2sE_{112}e_1$, $E_{112} = e_1E_{12} - rs^2E_{12}e_1$, and $e_1e_2 = E_{12} + s^3e_2e_1$, we have

$$\begin{aligned} E_{1112}e_2 &= e_1(E_{112}e_2) - r^2sE_{112}(e_1e_2) \\ &= r(r^2 - s^2)e_1E_{12}^2 + (rs)^3(e_1e_2)E_{112} - r^2sE_{112}(e_1e_2) \\ &= r(r^2 - s^2)e_1E_{12}^2 + (rs)^3E_{12}E_{112} + (rs^2)^3e_2e_1E_{112} - r^2sE_{112}E_{12} \\ &\qquad - (rs^2)^2(E_{112}e_2)e_1 \\ &= r(r^2 - s^2)E_{112}E_{12} + (rs)^2(r^2 - s^2)E_{12}e_1E_{12} + (rs)^3E_{12}E_{112} \\ &\qquad + (rs^2)^3e_2e_1E_{112} - r^2sE_{112}E_{12} - r^3s^4(r^2-s^2)E_{12}^2e_1 - r^5s^7e_2E_{112}e_1 \\ &= (rs^2)^3e_2E_{1112} - r(rs-r^2+s^2)E_{112}E_{12} + (rs)^2(r^2+rs-s^2)E_{12}E_{112}. \end{aligned}$$

To prove (14), we first note that

$$\begin{aligned} &[(r+s)((rs)^3E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2,\ f_1] \\ &\quad = (r+s)(rs)^3\big(E_{12}[E_{1112},\ f_1] + [E_{12},\ f_1]E_{1112}\big) \\ &\qquad\qquad - (r+s)\big(E_{1112}[E_{12},\ f_1] + [E_{1112},\ f_1]E_{12}\big) \\ &\qquad\qquad + r(r-s)\Delta(E_{112}[E_{112}, f_1] + [E_{112},\ f_1]E_{112}) \\ &\quad = -(r+s)(rs)^3\Delta\big(E_{12}E_{112} + s^3e_2E_{1112}\big)\omega_1' \\ &\qquad\qquad + (r+s)\Delta\big(E_{1112}e_2 + r^2sE_{112}E_{12}\big)\omega_1' \\ &\qquad\qquad - r(r-s)(r+s)^2\Delta\big(E_{112}E_{12} + rs^2E_{12}E_{112}\big)\omega_1', \end{aligned}$$

which vanishes by the preceding identity. A similar but longer computation (see [Hu and Shi 2006] for details) shows that the bracket

$$[(r+s)((rs)^3E_{12}E_{1112} - E_{1112}E_{12}) + r(r-s)\Delta E_{112}^2,\ f_2].$$

also vanishes. Then, through an argument similar to the one used in the deduction of [Benkart et al. 2006, Lemma 3.4], we get (14). □

By [Benkart et al. 2006, Lemma 3.4], Lemmas 3.5 and 3.6 imply:

Lemma 3.7. $$E_{1112}E_{112}-r^3E_{112}E_{1112}=0.$$

Lemma 3.8. *$\mathcal{T}_1$ preserves the (r,s)-Serre relations* $(\mathrm{G5})_1$, $(\mathrm{G6})_1$ *into its associated object $U_{s^{-1},r^{-1}}(G_2)$:*

$$\mathcal{T}_1(e_2)^2\mathcal{T}_1(e_1)-(r^3+s^3)\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)+(rs)^3\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)^2=0, \tag{15}$$

$$\mathcal{T}_1(f_1)\mathcal{T}_1(f_2)^2-(r^3+s^3)\mathcal{T}_1(f_2)\mathcal{T}_1(f_1)\mathcal{T}_1(f_2)+(rs)^3\mathcal{T}_1(f_2)^2\mathcal{T}_1(f_1)=0. \tag{16}$$

Proof. By direct calculation, we have

$$\begin{aligned}\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)&=\Big(-\frac{1}{s^3(r+s)\Delta}E_{1112}\Big)(-\omega_1'^{-1}f_1)\\&=s^3\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)-\frac{1}{rs^2(r+s)}E_{112}.\end{aligned} \tag{17}$$

Hence, to prove (15) is equivalent to prove

$$\mathcal{T}_1(e_2)E_{112}-r^3E_{112}\mathcal{T}_1(e_2)=0.$$

However, the latter is given by Lemma 3.7.

The proof of (16) is analogous. □

To prove that $\mathcal{T}_2$ preserves the Serre relations, we also need auxiliary lemmas. Write

$$E_{21}:=(ad_l e_2)(e_1)=e_2e_1-r^{-3}e_1e_2,$$

and note that $(\mathrm{G5})_1$ is equivalent to $(ad_l e_2)(E_{21})=e_2E_{21}-s^{-3}E_{21}e_2=0$, i.e., $E_{21}e_2=s^3e_2E_{21}$.

Lemma 3.9. $\big[\,e_1E_{21}^3-s\Delta E_{21}e_1E_{21}^2+rs^3\Delta E_{21}^2e_1E_{21}-(rs^2)^3E_{21}^3e_1,\ f_1\,\big]=0.$

Proof. Since $[E_{21},\ f_1]=r^{-3}\Delta e_2\omega_1$, $\omega_1E_{21}=rs^2E_{21}\omega_1$, $[E_{21}^2,\ f_1]=r^{-3}s^{-1}(r+s)\cdot\Delta E_{21}e_2\omega_1$, $\omega_1'E_{21}=r^2sE_{21}\omega_1'$, and $[E_{21}^3,\ f_1]=r^{-3}s^{-2}\Delta^2E_{21}^2e_2\omega_1$, we get

$$\begin{aligned}\Sigma_1&=\frac{\omega_1-\omega_1'}{r-s}E_{21}^3-(rs^2)^3E_{21}^3\frac{\omega_1-\omega_1'}{r-s}-s\Delta E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}^2+rs^3\Delta E_{21}^2\frac{\omega_1-\omega_1'}{r-s}E_{21}\\&=-(rs)^3\Delta E_{21}^3\omega_1'+rs^2\Delta E_{21}^2\omega_1'E_{21}\\&=0\end{aligned}$$

and

$$
\begin{aligned}
&\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_1 \,\big]\\
&\quad = e_1[E_{21}^3,\ f_1] + [e_1,\ f_1]E_{21}^3 - (rs^2)^3\big(E_{21}^3[e_1,\ f_1] + [E_{21}^3,\ f_1]e_1\big)\\
&\qquad - s\Delta\big(E_{21}e_1[E_{21}^2,\ f_1] + E_{21}[e_1,\ f_1]E_{21}^2 + [E_{21},\ f_1]e_1E_{21}^2\big)\\
&\qquad + rs^3\Delta\big(E_{21}^2e_1[E_{21},\ f_1] + E_{21}^2[e_1,\ f_1]E_{21} + [E_{21}^2,\ f_1]e_1E_{21}\big)\\
&\quad = r^{-3}s^{-2}\Delta^2e_1E_{21}^2e_2\omega_1 + \frac{\omega_1-\omega_1'}{r-s}E_{21}^3\\
&\qquad - s\Delta\Big(r^{-3}s^{-1}(r+s)\Delta E_{21}e_1E_{21}e_2\omega_1 + E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}^2 + r^{-3}\Delta e_2\omega_1e_1E_{21}^2\Big)\\
&\qquad + rs^3\Delta\Big(r^{-3}\Delta E_{21}^2e_1e_2\omega_1 + E_{21}^2\frac{\omega_1-\omega_1'}{r-s}E_{21} + r^{-3}s^{-1}(r+s)\Delta E_{21}e_2\omega_1e_1E_{21}\Big)\\
&\qquad - (rs^2)^3\Big(E_{21}^3\frac{\omega_1-\omega_1'}{r-s} + r^{-3}s^{-2}\Delta^2E_{21}^2e_2\omega_1e_1\Big)\\
&\quad = \Sigma_1 + (r^{-3}s^{-2}\Delta^2)\,\Sigma_2\,\omega_1 = (r^{-3}s^{-2}\Delta^2)\,\Sigma_2\,\omega_1,
\end{aligned}
$$

where

$$
\begin{aligned}
\Sigma_2 = e_1E_{21}^2e_2 - s^2(r+s)E_{21}e_1E_{21}e_2 - (rs^2)^3e_2e_1E_{21}^2&\\
+ rs^5E_{21}^2e_1e_2 + r^3s^5(r+s)E_{21}e_2e_1E_{21} - r^4s^5E_{21}^2e_2e_1&.
\end{aligned}
$$

We next show $\Sigma_2 = 0$. As $E_{21}e_2 = s^3e_2E_{21}$ and $e_2e_1 - r^{-3}e_1e_2 = E_{21}$, we get

$$
\begin{aligned}
\Sigma_2 &= \big(e_1E_{21}^2e_2 - (rs^2)^3e_2e_1E_{21}^2\big) + \big(rs^5E_{21}^2e_1e_2 - r^4s^5E_{21}^2e_2e_1\big)\\
&\qquad\qquad - s^2(r+s)E_{21}e_1E_{21}e_2 + r^3s^5(r+s)E_{21}e_2e_1E_{21}\\
&= -(rs^2)^3E_{21}^3 - r^4s^5E_{21}^3 + r^3s^5(r+s)E_{21}^3\\
&= 0.
\end{aligned}
$$

This completes the proof. □

Lemma 3.10. $\big[\, e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_2 \,\big] = 0.$

Proof. Noting that

$$
\begin{gathered}
[E_{21},\ f_2] = -r^{-3}\omega_2'e_1, \qquad E_{21}\omega_2' = r^3\omega_2'E_{21},\\
[E_{21}^2,\ f_2] = -r^{-3}\omega_2'(e_1E_{21} + r^3E_{21}e_1),\\
[E_{21}^3,\ f_2] = -r^{-3}\omega_2'(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1),
\end{gathered}
$$

we obtain

$$\begin{aligned}
&\big[\,e_1E_{21}^3 - s\Delta E_{21}e_1E_{21}^2 + rs^3\Delta E_{21}^2e_1E_{21} - (rs^2)^3E_{21}^3e_1,\ f_2\,\big]\\
&\quad = e_1[E_{21}^3,\ f_2] - s\Delta\big(E_{21}e_1[E_{21}^2,\ f_2] + [E_{21},\ f_2]e_1E_{21}^2\big)\\
&\qquad\qquad + rs^3\Delta\big(E_{21}^2e_1[E_{21},\ f_2] + [E_{21}^2,\ f_2]e_1E_{21}\big) - (rs^2)^3[E_{21}^3,\ f_2]e_1\\
&\quad = -r^{-3}\omega_2'\Big(s^3e_1\big(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1\big)\\
&\qquad\qquad - s\Delta\big((rs)^3E_{21}e_1\big(e_1E_{21} + r^3E_{21}e_1\big) + e_1^2E_{21}^2\big)\\
&\qquad\qquad + rs^3\Delta\big((r^2s)^3E_{21}^2e_1^2 + (e_1E_{21} + r^3E_{21}e_1)e_1E_{21}\big)\\
&\qquad\qquad\qquad - (rs^2)^3(e_1E_{21}^2 + r^3E_{21}e_1E_{21} + r^6E_{21}^2e_1)e_1\Big)\\
&\quad = -r^{-2}s\,\omega_2'\,S,
\end{aligned}$$

where

$$\begin{aligned}
S = (rs)^2(r^3{-}s^3)\big(e_1E_{21}^2e_1 + E_{21}e_1^2E_{21}\big) + s^2(2r^2{+}rs{+}s^2)(e_1E_{21})^2\\
- r^5s^3(2s^2{+}rs{+}r^2)(E_{21}e_1)^2 - (r{+}s)\big(e_1^2E_{21}^2 - (rs)^6E_{21}^2e_1^2\big).
\end{aligned}$$

It remains to prove that $S = 0$, which by [Benkart et al. 2006, Lemma 3.4] is equivalent to showing that $[S,\ f_1] = 0 = [S,\ f_2]$. To this end, we first observe:

Lemma 3.11. $e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3 = 0.$

Proof. It is easy to see that

$$e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3 = r^{-3}(\mathrm{ad}_l e_1)^4(e_2),$$

which is in fact the (r, s)-Serre relation $(\mathrm{G5})_2$ up to a factor r^{-3}. □

Now set $S_i := [S,\ f_i]$. Using the equations at the bottom of page 341, we obtain after some manipulations (see [Hu and Shi 2006] for details)

$$\begin{aligned}
S_2 &= (rs)^2(r^3{-}s^3)\big(e_1[E_{21}^2, f_2]e_1 + [E_{21}, f_2]e_1^2E_{21} + E_{21}e_1^2[E_{21}, f_2]\big)\\
&\qquad + s^2(2r^2{+}rs{+}s^2)\big(e_1[E_{21}, f_2]e_1E_{21} + e_1E_{21}e_1[E_{21}, f_2]\big)\\
&\qquad - r^5s^3(2s^2{+}rs{+}r^2)\big([E_{21}, f_2]e_1E_{21}e_1 + E_{21}e_1[E_{21}, f_2]e_1\big)\\
&\qquad - (r+s)\big(e_1^2[E_{21}^2, f_2] - (rs)^6[E_{21}^2, f_2]e_1^2\big)\\
&= -r^{-3}(rs)^2(r^3+s^3)\,\omega_2'\big(e_1^3E_{21} - s\Delta e_1^2E_{21}e_1 + rs^3\Delta e_1E_{21}e_1^2 - (rs^2)^3E_{21}e_1^3\big),
\end{aligned}$$

which vanishes by Lemma 3.11.

Next we prove that $S_1 = 0$. Using the formulas at the very beginning of the proof of Lemma 3.9 and noting that

$$[e_1^2,\ f_1] = \frac{r+s}{rs}\cdot\frac{s\omega_1 - r\omega_1'}{r-s}e_1,$$

we can express S_1 as the sum $A+B+C+D$, where

$$\begin{aligned}
A &:= (rs)^2\Delta(\omega_1-\omega_1')E_{21}^2e_1\\
&\qquad -r^5s^3(2s^2+rs+r^2)E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1+(rs)^5\frac{(r+s)^2}{r-s}E_{21}^2(s\omega_1-r\omega_1')e_1,\\
B &:= (rs)(r+s)\Delta E_{21}(s\omega_1-r\omega_1')e_1E_{21}\\
&\qquad +s^2(2r^2+rs+s^2)\frac{\omega_1-\omega_1'}{r-s}E_{21}e_1E_{21}-r^5s^3(2s^2+rs+r^2)E_{21}e_1E_{21}\frac{\omega_1-\omega_1'}{r-s},\\
C &:= (rs)^2\Delta e_1E_{21}^2(\omega_1-\omega_1')\\
&\qquad +s^2(2r^2+rs+s^2)e_1E_{21}\frac{\omega_1-\omega_1'}{r-s}E_{21}-\frac{(r+s)^2}{rs}\frac{s\omega_1-r\omega_1'}{r-s}e_1E_{21}^2,\\
D &:= \frac{\Delta}{r^3}\Big((rs)^2(r^3-s^3)\big(s^{-1}(r+s)e_1E_{21}e_2\omega_1e_1+e_2\omega_1e_1^2E_{21}+E_{21}e_1^2e_2\omega_1\big)\\
&\qquad +s^2(2r^2+rs+s^2)\big(e_1e_2\omega_1e_1E_{21}+e_1E_{21}e_1e_2\omega_1\big)\\
&\qquad -r^5s^3(2s^2+rs+r^2)\big(e_2\omega_1e_1E_{21}e_1+E_{21}e_1e_2\omega_1e_1\big)\\
&\qquad -(r+s)^2s^{-1}\big(e_1^2E_{21}e_2\omega_1-(rs)^6E_{21}e_2\omega_1e_1^2\big)\Big).
\end{aligned}$$

Noting that $\omega_1E_{21}=rs^2E_{21}\omega_1$ and $\omega_1'E_{21}=r^2sE_{21}\omega_1'$, we obtain the simplified expressions

$$\begin{aligned}
A &= -r^5s^4(r^3-s^3)E_{21}^2e_1\omega_1,\\
B &= 0,\\
C &= rs^2(r^3-s^3)e_1E_{21}^2\omega_1.
\end{aligned}$$

For the last summand, a calculation using the equalities $E_{21}e_2=s^3e_2E_{21}$, $r^{-3}e_1e_2=e_2e_1-E_{21}$ and $e_2e_1=E_{21}+r^{-3}e_1e_2$ leads to

$$D=(r^3-s^3)\left(r^5s^4E_{21}^2e_1-rs^2e_1E_{21}^2\right)\omega_1$$

(see [Hu and Shi 2006] for details), showing that $S_1=A+B+C+D=0$. This completes the proof of Lemma 3.10. □

The next identity is a consequence of Lemmas 3.9, 3.10 and [Benkart et al. 2006, Lemma 3.4].

Lemma 3.12. $e_1E_{21}^3-s\Delta E_{21}e_1E_{21}^2+rs^3\Delta E_{21}^2e_1E_{21}-(rs^2)^3E_{21}^3e_1=0.$

Lemma 3.13. *$\mathcal{T}_2$ preserves the (r,s)-Serre relations* (G5)$_2$, (G6)$_2$ *into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

Proof. For the fourth-degree (r,s)-Serre relation $(\mathrm{G5})_2$, we have to prove that

$$(rs)^6\mathcal{T}_2(e_1)^4\mathcal{T}_2(e_2)-(rs)^3(r+s)(r^2+s^2)\mathcal{T}_2(e_1)^3\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)\\ +(rs)(r^2+s^2)(r^2+rs+s^2)\mathcal{T}_2(e_1)^2\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^2\\ -(r+s)(r^2+s^2)\mathcal{T}_2(e_1)\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^3+\mathcal{T}_2(e_2)\mathcal{T}_2(e_1)^4$$

vanishes. By virtue of the commutation relation in (13), this is equivalent to

$$e_1\mathcal{T}_2(e_1)^3-s\Delta\mathcal{T}_2(e_1)e_1\mathcal{T}_2(e_1)^2+rs^3\Delta\mathcal{T}_2(e_1)^2e_1\mathcal{T}_2(e_1)-(rs^2)^3\mathcal{T}_2(e_1)^3e_1=0.$$

However, since $\mathcal{T}_2(e_1)=e_1e_2-r^3e_2e_1=(-r^3)E_{21}$, the above identity is exactly the one given by Lemma 3.12.

Similarly, we can verify that $\mathcal{T}_2$ preserves the (r,s)-Serre relation $(\mathrm{G6})_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$. □

Lemma 3.14. *$\mathcal{T}_1$ preserves the (r,s)-Serre relations $(\mathrm{G5})_2$, $(\mathrm{G6})_2$ into its associated object $U_{s^{-1},r^{-1}}(G_2)$.*

Proof. For the fourth-degree (r,s)-Serre relation $(\mathrm{G5})_2$, we have to prove that

$$(rs)^6\mathcal{T}_1(e_1)^4\mathcal{T}_1(e_2)-(rs)^3(r+s)(r^2+s^2)\mathcal{T}_1(e_1)^3\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)\\ +(rs)(r^2+s^2)(r^2+rs+s^2)\mathcal{T}_1(e_1)^2\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^2\\ -(r+s)(r^2+s^2)\mathcal{T}_1(e_1)\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^3+\mathcal{T}_1(e_2)\mathcal{T}_1(e_1)^4=0.$$

In view of the commutation relation in (17), this is equivalent to

$$E_{112}\mathcal{T}_1(e_1)^3-r\Delta\mathcal{T}_1(e_1)E_{112}\mathcal{T}_1(e_1)^2+r^3s\Delta\mathcal{T}_1(e_1)^2E_{112}\mathcal{T}_1(e_1)\\ -(r^2s)^3\mathcal{T}_1(e_1)^3E_{112}=0.$$

We can further reduce this condition to

$$E_{12}\mathcal{T}_1(e_1)^2-r^2(r+s)\mathcal{T}_1(e_1)E_{12}\mathcal{T}_1(e_1)+r^5s\mathcal{T}_1(e_1)^2E_{12}=0, \tag{18}$$

as a consequence of the commutative relation

$$E_{112}\mathcal{T}_1(e_1)=rs^2\mathcal{T}_1(e_1)E_{112}+r^{-1}s(r+s)^2E_{12},$$

itself arising from the equalities

$$[E_{112}f_1]=-(r+s)^2E_{12}\omega_1',\quad \omega_1'E_{112}=rs^2E_{112}\omega_1'.$$

Again, since $[E_{12},f_1]=-\Delta e_2\omega_1'$, we have

$$E_{12}\mathcal{T}_1(e_1)=r^2s\mathcal{T}_1(e_1)E_{12}+r^{-1}s\Delta e_2,$$

by which (18) is finally reduced to $e_2\mathcal{T}_1(e_1)=r^3\mathcal{T}_1(e_1)e_2$, since $\mathcal{T}_1(e_1)=-\omega_1'^{-1}f_1$.

The proof of the second part is similar. □

Theorem 3.15. *$\mathcal{T}_1$ and $\mathcal{T}_2$ are the Lusztig symmetries from $U_{r,s}(G_2)$ to its associated quantum group $U_{s^{-1},r^{-1}}(G_2)$ as $\mathbb{Q}$-isomorphisms, inducing the usual Lusztig symmetries as $\mathbb{Q}(q)$-automorphisms not only on the quantum group $U_q(G_2)$ of Drinfel'd–Jimbo type but also on the centralized quantum group $U_q^c(G_2)$, only when $r = q = s^{-1}$.* □

References

[Benkart et al. 2006] G. Benkart, S.-J. Kang, and K.-H. Lee, "On the centre of two-parameter quantum groups", *Proc. Roy. Soc. Edinburgh Sect. A* **136**:3 (2006), 445–472. MR 2007a:17019

[Bergeron et al. 2006] N. Bergeron, Y. Gao, and N. Hu, "Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups", *J. Algebra* **301**:1 (2006), 378–405. MR MR2230338

[Hu and Shi 2006] N. Hu and Q. Shi, "Two-parameter quantum group of exceptional type G_2 and Lusztig's symmetries", Preprint, 2006. math.QA/0601444

[Klimyk and Schmüdgen 1997] A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Springer, Berlin, 1997. MR 99f:17017

Received August 16, 2005.

NAIHONG HU
DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
ZHONGSHAN NORTH ROAD 3663
SHANGHAI 200062
CHINA
nhhu@math.ecnu.edu.cn

QIAN SHI
DEPARTMENT OF MATHEMATICS
EAST CHINA NORMAL UNIVERSITY
ZHONGSHAN NORTH ROAD 3663
SHANGHAI 200062
CHINA