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CONVERGENCE TO STEADY STATES FOR A ONE-DIMENSIONAL VISCOUS HAMILTON–JACOBI EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

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We investigate the convergence to steady states of the solutions to the one-dimensional viscous Hamilton–Jacobi equation $\partial_t u - \partial_x^2 u = |\partial_x u|^p$, where $(t, x) \in (0, \infty) \times (-1, 1)$ and $p \in (0, 1)$, with homogeneous Dirichlet boundary conditions. For that purpose, a Liapunov functional is constructed by the approach of Zelenyak (1968). Instantaneous extinction of $\partial_x u$ on a subinterval of $(-1, 1)$ is shown for suitable initial data.

1. Introduction

Nonnegative solutions to the one-dimensional viscous Hamilton–Jacobi equation

$$\partial_t u - \partial_x^2 u = a\, |\partial_x u|^p, \quad (t, x) \in (0, \infty) \times (-1, 1), \tag{1}$$

$$u(t, \pm 1) = 0, \quad t \in (0, \infty), \tag{2}$$

$$u(0) = u_0 \geq 0, \quad x \in (-1, 1), \tag{3}$$

exhibit a rich variety of qualitative behaviours, according to the sign of $a \in \{-1, 1\}$ and the values of $p \in (0, \infty)$. On the one hand, extinction in finite time (that is, there is $T_\star > 0$ such that $u(t) \equiv 0$ for $t \geq T_\star$) occurs for $a = -1$ and $p \in (0, 1)$, while $u(t)$ converges exponentially fast to zero as $t \to \infty$ if $a = -1$ and $p \geq 1$ [Benachour et al. 2007]. On the other hand, if $a = 1$ and $p > 2$, finite time gradient blow-up takes place for suitably large initial data [Souplet 2002] while convergence to zero of $u(t)$ as $t \to \infty$ still holds true for global solutions [Arrieta et al. 2004; Souplet and Zhang 2006]. In addition, all solutions are global for $a = 1$ and $p \in [1, 2]$ and converge to zero as $t \to \infty$ [Benachour et al. 2007; Souplet and Zhang 2006].

The case $a = 1$ and $p \in (0, 1)$ offers an interesting novelty and is the subject of the present paper. Indeed, in contrast to the previous cases, the initial-boundary value problem (1)–(3) has a one parameter family $(U_\vartheta)_{\vartheta \in [0,1]}$ of steady states when $a = 1$ and $p \in (0, 1)$ with $U_1 \equiv 0$ and U_ϑ is not constant if $\vartheta \in [0, 1)$. These steady

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states play an important role in the dynamics of solutions to (1)–(3): indeed, we will prove that any solution u to (1)–(3) converges as $t \to \infty$ towards a steady state, which is nontrivial if, for instance, the initial datum u_0 is nonnegative with a positive maximum. An interesting feature of U_ϑ for $\vartheta \in (0,1)$ is that they are constant on a subinterval of $(-1,1)$. This property is of course related to the fact that p ranges in $(0,1)$ and is reminiscent of the finite time extinction phenomenon already alluded to for nonnegative solutions when $a=-1$ and $p \in (0,1)$. It is then natural to wonder whether the nonlinear term $|\partial_x u|^p$ may induce a similar singular behaviour on the dynamics of u. More precisely, for a particular class of nonnegative initial data, we will show that the gradient $\partial_x u$ vanishes identically on $[T_\star, \infty) \times I$ for some $T_\star > 0$ and some subinterval I of $(-1,1)$. Let us point out here that, for nonnegative initial data, extinction in finite time cannot occur when $a=1$ and $p \in (0,1)$, for the comparison principle warrants that u is bounded from below by the solution to the linear heat equation with the same initial and boundary data.

From now on, we thus assume that

$$a = 1 \quad \text{and} \quad p \in (0,1), \tag{4}$$

and

$$u_0 \in Y := \left\{ w \in \mathscr{C}^1([-1,1]), \quad w(\pm 1) = 0 \right\}. \tag{5}$$

It then follows from [Benachour and Dabuleanu 2003, Theorem 3.1 and Proposition 4.1] that the initial-boundary value problem (1)–(3) has a unique classical solution

$$u \in \mathscr{C}([0,\infty) \times [-1,1]) \cap \mathscr{C}^{2,1}((0,\infty) \times (-1,1))$$

satisfying

$$\min_{[-1,1]} u_0 \le u(t,x) \le \max_{[-1,1]} u_0, \qquad (t,x) \in [0,\infty) \times [-1,1]. \tag{6}$$

In addition, setting

$$M(t) := \max_{x \in [-1,1]} u(t,x), \tag{7}$$

the comparison principle ensures that $t \mapsto M(t)$ is a nonincreasing function of time and we put

$$M_\infty := \lim_{t \to \infty} M(t) \in \Big[\min_{[-1,1]} u_0, \max_{[-1,1]} u_0 \Big]. \tag{8}$$

We recall that classical solutions to (1)–(3) enjoy the comparison principle; this may be proved by standard arguments, as in [Gilding et al. 2003, Theorem 4].

Remark 1. The initial-boundary value problem (1)–(3) is actually well-posed in a larger space than Y, which depends on p, and we refer to [Benachour and Dabuleanu 2003] for a more detailed account. Still, the solutions constructed in that reference belong to Y for any positive time. Since we are interested here in the large time behaviour, the assumption (5) that $u_0 \in Y$ is thus not restrictive.

For further use, we also introduce the following notations:

$$\alpha := \frac{2-p}{1-p} \quad \text{and} \quad \mathcal{M}_0 := \frac{(1-p)^\alpha}{2-p}. \tag{9}$$

We may now state our main result.

Theorem 2. *Consider $u_0 \in Y$ and denote by u the corresponding classical solution to (1)–(3). Then $M_\infty \in [0, \mathcal{M}_0]$ and there is a nonnegative stationary solution u_s to (1)–(2) such that*

$$\lim_{t\to\infty} \|u(t) - u_s\|_\infty = 0. \tag{10}$$

Furthermore, $u_s \not\equiv 0$ and $M_\infty > 0$ if

$$\int_{-1}^{1} u_0(x) \cos\left(\frac{\pi x}{2}\right) dx > 0. \tag{11}$$

It readily follows from the second assertion of Theorem 2 that the set of nontrivial and nonnegative steady states to (1)–(2) attracts all solutions to (1)–(3) starting from a nonnegative initial datum $u_0 \not\equiv 0$. Observe however that the set of nontrivial and nonnegative steady states to (1)–(2) also attracts sign-changing solutions u to (1)–(3) since there are sign-changing initial data fulfilling (11).

The proof of Theorem 2 requires several steps and is performed as follows: we first identify the stationary solutions to (1)–(2) in Section 2 and use them together with comparison arguments to establish that, if $u_0 \in Y$ is nonnegative with $u_0 \not\equiv 0$, then $M_\infty > 0$ and $\{u(t);\ t \geq 0\}$ is bounded in $\mathscr{C}^1([-1, 1])$ (Section 3). In Section 4, we employ the technique of [Zelenyak 1968] to construct a Liapunov functional for nonnegative solutions to (1)–(3). Let us mention here that this technique has also been used recently for related problems in [Arrieta et al. 2004; Simondon and Touré 1996]. For nonnegative initial data convergence towards a steady state then follows from the results of Section 3 and Section 4 by a LaSalle invariance principle argument. The large time behaviour of sign-changing initial data is next deduced from that of nonnegative solutions after observing that the negative part of any solution to (1)–(3) vanishes in a finite time (Section 6).

Remark 3. A further outcome of Theorem 2 is that the large behaviour of solutions to (1) on a bounded interval is more complex for homogeneous Dirichlet boundary conditions than for periodic and homogeneous Neumann boundary conditions. Indeed, for the latter boundary conditions, it follows from [Benachour and Dabuleanu 2005; Benachour et al. 2002] that there are $T_\star > 0$ and $m_\star \in \mathbb{R}$ such that $u(t) \equiv m_\star$ for $t \geq T_\star$ whatever the signs of a and u_0 are.

In Section 7, we prove the extinction in finite time of $\partial_x u$ on a subinterval of $(-1, 1)$ for a specific class of initial data. More precisely, we have the following result:

Theorem 4. *Assume further that there are* $m_0 \in (0, \mathcal{M}_0)$ *and* $\varepsilon > 0$ *such that*

$$m_0 - \mathcal{M}_0 \, |x|^\alpha + \varepsilon \, |x|^{1+\alpha} \leq u_0(x) \leq m_0, \quad x \in [-1, 1]. \tag{12}$$

Then, for each $t \in (0, \infty)$, *there is* $X(t) \in (0, 1)$ *such that*

$$u(t, x) = m_0 \quad \textit{for} \quad x \in (-X(t), X(t)).$$

Furthermore, if

$$\delta_0 := 1 - \left(\frac{m_0}{\mathcal{M}_0}\right)^{1/\alpha} \in (0, 1), \tag{13}$$

and $\delta \in (0, \delta_0)$, *there exists* $T(\delta) > 0$ *such that*

$$u(t, x) = m_0 \quad \textit{for} \quad (t, x) \in [T(\delta), \infty) \times [-\delta, \delta].$$

An example of initial datum in Y fulfilling (12) is the following: $u_0(x) = \mathcal{M}_0 - \varepsilon - \mathcal{M}_0 \, |x|^\alpha + \varepsilon \, |x|^\beta$ for $x \in [-1, 1]$, where $\beta \in (\alpha, \alpha + 1]$ and $\varepsilon \in (0, \alpha\mathcal{M}_0/\beta)$.

The second assertion of Theorem 4 shows that $\partial_x u$ vanishes identically after some time on a subinterval of $[-1, 1]$, a phenomenon which one could call *finite time incomplete extinction* in comparison to what occurs for periodic or homogeneous Neumann boundary conditions. But the first assertion of Theorem 4 reveals that the extinction mechanism is somewhat stronger since, even if $\partial_x u_0(x)$ vanishes only for $x = 0$, $\partial_x u$ vanishes instantaneously on a subinterval of $[-1, 1]$ with positive measure.

Another consequence of Theorem 4 and (6) is that $\|u(t)\|_\infty = m_0$ for every $t \geq 0$. Therefore, for an initial datum u_0 in Y satisfying (12), the corresponding solution u to (1)–(3) does not obey the strong maximum principle.

The proof of Theorem 4 relies on comparison arguments with travelling wave solutions to (1) and is similar to that of [Gilding 2005, Theorem 9], some care being needed to cope with the boundary conditions.

Notations. Throughout the paper, we denote by $r_+ := \max\{r, 0\}$ the positive part of the real number r. For $r \in \mathbb{R}$ and $s \in \mathbb{R}$, we put $r \vee s := \max\{r, s\}$ and $r \wedge s := \min\{r, s\}$. Also, for $q \in [1, \infty]$, $\|.\|_q$ denotes the $L^q(-1, 1)$-norm.

2. Nonnegative steady states

In this section, we look for nonnegative stationary solutions to (1), (2), that is, nonnegative functions $U \in \mathscr{C}^2([-1, 1])$ such that

$$\frac{d^2U}{dx^2} + \left|\frac{dU}{dx}\right|^p = 0, \quad x \in (-1, 1), \tag{14}$$

$$U(\pm 1) = 0. \tag{15}$$

Proposition 5. *Let $U \in \mathscr{C}^2([-1, 1])$ be a nonnegative solution to (14), (15). Then there is $\vartheta \in [0, 1]$ such that $U = U_\vartheta$, where*

$$U_\vartheta(x) := \mathcal{M}_0 \left[(1-\vartheta)^\alpha - (|x| - \vartheta)_+^\alpha\right], \quad x \in [-1, 1].$$

Observe that U_ϑ is constant on $[-\vartheta, \vartheta]$ for each $\vartheta \in (0, 1)$ and that $U_1 \equiv 0$.

Proof. Let $U \in \mathscr{C}^2([-1, 1])$ be a nonnegative solution to (14), (15). Then U is concave by (14) and we infer from the nonnegativity of U and the boundary conditions (15) that $dU/dx(-1) \geq 0$ and $dU/dx(1) \leq 0$.

If $dU/dx(-1) = 0$, the concavity of U entails that U is a nonincreasing function in $(-1, 1)$. Consequently, $U \equiv 0 = U_1$ to comply with the boundary conditions (15).

Similarly, if $dU/dx(1) = 0$, it follows from the concavity of U that U is nondecreasing on $(-1, 1)$, whence $U \equiv 0 = U_1$ by (15).

We finally consider the case where $dU/dx(-1) > 0$ and $dU/dx(1) < 0$ and put

$$x_I := \sup\{X \in (-1, 1) \text{ such that } dU/dx(x) > 0 \text{ on } [-1, X)\},$$
$$x_S := \inf\{X \in (-1, 1) \text{ such that } dU/dx(x) < 0 \text{ on } (X, 1]\}.$$

Owing to the continuity of dU/dx, we have $-1 < x_I \leq x_S < 1$ and $dU/dx(x) = 0$ for $x \in [x_I, x_S]$ by the concavity of U. Direct integration of (14) then entails that there are two constants A and B such that

$$\left|\frac{dU}{dx}(x)\right|^{-p} \frac{dU}{dx}(x) + (1-p)\, x = \begin{cases} A & \text{if } x \in (x_S, 1], \\ B & \text{if } x \in [-1, x_I). \end{cases} \tag{16}$$

Since $p \in (0, 1)$ and dU/dx vanishes for $x \in \{x_I, x_S\}$, we may let $x \to x_I$ and $x \to x_S$ in (16) to deduce that $A = (1-p)\, x_S$ and $B = (1-p)\, x_I$. We next integrate (16) to obtain that there are two constants C_I and C_S such that

$$U(x) = \begin{cases} C_S - \mathcal{M}_0\, (x - x_S)^\alpha & \text{if } x \in (x_S, 1], \\ C_I - \mathcal{M}_0\, (x_I - x)^\alpha & \text{if } x \in [-1, x_I). \end{cases}$$

Requiring the boundary conditions (15) to be fulfilled provides the values of C_I and C_S, whence

$$U(x) = \begin{cases} \mathcal{M}_0\ (1-x_S)^\alpha - \mathcal{M}_0\ (x-x_S)^\alpha & \text{if} \quad x \in (x_S, 1], \\ \mathcal{M}_0\ (x_I+1)^\alpha - \mathcal{M}_0\ (x_I-x)^\alpha & \text{if} \quad x \in [-1, x_I). \end{cases}$$

Now, since dU/dx vanishes for $x \in [x_I, x_S]$, we shall have $U(x_S) = U(x_I)$, which implies that $1 - x_S = x_I + 1$, whence $x_S = -x_I$. Thus, necessarily, $x_S \in [0, 1]$, from which the equality $U = U_{x_S}$ readily follows. □

It is worth mentioning that $\|U_\vartheta\|_\infty \le \mathcal{M}_0$ for each $\vartheta \in [0, 1]$. Combining this property with the convergence to a steady state to be proved in Section 5, we will conclude that $M_\infty \le \mathcal{M}_0$.

Remark 6. Proposition 5 shows in particular that there is nonuniqueness of classical solutions to (14), (15). A similar construction is performed in [Alaa and Pierre 1993; Lions 1985] for the boundary-value problem

$$-\Delta u = |\nabla u|^p \ \text{ in } \ B(0,1), \qquad u = 0 \ \text{ on } \ \partial B(0,1),$$

where $B(0, 1)$ denotes the open unit ball of $\mathbb{R}^N$, $N > 1$, to establish the nonuniqueness of weak solutions for $p > N/(N-1)$.

3. Some properties of $\{u(t)\ ;\ t \ge 0\}$

Introducing the positive cone $Y_+ := \{w \in Y \ \text{ such that } \ w \ge 0\}$ of Y, we first prove that $M_\infty > 0$ for $u_0 \in Y_+$, $u_0 \not\equiv 0$, by constructing suitable subsolutions to (1)–(3) with the help of U_0.

Lemma 7. *Let $u_0 \in Y_+$ and denote by u the corresponding classical solution to (1)–(3). If $u_0 \not\equiv 0$, we have $M_\infty > 0$.*

Proof. Since $u_0 \not\equiv 0$, there are $x_0 \in (-1, 1)$, $\delta \in (0, 1)$ and $m > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (-1, 1)$ and

$$u_0(x) \ge m \quad \text{for} \quad x \in (x_0 - \delta, x_0 + \delta). \tag{17}$$

We put $x_1 := (x_0 - 1) \vee (-1)$, $x_2 := (x_0 + 1) \wedge 1$, $J := [x_1, x_2]$,

$$\lambda := 1 \wedge \frac{m}{\mathcal{M}_0 - U_0(\delta)},$$

and $v(x) := \lambda\ (U_0(x - x_0) - U_0(\delta))$ for $x \in J$.

On the one hand, it follows from (1) and (14) that

$$\partial_t v - \partial_x^2 v - |\partial_x v|^p = (\lambda - \lambda^p)\ |\partial_x U_0(. - x_0)|^p \le 0 = \partial_t u - \partial_x^2 u - |\partial_x u|^p$$

on $[0,\infty)\times J$. On the other hand, the nonnegativity of u_0 and the maximum principle entail the nonnegativity of u which then warrants that

$$v(x_1) \le v(x_0-\delta) = 0 \le u(t,x_1),$$
$$v(x_2) \le v(x_0+\delta) = 0 \le u(t,x_2),$$

while the choice of λ entails that

$$v(x) \le \lambda\ (\mathcal{M}_0 - U_0(\delta)) \le m \le u_0(x) \quad \text{for} \quad x\in(x_0-\delta,x_0+\delta),$$
$$v(x) \le v(x_0\pm\delta) = 0 \le u_0(x) \quad \text{for} \quad x\in J\setminus(x_0-\delta,x_0+\delta).$$

We then infer from the comparison principle that $u(t,x)\ge v(x)$ for $(t,x)\in[0,\infty)\times J$. In particular, $M(t)=\|u(t)\|_\infty \ge u(t,x_0)\ge v(x_0)=\lambda\ (\mathcal{M}_0-U_0(\delta))$ for each $t\ge 0$, whence $M_\infty \ge \lambda\ (\mathcal{M}_0-U_0(\delta))>0$. □

We now turn to the question of global boundedness of the trajectory $\{u(t)\ ;\ t\ge 0\}$ in $\mathcal{C}^1([-1,1])$.

Lemma 8. *Let $u_0\in Y_+$ and denote by u the corresponding classical solution to (1)–(3). There is a constant $\Lambda>0$ depending only on $\|u_0\|_{W^{1,\infty}(-1,1)}$ and p such that*

$$\|u(t)\|_{W^{1,\infty}(-1,1)} \le \Lambda \quad \textit{for} \quad t\ge 0. \tag{18}$$

Proof. We first recall that $\{u(t)\ ;\ t\ge 0\}$ is bounded in $L^\infty(-1,1)$ by (6) and we are left with the proof that $\{\partial_x u(t)\ ;\ t\ge 0\}$ is bounded in $L^\infty(-1,1)$. For that purpose, we choose $\lambda>1$ such that

$$\lambda \ge \left[\left(\frac{2}{1-p}\right)^{1/(1-p)} \|\partial_x u_0\|_\infty\right] \vee \left[\frac{\|u_0\|_\infty}{(1-2^{-\alpha})\ \mathcal{M}_0}\right]. \tag{19}$$

Putting $v:=\lambda U_0$, we first notice that the condition $\lambda>1$ ensures that

$$\partial_t v - \partial_x^2 v - |\partial_x v|^p = (\lambda-\lambda^p)\ |\partial_x U_0|^p \ge 0 \quad \text{in} \quad (0,\infty)\times(-1,1),$$

while $v(\pm 1)=u(t,\pm 1)=0$ for each $t\ge 0$. Next, on the one hand, it follows from (19) and the monotonicity properties of U_0 that, if $x\in(-1/2,1/2)$, we have

$$v(x) = \lambda\ U_0(x) \ge \lambda\ U_0(1/2) = \lambda\ \mathcal{M}_0\ (1-2^{-\alpha}) \ge \|u_0\|_\infty \ge u_0(x).$$

On the other hand, if $x\in[1/2,1]$, we have by (19) that

$$v(x) = \lambda\ (U_0(x)-U_0(1)) = \lambda \int_x^1 \left|\frac{dU_0}{dx}(y)\right|\, dy = \alpha\ \lambda\ \mathcal{M}_0 \int_x^1 y^{1/(1-p)}\, dy$$
$$\ge \alpha\ \lambda\ \mathcal{M}_0 \int_x^1 2^{-1/(1-p)}\, dy \ge \int_x^1 \|\partial_x u_0\|_\infty\, dy \ge \int_x^1 |\partial_x u_0(y)| dy$$
$$\ge u_0(x).$$

A similar computation shows that $v(x) \geq u_0(x)$ also holds true for $x \in [-1, -1/2]$. Therefore, $v \geq u_0$ in $[-1, 1]$ and the previous analysis allows us to apply the comparison principle and conclude that $u(t, x) \leq v(x)$ for $(t, x) \in [0, \infty) \times [-1, 1]$. In particular, if $t \geq 0$ and $x \in (0, 1)$, we have

$$\frac{u(t, x) - u(t, 1)}{x - 1} = \frac{u(t, x)}{x - 1} \geq \frac{v(x)}{x - 1} = \frac{v(x) - v(1)}{x - 1}.$$

Letting $x \to 1$, we deduce that $\partial_x u(t, 1) \geq \partial_x v(1) = -\lambda\ (1 - p)^{1/(1-p)}$. Since $u_0 \geq 0$, the comparison principle ensures that $u(t, x) \geq 0 = u(t, 1)$ for $x \in (0, 1)$, so that we also have $\partial_x u(t, 1) \leq 0$. Arguing in a similar way for $x = -1$, we end up with

$$|\partial_x u(t, \pm 1)| \leq \lambda\ (1 - p)^{1/(1-p)} \quad \text{for} \quad t \geq 0. \tag{20}$$

We now put $k := \|\partial_x u_0\|_\infty \vee \lambda\ (1 - p)^{1/(1-p)}$, $z := \partial_x u$ and $\mathcal{R} := \{(t, x) \in (0, \infty) \times (-1, 1)\ ,\ z(t, x) \neq 0\}$. In the neighbourhood of each point (t_0, x_0) of $\mathcal{R}$, the function $|\partial_x u|^p$ is smooth, and classical parabolic regularity theory implies that z is $\mathcal{C}^{1,2}$ in a neighbourhood of (t_0, x_0) and satisfies

$$\partial_t z(t, x) - \partial_x^2 z(t, x) = p\ |z(t, x)|^{p-2}\ z(t, x)\ \partial_x z(t, x).$$

Since $\{(t, x) \in (0, \infty) \times (-1, 1)\ ,\ z(t, x) > k\} \subset \mathcal{R}$, we deduce from the previous identity and (20) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(z - k)_+\|_2^2 &= \big[(z - k)_+\ \partial_x z\big]_{x=-1}^{x=1} - \int_{-1}^{1} |\partial_x (z - k)_+|^2\ dx \\ &\qquad + \left[\left(\frac{p}{p+1}\ z - k\right)\ |z|^p\ \frac{(z - k)_+}{|z - k|}\right]_{x=-1}^{x=1} \\ &= -\int_{-1}^{1} |\partial_x (z - k)_+|^2\ dx,\end{aligned}$$

whence

$$\|(z(t) - k)_+\|_2^2 \leq \|(z(0) - k)_+\|_2^2 = 0,$$

the last equality being true thanks to the choice of k. Consequently, $\partial_x u(t, x) = z(t, x) \leq k$ in $[0, \infty) \times [-1, 1]$. By a similar argument, we also establish that $\partial_x u(t, x) = z(t, x) \geq -k$ in $[0, \infty) \times [-1, 1]$. Therefore,

$$|\partial_x u(t, x)| \leq \|\partial_x u_0\|_\infty \vee \lambda\ (1 - p)^{1/(1-p)}$$

for $(t, x) \in [0, \infty) \times [-1, 1]$, which completes the proof of Lemma 8. □

4. A Liapunov functional

We now construct a Liapunov functional for nonnegative solutions to (1)–(3) with the help of the technique developed in [Zelenyak 1968]. Let $u_0 \in Y_+$ and denote by u the corresponding classical solution to (1)–(3) which is also nonnegative by the maximum principle. We look for a pair of functions Φ and $\varrho \geq 0$ such that

$$\frac{d}{dt}\int_{-1}^{1} \Phi\left(u, \partial_x u\right) dx = \int_{-1}^{1} \varrho\left(u, \partial_x u\right)\ \left|\partial_t u\right|^2 dx. \tag{21}$$

Since $\partial_t u(t, \pm 1) = 0$ by (2), the first term of the right-hand side of this equality also reads

$$\begin{aligned}\frac{d}{dt}\int_{-1}^{1} &\Phi\left(u, \partial_x u\right) dx\\ &= \int_{-1}^{1} \left[\partial_1\Phi\left(u, \partial_x u\right)\ \partial_t u + \partial_2\Phi\left(u, \partial_x u\right)\ \partial_x\partial_t u\right] dx\\ &= \int_{-1}^{1} \left[\partial_1\Phi\left(u, \partial_x u\right) - \partial_1\partial_2\Phi\left(u, \partial_x u\right)\ \partial_x u - \partial_2^2\Phi\left(u, \partial_x u\right)\ \partial_x^2 u\right]\ \partial_t u\, dx,\end{aligned}$$

and it is then natural to require that

$$\begin{aligned}\Big[\partial_1\Phi\left(u, \partial_x u\right) - \partial_1\partial_2\Phi\left(u, \partial_x u\right)\ \partial_x u - \partial_2^2\Phi\left(u, \partial_x u\right)\ \partial_x^2 u\Big]&\\ &= \varrho\left(u, \partial_x u\right)\ \partial_t u\\ &= \varrho\left(u, \partial_x u\right)\ \left(\left|\partial_x u\right|^p + \partial_x^2 u\right)\end{aligned}$$

for (21) to hold true. Following [Zelenyak 1968], we realize that a sufficient condition for the previous equality to be valid is

$$\partial_1\Phi\left(u, \partial_x u\right) - \partial_1\partial_2\Phi\left(u, \partial_x u\right)\ \partial_x u = \varrho\left(u, \partial_x u\right)\ \left|\partial_x u\right|^p, \tag{22}$$

$$-\partial_2^2\Phi\left(u, \partial_x u\right) = \varrho\left(u, \partial_x u\right). \tag{23}$$

Performing the computations as in [Zelenyak 1968], we see that the functions

$$\Phi\left(u, \partial_x u\right) := u - \frac{\left|\partial_x u\right|^{2-p}}{(2-p)(1-p)} \quad \text{and} \quad \varrho\left(u, \partial_x u\right) := \left|\partial_x u\right|^{-p}$$

solve the differential system (22), (23). However, ϱ is singular when $\partial_x u$ vanishes and it is not clear how to give a meaning to (21) for such a choice of functions Φ and ϱ. Nevertherless, we have the following weaker result which turns out to be sufficient for our purposes.

Proposition 9. *For each $t > 0$ and $\delta \in (0, 1]$, we have*

$$\frac{d}{dt}\int_{-1}^{1}\left(\frac{|\partial_x u(t,x)|^{2-p}}{(2-p)(1-p)} - u(t,x)\right)dx + \int_{-1}^{1}\frac{|\partial_t u|^2}{\left(|\partial_x u|^2+\delta^2\right)^{p/2}}\,dx \le 0. \tag{24}$$

Proof. We fix $\delta \in (0, 1]$ and define ψ_ε by

$$\psi_\varepsilon(0) = \psi_\varepsilon'(0) = 0 \quad \text{and} \quad \psi_\varepsilon''(r) = (|r| \vee \varepsilon)^{-p}, \quad r \in \mathbb{R}$$

for $\varepsilon \in (0, \delta)$. We infer from (1) and (2) that

$$\begin{aligned}
\frac{d}{dt}\int_{-1}^{1}&\left[\psi_\varepsilon(\partial_x u) - u\right]dx \\
&= \int_{-1}^{1}\left[\psi_\varepsilon'(\partial_x u)\ \partial_x\partial_t u - \partial_t u\right]dx \\
&= \left[\psi_\varepsilon'(\partial_x u)\ \partial_t u\right]_{x=-1}^{x=1} - \int_{-1}^{1}\left[\psi_\varepsilon''(\partial_x u)\ \partial_x^2 u + 1\right]\partial_t u\,dx \\
&= -\int_{-1}^{1}\psi_\varepsilon''(\partial_x u)\ \left(\partial_x^2 u + (|\partial_x u| \vee \varepsilon)^p\right)\partial_t u\,dx \\
&= -\int_{-1}^{1}\psi_\varepsilon''(\partial_x u)\ \left(\partial_t u + (|\partial_x u| \vee \varepsilon)^p - |\partial_x u|^p\right)\partial_t u\,dx \\
&= -\int_{-1}^{1}\psi_\varepsilon''(\partial_x u)\ |\partial_t u|^2\,dx - \int_{-1}^{1}\left(1 - \frac{|\partial_x u|^p}{\varepsilon^p}\right)_+\partial_t u\,dx.
\end{aligned}$$

On the one hand, since $\varepsilon \in (0, \delta)$, we have

$$|\partial_x u| \vee \varepsilon \le \left(|\partial_x u|^2 + \delta^2\right)^{1/2},$$

so that

$$\int_{-1}^{1}\psi_\varepsilon''(\partial_x u)\ |\partial_t u|^2\,dx \ge \int_{-1}^{1}\frac{|\partial_t u|^2}{\left(|\partial_x u|^2+\delta^2\right)^{p/2}}\,dx.$$

On the other hand, introducing

$$\xi(r) := \begin{cases} r - \dfrac{|r|^p r}{(p+1)\varepsilon^p} & \text{if } |r| \le \varepsilon, \\[2ex] \dfrac{p\varepsilon}{p+1}\,\dfrac{r}{|r|} & \text{if } |r| \ge \varepsilon, \end{cases}$$

we have $\xi'(r) = (1 - |r|^p/\varepsilon^p)_+$ and $|\xi(r)| \le \varepsilon$. Therefore, thanks to (1),

$$\begin{aligned}\left|\int_{-1}^{1}\left(1-\frac{|\partial_x u|^p}{\varepsilon^p}\right)_+ \partial_t u\, dx\right| &\le \left|\int_{-1}^{1}\left(1-\frac{|\partial_x u|^p}{\varepsilon^p}\right)_+ \partial_x^2 u\, dx\right| + \varepsilon^p \int_{-1}^{1}\left(1-\frac{|\partial_x u|^p}{\varepsilon^p}\right)_+ dx \\ &\le \left|\int_{-1}^{1} \partial_x \xi\,(\partial_x u)\, dx\right| + 2\,\varepsilon^p \\ &\le |\xi(\partial_x u(t,1))| + |\xi(\partial_x u(t,-1))| + 2\,\varepsilon^p \le 4\varepsilon^p.\end{aligned}$$

Consequently, for each $\varepsilon \in (0, \delta)$, we have

$$\frac{d}{dt}\int_{-1}^{1} [\psi_\varepsilon(\partial_x u) - u]\, dx + \int_{-1}^{1} \frac{|\partial_t u|^2}{\left(|\partial_x u|^2 + \delta^2\right)^{p/2}}\, dx \le 4\varepsilon^p. \tag{25}$$

It remains to pass to the limit in (25) as $\varepsilon \to 0$. For that purpose, we notice that

$$\left|\psi_\varepsilon'(r) - \frac{|r|^{-p} r}{1-p}\right| \le \frac{p}{1-p}\ \varepsilon^{1-p}$$

for $r \in \mathbb{R}$, so that (ψ_ε) converges uniformly towards $r \mapsto |r|^{2-p}/((2-p)(1-p))$ on compact subsets of $\mathbb{R}$. Recalling that $\partial_x u(t)$ belongs to $L^\infty(-1, 1)$ by Lemma 8, we may let $\varepsilon \to 0$ in (25) and obtain (24). □

Remark 10. It turns out that, at least formally, the functional

$$w \mapsto \int_{-1}^{1}\left(\frac{|\partial_x w(x)|^{2-p}}{(2-p)(1-p)} - w(x)\right) dx$$

is also a Liapunov functional for (1)–(3) when $p \in (1, 2)$, while

$$w \mapsto \int_{-1}^{1} (|\partial_x w(x)|\ \ln(|\partial_x w(x)|) - |\partial_x w(x)| - w(x))\, dx$$

is a Liapunov functional for (1)–(3) when $p = 1$. For $p > 2$, (1)–(3) still have Liapunov functionals but of a different kind [Arrieta et al. 2004].

Corollary 11. *We have*

$$\int_0^\infty \int_{-1}^{1} |\partial_t u(t,x)|^2\, dx\, dt < \infty. \tag{26}$$

Proof. Let $T > 0$. We integrate (24) with $\delta = 1$ over $(0, T)$ and use (18) and the nonnegativity of u to obtain

$$\begin{aligned}
\int_0^T \int_{-1}^1 \frac{|\partial_t u(t,x)|^2}{\left(1+\Lambda^2\right)^{p/2}}\,dx\,dt
&\le \int_0^T \int_{-1}^1 \frac{|\partial_t u(t,x)|^2}{\left(|\partial_x u(t,x)|^2+1\right)^{p/2}}\,dx\,dt\\
&\le \int_{-1}^1 \left(\frac{|\partial_x u(0,x)|^{2-p}}{(2-p)(1-p)} - u(0,x)\right) dx - \int_{-1}^1 \left(\frac{|\partial_x u(T,x)|^{2-p}}{(2-p)(1-p)} - u(T,x)\right) dx\\
&\le \frac{2\,\|\partial_x u_0\|_\infty^{2-p}}{(2-p)(1-p)} + \int_{-1}^1 u(T,x)\,dx \le \frac{2\,\|\partial_x u_0\|_\infty^{2-p}}{(2-p)(1-p)} + 2\,\Lambda.
\end{aligned}$$

Since the right-hand side does not depend on $T > 0$, we deduce (26). □

5. Convergence to steady states

Proof of Theorem 2: nonnegative initial data. Let $u_0 \in Y_+$, $u_0 \not\equiv 0$, and denote by u the corresponding classical solution to (1)–(3). We consider an increasing sequence $(t_n)_{n\ge1}$ of positive real numbers such that $t_n \to \infty$ as $n \to \infty$ and define a sequence of functions $(u_n)_{n\ge1}$ by $u_n(t,x) := u(t_n + t, x)$ for $(t,x) \in [0,1] \times [-1,1]$ and $n \ge 1$. We next denote by g_n the solution to

$$\partial_t g_n - \partial_x^2 g_n = 0, \quad (t,x) \in (0,1) \times (-1,1), \tag{27}$$

$$g_n(t,\pm1) = 0, \quad t \in (0,1), \tag{28}$$

$$g_n(0) = u_n(0) = u(t_n), \quad x \in (-1,1), \tag{29}$$

and put $h_n = u_n - g_n$. Then h_n is a solution to

$$\partial_t h_n - \partial_x^2 h_n = |\partial_x u_n|^p, \quad (t,x) \in (0,1) \times (-1,1), \tag{30}$$

$$h_n(t,\pm1) = 0, \quad t \in (0,1), \tag{31}$$

$$h_n(0) = 0, \quad x \in (-1,1). \tag{32}$$

By Lemma 8, the sequence $(|\partial_x u_n|^p)$ is bounded in $L^q((0,1) \times (-1,1))$ for every $q \in (1,\infty)$. Since h_n is a solution to (30)–(32), we infer from [Ladyženskaja et al. 1968, Theorem IV.9.1] that (h_n) is bounded in $\{w \in L^q(0,1; W^{2,q}(-1,1))\,,\ \partial_t w \in L^q((0,1) \times (-1,1))\}$ for every $q \in (1,\infty)$. We may then use [Ladyženskaja et al. 1968, Lemma II.3.3] with $q = 4$ to deduce that there is $\beta \in (0,1)$ such that (h_n) and $(\partial_x h_n)$ are bounded in $\mathscr{C}^{\beta/2,\beta}([0,1] \times [-1,1])$. This last property together with the Arzelà–Ascoli theorem entail that (h_n) and $(\partial_x h_n)$ are relatively compact in $\mathscr{C}([0,1] \times [-1,1])$.

At the same time, it follows from Lemma 8 and classical regularity properties of the heat equation that (g_n) is relatively compact in $\mathscr{C}([0,1]\times[-1,1])$, while $(\partial_x g_n)$ is relatively compact in $\mathscr{C}([\tau,1]\times[-1,1])$ for each $\tau\in(0,1)$. Consequently, there are a subsequence of (u_n) (not relabeled) and $U\in\mathscr{C}([0,1]\times[-1,1])$ such that $\partial_x U\in\mathscr{C}((0,1]\times[-1,1])$ and

$$\begin{aligned} u_n &\longrightarrow U \quad \text{in } \mathscr{C}([0,1]\times[-1,1]),\\ \partial_x u_n &\longrightarrow \partial_x U \quad \text{in } \mathscr{C}([\tau,1]\times[-1,1]) \end{aligned} \tag{33}$$

for every $\tau\in(0,1)$.

Now, since (u_n) satisfies (1), (2), a straightforward consequence of (33) is that

$$\partial_t U - \partial_x^2 U = |\partial_x U|^p \quad \text{in } \mathscr{D}'((0,1)\times(-1,1)). \tag{34}$$

Furthermore, it follows from Corollary 11 that

$$\lim_{n\to\infty}\int_0^1\int_{-1}^1 |\partial_t u_n|^2\,dx\,dt = \lim_{n\to\infty}\int_{t_n}^{1+t_n}\int_{-1}^1 |\partial_t u|^2\,dx\,dt = 0.$$

By a weak lower semicontinuity argument, we infer from (33) and the previous identity that $\partial_t U = 0$. Then U does not depend on time and thus belongs to $\mathscr{C}^1([-1,1])$. Furthermore, recalling (34), we conclude that $\partial_x^2 U + |\partial_x U|^p = 0$ in $\mathscr{D}'(-1,1)$. The already established regularity of U implies that $U\in\mathscr{C}^2([-1,1])$ and solves (14), (15). Consequently, by Proposition 5, there exists $\vartheta\in[0,1]$ such that $U=U_\vartheta$ and $(u_n(0))=(u(t_n))$ converges towards U_ϑ in $\mathscr{C}([-1,1])$ as $n\to\infty$ by (33). In particular, recalling that $M(t)$ is defined by (7), we have

$$\mathcal{M}_0\,(1-\vartheta)^\alpha = \|U_\vartheta\|_\infty = \lim_{n\to\infty}\|u(t_n)\|_\infty = \lim_{n\to\infty} M(t_n) = M_\infty,$$

whence $M_\infty \le \mathcal{M}_0$ and

$$\vartheta = 1 - \left(\frac{M_\infty}{\mathcal{M}_0}\right)^{1/\alpha}. \tag{35}$$

Since this identity determines ϑ in a unique way, we deduce that the set of cluster points of $\{u(t)\,;\ t\ge 0\}$ is reduced to a single point $\{U_\vartheta\}$ with ϑ given by (35). The set $\{u(t)\,;\ t\ge0\}$ being relatively compact in $\mathscr{C}([-1,1])$ by Lemma 8 and the Arzelà–Ascoli theorem, we finally conclude that $\|u(t)-U_\vartheta\|_\infty\to 0$ as $t\to\infty$, whence (10). In addition, since $u_0\not\equiv 0$, Lemma 7 guarantees that $\vartheta<1$, so that U_ϑ is indeed a nontrivial steady state to (1)–(3). We have thus proved that,

(36) if $u_0\in Y_+$, $u_0\not\equiv 0$, then $M_\infty>0$ and there is $\vartheta\in[0,1)$ such that $\|u(t)-U_\vartheta\|_\infty \to 0$ as $t\to\infty$,

and Theorem 2 holds true for nonnegative initial data. □

6. Sign-changing solutions

We now show that the family $(U_\vartheta)_{\vartheta\in[0,1]}$ of nonnegative steady states to (1)–(2) constructed in Proposition 5 also describes the large time behaviour of sign-changing solutions to (1)–(3). For that purpose, we first establish that any solution to (1)–(3) becomes nonnegative after a finite time.

Lemma 12. *Consider $u_0 \in Y$ and denote by u the corresponding classical solution to (1)–(3). Then there is $T_\star > 0$ such that $u(t,x) \ge 0$ for $(t,x) \in [T_\star,\infty)\times[-1,1]$. Moreover, if $u_0 \le 0$, then $u(t,x) = 0$ for $(t,x) \in [T_\star,\infty)\times[-1,1]$.*

Proof. We put $\tilde{u}_0(x) = 0 \wedge u_0(x)$ for $x \in [-1,1]$ and $\tilde{u}_0(x) = 0$ for $x \in \mathbb{R}\setminus[-1,1]$. Since $\tilde{u}_0$ is a nonpositive, bounded and continuous function in $\mathbb{R}$, we infer from [Gilding et al. 2003, Theorem 3] that there is a unique classical solution $\tilde{u} \in \mathscr{C}([0,\infty)\times\mathbb{R}) \cap \mathscr{C}^{1,2}((0,\infty)\times\mathbb{R}))$ to the Cauchy problem

$$\partial_t \tilde{u} - \partial_x^2 \tilde{u} = a\,|\partial_x \tilde{u}|^p, \quad (t,x) \in (0,\infty)\times\mathbb{R}, \tag{37}$$

$$\tilde{u}(0) = \tilde{u}_0, \quad x \in \mathbb{R}. \tag{38}$$

Furthermore, $\tilde{u}$ is nonpositive in $(0,\infty)\times\mathbb{R}$ and is thus clearly a subsolution to (1)–(3) since $\tilde{u}_0 \le u_0$. The comparison principle then entails that

$$\tilde{u}(t,x) \le u(t,x) \quad \text{for} \quad (t,x) \in [0,\infty)\times[-1,1].$$

But, since $\tilde{u}_0$ is a nonpositive, bounded and continuous function with compact support in $\mathbb{R}$, it follows from [Benachour et al. 2002; Gilding 2005] that $\tilde{u}$ enjoys the property of finite time extinction, that is, there is $T_\star > 0$ such that

$$\tilde{u}(t,x) = 0 \quad \text{for} \quad (t,x) \in [T_\star,\infty)\times\mathbb{R}.$$

Combining these two facts yield the first assertion of Lemma 12. Next, if $u_0 \le 0$, we have also $u \le 0$ in $[0,\infty)\times[-1,1]$ by (6) and u thus identically vanishes in $[T_\star,\infty)\times[-1,1]$. □

Proof of Theorem 2: sign-changing initial data. By Lemma 12, there is $T_\star > 0$ such that $u(T_\star,x) \ge 0$ for $x \in [-1,1]$. Then either $u(T_\star) \equiv 0$ and thus $u(t) \equiv 0$ for $t \ge T_\star$, and $u(t)$ converges towards U_1 as $t \to \infty$. Or $u(T_\star) \not\equiv 0$ and we infer from (36) that there is $\vartheta \in [0,1)$ such that $u(t+T_\star)$ converges towards U_ϑ as $t \to \infty$, which completes the proof of the first statement of Theorem 2.

Assume next that u_0 fulfils (11). Putting $\varphi_1(x) := \cos(\pi x/2)$ for $x \in [-1,1]$ and $\lambda_1 := \pi^2/4$, we recall that $-d^2\varphi_1/dx^2 = \lambda_1\varphi_1$ in $(-1,1)$ with $\varphi_1(\pm 1) = 0$. We infer from (1), (11) and the nonnegativity of φ_1 and $|\partial_x u|^p$ that

$$\int_{-1}^{1} u(t,x)\,\varphi_1(x)\,dx \ge e^{-\lambda_1 t}\int_{-1}^{1} u_0(x)\,\varphi_1(x)\,dx > 0$$

for $t \geq 0$. In particular, with the previous notations, we have $u(T_\star) \geq 0$ with

$$\int_{-1}^{1} u(T_\star, x)\, \varphi_1(x)\, dx > 0,$$

which, together with the positivity of φ_1 on $(-1, 1)$, ensures that $u(T_\star)$ is nonnegative with $u(T_\star) \not\equiv 0$. Arguing as before, we infer from (36) that there is $\vartheta \in [0, 1)$ such that $u(t)$ converges towards U_ϑ as $t \to \infty$, which completes the proof of the second statement of Theorem 2. □

7. Partial extinction of $\partial_x u$ in finite time

Before proceeding with the proof of Theorem 4, we recall that, if $\sigma \in (0, \infty)$ and $\mu \in \mathbb{R}$, the function $(t, x) \mapsto \mu + W_\sigma(x - \sigma t)$ is a travelling wave solution to $\partial_t w - \partial_x^2 w = |\partial_x w|^p$ in $(0, \infty) \times \mathbb{R}$ (see [Gilding and Kersner 2004, Chapter 13], for instance), where

$$W_\sigma(\xi) := -\sigma^{-1/(1-p)} \int_0^{\xi} \left(1 - e^{-\sigma(1-p)\eta}\right)_+^{1/(1-p)} d\eta, \quad \xi \in \mathbb{R}. \tag{39}$$

Introducing $W_0(\xi) = -\mathcal{M}_0\, \xi_+^{\alpha}$ for $\xi \in \mathbb{R}$, we claim that

$$0 \leq W_\sigma(\xi) - W_0(\xi) \leq \sigma\, \kappa_p\, \xi_+^{1+\alpha}, \quad \xi \in \mathbb{R}, \tag{40}$$

with $\kappa_p := (1-p)^\alpha/(2(3-2p))$. Indeed, introducing $\zeta(r) := (r - 1 + e^{-r})/r^2$ and $\zeta_1(r) := r\zeta(r)$ for $r \geq 0$, we have for $\xi \geq 0$

$$W_\sigma(\xi) - W_0(\xi) = \int_0^{\xi} ((1-p)\eta)^{1/(1-p)} \left\{1 - (1 - \zeta_1(\sigma(1-p)\eta))^{1/(1-p)}\right\} d\eta.$$

We deduce from the elementary inequalities $0 \leq \zeta_1(r) \leq 1$ for $r \geq 0$ and

$$(1-r)^{1/(1-p)} \geq 1 - \frac{r}{1-p}, \quad r \in [0, 1],$$

that $W_\sigma(\xi) - W_0(\xi) \geq 0$ and

$$W_\sigma(\xi) - W_0(\xi) \leq \int_0^{\xi} ((1-p)\eta)^{1/(1-p)}\, \frac{\zeta_1(\sigma(1-p)\eta)}{1-p}\, d\eta.$$

We next use the fact that $\zeta(r) \leq 1/2$ for $r \geq 0$ to complete the proof of (40).

Proof of Theorem 4. As mentioned, the proof is similar to that of [Gilding 2005, Theorem 9], the main difference being due to the boundary conditions. We nevertheless reproduce the whole argument here for the sake of completeness. We first observe that (12) implies that $u_0(x) \geq m_0 - \mathcal{M}_0 + U_0(x)$ for $x \in [-1, 1]$ and that

$m_0 - \mathcal{M}_0 + U_0$ is a subsolution to (1) with $m_0 - \mathcal{M}_0 + U_0(\pm 1) \le 0$. We then infer from the comparison principle and (6) that

$$m_0 - \mathcal{M}_0 + U_0(x) \le u(t,x) \le m_0 \quad \text{for} \quad (t,x) \in [0,\infty) \times [-1,1]. \tag{41}$$

In particular,

$$u(t,0) = m_0 \quad \text{for} \quad t \in [0,\infty). \tag{42}$$

We now consider $\sigma \in (0, \varepsilon/\kappa_p)$ and put $w_\sigma(t,x) = m_0 + W_\sigma(x - \sigma t)$ for $(t,x) \in [0,\infty) \times \mathbb{R}$ (recall that ε and m_0 are both defined in (12)). We readily have that

$$\partial_t w_\sigma - \partial_x^2 w_\sigma - |\partial_x w_\sigma|^p = 0 = \partial_t u - \partial_x^2 u - |\partial_x u|^p \quad \text{in} \quad (0,\infty) \times (0,1) \tag{43}$$

with

$$w_\sigma(t,0) = m_0 = u(t,0), \qquad t \ge 0, \tag{44}$$

by (39) and (42). In addition, we infer from (12), (40) and the choice of σ that, for $x \in [0,1]$,

$$\begin{aligned} w_\sigma(0,x) &= m_0 + W_\sigma(x) = m_0 + W_0(x) + W_\sigma(x) - W_0(x) \\ &\le m_0 - \mathcal{M}_0\, x^\alpha + \sigma\, \kappa_p\, x^{1+\alpha} \le m_0 - \mathcal{M}_0\, x^\alpha + \varepsilon\, x^{1+\alpha} \\ &\le u_0(x). \end{aligned} \tag{45}$$

Finally, if $\delta \in (0,\delta_0)$ and $t \in [0, \delta/\sigma]$, it follows from (40) that

$$\begin{aligned} w_\sigma(t,1) &= m_0 + W_\sigma(1 - \sigma t) \\ &= m_0 + W_0(1-\sigma t) + W_\sigma(1-\sigma t) - W_0(1-\sigma t) \\ &\le m_0 - \mathcal{M}_0\, (1-\sigma t)^\alpha + \sigma\, \kappa_p\, (1-\sigma t)^{1+\alpha} \\ &\le \mathcal{M}_0\, \left((1-\delta_0)^\alpha - (1-\delta)^\alpha \right) + \sigma\, \kappa_p \\ &\le 0 \end{aligned} \tag{46}$$

as soon as σ is sufficiently small. Owing to (43), (44), (45) and (46), there is σ_δ depending only on p, m_0, ε and δ such that, if $\sigma \in (0, \sigma_\delta)$, we may apply the comparison principle on $[0,\delta/\sigma] \times [0,1]$ to deduce that

$$w_\sigma(t,x) \le u(t,x), \qquad (t,x) \in [0,\delta/\sigma] \times [0,1]. \tag{47}$$

Recalling (41), we conclude from (47) that, if $\sigma \in (0,\sigma_\delta)$,

$$u(t,x) = m_0 \quad \text{for} \quad t \in [0,\delta/\sigma] \quad \text{and} \quad x \in [0, \sigma t]. \tag{48}$$

A first consequence of (47) is that, if $t > 0$, we may find σ small enough such that $\sigma \in (0,\sigma_\delta)$ and $t \in [0,\delta/\sigma]$. It then follows from (48) that $u(t,x) = m_0$ for $x \in [0, X(t)]$ with $X(t) := \sigma\, t$.

As a second consequence of (47), we note that, if $t \geq T(\delta) := \delta/\sigma_\delta$, there is $\sigma \in (0, \sigma_\delta)$ such that $t = \delta/\sigma$. Then $u(t,x) = m_0$ for $x \in [0, \delta]$ by (48).

To complete the proof of Theorem 4, it suffices to notice that $v : (t,x) \mapsto u(t,-x)$ also solves (1)–(2) with initial datum $x \mapsto u_0(-x)$ which satisfies (12). Then, v also enjoys the above two properties from which we deduce that we have also $u(t,x) = m_0$ for $x \in [-X(t), 0]$ for every $t > 0$ and $u(t,x) = m_0$ for $x \in [-\delta, 0]$ and $t \geq T(\delta)$, thus completing the proof of Theorem 4. □

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