CONVERGENCE TO STEADY STATES FOR A ONE-DIMENSIONAL VISCOUS HAMILTON–JACOBI EQUATION WITH DIRICHLET BOUNDARY CONDITIONS

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We investigate the convergence to steady states of the solutions to the one-dimensional viscous Hamilton–Jacobi equation \( \partial_t u - \partial_x^2 u = |\partial_x u|^p \), where \((t, x) \in (0, \infty) \times (-1, 1)\) and \(p \in (0, 1)\), with homogeneous Dirichlet boundary conditions. For that purpose, a Lyapunov functional is constructed by the approach of Zelenyak (1968). Instantaneous extinction of \( \partial_x u \) on a subinterval of \((-1, 1)\) is shown for suitable initial data.

### 1. Introduction

Nonnegative solutions to the one-dimensional viscous Hamilton–Jacobi equation

\[
\begin{align*}
\partial_t u - \partial_x^2 u &= a \, |\partial_x u|^p, \quad (t, x) \in (0, \infty) \times (-1, 1), \\
u(t, \pm 1) &= 0, \quad t \in (0, \infty), \\
u(0) &= u_0 \geq 0, \quad x \in (-1, 1),
\end{align*}
\]

exhibit a rich variety of qualitative behaviours, according to the sign of \(a \in \{-1, 1\}\) and the values of \(p \in (0, \infty)\). On the one hand, extinction in finite time (that is, there is \(T_* > 0\) such that \(u(t) \equiv 0\) for \(t \geq T_*\)) occurs for \(a = -1\) and \(p \in (0, 1)\), while \(u(t)\) converges exponentially fast to zero as \(t \to \infty\) if \(a = -1\) and \(p \geq 1\) [Benachour et al. 2007]. On the other hand, if \(a = 1\) and \(p > 2\), finite time gradient blow-up takes place for suitably large initial data [Souplet 2002] while convergence to zero of \(u(t)\) as \(t \to \infty\) still holds true for global solutions [Arrieta et al. 2004; Souplet and Zhang 2006]. In addition, all solutions are global for \(a = 1\) and \(p \in [1, 2]\) and converge to zero as \(t \to \infty\) [Benachour et al. 2007; Souplet and Zhang 2006].

The case \(a = 1\) and \(p \in (0, 1)\) offers an interesting novelty and is the subject of the present paper. Indeed, in contrast to the previous cases, the initial-boundary value problem (1)–(3) has a one parameter family \((U_{\theta})_{\theta \in [0, 1]}\) of steady states when \(a = 1\) and \(p \in (0, 1)\) with \(U_1 \equiv 0\) and \(U_{\theta}\) is not constant if \(\theta \in [0, 1)\). These steady

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states play an important role in the dynamics of solutions to (1)–(3): indeed, we will prove that any solution \( u \) to (1)–(3) converges as \( t \to \infty \) towards a steady state, which is nontrivial if, for instance, the initial datum \( u_0 \) is nonnegative with a positive maximum. An interesting feature of \( U_\vartheta \) for \( \vartheta \in (0, 1) \) is that they are constant on a subinterval of \((-1, 1)\). This property is of course related to the fact that \( p \) ranges in \((0, 1)\) and is reminiscent of the finite time extinction phenomenon already alluded to for nonnegative solutions when \( a = -1 \) and \( p \in (0, 1) \). It is then natural to wonder whether the nonlinear term \( |\partial_x u|^p \) may induce a similar singular behaviour on the dynamics of \( u \). More precisely, for a particular class of nonnegative initial data, we will show that the gradient \( \partial_x u \) vanishes identically on \([T_\star, \infty) \times I\) for some \( T_\star > 0 \) and some subinterval \( I \) of \((-1, 1)\). Let us point out here that, for nonnegative initial data, extinction in finite time cannot occur when \( a = 1 \) and \( p \in (0, 1) \), for the comparison principle warrants that \( u \) is bounded from below by the solution to the linear heat equation with the same initial and boundary data.

From now on, we thus assume that

\[
(4) \quad a = 1 \quad \text{and} \quad p \in (0, 1),
\]

and

\[
(5) \quad u_0 \in Y := \{ w \in \mathcal{C}^1([-1, 1]), \quad w(\pm 1) = 0 \}.
\]

It then follows from [Benachour and Dabuleanu 2003, Theorem 3.1 and Proposition 4.1] that the initial-boundary value problem (1)–(3) has a unique classical solution

\[
\begin{array}{c}
\text{u} \in \mathcal{C}([0, \infty) \times [-1, 1]) \cap \mathcal{C}^2((0, \infty) \times (-1, 1))
\end{array}
\]

satisfying

\[
(6) \quad \min_{[-1,1]} u_0 \leq u(t, x) \leq \max_{[-1,1]} u_0, \quad (t, x) \in [0, \infty) \times [-1, 1].
\]

In addition, setting

\[
(7) \quad M(t) := \max_{x \in [-1,1]} u(t, x),
\]

the comparison principle ensures that \( t \mapsto M(t) \) is a nonincreasing function of time and we put

\[
(8) \quad M_\infty := \lim_{t \to \infty} M(t) \in \left[ \min_{[-1,1]} u_0, \max_{[-1,1]} u_0 \right].
\]

We recall that classical solutions to (1)–(3) enjoy the comparison principle; this may be proved by standard arguments, as in [Gilding et al. 2003, Theorem 4].
**Remark 1.** The initial-boundary value problem (1)–(3) is actually well-posed in a larger space than $Y$, which depends on $p$, and we refer to [Benachour and Dabuleanu 2003] for a more detailed account. Still, the solutions constructed in that reference belong to $Y$ for any positive time. Since we are interested here in the large time behaviour, the assumption (5) that $u_0 \in Y$ is thus not restrictive.

For further use, we also introduce the following notations:

$$\alpha := \frac{2 - p}{1 - p} \quad \text{and} \quad M_0 := \frac{(1 - p)^\alpha}{2 - p}.$$  

We may now state our main result.

**Theorem 2.** Consider $u_0 \in Y$ and denote by $u$ the corresponding classical solution to (1)–(3). Then $M_\infty \in [0, M_0]$ and there is a nonnegative stationary solution $u_s$ to (1)–(2) such that

$$\lim_{t \to \infty} \| u(t) - u_s \|_\infty = 0.$$  

Furthermore, $u_s \not\equiv 0$ and $M_\infty > 0$ if

$$\int_{-1}^{1} u_0(x) \cos \left( \frac{\pi x}{2} \right) dx > 0.$$  

It readily follows from the second assertion of Theorem 2 that the set of nontrivial and nonnegative steady states to (1)–(2) attracts all solutions to (1)–(3) starting from a nonnegative initial datum $u_0 \not\equiv 0$. Observe however that the set of nontrivial and nonnegative steady states to (1)–(2) also attracts sign-changing solutions $u$ to (1)–(3) since there are sign-changing initial data fulfilling (11).

The proof of Theorem 2 requires several steps and is performed as follows: we first identify the stationary solutions to (1)–(2) in Section 2 and use them together with comparison arguments to establish that, if $u_0 \in Y$ is nonnegative with $u_0 \not\equiv 0$, then $M_\infty > 0$ and $\{u(t); \ t \geq 0\}$ is bounded in $C^1([-1, 1])$ (Section 3). In Section 4, we employ the technique of [Zelenyak 1968] to construct a Liapunov functional for nonnegative solutions to (1)–(3). Let us mention here that this technique has also been used recently for related problems in [Arrieta et al. 2004; Simondon and Touré 1996]. For nonnegative initial data convergence towards a steady state then follows from the results of Section 3 and Section 4 by a LaSalle invariance principle argument. The large time behaviour of sign-changing initial data is next deduced from that of nonnegative solutions after observing that the negative part of any solution to (1)–(3) vanishes in a finite time (Section 6).

**Remark 3.** A further outcome of Theorem 2 is that the large behaviour of solutions to (1) on a bounded interval is more complex for homogeneous Dirichlet boundary
conditions than for periodic and homogeneous Neumann boundary conditions. Indeed, for the latter boundary conditions, it follows from [Benachour and Dabuleanu 2005; Benachour et al. 2002] that there are $T_* > 0$ and $m_* \in \mathbb{R}$ such that $u(t) \equiv m_*$ for $t \geq T_*$ whatever the signs of $a$ and $u_0$ are.

In Section 7, we prove the extinction in finite time of $\partial_\nu u$ on a subinterval of $(-1, 1)$ for a specific class of initial data. More precisely, we have the following result:

**Theorem 4.** Assume further that there are $m_0 \in (0, M_0)$ and $\varepsilon > 0$ such that

\begin{equation}
    m_0 - M_0 |x|^{\alpha} + \varepsilon |x|^{1+\alpha} \leq u_0(x) \leq m_0, \quad x \in [-1, 1].
\end{equation}

Then, for each $t \in (0, \infty)$, there is $X(t) \in (0, 1)$ such that

\[ u(t, x) = m_0 \quad \text{for} \quad x \in (-X(t), X(t)). \]

Furthermore, if

\begin{equation}
    \delta_0 := 1 - \left( \frac{m_0}{M_0} \right)^{1/\alpha} \in (0, 1),
\end{equation}

and $\delta \in (0, \delta_0)$, there exists $T(\delta) > 0$ such that

\[ u(t, x) = m_0 \quad \text{for} \quad (t, x) \in [T(\delta), \infty) \times [-\delta, \delta]. \]

An example of initial datum in $Y$ fulfilling (12) is the following: $u_0(x) = M_0 - \varepsilon - M_0 |x|^{\alpha} + \varepsilon |x|^{\beta}$ for $x \in [-1, 1]$, where $\beta \in (\alpha, \alpha + 1]$ and $\varepsilon \in (0, \alpha M_0/\beta)$.

The second assertion of Theorem 4 shows that $\partial_\nu u$ vanishes identically after some time on a subinterval of $[-1, 1]$, a phenomenon which one could call *finite time incomplete extinction* in comparison to what occurs for periodic or homogeneous Neumann boundary conditions. But the first assertion of Theorem 4 reveals that the extinction mechanism is somewhat stronger since, even if $\partial_\nu u_0(x)$ vanishes only for $x = 0$, $\partial_\nu u$ vanishes instantaneously on a subinterval of $[-1, 1]$ with positive measure.

Another consequence of Theorem 4 and (6) is that $\|u(t)\|_\infty = m_0$ for every $t \geq 0$. Therefore, for an initial datum $u_0$ in $Y$ satisfying (12), the corresponding solution $u$ to (1)–(3) does not obey the strong maximum principle.

The proof of Theorem 4 relies on comparison arguments with travelling wave solutions to (1) and is similar to that of [Gilding 2005, Theorem 9], some care being needed to cope with the boundary conditions.
Let $U$ and we infer from the nonnegativity of to deduce that

$$d^2U \over dx^2 + |dU| = 0, \quad x \in (-1, 1),$$

$$U(\pm 1) = 0.$$

**Proposition 5.** Let $U \in C^2([-1, 1])$ be a nonnegative solution to (14), (15). Then there is $\theta \in [0, 1]$ such that $U = U_\theta$, where

$$U_\theta(x) := M_0 \left[ (1 - \theta)^a - (|x| - \theta)^a_+ \right], \quad x \in [-1, 1].$$

Observe that $U_\theta$ is constant on $[-\theta, \theta]$ for each $\theta \in (0, 1)$ and that $U_1 \equiv 0$.

**Proof.** Let $U \in C^2([-1, 1])$ be a nonnegative solution to (14), (15). Then $U$ is concave by (14) and we infer from the nonnegativity of $U$ and the boundary conditions (15) that $dU/dx(-1) \geq 0$ and $dU/dx(1) \leq 0$.

If $dU/dx(-1) = 0$, the concavity of $U$ entails that $U$ is a nonincreasing function in $(-1, 1)$. Consequently, $U \equiv 0 = U_1$ to comply with the boundary conditions (15).

Similarly, if $dU/dx(1) = 0$, it follows from the concavity of $U$ that $U$ is nondecreasing on $(-1, 1)$, whence $U \equiv 0 = U_1$ by (15).

We finally consider the case where $dU/dx(-1) > 0$ and $dU/dx(1) < 0$ and put

$$I_1 := \sup \{ X \in (-1, 1) \text{ such that } dU/dx(x) > 0 \text{ on } [-1, X],$$

$$I_S := \inf \{ X \in (-1, 1) \text{ such that } dU/dx(x) < 0 \text{ on } (X, 1].$$

Owing to the continuity of $dU/dx$, we have $-1 < I_1 \leq I_S < 1$ and $dU/dx(x) = 0$ for $x \in [I_1, I_S]$ by the concavity of $U$. Direct integration of (14) then entails that there are two constants $A$ and $B$ such that

$$\left| \frac{dU}{dx} (x) \right|^p \frac{dU}{dx} (x) + (1 - p) x = \begin{cases} A & \text{if } x \in (I_S, 1), \\ B & \text{if } x \in [-1, I_1). \end{cases}$$

Since $p \in (0, 1)$ and $dU/dx$ vanishes for $x \in [I_1, I_S]$, we may let $x \to I_1$ and $x \to I_S$ in (16) to deduce that $A = (1 - p) I_S$ and $B = (1 - p) I_1$. We next integrate (16) to obtain that there are two constants $C_I$ and $C_S$ such that

$$U(x) = \begin{cases} C_S - M_0 (x - I_S)^a & \text{if } x \in (I_S, 1], \\ C_I - M_0 (I_1 - x)^a & \text{if } x \in [-1, I_1). \end{cases}$$
Requiring the boundary conditions (15) to be fulfilled provides the values of $C_I$ and $C_S$, whence

$$U(x) = \begin{cases} \mathcal{M}_0 (1 - x_S)^\alpha - \mathcal{M}_0 (x - x_S)^\alpha & \text{if } x \in (x_S, 1], \\ \mathcal{M}_0 (x_I + 1)^\alpha - \mathcal{M}_0 (x_I - x)^\alpha & \text{if } x \in [-1, x_I). \end{cases}$$

Now, since $dU/dx$ vanishes for $x \in [x_I, x_S]$, we shall have $U(x_S) = U(x_I)$, which implies that $1 - x_S = x_I + 1$, whence $x_S = x_I$. Thus, necessarily, $x_S \in [0, 1]$, from which the equality $U = U_{x_S}$ readily follows. \hfill \Box

It is worth mentioning that $\|U_0\|_\infty \leq \mathcal{M}_0$ for each $\vartheta \in [0, 1]$. Combining this property with the convergence to a steady state to be proved in Section 5, we will conclude that $\mathcal{M}_\infty \leq \mathcal{M}_0$.

**Remark 6.** Proposition 5 shows in particular that there is nonuniqueness of classical solutions to (14), (15). A similar construction is performed in [Alaa and Pierre 1993; Lions 1985] for the boundary-value problem

$$-\Delta u = |\nabla u|^p \text{ in } B(0, 1), \quad u = 0 \text{ on } \partial B(0, 1),$$

where $B(0, 1)$ denotes the open unit ball of $\mathbb{R}^N$, $N > 1$, to establish the nonuniqueness of weak solutions for $p > N/(N-1)$.

### 3. Some properties of $\{u(t) ; t \geq 0\}$

Introducing the positive cone $Y_+ := \{w \in Y \text{ such that } w \geq 0\}$ of $Y$, we first prove that $\mathcal{M}_\infty > 0$ for $u_0 \in Y_+$, $u_0 \neq 0$, by constructing suitable subsolutions to (1)–(3) with the help of $U_0$.

**Lemma 7.** Let $u_0 \in Y_+$ and denote by $u$ the corresponding classical solution to (1)–(3). If $u_0 \neq 0$, we have $\mathcal{M}_\infty > 0$.

**Proof.** Since $u_0 \neq 0$, there are $x_0 \in (-1, 1)$, $\delta \in (0, 1)$ and $m > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (-1, 1)$ and

$$u_0(x) \geq m \text{ for } x \in (x_0 - \delta, x_0 + \delta). \tag{17}$$

We put $x_1 := (x_0 - 1) \vee (-1)$, $x_2 := (x_0 + 1) \wedge 1$, $J := [x_1, x_2]$, $\lambda := 1 \wedge \frac{m}{\mathcal{M}_0 - U_0(\delta)}$, and $v(x) := \lambda (U_0(x - x_0) - U_0(\delta))$ for $x \in J$.

On the one hand, it follows from (1) and (14) that

$$\partial_t v - \partial_x^2 v - |\partial_x v|^p = (\lambda - \lambda^p) |\partial_x U_0(\cdot - x_0)|^p \leq 0 = \partial_t u - \partial_x^2 u - |\partial_x u|^p.$$
on \([0, \infty) \times J\). On the other hand, the nonnegativity of \(u_0\) and the maximum principle entail the nonnegativity of \(u\) which then warrants that

\[
\begin{align*}
 v(x_1) &\leq v(x_0 - \delta) = 0 \leq u(t, x_1), \\
v(x_2) &\leq v(x_0 + \delta) = 0 \leq u(t, x_2),
\end{align*}
\]

while the choice of \(\lambda\) entails that

\[
\begin{align*}
 v(x) &\leq \lambda \ (M_0 - U_0(\delta)) \leq m \leq u_0(x) \quad \text{for} \quad x \in (x_0 - \delta, x_0 + \delta), \\
v(x) &\leq v(x_0 \pm \delta) = 0 \leq u_0(x) \quad \text{for} \quad x \in J \setminus (x_0 - \delta, x_0 + \delta).
\end{align*}
\]

We then infer from the comparison principle that \(u(t, x) \geq v(x)\) for \((t, x) \in [0, \infty) \times J\). In particular, \(M(t) = \|u(t)\|_\infty \geq u(t, x_0) \geq v(x_0) = \lambda \ (M_0 - U_0(\delta))\) for each \(t \geq 0\), whence \(M_\infty \geq \lambda \ (M_0 - U_0(\delta)) > 0\). \(\square\)

We now turn to the question of global boundedness of the trajectory \(\{u(t) : t \geq 0\}\) in \(W^1(\mathbb{R})\).}

**Lemma 8.** Let \(u_0 \in Y_+\) and denote by \(u\) the corresponding classical solution to (1)–(3). There is a constant \(\Lambda > 0\) depending only on \(\|u_0\|_{W^{1, \infty}(-1, 1)}\) and \(p\) such that

\[
\|u(t)\|_{W^{1, \infty}(-1, 1)} \leq \Lambda \quad \text{for} \quad t \geq 0.
\]

**Proof.** We first recall that \(\{u(t) : t \geq 0\}\) is bounded in \(L^\infty(-1, 1)\) by (6) and we are left with the proof that \(\{\partial_x u(t) : t \geq 0\}\) is bounded in \(L^\infty(-1, 1)\). For that purpose, we choose \(\lambda > 1\) such that

\[
\lambda \geq \left[\left(\frac{2}{1 - p}\right)^{1/(1 - p)} \|\partial_x u_0\|_\infty\right] \vee \left[\frac{\|u_0\|_\infty}{(1 - 2^{-\alpha}) \ M_0}\right].
\]

Putting \(v := \lambda u_0\), we first notice that the condition \(\lambda > 1\) ensures that

\[
\partial_x v - \partial_x^2 v - |\partial_x v|^p = (\lambda - \lambda^p) \ |\partial_x u_0|^p \geq 0 \quad \text{in} \quad (0, \infty) \times (-1, 1),
\]

while \(v(\pm 1) = u(t, \pm 1) = 0\) for each \(t \geq 0\). Next, on the one hand, it follows from (19) and the monotonicity properties of \(U_0\) that, if \(x \in (-1/2, 1/2)\), we have

\[
v(x) = \lambda \ U_0(x) \geq \lambda \ U_0(1/2) = \lambda \ M_0 \ (1 - 2^{-\alpha}) \geq \|u_0\|_\infty \geq u_0(x).
\]

On the other hand, if \(x \in [1/2, 1]\), we have by (19) that

\[
\begin{align*}
v(x) &= \lambda \ (U_0(x) - U_0(1)) = \lambda \int_x^1 \left|\frac{dU_0}{dx}(y)\right| \ dy = \alpha \ \lambda \ M_0 \int_x^1 y^{1/(1 - p)} \ dy \\
&\geq \alpha \ \lambda \ M_0 \int_x^1 2^{-1/(1 - p)} \ dy \geq \int_x^1 \|\partial_x u_0\|_\infty \ dy \geq \int_x^1 |\partial_x u_0(y)| \ dy \\
&\geq u_0(x).
\end{align*}
\]
A similar computation shows that \( v(x) \geq u_0(x) \) also holds true for \( x \in [-1, -1/2] \). Therefore, \( v \geq u_0 \) in \([-1, 1]\) and the previous analysis allows us to apply the comparison principle and conclude that \( u(t, x) \leq v(x) \) for \((t, x) \in [0, \infty) \times [-1, 1]\). In particular, if \( t \geq 0 \) and \( x \in (0, 1) \), we have

\[
\frac{u(t, x) - u(t, 1)}{x - 1} = \frac{u(t, x)}{x} - \frac{u(t, 1)}{1} \geq \frac{v(x)}{x} - \frac{v(1)}{1} = \frac{v(x) - v(1)}{x - 1}.
\]

Letting \( x \to 1 \), we deduce that \( \partial_x u(t, 1) \geq \partial_x v(1) = -\lambda (1 - p)^{1/(1 - p)} \). Since \( u_0 \geq 0 \), the comparison principle ensures that \( u(t, x) \geq 0 = u(t, 1) \) for \( x \in (0, 1) \), so that we also have \( \partial_x u(t, 1) \leq 0 \). Arguing in a similar way for \( x = -1 \), we end up with

\[
|\partial_x u(t, \pm 1)| \leq \lambda (1 - p)^{1/(1 - p)} \quad \text{for} \quad t \geq 0.
\]

We now put \( k := \|\partial_x u_0\|_\infty \lor \lambda (1 - p)^{1/(1 - p)} \), \( z := \partial_x u \) and \( \mathcal{R} := \{(t, x) \in (0, \infty) \times (-1, 1) \mid z(t, x) \neq 0\} \). In the neighbourhood of each point \((t_0, x_0)\) of \( \mathcal{R} \), the function \( |\partial_x u|^p \) is smooth, and classical parabolic regularity theory implies that \( \partial_x z \) is \( C^{1,2} \) in a neighbourhood of \((t_0, x_0)\) and satisfies

\[
\partial_x z(t, x) - \partial_x z(t, x) = p |z(t, x)|^{p-2} z(t, x) \partial_x z(t, x).
\]

Since \( \{(t, x) \in (0, \infty) \times (-1, 1) \mid z(t, x) > k\} \subset \mathcal{R} \), we deduce from the previous identity and (20) that

\[
\frac{1}{2} \frac{d}{dt} \|z(t) - k\|_2^2 + \int_{-1}^1 |\partial_x z(t, x)|^2 \, dx = - \int_{-1}^1 |\partial_x (z(t, x) - k)|^2 \, dx,
\]

whence

\[
\|z(t) - k\|_2^2 \leq \|z(0) - k\|_2^2 = 0,
\]

the last equality being true thanks to the choice of \( k \). Consequently, \( \partial_x u(t, x) = z(t, x) \leq k \) in \([0, \infty) \times [-1, 1]\). By a similar argument, we also establish that \( \partial_x u(t, x) = z(t, x) \geq -k \) in \([0, \infty) \times [-1, 1]\). Therefore,

\[
|\partial_x u(t, x)| \leq \|\partial_x u_0\|_{\infty} \lor \lambda (1 - p)^{1/(1 - p)}
\]

for \((t, x) \in [0, \infty) \times [-1, 1]\), which completes the proof of Lemma 8. \( \square \)
4. A Liapunov functional

We now construct a Liapunov functional for nonnegative solutions to (1)–(3) with the help of the technique developed in [Zelenyak 1968]. Let $u_0 \in Y_+$ and denote by $u$ the corresponding classical solution to (1)–(3) which is also nonnegative by the maximum principle. We look for a pair of functions $\Phi$ and $\varrho \geq 0$ such that

$$
\frac{d}{dt} \int_{-1}^{1} \Phi(u, \partial_x u) \, dx = \int_{-1}^{1} \varrho(u, \partial_x u) |\partial_t u|^2 \, dx.
$$

Since $\partial_t u(t, \pm 1) = 0$ by (2), the first term of the right-hand side of this equality also reads

$$
\frac{d}{dt} \int_{-1}^{1} \Phi(u, \partial_x u) \, dx = \int_{-1}^{1} \left[ \partial_1 \Phi(u, \partial_x u) \partial_t u + \partial_2 \Phi(u, \partial_x u) \partial_x \partial_t u \right] \, dx
$$

$$
= \int_{-1}^{1} \left[ \partial_1 \Phi(u, \partial_x u) - \partial_1 \partial_2 \Phi(u, \partial_x u) \partial_x u - \partial_2^2 \Phi(u, \partial_x u) \partial_x^2 u \right] \partial_t u \, dx,
$$

and it is then natural to require that

$$
\left[ \partial_1 \Phi(u, \partial_x u) - \partial_1 \partial_2 \Phi(u, \partial_x u) \partial_x u - \partial_2^2 \Phi(u, \partial_x u) \partial_x^2 u \right] = \varrho(u, \partial_x u) \partial_t u
$$

$$
= \varrho(u, \partial_x u) \left(|\partial_x u|^p + \partial_x^2 u\right)
$$

for (21) to hold true. Following [Zelenyak 1968], we realize that a sufficient condition for the previous equality to be valid is

$$
\partial_1 \Phi(u, \partial_x u) - \partial_1 \partial_2 \Phi(u, \partial_x u) \partial_x u = \varrho(u, \partial_x u) |\partial_t u|^p.
$$

(22)

$$
-\partial_2^2 \Phi(u, \partial_x u) = \varrho(u, \partial_x u).
$$

(23)

Performing the computations as in [Zelenyak 1968], we see that the functions

$$
\Phi(u, \partial_x u) := u - \frac{|\partial_x u|^{2-p}}{(2-p)(1-p)} \quad \text{and} \quad \varrho(u, \partial_x u) := |\partial_x u|^{-p}
$$

solve the differential system (22), (23). However, $\varrho$ is singular when $\partial_x u$ vanishes and it is not clear how to give a meaning to (21) for such a choice of functions $\Phi$ and $\varrho$. Nevertheless, we have the following weaker result which turns out to be sufficient for our purposes.
Proposition 9. For each $t > 0$ and $\delta \in (0, 1]$, we have

$$
\frac{d}{dt} \int_{-1}^{1} \left( \frac{|\partial_x u(t, x)|^{2-p}}{(2-p)(1-p)} - u(t, x) \right) dx + \int_{-1}^{1} \frac{|\partial_x u|^2}{(|\partial_x u|^2 + \delta^2)^{p/2}} dx \leq 0.
$$

Proof. We fix $\delta \in (0, 1]$ and define $\psi_\varepsilon$ by

$$
\psi_\varepsilon(0) = \psi_\varepsilon'(0) = 0 \quad \text{and} \quad \psi_\varepsilon''(r) = (|r| \vee \varepsilon)^{-p}, \quad r \in \mathbb{R}
$$

for $\varepsilon \in (0, \delta)$. We infer from (1) and (2) that

$$
\frac{d}{dt} \int_{-1}^{1} [\psi_\varepsilon(\partial_x u) - u] \ dx
$$

$$
= \int_{-1}^{1} \left[ \psi_\varepsilon'(\partial_x u) \partial_t \partial_x u - \partial_t u \right] \ dx
$$

$$
= \left[ \psi_\varepsilon'(\partial_x u) \partial_t u \right]_{x=-1}^{1} - \int_{-1}^{1} \left[ \psi_\varepsilon''(\partial_x u) \partial_x^2 u + 1 \right] \partial_t u \ dx
$$

$$
= - \int_{-1}^{1} \psi_\varepsilon''(\partial_x u) \left( \partial_x^2 u + (|\partial_x u| \vee \varepsilon)^p \right) \partial_t u \ dx
$$

$$
= - \int_{-1}^{1} \psi_\varepsilon''(\partial_x u) \left( \partial_t u + (|\partial_x u| \vee \varepsilon)^p - |\partial_x u|^p \right) \partial_t u \ dx
$$

$$
= - \int_{-1}^{1} \psi_\varepsilon''(\partial_x u) \partial_t u \ dx - \int_{-1}^{1} \left( 1 - \frac{|\partial_x u|^p}{\varepsilon^p} \right) \partial_t u \ dx.
$$

On the one hand, since $\varepsilon \in (0, \delta)$, we have

$$
|\partial_x u| \vee \varepsilon \leq \left( |\partial_x u|^2 + \delta^2 \right)^{1/2},
$$

so that

$$
\int_{-1}^{1} \psi_\varepsilon''(\partial_x u) \partial_t u \ dx \geq \int_{-1}^{1} \left( 1 - \frac{|\partial_x u|^p}{\varepsilon^p} \right) \partial_t u \ dx.
$$

On the other hand, introducing

$$
\xi(r) := \begin{cases} 
    r - \frac{|r|^p}{(p+1)\varepsilon^p} & \text{if } |r| \leq \varepsilon, \\
    \frac{p\varepsilon}{p+1} \frac{r}{|r|} & \text{if } |r| \geq \varepsilon,
\end{cases}
$$

we have $\xi'(r) = (1 - |r|^p / \varepsilon^p)_+$ and $|\xi(r)| \leq \varepsilon$. Therefore, thanks to (1),

\[
\int_{-1}^{1} \left( 1 - \frac{|\partial_x u|^p}{\varepsilon^p} \right) \partial_t u \, dx \leq \int_{-1}^{1} \left( 1 - \frac{|\partial_x u|^p}{\varepsilon^p} \right) \partial_x^2 u \, dx + \varepsilon^p \int_{-1}^{1} \left( 1 - \frac{|\partial_x u|^p}{\varepsilon^p} \right) \, dx \\
\leq \int_{-1}^{1} \partial_t \xi (\partial_x u) \, dx + 2 \varepsilon^p \\
\leq |\xi(\partial_x u(t, 1))| + |\xi(\partial_x u(t, -1))| + 2 \varepsilon^p \leq 4 \varepsilon^p.
\]

Consequently, for each $\varepsilon \in (0, \delta)$, we have

\[
(25) \quad \frac{d}{dt} \int_{-1}^{1} [\psi_\varepsilon (\partial_x u) - u] \, dx + \int_{-1}^{1} \frac{|\partial_x u|^2}{(|\partial_x u|^2 + \delta^2)^{p/2}} \, dx \leq 4 \varepsilon^p.
\]

It remains to pass to the limit in (25) as $\varepsilon \to 0$. For that purpose, we notice that

\[
\left| \psi_\varepsilon' (r) - \frac{|r|^{-p} r}{1 - p} \right| \leq \frac{p}{1 - p} \varepsilon^{1-p}
\]

for $r \in \mathbb{R}$, so that $(\psi_\varepsilon)$ converges uniformly towards $r \mapsto |r|^{2-p}/((2-p)(1-p))$ on compact subsets of $\mathbb{R}$. Recalling that $\partial_x u(t)$ belongs to $L^\infty(-1, 1)$ by Lemma 8, we may let $\varepsilon \to 0$ in (25) and obtain (24). \hfill \Box

**Remark 10.** It turns out that, at least formally, the functional

\[
w \mapsto \int_{-1}^{1} \left( \frac{|\partial_x w(x)|^{2-p}}{(2-p)(1-p)} - w(x) \right) \, dx
\]

is also a Liapunov functional for (1)–(3) when $p \in (1, 2)$, while

\[
w \mapsto \int_{-1}^{1} (|\partial_x w(x)| \ln (|\partial_x w(x)|) - |\partial_x w(x)| - w(x)) \, dx
\]

is a Liapunov functional for (1)–(3) when $p = 1$. For $p > 2$, (1)–(3) still have Liapunov functionals but of a different kind [Arrieta et al. 2004].

**Corollary 11.** We have

\[
(26) \quad \int_{0}^{\infty} \int_{-1}^{1} \left| \partial_t u(t, x) \right|^2 \, dx \, dt < \infty.
\]
Proof: Let \( T > 0 \). We integrate (24) with \( \delta = 1 \) over \((0, T)\) and use (18) and the nonnegativity of \( u \) to obtain
\[
\int_0^T \int_{-1}^1 \frac{\left| \partial_t u(t, x) \right|^2}{(1 + \Lambda^2)^{p/2}} \, dx \, dt \\
\leq \int_0^T \int_{-1}^1 \frac{\left| \partial_t u(t, x) \right|^2}{(\partial^2 u(t, x))^2 + 1} \, dx \, dt \\
\leq \int_{-1}^1 \left( \frac{|\partial_x u(0, x)|^{2-p}}{(2-p)(1-p)} - u(0, x) \right) \, dx - \int_{-1}^1 \left( \frac{|\partial_x u(T, x)|^{2-p}}{(2-p)(1-p)} - u(T, x) \right) \, dx \\
\leq 2 \|\partial_x u_0\|_\infty^{2-p} + \int_{-1}^1 u(T, x) \, dx \leq 2 \|\partial_x u_0\|_\infty^{2-p} + 2 \Lambda.
\]
Since the right-hand side does not depend on \( T > 0 \), we deduce (26). \( \square \)

5. Convergence to steady states

Proof of Theorem 2: nonnegative initial data. Let \( u_0 \in Y_+ \), \( u_0 \neq 0 \), and denote by \( u \) the corresponding classical solution to (1)–(3). We consider an increasing sequence \((t_n)_{n \geq 1}\) of positive real numbers such that \( t_n \rightarrow \infty \) as \( n \rightarrow \infty \) and define a sequence of functions \((u_n)_{n \geq 1}\) by \( u_n(t, x) := u(t_n + t, x) \) for \((t, x) \in [0, 1] \times [-1, 1]\) and \( n \geq 1 \). We next denote by \( g_n \) the solution to
\[
\partial_t g_n - \partial^2_x g_n = 0, \quad (t, x) \in (0, 1) \times (-1, 1), \\
g_n(t, \pm 1) = 0, \quad t \in (0, 1), \\
g_n(0) = u_n(0) = u(t_n), \quad x \in (-1, 1),
\]
and put \( h_n = u_n - g_n \). Then \( h_n \) is a solution to
\[
\partial_t h_n - \partial^2_x h_n = |\partial_x u_n|^p, \quad (t, x) \in (0, 1) \times (-1, 1), \\
h_n(t, \pm 1) = 0, \quad t \in (0, 1), \\
h_n(0) = 0, \quad x \in (-1, 1).
\]
By Lemma 8, the sequence \((|\partial_x u_n|^p)\) is bounded in \( L^q((0, 1) \times (-1, 1)) \) for every \( q \in (1, \infty) \). Since \( h_n \) is a solution to (30)–(32), we infer from [Ladyženskaja et al. 1968, Theorem IV.9.1] that \( (h_n) \) is bounded in \( \{w \in L^q(0, 1; W^{2,q}(-1, 1)) \, \partial_t w \in \, L^q((0, 1) \times (-1, 1)) \} \) for every \( q \in (1, \infty) \). We may then use [Ladyženskaja et al. 1968, Lemma II.3.3] with \( q = 4 \) to deduce that there is \( \beta \in (0, 1) \) such that \( (h_n) \) and \( (\partial_t h_n) \) are bounded in \( E^{\beta/2}((0, 1) \times [-1, 1]) \). This last property together with the Arzelà–Ascoli theorem entail that \( (h_n) \) and \( (\partial_t h_n) \) are relatively compact in \( C((0, 1) \times [-1, 1]) \).
At the same time, it follows from Lemma 8 and classical regularity properties of the heat equation that \((g_n)\) is relatively compact in \(\C([0, 1] \times [-1, 1])\), while \((\partial_\tau g_n)\) is relatively compact in \(\C([\tau, 1] \times [-1, 1])\) for each \(\tau \in (0, 1)\). Consequently, there is a subsequence of \((u_n)\) (not relabeled) and \(U \in \C([0, 1] \times [-1, 1])\) such that
\[
\partial_\tau U \in \C((0, 1] \times [-1, 1]),
\]
and the previous identity that
\[
\partial_\tau u_n \rightarrow \partial_\tau U \quad \text{in} \quad \C([\tau, 1] \times [-1, 1])
\]
for every \(\tau \in (0, 1)\).

Now, since \((u_n)\) satisfies (1), (2), a straightforward consequence of (33) is that
\[
\partial_\tau U - \partial_\tau^2 U = |\partial_\tau U|^p \quad \text{in} \quad \mathcal{D}'((0, 1) \times (-1, 1)).
\]

Furthermore, it follows from Corollary 11 that
\[
\lim_{n \to \infty} \int_0^1 \int_{-1}^1 |\partial_\tau u_n|^2 \, dx \, dt = \lim_{n \to \infty} \int_{t_n}^{1+t_n} \int_{-1}^1 |\partial_\tau u|^2 \, dx \, dt = 0.
\]
By a weak lower semicontinuity argument, we infer from (33) and the previous identity that \(\partial_\tau U = 0\). Then \(U\) does not depend on time and thus belongs to \(\C^1([-1, 1])\). Furthermore, recalling (34), we conclude that \(\partial_\tau^2 U + |\partial_\tau U|^p = 0\) in \(\mathcal{D}'(-1, 1)\). The already established regularity of \(U\) implies that \(U \in \mathcal{C}^2([-1, 1])\) and solves (14), (15). Consequently, by Proposition 5, there exists \(\vartheta \in [0, 1]\) such that \(U = U_{\vartheta}\) and \((u_n(0)) = (u(t_n))\) converges towards \(U_{\vartheta}\) in \(\C([-1, 1])\) as \(n \to \infty\) by (33). In particular, recalling that \(M(t)\) is defined by (7), we have
\[
M_0 \left(1 - \vartheta\right)^\alpha = \|U_{\vartheta}\|_\infty = \lim_{n \to \infty} \|u(t_n)\|_\infty = \lim_{n \to \infty} M(t_n) = M_\infty,
\]
whence \(M_\infty \leq M_0\) and
\[
\vartheta = 1 - \left(\frac{M_\infty}{M_0}\right)^{1/\alpha}.
\]
Since this identity determines \(\vartheta\) in a unique way, we deduce that the set of cluster points of \([u(t) : t \geq 0]\) is reduced to a single point \([U_{\vartheta}]\) with \(\vartheta\) given by (35). The set \([u(t) : t \geq 0]\) being relatively compact in \(\mathcal{C}([-1, 1])\) by Lemma 8 and the Arzelà–Ascoli theorem, we finally conclude that \(\|u(t) - U_{\vartheta}\|_\infty \to 0\) as \(t \to \infty\), whence (10). In addition, since \(u_0 \neq 0\), Lemma 7 guarantees that \(\vartheta < 1\), so that \(U_{\vartheta}\) is indeed a nontrivial steady state to (1)–(3). We have thus proved that,
\[
\text{if } u_0 \in Y_+, \text{ then } M_\infty > 0 \text{ and there is } \vartheta \in [0, 1) \text{ such that } \|u(t) - U_{\vartheta}\|_\infty \to 0 \text{ as } t \to \infty,
\]
and Theorem 2 holds true for nonnegative initial data. \(\square\)
6. Sign-changing solutions

We now show that the family \((U_\theta)_{\theta \in [0,1]}\) of nonnegative steady states to (1)--(2) constructed in Proposition 5 also describes the large time behaviour of sign-changing solutions to (1)--(3). For that purpose, we first establish that any solution to (1)--(3) becomes nonnegative after a finite time.

**Lemma 12.** Consider \(u_0 \in Y\) and denote by \(u\) the corresponding classical solution to (1)--(3). Then there is \(T_* > 0\) such that \(u(t, x) \geq 0\) for \((t, x) \in [T_*, \infty) \times [-1, 1]\). Moreover, if \(u_0 \leq 0\), then \(u(t, x) = 0\) for \((t, x) \in [T_*, \infty) \times [-1, 1]\).

**Proof.** We put \(\tilde{u}_0(x) = 0 \wedge u_0(x)\) for \(x \in [-1, 1]\) and \(\tilde{u}_0(x) = 0\) for \(x \in \mathbb{R} \setminus [-1, 1]\). Since \(\tilde{u}_0\) is a nonpositive, bounded and continuous function in \(\mathbb{R}\), we infer from [Gilding et al. 2003, Theorem 3] that there is a unique classical solution \(\tilde{u} \in \mathcal{C}([0, \infty) \times \mathbb{R}) \cap \mathcal{C}^{1,2}((0, \infty) \times \mathbb{R})\) to the Cauchy problem

\[
\partial_t \tilde{u} - \partial^2_x \tilde{u} = a |\partial_x \tilde{u}|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R},
\]

\[
\tilde{u}(0) = \tilde{u}_0, \quad x \in \mathbb{R}.
\]

Furthermore, \(\tilde{u}\) is nonpositive in \((0, \infty) \times \mathbb{R}\) and is thus clearly a subsolution to (1)--(3) since \(\tilde{u}_0 \leq u_0\). The comparison principle then entails that

\[
\tilde{u}(t, x) \leq u(t, x) \quad \text{for} \quad (t, x) \in [0, \infty) \times [-1, 1].
\]

But, since \(\tilde{u}_0\) is a nonpositive, bounded and continuous function with compact support in \(\mathbb{R}\), it follows from [Benachour et al. 2002; Gilding 2005] that \(\tilde{u}\) enjoys the property of finite time extinction, that is, there is \(T_* > 0\) such that

\[
\tilde{u}(t, x) = 0 \quad \text{for} \quad (t, x) \in [T_*, \infty) \times \mathbb{R}.
\]

Combining these two facts yield the first assertion of Lemma 12. Next, if \(u_0 \leq 0\), we have also \(u \leq 0\) in \([0, \infty) \times [-1, 1]\) by (6) and \(u\) thus identically vanishes in \([T_*, \infty) \times [-1, 1]\). \(\square\)

**Proof of Theorem 2: sign-changing initial data.** By Lemma 12, there is \(T_* > 0\) such that \(u(T_*, x) \geq 0\) for \(x \in [-1, 1]\). Then either \(u(T_*) \equiv 0\) and thus \(u(t) \equiv 0\) for \(t \geq T_*\), and \(u(t)\) converges towards \(U_1\) as \(t \to \infty\). Or \(u(T_*) \not\equiv 0\) and we infer from (36) that there is \(\vartheta \in (0, 1)\) such that \(u(t + T_*)\) converges towards \(U_\vartheta\) as \(t \to \infty\), which completes the proof of the first statement of Theorem 2.

Assume next that \(u_0\) fulfils (11). Putting \(\varphi_1(x) := \cos(\pi x/2)\) for \(x \in [-1, 1]\) and \(\lambda_1 := \pi^2/4\), we recall that \(-d^2\varphi_1/dx^2 = \lambda_1 \varphi_1\) in \((-1, 1)\) with \(\varphi_1(\pm 1) = 0\). We infer from (1), (11) and the nonnegativity of \(\varphi_1\) and \(|\partial_x u|^p\) that

\[
\int_{-1}^{1} u(t, x) \varphi_1(x) \, dx \geq e^{-\lambda_1 t} \int_{-1}^{1} u_0(x) \varphi_1(x) \, dx > 0
\]
for \( t \geq 0 \). In particular, with the previous notations, we have \( u(T_*) \geq 0 \) with

\[
\int_{-1}^{1} u(T_*, x) \varphi_1(x) \, dx > 0,
\]

which, together with the positivity of \( \varphi_1 \) on \((-1, 1)\), ensures that \( u(T_*) \) is nonnegative with \( u(T_*) \neq 0 \). Arguing as before, we infer from (36) that there is \( \vartheta \in [0, 1) \) such that \( u(T_*) \) is nonnegative with \( u(T_*) \neq 0 \). Arguing as before, we infer from (36) that there is \( \vartheta \in [0, 1) \) such that \( u(T_*) \) converges towards \( U_\vartheta \) as \( t \to \infty \), which completes the proof of the second statement of Theorem 2.

\[ \square \]

7. Partial extinction of \( \partial_x u \) in finite time

Before proceeding with the proof of Theorem 4, we recall that, if \( \sigma \in (0, \infty) \) and \( \mu \in \mathbb{R} \), the function \( (t, x) \mapsto \mu + W_\sigma(x - \sigma t) \) is a travelling wave solution to

\[
\partial_t w - \partial_x^2 w = |\partial_x w|^p \quad \text{in} \quad (0, \infty) \times \mathbb{R} \]

(see [Gilding and Kersner 2004, Chapter 13], for instance), where

\[
W_\sigma(\xi) := -\sigma^{-1/(1-p)} \int_0^\xi \left(1 - e^{-\sigma(1-p)\eta}\right)^{1/(1-p)} \frac{1}{1 - \zeta_1(\sigma(1-p)\eta)} \, d\eta, \quad \xi \in \mathbb{R}.
\]

Introducing \( W_0(\xi) = -M_0 \xi^{\alpha} \) for \( \xi \in \mathbb{R} \), we claim that

\[
0 \leq W_\sigma(\xi) - W_0(\xi) \leq \sigma \kappa_p \xi^{1+\alpha}, \quad \xi \in \mathbb{R},
\]

with \( \kappa_p := (1 - p)^\alpha/(2(3 - 2p)) \). Indeed, introducing \( \zeta(r) := (r - 1 + e^{-r})/r^2 \) and \( \zeta_1(r) := r\zeta(r) \) for \( r \geq 0 \), we have for \( \xi \geq 0 \)

\[
W_\sigma(\xi) - W_0(\xi) = \int_0^\xi \left( (1 - p)\eta \right)^{1/(1-p)} \left\{ 1 - \frac{1}{1 - \zeta_1(\sigma(1-p)\eta)} \right\} \, d\eta.
\]

We deduce from the elementary inequalities \( 0 \leq \zeta_1(r) \leq 1 \) for \( r \geq 0 \) and

\[
(1 - r)^{1/(1-p)} \geq 1 - \frac{r}{1 - p}, \quad r \in [0, 1],
\]

that \( W_\sigma(\xi) - W_0(\xi) \geq 0 \) and

\[
W_\sigma(\xi) - W_0(\xi) \leq \int_0^\xi \left( (1 - p)\eta \right)^{1/(1-p)} \frac{\zeta_1(\sigma(1-p)\eta)}{1 - p} \, d\eta.
\]

We next use the fact that \( \zeta(r) \leq 1/2 \) for \( r \geq 0 \) to complete the proof of (40).

Proof of Theorem 4. As mentioned, the proof is similar to that of [Gilding 2005, Theorem 9], the main difference being due to the boundary conditions. We nevertheless reproduce the whole argument here for the sake of completeness. We first observe that (12) implies that \( u_0(x) \geq m_0 - M_0 + U_0(x) \) for \( x \in [-1, 1] \) and that
$m_0 - M_0 + U_0$ is a subsolution to (1) with $m_0 - M_0 + U_0(\pm 1) \leq 0$. We then infer from the comparison principle and (6) that

$$m_0 - M_0 + U_0(x) \leq u(t, x) \leq m_0 \quad \text{for} \quad (t, x) \in [0, \infty) \times [-1, 1].$$

In particular,

$$u(t, 0) = m_0 \quad \text{for} \quad t \in [0, \infty).$$

We now consider $\sigma \in (0, \varepsilon/k_p)$ and put $w_\sigma(t, x) = m_0 + W_\sigma(x - \sigma t)$ for $(t, x) \in [0, \infty) \times \mathbb{R}$ (recall that $\varepsilon$ and $m_0$ are both defined in (12)). We readily have that

$$\frac{\partial w_\sigma}{\partial t} - \partial^2_x w_\sigma - |\partial_x w_\sigma|^p = 0 = \frac{\partial u}{\partial t} - \partial^2_x u - |\partial_x u|^p \quad \text{in} \quad (0, \infty) \times (0, 1)$$

with

$$w_\sigma(t, 0) = m_0 = u(t, 0), \quad t \geq 0,$$

by (39) and (42). In addition, we infer from (12), (40) and the choice of $\sigma$ that, for $x \in [0, 1]$,

$$w_\sigma(0, x) = m_0 + W_\sigma(x) = m_0 + W_0(x) + W_\sigma(x - W_0(x))$$

$$\leq m_0 - M_0 x^\alpha + \sigma k_p x^{1+\alpha} \leq m_0 - M_0 x^\alpha + \varepsilon x^{1+\alpha}$$

$$\leq u_0(x).$$

Finally, if $\delta \in (0, \delta_0)$ and $t \in [0, \delta/\sigma]$, it follows from (40) that

$$w_\sigma(t, 1) = m_0 + W_\sigma(1 - \sigma t)$$

$$= m_0 + W_0(1 - \sigma t) + W_\sigma(1 - \sigma t) - W_0(1 - \sigma t)$$

$$\leq m_0 - M_0 (1 - \sigma t)^\alpha + \sigma k_p (1 - \sigma t)^{1+\alpha}$$

$$\leq M_0 ((1 - \delta_0)^\alpha - (1 - \delta)^\alpha) + \sigma k_p$$

$$\leq 0$$

as soon as $\sigma$ is sufficiently small. Owing to (43), (44), (45) and (46), there is $\sigma_3$ depending only on $p$, $m_0$, $\varepsilon$ and $\delta$ such that, if $\sigma \in (0, \sigma_3)$, we may apply the comparison principle on $[0, \delta/\sigma] \times [0, 1]$ to deduce that

$$w_\sigma(t, x) \leq u(t, x), \quad (t, x) \in \left[0, \frac{\delta}{\sigma}\right] \times [0, 1].$$

Recalling (41), we conclude from (47) that, if $\sigma \in (0, \sigma_3)$,

$$u(t, x) = m_0 \quad \text{for} \quad t \in [0, \delta/\sigma] \quad \text{and} \quad x \in [0, \sigma t].$$

A first consequence of (47) is that, if $t > 0$, we may find $\sigma$ small enough such that $\sigma \in (0, \sigma_3)$ and $t \in \left[0, \frac{\delta}{\sigma}\right]$. It then follows from (48) that $u(t, x) = m_0$ for $x \in [0, X(t)]$ with $X(t) := \sigma t$. 

As a second consequence of (47), we note that, if \( t \geq T(\delta) := \delta / \sigma \), there is \( \sigma \in (0, \sigma_3) \) such that \( t = \delta / \sigma \). Then \( u(t, x) = m_0 \) for \( x \in [0, \delta] \) by (48).

To complete the proof of Theorem 4, it suffices to notice that \( v : (t, x) \mapsto u(t, -x) \) also solves (1)–(2) with initial datum \( x \mapsto u_0(-x) \) which satisfies (12). Then, \( v \) also enjoys the above two properties from which we deduce that we have also \( u(t, x) = m_0 \) for \( x \in [-X(t), 0] \) for every \( t > 0 \) and \( u(t, x) = m_0 \) for \( x \in [-\delta, 0] \) and \( t \geq T(\delta) \), thus completing the proof of Theorem 4. \( \square \)

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