THE FIXED POINT SUBALGEBRA
OF A LATTICE VERTEX OPERATOR ALGEBRA
BY AN AUTOMORPHISM OF ORDER THREE

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We study the subalgebra of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ consisting of the fixed points of an automorphism which is induced from an order-three isometry of the root lattice $A_2$. We classify the simple modules for the subalgebra. The rationality and the $C_2$-cofiniteness are also established.

1. Introduction

The space of fixed points of an automorphism group of finite order in a vertex operator algebra is a vertex operator subalgebra. The study of such subalgebras and their modules is called orbifold theory. It is a rich field both in conformal field theory and in the theory of vertex operator algebras. However, orbifold theory is difficult to study in general. One reason is that the subalgebra of fixed points usually has more complicated structure than the original vertex operator algebra.

The first example of orbifold theory in vertex operator algebras is the moonshine module $V^\natural$ by Frenkel, Lepowsky, and Meurman [Frenkel et al. 1988], constructed as an extension of $V^+_A$ by its simple module $V^{T+}_A$, where $V^+_A$ is the space of fixed points of an automorphism $\theta$ of order two in the Leech lattice vertex operator algebra $V_A$. This construction is called a $2B$-orbifold construction because $\theta$ corresponds to a $2B$ involution of the monster simple group. More generally, Frenkel et al. defined a vertex operator algebra $V_L$ associated with an arbitrary positive definite even lattice $L$. These lattice vertex operator algebras provide a large family of vertex operator algebras. Such a lattice vertex operator algebra admits an automorphism $\theta$ of order two, which is a lift of the isometry $\alpha \mapsto -\alpha$ of the underlying lattice $L$. Orbifold theory for the fixed point subalgebra $V^{T+}_L$ of $\theta$ has been developed extensively. The simple $V^{T+}_L$-modules have been classified.
[Abe and Dong 2004], the fusion rules have been determined [Abe et al. 2005], and it has been established that $V_L^+$ is $C_2$-cofinite [Abe et al. 2004; Yamskulna 2004].

Here we study the fixed point subalgebra by an automorphism of order three for a certain lattice vertex operator algebra. Namely, let $L = \sqrt{2} A_2$ be $\sqrt{2}$ times an ordinary root lattice of type $A_2$ and let $\tau$ be an isometry of the root lattice of type $A_2$ induced from an order-three permutation on the set of positive roots. We classify the simple modules for the subalgebra $V_L^\tau$ of fixed points by $\tau$. Moreover, we show that $V_L^\tau$ is rational and $C_2$-cofinite.

In [Dong et al. 2004; Kitazume et al. 2003] we have already discussed the vertex operator algebra $V_L^\tau$. It was shown that $V_L^\tau = M^0 \oplus W^0$ is a direct sum of a subalgebra $M^0$ and its simple highest-weight module $W^0$. Actually, $M^0$ is a tensor product of a $W_3$ algebra of central charge $6/5$ and a $W_3$ algebra of central charge $4/5$. The property of a $W_3$ algebra of central charge $6/5$ as the first component of the tensor product $M^0$ was investigated in [Dong et al. 2004]. It is generated by the Virasoro element $\tilde{\omega}^1$ and a weight-three vector $J$. The second component of $M^0$, a $W_3$ algebra of central charge $4/5$, was studied in [Kitazume et al. 2000b]. It is generated by the Virasoro element $\tilde{\omega}^2$ and a weight-three vector $K$. Each of these $W_3$ algebras possesses a symmetry of order three. The order-three symmetry of the second $W_3$ algebra is related to the $\mathbb{Z}_3$ part of $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, where $L^\perp$ denotes the dual lattice of $L$. As an $M^0$-module, $W^0$ is generated by a highest-weight vector $P$ of weight 2. Thus the vertex operator algebra $V_L^\tau$ is generated by the five elements $\tilde{\omega}^1, \tilde{\omega}^2, J, K, \text{ and } P$.

There are 12 inequivalent simple $V_L$-modules, which correspond to the cosets of $L$ in its dual lattice $L^\perp$ [Dong 1993]. Let $(U, Y_U)$ be a simple $V_L$-module. One can define a new simple $V_L$-module $(U \circ \tau, Y_{U \circ \tau})$ by $U \circ \tau = U$ as vector spaces and $Y_{U \circ \tau}(v, z) = Y_U(\tau v, z)$ for $v \in V_L$. Then $U \mapsto U \circ \tau$ is a permutation on the set of simple $V_L$-modules. In the case where $U$ and $U \circ \tau$ are equivalent $V_L$-modules, $U$ is said to be $\tau$-stable. If $U$ is $\tau$-stable, the eigenspace $U(\varepsilon)$ of $\tau$ with eigenvalue $\xi^\varepsilon$, where $\xi = \exp(2\pi \sqrt{-1}/3)$, $\varepsilon = 0, 1, 2$, is a simple $V_L^\tau$-module, while if $U$ belongs to a $\tau$-orbit of length three, $U$ itself is a simple $V_L^\tau$-module and the three members in the $\tau$-orbit are equivalent [Dong and Yamskulna 2002, Theorem 6.14]. Among the 12 inequivalent simple $V_L$-modules, three are $\tau$-stable and the remaining nine are divided into three orbits. In this way we obtain 12 simple $V_L^\tau$-modules. It is known that there are three inequivalent simple $\tau$-twisted $V_L$-modules and three inequivalent simple $\tau^2$-twisted $V_L$-modules. We denote them respectively by

$$(1-1) \quad V_L^j(\tau) := V_L^{T\tau^j}(\tau), \quad V_L^j(\tau^2) := V_L^{T\tau^2j}(\tau^2), \quad j = 0, 1, 2.$$ 

The automorphism $\tau$ acts on these $\tau$-twisted or $\tau^2$-twisted $V_L$-modules and each eigenspace of $\tau$ is a simple $V_L^\tau$-module [Miyamoto and Tanabe 2004, Theorem 2].
There are 18 such simple $V_L^\tau$-modules, all of them inequivalent. Hence there are at least 30 inequivalent simple $V_L^\tau$-modules.

The main part of our argument is to show that every simple $V_L^\tau$-module is isomorphic to one of these 30 simple $V_L^\tau$-modules. Recall that $V_L^\tau = M^0 \oplus W^0$ and that $M^0$ is a tensor product of two $W_3$ algebras. The $W_3$ algebra of central charge $6/5$ (resp. $4/5$) possesses 20 (resp. 6) inequivalent simple modules. Thus there are 120 inequivalent simple $M^0$-modules. It turns out that among these simple $M^0$-modules, 60 of them cannot appear as an $M^0$-submodule in any simple $V_L^\tau$-module and that each simple $V_L^\tau$-module is a direct sum of two of the remaining 60 simple $M^0$-modules. We note that $W^0$ is not a simple current $M^0$-module. Thus $V_L^\tau$ is a nonsimple current extension of $M^0$. A discussion on simple modules for another nonsimple current extension of a certain vertex operator algebra can be found in [Lam et al. 2005, Appendix C].

The organization of this paper is as follows. In Section 2 we review various notions about untwisted or twisted modules for vertex operator algebras, together with some basic tools which will be used in later sections. In Section 3 we fix notation for the vertex operator algebra $V_L^\tau$ and collect its properties. We clarify an argument on the simplicity of $M^0_\tau(\tau^i)$ and $W^0_\tau(\tau^i), i = 1, 2$, in [Kitazume et al. 2003, Proposition 6.8]. Furthermore, we correct some misprints in [Kitazume et al. 2003, (6.46)] and in an equation of [Dong et al. 2004, page 265] concerning a decomposition of a simple $\tau$-twisted $V_L^\tau$-module $V_L^\tau(\tau), j = 1, 2$ as a $\tau$-twisted $M^0_k \otimes M^0_t$-module (see Remark 3.5). In Section 4 we discuss the structure of the 30 known simple $V_L^\tau$-modules. In particular, we calculate the action of $o(\tilde{\omega}^1), o(\tilde{\omega}^2), o(J), o(K),$ and $o(P)$ on the top level of these simple modules. Finally, in Section 5 we complete the classification of simple $V_L^\tau$-modules. We also show the rationality of $V_L^\tau$.

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2. Preliminaries

We recall some notation for untwisted or twisted modules for a vertex operator algebra. We also review the twisted version of Zhu’s theory. A basic reference to twisted modules is [Dong et al. 1998a]. For untwisted modules, see also [Lepowsky and Li 2004]. Let $(V, Y, 1, \omega)$ be a vertex operator algebra and $g$ be an automorphism of $V$ of finite order $T$. Set $V' = \{v \in V \mid gv = e^{2\pi \sqrt{-1}v/T}v\}$, so that $V = \bigoplus_{r \in \mathbb{Z}/T\mathbb{Z}} V'^r$. 
Definition 2.1. A weak g-twisted $V$-module $M$ is a vector space equipped with a linear map

$$Y_M(\cdot, z) : v \in V \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v_n z^{-n-1} \in (\text{End} M)[z]$$

satisfying the following conditions.

1. $Y_M(v, z) = \sum_{n \in \mathbb{Q}/T+\mathbb{Z}} v_n z^{-n-1}$ for $v \in V^r$.
2. $v_n w = 0$ if $n \gg 0$, where $v \in V$ and $w \in M$.
3. $Y_M(1, z) = \text{id}_M$.
4. For $u \in V^r$ and $v \in V$, the $g$-twisted Jacobi identity holds:

$$z_0^{-1} \delta(\frac{z_1-z_2}{z_0}) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta(\frac{z_2-z_1}{z_0}) Y_M(v, z_2) Y_M(u, z_1) = z_2^{-1} \delta(\frac{z_1-z_0}{z_2}) \delta(\frac{z_2-z_0}{z_2}) Y_M(u, (z_0, z_2), v, z_2).$$

Compare the coefficients of $z_0^{-l-1} z_1^{-m-1} z_2^{-n-1}$ in both sides of (2-1) for $u \in V^r$, $v \in V^r$, $l, m \in \mathbb{Z}$, and $n \in \frac{l}{2} + \mathbb{Z}$. Then we obtain

$$\sum_{i=0}^{\infty} \binom{m}{i} (u_{i+1} v)_{m+n-i} = \sum_{i=0}^{\infty} (-1)^i \binom{l}{i} (u_{i+m-i} v_{n+i} - (-1)^i v_{i+n-i} u_{m+i}).$$

In the case $l = 0$, this reduces to

$$[u_m, v_n] = \sum_{i=0}^{\infty} \binom{m}{i} (u_i v)_{m+n-i}.$$

The Virasoro element $\omega$ is contained in $V^0$. Let $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$. Then

$$[L(m), L(n)] = (m - n) L(m+n) + \frac{1}{12}(m^3 - m) \delta_{m+n, 0}(\text{rank} V),$$

$$\frac{d}{dz} Y_M(v, z) = Y_M(L(-1) v, z)$$

for $v \in V$; see [Dong et al. 1998a, (3.8), (3.9)].

An important consequence of (2-1) is the associativity formula

$$z_0 + z_2)^{k+r/T} Y_M(u, z_0 + z_2) Y_M(v, z_2) w = (z_2 + z_0)^{k+r/T} Y_M(Y(u, z_0) v, z_2) w$$

(see [Dong et al. 1998a, (3.5)]), where $u \in V^r$, $v \in V$, $w \in M$, and $k$ is a nonnegative integer such that $z^{k+r/T} Y_M(u, z) w \in M[z]$]

Let $(M, Y_M)$ and $(N, Y_N)$ be weak $g$-twisted $V$-modules. A homomorphism of $M$ to $N$ is a linear map $f : M \to N$ such that $f Y_M(v, z) = Y_N(v, z) f$ for all $v \in V$.

Let $\mathbb{N}$ be the set of nonnegative integers.
Definition 2.2. A $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module $M$ is a weak $g$-twisted $V$-module with a $\frac{1}{T}\mathbb{N}$-grading $M = \bigoplus_{n \in \frac{1}{T}\mathbb{N}} M_{(n)}$ such that

$$v_m M_{(n)} \subset M_{(n + \text{wt}(v) - m - 1)}$$

for any homogeneous vectors $v \in V$.

A $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module here is called an admissible $g$-twisted $V$-module in [Dong et al. 1998a]. Without loss we can shift the grading of a $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module $M$ so that $M_{(0)} \neq 0$ if $M \neq 0$. We call such an $M_{(0)}$ the top level of $M$.

Definition 2.3. A $g$-twisted $V$-module $M$ is a weak $g$-twisted $V$-module with a $\mathbb{C}$-grading $M = \bigoplus_{\lambda \in \mathbb{C}} M_{\lambda}$, where $M_{\lambda} = \{w \in M \mid L(0)w = \lambda w\}$. Moreover, each $M_{\lambda}$ is a finite dimensional space and for any fixed $\lambda$, $M_{\lambda + n/T} = 0$ for all sufficiently small integers $n$.

A $g$-twisted $V$-module is sometimes called an ordinary $g$-twisted $V$-module. By [Dong et al. 1998a, Lemma 3.4], any $g$-twisted $V$-module is a $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module. Indeed, assume that $M$ is a $g$-twisted $V$-module. For each $\lambda \in \mathbb{C}$ with $M_{\lambda} \neq 0$, let $\lambda_0 = \lambda + m/T$ be such that $m \in \mathbb{Z}$ is minimal subject to $M_{\lambda_0} \neq 0$. Let $\Lambda$ be the set of all such $\lambda_0$ and let $M_{(n)} = \bigoplus_{\lambda \in \Lambda} M_{n + \lambda}$. Then $M_{(n)}$ satisfies the condition in Definition 2.2. Thus we have the following inclusions.

\[
\{g\text{-twisted }V\text{-modules}\} \subset \left\{\frac{1}{T}\mathbb{N}\text{-graded weak }g\text{-twisted }V\text{-modules}\right\}
\subset \{\text{weak }g\text{-twisted }V\text{-modules}\}
\]

Definition 2.4. A vertex operator algebra $V$ is said to be $g$-rational if every $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module is semisimple, that is, a direct sum of simple $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-modules.

Let $M$ be a weak $g$-twisted $V$-module. The next lemma is a twisted version of [Li 2001, Lemma 3.12]. In fact, using the associativity formula (2-4) we can prove it by essentially the same argument as in [Li 2001].

Lemma 2.5. Let $u \in V'$, $v \in V^s$, $w \in M$, and $k$ be a nonnegative integer such that $z^{k+r/T}Y_M(u, z)w \in M[[z]]$. Let $p \in \frac{1}{T} + \mathbb{Z}$, $q \in \frac{r}{T} + \mathbb{Z}$, and $N$ be a nonnegative integer such that $z^{N+1+q}Y_M(v, z)w \in M[[z]]$. Then

$$u_p v_q w = \sum_{i=0}^{N} \sum_{j=0}^{\infty} \binom{p-k-r/T}{i} \binom{k+r/T}{j} (u_{p-k-r/T-i+j} v)_{q+k+r/T+i-j} w.$$ 

Conversely, $(u_p v_q)_{q} w$ can be written as a linear combination of some vectors of the form $u_i v_j w$. 

Lemma 2.6. Let \( u \in V' \), \( v \in V^i \), \( w \in M \). Then for \( p \in \mathbb{Z} \) and \( q \in \frac{r}{r+1} + \mathbb{Z} \), the vector \((u_p v)_q w\) is a linear combination of \( u_i v_j w \) with \( i \in \frac{r}{r+1} + \mathbb{Z} \) and \( j \in \frac{r}{r+1} + \mathbb{Z} \).

Proof: Let \( X = \operatorname{span}\{u_i v_j w \mid i \in \frac{r}{r+1} + \mathbb{Z} \text{, } j \in \frac{r}{r+1} + \mathbb{Z} \} \). We use (2.2). Take \( m \in \frac{r}{r+1} + \mathbb{Z} \) such that \( u_{m+i} w = 0 \) for \( i \geq 0 \). Let \( N \in \mathbb{Z} \) be such that \( u_{N+i} v = 0 \) for \( i > 0 \). If \( p > N \), then \( u_p v = 0 \) and the assertion is trivial. Assume that \( p \leq N \). For \( j = 0, 1, \ldots, N - p \), let \( l = p + j \) and \( n = q - m - j \) in (2.2). Then

\[
\sum_{i=0}^{\infty} \binom{m}{i} (u_{p+j+i} v)_{q-j-i} w = \sum_{i=0}^{\infty} (-1)^i \binom{p+j}{i} u_{p+m+j-i} v_{q-m-j+i} w.
\]

The right hand side of this equation is contained in \( X \). Consider the left hand side for each of \( j = N - p, N - p - 1, \ldots, 1, 0 \). Then we see that \((u_N v)_{q-N+p} w \in X\), \((u_{N-1} v)_{q-N+p+1} w \in X, \ldots\), and \((u_p v)_q w \in X\). \(\square\)

For subsets \( A, B \) of \( V \) and a subset \( X \) of \( M \), set \( A \cdot X = \operatorname{span}\{u_n w \mid u \in A, w \in X, n \in \frac{r}{r+1} \mathbb{Z} \} \) and \( A \cdot B = \operatorname{span}\{u_n v \mid u \in A, v \in B, n \in \mathbb{Z} \} \). Then it follows from (2.6) that \( A \cdot (B \cdot X) \subset (A \cdot B) \cdot X \) (see also [Yamauchi 2004, (2.2)])). For a vector \( w \in M \), this in particular implies that \( V \cdot w \) is a weak \( g \)-twisted \( V \)-submodule of \( M \). If \( w \) is an eigenvector for \( L(0) \), then \( V \cdot w \) is a direct sum of eigenspaces for \( L(0) \). Each eigenspace is not necessarily of finite dimension. Thus \( V \cdot w \) is not a \( g \)-twisted module in general. This subject was discussed in [Abe et al. 2004; Buhl 2002; Yamauchi 2004]. We will review it later in this section.

Zhu [1996] introduced an associative algebra \( A(V) \) called the Zhu algebra for a vertex operator algebra \( V \), which plays a crucial role in the study of representations for \( V \). Later, Dong, Li and Mason [Dong et al. 1998a] constructed an associative algebra \( A_g(V) \) called the \( g \)-twisted Zhu algebra in order to generalize Zhu’s theory to \( g \)-twisted representations for \( V \). The definition of \( A_g(V) \) is similar to that of \( A(V) \). Let \( V \), \( g \), \( T \), and \( V' \) be as before. Roughly speaking, \( A_g(V) = V/O_g(V) \) is a quotient space of \( V \) with a binary operation \(*_g\). It is in fact an associative algebra with respect to \(*_g\). If \( r \neq 0 \), then \( V' \subset O_g(V) \). Thus \( A_g(V) = (V^0 + O_g(V))/O_g(V) \). For the case \( g = 1 \), see (5.1) in Section 5.

A certain Lie algebra \( V[g] \) was considered in [Dong et al. 1998a]. Any weak \( g \)-twisted \( V \)-module is a module for the Lie algebra \( V[g] \) (see Lemma 5.1 of that reference). Moreover, for a \( V[g] \)-module \( M \), the space \( \Omega(M) \) of lowest-weight vectors with respect to \( V[g] \) was defined. If \( M \) is a weak \( g \)-twisted \( V \)-modules, then \( \Omega(M) \) is the set of \( w \in M \) such that \( v_{\text{wt}(v)-1+n} w = 0 \) for all homogeneous vectors \( v \in V \) and \( 0 < n \in \frac{1}{T} \mathbb{Z} \). The map \( v \mapsto \alpha(v) \) for homogeneous vectors \( v \in V^0 \) induces a representation of the associative algebra \( A_g(V) \) on \( \Omega(M) \), where \( \alpha(v) = v_{\text{wt}(v)-1} \). If \( M \) is a \( \frac{1}{T} \mathbb{N} \)-graded weak \( g \)-twisted \( V \)-module, then the top level
$M(0)$ is contained in $\Omega(M)$. In the case where $M$ is a simple $\frac{1}{7} \mathbb{Z}$-graded weak $g$-twisted $V$-module, $M(0) = \Omega(M)$ and $M(0)$ is a simple $A_g(V)$-module (see [Dong et al. 1998a, Proposition 5.4]).

For any $A_g(V)$-module $U$, a certain $\frac{1}{7} \mathbb{Z}$-graded $V[g]$-module $M(U)$ such that $M(U)(0) = U$ was defined (see [Dong et al. 1998a, (6.1)]). Let $W$ be the subspace of $M(U)$ spanned by the coefficients of

$$(z_0 + z_2)^{\text{wt}(u) - 1 + \delta_u + r/T} Y_M(u, z_0 + z_2) Y_M(v, z_2) - (z_2 + z_0)^{\text{wt}(u) - 1 + \delta_u + r/T} Y_M(Y(u, z_0)v, z_2)$$

for all homogeneous $u \in V'$, $v \in V$, $w \in U$ (see [Dong et al. 1998a, (6.3)]). Set $\tilde{M}(U) = M(U)/U(V[g])W$, which is a quotient module of $M(U)$ by the $V[g]$-submodule generated by $W$.

The following results will be necessary in Sections 3 and 5.

**Theorem 2.7** [Dong et al. 1998a, Theorem 6.2]. $\tilde{M}(U)$ is a $\frac{1}{7} \mathbb{Z}$-graded weak $g$-twisted $V$-module such that its top level $M(U)(0)$ is equal to $U$ and such that it has the following universal property: for any weak $g$-twisted $V$-module $M$ and any homomorphism $\varphi : U \rightarrow \Omega(M)$ of $A_g(V)$-modules, there is a unique homomorphism $\bar{\varphi} : \tilde{M}(U) \rightarrow M$ of weak $g$-twisted $V$-modules which is an extension of $\varphi$.

Let $J$ be the sum of all $\frac{1}{7} \mathbb{Z}$-graded $V[g]$-submodules of $M(U)$ which intersect trivially with $U$. Since $M(U)(0) = U$, it is a unique $\frac{1}{7} \mathbb{Z}$-graded $V[g]$-submodule of $M(U)$ being maximal subject to $J \cap U = 0$. The principal point is that $U(V[g])W \subset J$. Set $L(U) = M(U)/J$.

**Theorem 2.8** [Dong et al. 1998a, Theorem 6.3]. $L(U)$ is a $\frac{1}{7} \mathbb{Z}$-graded weak $g$-twisted $V$-module such that $\Omega(L(U)) \cong U$ as $A_g(V)$-modules.

**Remark 2.9.** If $M$ is a $\frac{1}{7} \mathbb{Z}$-graded weak $g$-twisted $V$-module and $\varphi : U \rightarrow M(0)$ is a homomorphism of $A_g(V)$-modules, then the homomorphism $\bar{\varphi} : M(U) \rightarrow M$ of weak $g$-twisted $V$-modules in Theorem 2.7 preserves the $\frac{1}{7} \mathbb{Z}$-grading. Indeed, $\tilde{M}(U) = \text{span}\{v_nU \mid v \in V, n \in \frac{1}{7} \mathbb{Z}\}$ by (2-6), since $\tilde{M}(U)$ is generated by $U$ as a $\frac{1}{7} \mathbb{Z}$-graded weak $g$-twisted $V$-module. By (2-5), $v_{\text{wt}(v) - 1 - n} \tilde{M}(U)(0) \subset \tilde{M}(U)(0)$ for any homogeneous $v \in V$ and $n \in \frac{1}{7} \mathbb{Z}$. Since $M(U)(0) = U$, it follows that $\tilde{M}(U)(0)$ is spanned by $v_{\text{wt}(v) - 1 - n}U$ for all homogeneous $v \in V$. Now, $\bar{\varphi}(v_{\text{wt}(v) - 1 - n}U) = v_{\text{wt}(v) - 1 - n}\bar{\varphi}(U)$ is contained in $v_{\text{wt}(v) - 1 - n}M(0) \subset M(0)$. Hence $\bar{\varphi}(\tilde{M}(U)(0)) \subset M(0)$ as required. In the case where both of $M(U)$ and $M$ are ordinary $g$-twisted $V$-modules, $\bar{\varphi}$ becomes a homomorphism of ordinary $g$-twisted $V$-modules since $\bar{\varphi}$ commutes with $L(0)$.
Lemma 2.10. Let $U$ be an $A_g(V)$-module. Let $S$ be a $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module such that it is generated by its top level $S_{(0)}$ and such that $S_{(0)}$ is isomorphic to $U$ as an $A_g(V)$-module. Then there is a surjective homomorphism $S \rightarrow L(U)$ of weak $g$-twisted $V$-modules which preserves the $\frac{1}{T}\mathbb{N}$-grading.

Proof. By Theorem 2.7 and Remark 2.9, an isomorphism $\varphi : U \rightarrow S_{(0)}$ of $A_g(U)$-modules can be extended to a surjective homomorphism $\bar{\varphi} : \bar{M}(U) \rightarrow S$ of weak $g$-twisted $V$-modules which preserves the $\frac{1}{T}\mathbb{N}$-grading. The kernel $\text{Ker} \bar{\varphi}$ of $\bar{\varphi}$ intersects trivially with $\bar{M}(U)_{(0)}$ and so is contained in $\bigoplus_{0<n\in\mathbb{N}}\bar{M}(U)_{(n)}$. Let $I$ be a $\frac{1}{T}\mathbb{N}$-graded $V[g]$-submodule of $M(U)$ such that $\text{Ker} \bar{\varphi} = I/\bar{M}(U[g])W$. Then $I \cap U = 0$. This implies that $I \subseteq J$. Hence $L(U) = M(U)/J$ is a homomorphic image of $M(U)/I \cong S$. \hfill $\square$

Theorem 2.11 [Dong et al. 1998a, Theorem 7.2]. L is a functor from the category of simple $A_g(V)$-modules to the category of simple $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-modules such that $\Omega \circ L = \text{id}$ and $L \circ \Omega = \text{id}$.

Theorem 2.12 [Dong et al. 1998a, Theorem 8.1]. Let $V$ be a $g$-rational vertex operator algebra.

(1) $A_g(V)$ is a finite dimensional semisimple associative algebra.

(2) $V$ has only finitely many isomorphism classes of simple $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-modules.

(3) Every simple $\frac{1}{T}\mathbb{N}$-graded weak $g$-twisted $V$-module is an ordinary $g$-twisted $V$-module.

In case of $g = 1$, the above argument reduces to the untwisted case. In particular, $A_g(V)$ is identical with the original Zhu algebra $A(V)$ if $g = 1$.

There is an important intrinsic property of a vertex operator algebra, namely, the $C_2$-cofiniteness. Let $C_2(V) = \text{span}\{u_{-2}v | u, v \in V\}$. More generally, we set $C_2(M) = \text{span}\{u_{-2}w | u \in V, w \in M\}$ for a weak $V$-module $M$. If the dimension of the quotient space $V/C_2(V)$ is finite, $V$ is said to be $C_2$-cofinite. Similarly, a weak $V$-module $M$ is said to be $C_2$-cofinite if $M/C_2(M)$ is of finite dimension. The notion of $C_2$-cofiniteness of a vertex operator algebra was first introduced by Zhu [1996]. The subspace $C_2(M)$ of a weak $V$-module $M$ was studied in [Li 1999b]. We refer the reader to [Nagatomo and Tsuchiya 2005] also.

Theorem 2.13 [Dong et al. 2000, Proposition 3.6]. If $V$ is $C_2$-cofinite, then $A_g(V)$ is of finite dimension.

If $V = \bigoplus_{n=1}^{\infty} V_n$ and $V_0 = \mathbb{C} \mathbf{1}$, then $V$ is said to be of CFT type. Here $V_n$ denotes the homogeneous subspace of weight $n$, that is, the eigenspace of $L(0) = \omega_1$ with eigenvalue $n$. 

Theorem 2.14 [Yamauchi 2004, Lemma 3.3]. Suppose $V$ is $C_2$-cofinite and of CFT type. Choose a finite dimensional $L(0)$-invariant and $g$-invariant subspace $U$ of $V$ such that $V = U + C_2(V)$. Let $W$ be a weak $g$-twisted $V$-module generated by a vector $w$. Then $W$ is spanned by the vectors of the form $u_1^{n_1}u_2^{n_2} \cdots u_k^{n_k}w$ with $n_1 > n_2 > \cdots > n_k > -N$ and $u_i \in U$, $i = 1, 2, \ldots, k$, where $N \in \mathbb{Z}$ is a constant such that $u_mw = 0$ for all $u \in U$ and $m \geq N$.

Theorem 2.15 [Yamauchi 2004, Corollaries 3.8 and 3.9]. Suppose $V$ is $C_2$-cofinite and of CFT type. Then the following assertions hold.

1. Every weak $g$-twisted $V$-module is a $\frac{1}{j} \mathbb{Z}$-graded weak $g$-twisted $V$-module.

2. Every simple weak $g$-twisted $V$-module is a simple ordinary $g$-twisted $V$-module.

Remark 2.16. Suppose $V$ is $C_2$-cofinite and of CFT type. Let $M$ be a weak $g$-twisted $V$-module and $w^1, \ldots, w^k$ be eigenvectors of $L(0)$ in $M$. Then the weak $g$-twisted $V$-submodule $W$ generated by $w^1, \ldots, w^k$ is an ordinary $g$-twisted $V$-module. Indeed, $W$ is a direct sum of eigenspaces for $L(0)$ and each homogeneous subspace is of finite dimension by Theorem 2.14.

For the untwisted case, that is, the case $g = 1$, we refer the reader to [Abe et al. 2004; Buhl 2002; Dong et al. 1997; Li 1999b]. A spanning set for a vertex operator algebra was first studied in [Gaberdiel and Neitzke 2003, Proposition 8].

3. The fixed point subalgebra $(V_{\sqrt{2}A_2})^\Gamma$

In this section we fix notation. We tend to follow the notation in [Dong et al. 2004; Kitazume et al. 2000a; Kitazume et al. 2003] unless otherwise specified. We also recall certain properties of the lattice vertex operator algebra $V_{\sqrt{2}A_2}$ associated with $\sqrt{2}$ times an ordinary root lattice of type $A_2$ and its subalgebras (see [Dong et al. 2004; Kitazume et al. 2000a; Kitazume et al. 2003; Kitazume et al. 2000b]).

Let $\alpha_1, \alpha_2$ be the simple roots of type $A_2$ and set $\alpha_0 = - (\alpha_1 + \alpha_2)$. Thus $\langle \alpha_1, \alpha_1 \rangle = 2$ and $\langle \alpha_1, \alpha_j \rangle = -1$ if $i \neq j$. Set $\beta_i = \sqrt{2} \alpha_i$ and let $L = \mathbb{Z} \beta_1 + \mathbb{Z} \beta_2$ be the lattice spanned by $\beta_1$ and $\beta_2$. We denote the cosets of $L$ in its dual lattice $L^\perp = \{ \alpha \in \mathbb{Q} \otimes \mathbb{Z} \mid \langle \alpha, L \rangle \in \mathbb{Z} \}$ as follows.

\[ L^\perp = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L, \]

\[ L_0 = L, \quad L_a = \frac{\beta_2}{2} + L, \quad L_b = \frac{\beta_0}{2} + L, \quad L_c = \frac{\beta_1}{2} + L, \]

\[ L^{(i, j)} = L_i + L_j \]

for $i = 0, a, b, c$ and $j = 0, 1, 2$, where $\{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is Klein’s four-group. Note that $L^{(i, j)}, i \in \{0, a, b, c\}, j \in \{0, 1, 2\}$ are all the cosets of $L$ in $L^\perp$ and $L^\perp / L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. 

We adopt the standard notation for the vertex operator algebra $(V_L, Y(\cdot, z))$ associated with the lattice $L$ (see [Frenkel et al. 1988]). In particular, $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L$ is an abelian Lie algebra, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c$ is the corresponding affine Lie algebra, $M(1) = \mathbb{C}[\alpha(n); \alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique simple $\hat{\mathfrak{h}}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n > 0$ and $c = 1$. As a vector space $V_L = M(1) \otimes \mathbb{C}[L]$ and for each $v \in V_L$, a vertex operator $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \text{End}(V_L)[[z, z^{-1}]]$ is defined. The vector $1 = 1 \otimes 1$ is called the vacuum vector. In our case $\langle \alpha, \beta \rangle \in 2\mathbb{Z}$ for any $\alpha, \beta \in L$. Thus the twisted group algebra $\mathbb{C}[L]$ of [Frenkel et al. 1988] is naturally isomorphic to the ordinary group algebra $\mathbb{C}[L]$.

There are exactly 12 inequivalent simple $V_L$-modules, which are represented by $V_{L(\iota, j)}, i = 0, \alpha, \beta$ and $j = 0, 1, 2$ (see [Dong 1993]). We use the symbol $e^\alpha, \alpha \in L^\perp$ to denote a basis of $\mathbb{C}[L^\perp]$.

We consider the following three isometries of $(L, \langle \cdot, \cdot \rangle)$.

\begin{align}
\tau : \beta_1 & \to \beta_2 \to \beta_0 \to \beta_1, \\
\sigma : \beta_1 & \to \beta_2, \quad \beta_2 \to \beta_1, \\
\theta : \beta_i & \to -\beta_i, \quad i = 1, 2.
\end{align}

Note that $\tau$ is fixed-point-free and of order 3. The isometries $\tau, \sigma$, and $\theta$ of $L$ can be extended linearly to isometries of $L^\perp$. Moreover, the isometry $\tau$ lifts naturally to an automorphism of $V_L$:

$$
\alpha^1(-n_1) \cdots \alpha^k(-n_k)e^\beta \mapsto (\tau \alpha^1(-n_1) \cdots (\tau \alpha^k(-n_k))e^{\tau \beta}.
$$

By abuse of notation, we denote it by $\tau$ also. We can consider the action of $\tau$ on $V_{L(\iota, j)}$ in a similar way. We apply the same argument to $\sigma$ and $\theta$. Our purpose is the classification of simple modules for the fixed point subalgebra $V_L^\tau = \{ v \in V_L \mid \tau v = v \}$ of $V_L$ by the automorphism $\tau$.

For a simple $V_L$-module $(U, Y_U)$, let $(U \circ \tau, Y_{U\circ \tau})$ be a new $V_L$-module such that $U \circ \tau = U$ as vector spaces and $Y_{U\circ \tau}(v, z) = Y_U(\tau v, z)$ for $v \in V_L$ [Dong et al. 2000]. Then $U \mapsto U \circ \tau$ induces a permutation on the set of simple $V_L$-modules. If $U$ and $U \circ \tau$ are equivalent $V_L$-modules, $U$ is said to be $\tau$-stable. The following lemma is a straightforward consequence of the definition of $V_{L(\iota, j)}$.

**Lemma 3.1.** (1) $V_{L(\iota, j)}$, $j = 0, 1, 2$ are $\tau$-stable.

(2) $V_{L(\iota, j)} \circ \tau = V_{L(\iota, j)}$, $V_{L(\iota, j)} \circ \tau = V_{L(\iota, j)}$, and $V_{L(\iota, j)} \circ \tau = V_{L(\iota, j)}$, $j = 0, 1, 2$.

A family of simple twisted modules for lattice vertex operator algebras was constructed in [Dong and Lepowsky 1996; Lepowsky 1985]. Following Dong and Lepowsky, three inequivalent simple $\tau$-twisted $V_L$-modules $(V_L^1(\tau), Y^1(\cdot, z)), j = 0, 1, 2$ were studied in [Dong et al. 2004; Kitazume et al. 2003]. By the preceding lemma and [Dong et al. 2000, Theorem 10.2], we know that $(V_L^1(\tau), Y^1(\cdot, z)),$
$j = 0, 1, 2$, are all the inequivalent simple $\tau$-twisted $V_L$-modules. Similarly, there are exactly three inequivalent simple $\tau^2$-twisted $V_L$-modules $(V_L^j(\tau^2), Y^{\tau^2}(\cdot, z))$, $j = 0, 1, 2$.

We use the same notation for $(V_L^j(\tau), Y^\tau(\cdot, z))$ and $(V_L^j(\tau^2), Y^{\tau^2}(\cdot, z))$ as in [Dong et al. 2004, Section 4]. Thus

$$V_L^j(\tau) = S[\tau] \otimes T_{\chi_j},$$

where $T_{\chi_j}$, $j = 0, 1, 2$, is the one-dimensional representation of a certain central extension of $L$ affording the character $\chi_j$. Let

$$h_1 = \frac{1}{3}(\beta_1 + \xi^2 \beta_2 + \xi \beta_0), \quad h_2 = \frac{1}{3}(\beta_1 + \xi \beta_2 + \xi^2 \beta_0).$$

Then $\tau h_i = \xi^i h_i$, $(h_1, h_1) = (h_2, h_2) = 0$, and $(h_1, h_2) = 2$. Moreover, $\beta_1 = \xi^{-1} h_1 + \xi^2 (i-1) h_2$, $i = 0, 1, 2$. As a vector space, $S[\tau]$ is isomorphic to a polynomial algebra with variables $h_1(1/3 + n)$, $h_2(2/3 + n)$, $n \in \mathbb{Z}_{\leq 0}$. The isometry $\tau$ acts on $S[\tau]$ by $\tau h_j = \xi^j h_j$. We define the action of $\tau$ on $T_{\chi_j}$ to be the identity. The weight in $S[\tau]$ is given by $\text{wt} h_i(i/3 + n) = -i/3 - n$, $i = 1, 2$ and $\text{wt} 1 = 1/9$. The weight of any element of $T_{\chi_j}$ is defined to be 0. Note that the weight in $V_L^j(\tau)$ is identical with the eigenvalue for the action of the coefficient of $z^{-2}$ in the $\tau$-twisted vertex operator $Y^\tau(\omega, z)$, where $\omega$ denotes the Virasoro element of $V_L$.

The simple $\tau^2$-twisted $V_L$-modules $(V_L^j(\tau^2), Y^{\tau^2}(\cdot, z))$, $j = 0, 1, 2$ are

$$V_L^j(\tau^2) = S[\tau^2] \otimes T_{\chi'_j},$$

where $T_{\chi'_j}$, $j = 0, 1, 2$, are the one-dimensional representations of a certain central extension of $L$ affording the character $\chi'_j$. Moreover, $S[\tau^2]$ is isomorphic to a polynomial algebra with variables $h'_1(1/3 + n)$, $h'_2(2/3 + n)$, $n \in \mathbb{Z}_{\leq 0}$ as a vector space, where $h'_1 = h_2$ and $h'_2 = h_1$. Thus $\tau^2 h'_i = \xi^i h'_i$, $i = 1, 2$. The action of $\tau$ on $S[\tau^2]$ is given by $\tau h'_i = \xi^i h'_i$, $i = 1, 2$. The action of $\tau$ on $T_{\chi'_j}$ is defined to be the identity. The weight in $S[\tau^2]$ is given by $\text{wt} h'_i(i/3 + n) = -i/3 - n$, $i = 1, 2$ and $\text{wt} 1 = 1/9$. The weight of any element of $T_{\chi'_j}$ is defined to be 0. The weight in $V_L^j(\tau^2)$ is identical with the eigenvalue for the action of the coefficient of $z^{-2}$ in the $\tau^2$-twisted vertex operator $Y^{\tau^2}(\omega, z)$.

By Lemma 3.1, [Dong and Mason 1997, Theorem 4.4], and [Dong and Yamskulna 2002, Theorem 6.14],

$$V_{L,[0,j]}(\epsilon) = \{ v \in V_{L,[0,j]} \mid \tau v = \xi^\epsilon v \}, \quad j, \epsilon = 0, 1, 2$$

are inequivalent simple $V_L^j$-modules. For each of $j = 0, 1, 2$, we have that $V_{L,[0,j]}$, $i = a, b, c$ are equivalent simple $V_L^j$-modules. Moreover, $V_{L,[\epsilon,j]}$, $j = 0, 1, 2$ are inequivalent simple $V_L^j$-modules. From [Miyamoto and Tanabe 2004, Theorem 2],
it follows that
\[ V_L^j(\tau)(\varepsilon) = \{ v \in V_L^j(\tau) \mid \tau v = \xi^\varepsilon v \}, \quad j, \varepsilon = 0, 1, 2 \]
are inequivalent simple \( V_L^j \)-modules. Similar assertions hold for simple \( \tau^2 \)-twisted modules, namely,
\[ V_L^j(\tau^2)(\varepsilon) = \{ v \in V_L^j(\tau^2) \mid \tau^2 v = \xi^\varepsilon v \}, \quad j, \varepsilon = 0, 1, 2 \]
are inequivalent simple \( V_L^j \)-modules. In this way we obtain 30 simple \( V_L^j \)-modules. These 30 simple \( V_L^j \)-modules are inequivalent by [Miyamoto and Tanabe 2004, Theorem 2]. We summarize the result as follows.

**Lemma 3.2.** The following 30 simple \( V_L^j \)-modules are inequivalent.

1. \( V_L^{(0,0)}(\varepsilon), j, \varepsilon = 0, 1, 2, \)
2. \( V_L^{(1,0)}(\varepsilon), j = 0, 1, 2, \)
3. \( V_L^{(2,0)}(\varepsilon), j, \varepsilon = 0, 1, 2, \)
4. \( V_L^{(3,0)}(\varepsilon), j, \varepsilon = 0, 1, 2. \)

We consider the structure of \( V_L^j \) in detail. Set
\[ x(\alpha) = e^{\sqrt{3}\alpha} + e^{-\sqrt{3}\alpha}, \quad y(\alpha) = e^{\sqrt{2}\alpha} - e^{-\sqrt{2}\alpha}, \quad w(\alpha) = \frac{1}{2}\alpha(-1)^2 - x(\alpha) \]
for \( \alpha \in \{ \pm \alpha_0, \pm \alpha_1, \pm \alpha_2 \} \) and let
\[ \omega = \frac{1}{6}(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2), \]
\[ \tilde{\omega}^1 = \frac{1}{3}(w(\alpha_1) + w(\alpha_2) + w(\alpha_0)), \quad \tilde{\omega}^2 = \omega - \tilde{\omega}^1, \]
\[ \omega^1 = \frac{1}{4}w(\alpha_1), \quad \omega^2 = \tilde{\omega}^1 - \omega^1. \]

Then \( \omega \) is the Virasoro element of \( V_L \) and \( \tilde{\omega}^1, \tilde{\omega}^2 \) are mutually orthogonal conformal vectors of central charge 6/5, 4/5 respectively. The subalgebra \( \text{Vir}(\tilde{\omega}^j) \) generated by \( \tilde{\omega}^j \) is isomorphic to the Virasoro vertex operator algebra of given central charge, namely, \( \text{Vir}(\tilde{\omega}^1) \cong L(6/5, 0) \) and \( \text{Vir}(\tilde{\omega}^2) \cong L(4/5, 0) \). Moreover, \( \tilde{\omega}^1 \) is a sum of two conformal vectors \( \omega^1 \) and \( \omega^2 \) of central charge 1/2 and 7/10 respectively and \( \omega^1, \omega^2 \) and \( \tilde{\omega}^2 \) are mutually orthogonal. Note that \( \tilde{\omega}^2 \) was denoted by \( \omega^3 \) in [Dong et al. 2004; Kitazume et al. 2000a; Kitazume et al. 2003; Kitazume et al. 2000b]. Such a decomposition of the Virasoro element of a lattice vertex operator algebra into a sum of mutually orthogonal conformal vectors was first studied in [Dong et al. 1998b].

Set
\[ M_k^i = \{ v \in V_{L_i} \mid (\tilde{\omega}^2)_1 v = 0 \}, \quad W_k^i = \{ v \in V_{L_i} \mid (\tilde{\omega}^2)_1 v = \frac{2}{3} v \}, \quad i = 0, a, b, c, \]
\[ M^j_i = \{ v \in V_{L,j} \mid (\omega^1)_1 v = (\omega^2)_1 v = 0 \}, \]
\[ W^j_i = \{ v \in V_{L,j} \mid (\omega^1)_1 v = 0, \ (\omega^2)_1 v = \frac{3}{2} v \}, \quad j = 0, 1, 2. \]

Then \( M^0_i \) and \( M^0_j \) are simple vertex operator algebras. Moreover, \( \{ M^j_i, W^j_i; i = 0, a, b, c \} \) and \( \{ M^j_i, W^j_i; j = 0, 1, 2 \} \) are complete sets of representatives of isomorphism classes of simple modules for \( M^0_i \) and \( M^0_j \), respectively (see [Kitazume et al. 2000a; Kitazume et al. 2000b; Lam and Yamada 2000]). As \( \text{Vir}(\omega^1) \otimes \text{Vir}(\omega^2) \)-modules,

\begin{align}
M^0_k & \cong (L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, 0)) \oplus (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2})) , \\
M^a_k & \cong M^0_k \cong L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{7}{10}), \\
M^c_k & \cong (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, 0)) \oplus (L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, \frac{3}{2})) , \\
W^0_k & \cong (L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, \frac{3}{2})) \oplus (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{7}{10})), \\
W^a_k & \cong W^0_k \cong L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{7}{10}), \\
W^c_k & \cong (L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{7}{10}, \frac{3}{2})) \oplus (L(\frac{1}{2}, 0) \otimes L(\frac{7}{10}, \frac{1}{10})),
\end{align}

and as \( \text{Vir}(\omega^2) \)-modules,

\begin{align}
M^0_i & \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3), & M^1_i & \cong M^2_i \cong L(\frac{4}{5}, \frac{2}{3}), \\
W^0_i & \cong L(\frac{4}{5}, \frac{2}{3}) \oplus L(\frac{4}{5}, \frac{7}{5}), & W^1_i & \cong W^2_i \cong L(\frac{4}{5}, \frac{11}{15}).
\end{align}

Furthermore,

\begin{align}
(3-4) & \quad V_{L,\omega^1} \cong (M^0_i \otimes M^0_j) \oplus (W^0_i \otimes W^0_j)
\end{align}

as \( M^0_k \otimes M^0_j \)-modules. In particular,

\begin{align}
(3-5) & \quad V_{L} \cong (M^0_i \otimes M^0_j) \oplus (W^0_i \otimes W^0_j).
\end{align}

Note that \( M^j_i = \{ v \in V_{L,i} \mid (\tilde{\omega}^1)_1 v = 0 \} \) and that \( M^0_k, W^0_k \) and \( M^1_i, j = 0, 1, 2 \) are \( \tau \)-invariant. However, \( W^0_i, j = 0, 1, 2 \) are not \( \tau \)-invariant.

The fusion rules for \( M^0_i \) and \( M^0_j \) were determined in [Lam and Yamada 2000] and [Miyamoto 2001], respectively. They are

\begin{align}
(3-6) & \quad M^i_k \times M^j_k = M^{i+j}_k, \quad M^i_k \times W^j_k = W^{i+j}_k, \quad W^i_k \times W^j_k = M^{i+j}_k + W^{i+j}_k
\end{align}

for \( i, j = 0, a, b, c \) and

\begin{align}
(3-7) & \quad M^i_j \times M^j_i = M^{i+j}_i, \quad M^i_j \times W^j_i = W^{i+j}_i, \quad W^i_j \times W^j_i = M^{i+j}_i + W^{i+j}_i
\end{align}

for \( i, j = 0, 1, 2 \).
The following two weight-three vectors are important.

\[ J = w(\alpha_1) w(\alpha_2) - w(\alpha_2) w(\alpha_1) = -\frac{1}{6} \left( \beta_1(-2)(\beta_2 - \beta_0)(-1) + \beta_2(-2)(\beta_0 - \beta_1)(-1) + \beta_0(-2)(\beta_1 - \beta_2)(-1) \right) \]

\[ - (\beta_2 - \beta_0)(-1) y(\alpha_1) - (\beta_0 - \beta_1)(-1) y(\alpha_2) - (\beta_1 - \beta_2)(-1) y(\alpha_0), \]

\[ K = -\frac{1}{9} (\beta_1 - \beta_2)(-1)(\beta_2 - \beta_0)(-1)(\beta_0 - \beta_1)(-1) \]

\[ + (\beta_2 - \beta_0)(-1)x(\alpha_1) + (\beta_0 - \beta_1)(-1)x(\alpha_2) + (\beta_1 - \beta_2)(-1)x(\alpha_0). \]

Let \( M(0) = (M^0_\tau)^c = \{ u \in M^0_\tau \mid \tau u = u \} \). The vertex operator algebra \( M(0) \) was studied in [Dong et al. 2004]. Among other things, the classification of simple modules, the rationality and the \( C_2 \)-cofiniteness for \( M(0) \) were established. It is known that \( M(0) \) is a \( W_3 \) algebra of central charge 6/5 with the Virasoro element \( \tilde{\omega}^1 \). In fact, \( M(0) \) is generated by \( \tilde{\omega}^1 \) and \( J \). The following equations hold [Dong et al. 2004, (3.1)].

\[ J_S J = -84 \cdot 1, \]

\[ J_A J = 0, \]

\[ J_3 J = -420 \tilde{\omega}^1, \]

\[ J_2 J = -210(\tilde{\omega}^1)_0 \tilde{\omega}^1, \]

\[ J_1 J = 9(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0 \tilde{\omega}^1 - 240(\tilde{\omega}^1)_{-1} \tilde{\omega}^1, \]

\[ J_0 J = 22(\tilde{\omega}^1)_0(\tilde{\omega}^1)_0 \tilde{\omega}^1 - 120(\tilde{\omega}^1)_0(\tilde{\omega}^1)_{-1} \tilde{\omega}^1. \]

Let \( L^1(n) = (\tilde{\omega}^1)_{n+1} \) and \( J(n) = J_{n+2} \) for \( n \in \mathbb{Z} \), so that the weight of these operators is \( \text{wt } L^1(n) = \text{wt } J(n) = -n \). Then

\[ [L^1(m), L^1(n)] = (m-n)L^1(m+n) + \frac{m^3 - m}{12} \cdot 6 \cdot 5 \cdot \delta_{m+n,0}. \]

\[ [L^1(m), J(n)] = (2m-n)J(m+n), \]

\[ [J(m), J(n)] = (m-n)(22(m+n+2)(m+n+3) + 35(m+2)(n+2))L^1(m+n) - 120(m-n) \left( \sum_{k \geq -2} L^1(k)L^1(m+n-k) + \sum_{k \geq -1} L^1(m+n-k)L^1(k) \right) - \frac{7}{10}m(m^2-1)(m^2-4)\delta_{m+n,0}. \]

The vertex operator algebra \( M^0_\tau \) is known as a 3-State Potts model. It is a \( W_3 \) algebra of central charge 4/5 with the Virasoro element \( \tilde{\omega}^2 \) and is generated by
\[\tilde{\omega}^2\text{ and }K.\] Both of \(\tilde{\omega}^2\) and \(K\) are fixed by \(\tau\), so that \(\tau\) is the identity on \(M_1^0\). The rationality of \(M_1^0\) was established in [Kitazume et al. 2000b] and the \(C_2\)-cofiniteness of \(M_1^0\) follows from [Buhl 2002]. By a direct calculation, we can verify that

\[
\begin{align*}
K_2K &= 104 \cdot 1, \\
K_3K &= 0, \\
K_4K &= 780\tilde{\omega}^2,
\end{align*}
\]

(3-12)

\[
\begin{align*}
K_2K &= 390(\tilde{\omega}^2)_0\tilde{\omega}^2, \\
K_1K &= -27(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0\tilde{\omega}^2 + 480(\tilde{\omega}^2)_{-1}\tilde{\omega}^2, \\
K_0K &= -46(\tilde{\omega}^2)_0(\tilde{\omega}^2)_0\tilde{\omega}^2 + 240(\tilde{\omega}^2)_0(\tilde{\omega}^2)_{-1}\tilde{\omega}^2.
\end{align*}
\]

Let \(L^2(n) = (\tilde{\omega}^2)_{n+1}\) and \(K(n) = K_{n+2}\) for \(n \in \mathbb{Z}\). Then

\[
[L^2(m), L^2(n)] = (m-n)L^2(m+n) + \frac{m^3-m}{12} \cdot \frac{4}{5} \cdot \delta_{m+n,0},
\]

(3-13)

\[
[L^2(m), K(n)] = (2m-n)K(m+n),
\]

(3-14)

\[
[K(m), K(n)] = -(m-n)(46(m+n+2)(m+n+3) + 65(m+2)(n+2))L^2(m+n)
\]

\[
+ 240(m-n)\left(\sum_{k \leq -2} L^2(k)L^2(m+n-k) + \sum_{k \geq -1} L^2(m+n-k)L^2(k)\right)
\]

\[
+ \frac{12}{15} m(m^2-1)(m^2-4)\delta_{m+n,0}.
\]

Remark 3.3. Let \(L_n = L^1(n)\), \(W_n = \sqrt{-1/270}J(n)\), and \(c = 6/5\). Then the commutation relations above coincide with (2.1) and (2.2) of [Bouwknegt et al. 1996]. The same commutation relations also hold if we set \(L_n = L^2(n)\), \(W_n = K(n)/\sqrt{390}\), and \(c = 4/5\).

Let us review the 20 inequivalent simple \(M(0)\)-modules studied in [Dong et al. 2004]. Among those simple \(M(0)\)-modules, eight of them appear in simple \(M_1^0\)-modules, namely,

\[
M(\varepsilon) = \{u \in M_1^0 \mid \tau u = \xi^\varepsilon u\}, \quad W(\varepsilon) = \{u \in W_1^0 \mid \tau u = \xi^\varepsilon u\}
\]

for \(\varepsilon = 0, 1, 2\), \(M_\varepsilon^1\) and \(W_\varepsilon^1\). The remaining 12 simple \(M(0)\)-modules appear in simple \(\tau\)-twisted or \(\tau^2\)-twisted \(V_L\)-modules. Let

\[
M_T(\tau)(\varepsilon) = \{u \in V_0^0(\tau) \mid (\tilde{\omega}^2)_1u = 0, \ \tau u = \xi^\varepsilon u\},
\]

\[
W_T(\tau)(\varepsilon) = \{u \in V_0^0(\tau) \mid (\tilde{\omega}^2)_1u = \xi^\varepsilon u, \ \tau u = \xi^\varepsilon u\}.
\]
Then $M_T(\tau)(\epsilon)$, $W_T(\tau)(\epsilon)$, $\epsilon = 0, 1, 2$ are inequivalent simple $M(0)$-modules. Similarly,

\[
M_T(\tau^2)(\epsilon) = \{u \in V_L^0(\tau^2) \mid (\tilde{\omega}^2)_1 u = 0, \tau^2 u = \xi^\epsilon u\},
\]

\[
W_T(\tau^2)(\epsilon) = \{u \in V_L^0(\tau^2) \mid (\tilde{\omega}^2)_1 u = \frac{2}{\tau} u, \tau^2 u = \xi^\epsilon u\}
\]

for $\epsilon = 0, 1, 2$ are inequivalent simple $M(0)$-modules. In [Dong et al. 2004], it was shown that $M(\epsilon)$, $W(\epsilon)$, $M_\epsilon^0$, $W_\epsilon^0$, $M_T(\tau)(\epsilon)$, $W_T(\tau)(\epsilon)$, $M_T(\tau^2)(\epsilon)$, and $W_T(\tau^2)(\epsilon)$, $\epsilon = 0, 1, 2$ form a complete set of representatives of isomorphism classes of simple $M(0)$-modules.

Let us describe the structure of the fixed point subalgebra $V_L^\tau$. By the definition of $M(0)$ and $M_0^0$, we see that $V_L^\tau \supset M(0) \boxtimes M_0^0$. Since both of $M(0)$ and $M_0^0$ are rational, $M(0) \boxtimes M_0^0$ is also rational. Thus $V_L(\epsilon) = \{u \in V_L \mid \tau u = \xi^\epsilon u\}$, $\epsilon = 0, 1, 2$ can be decomposed into a direct sum of simple modules for $M(0) \boxtimes M_0^0$. Any simple module for $M(0) \boxtimes M_0^0$ is of the form $A \boxtimes B$, where $A$ and $B$ are simple modules for $M(0)$ and $M_0^0$, respectively. By (3-5), it follows that $B \cong M_0^0$ or $W_0^0$. Moreover, $V_L(\epsilon)$ contains the simple $M(0)$-modules $M(\epsilon)$ and $W(\epsilon)$. The eigenvalues of $(\tilde{\omega}^1)_1$ in $M(\epsilon)$ (resp. $W(\epsilon)$) are integers (resp. of the form $3/5 + n$, $n \in \mathbb{Z}$), while the eigenvalues of $(\tilde{\omega}^2)_1$ in $M_0^0$ (resp. $W_0^0$) are integers (resp. of the form $2/5 + n$, $n \in \mathbb{Z}$). Since the eigenvalues of $\omega_1 = (\tilde{\omega}^1)_1 + (\tilde{\omega}^2)_1$ in $V_L$ are integers, we conclude that

(3-16) \quad $V_L(\epsilon) \cong (M(\epsilon) \boxtimes M_0^0) \oplus (W(\epsilon) \boxtimes W_0^0)$

as $M(0) \boxtimes M_0^0$-modules, $\epsilon = 0, 1, 2$. In particular,

(3-17) \quad $V_L^\tau \cong (M(0) \boxtimes M_0^0) \oplus (W(0) \boxtimes W_0^0)$.

From now on we set $M^0 = M(0) \boxtimes M_0^0$ and $W^0 = W(0) \boxtimes W_0^0$. Thus $V_L^\tau = V_L(0) \cong M^0 \oplus W^0$. Let

\[
P = y(\alpha_1) + y(\alpha_2) + y(\alpha_0).
\]

Then we can verify that $(\tilde{\omega}^1)_n P = (\tilde{\omega}^2)_n P = 0$ for $n \geq 2$, $(\tilde{\omega}^1)_1 P = (8/5) P$, and $(\tilde{\omega}^2)_1 P = (2/5) P$. Moreover, $J_n P = K_n P = 0$ for $n \geq 2$. Thus $W^0$ is a simple $M^0$-module with $P$ a highest-weight vector of weight $(8/5, 2/5)$. The vertex operator algebra $V_L^\tau$ is generated by $\tilde{\omega}^1, \tilde{\omega}^2, J, K$ and $P$.

**Theorem 3.4.** $V_L^\tau$ is a simple $C_2$-cofinite vertex operator algebra.

**Proof.** We know that $M(0)$ and $M_0^0$ are $C_2$-cofinite. Thus $M^0$ is also $C_2$-cofinite. Since $W^0$ is generated by $P$ as an $M^0$-module, it follows from [Buhl 2002] that $V_L^\tau$ is $C_2$-cofinite. By [Dong and Mason 1997, Theorem 4.4], $V_L^\tau$ is simple. \qed
Following the outline of the argument in [Dong et al. 2004; Kitazume et al. 2003], we discuss the structure of the simple $\tau$-twisted $V_L$-modules $V^j_L(\tau)$, $j = 0, 1, 2$ as $\tau$-twisted $M^0_k \otimes M^0_l$-modules. Furthermore, we correct an error in [Dong et al. 2004; Kitazume et al. 2003] concerning a decomposition of $V^j_L(\tau)$ for $j = 1, 2$. We first consider $V^0_L(\tau)$. Let $0 \neq v \in T_{X_0}$ and $1$ be the identity of $S[\tau]$. Then $1 \otimes v \in S[\tau] \otimes T_{X_0} = V^0_L(\tau)$. Since $M^0_k \subset V^0_L$, we can decompose $V^0_L(\tau)$ into a direct sum of simple $M^0_k$-modules. By a direct calculation, we can verify that

\[(\tilde{\omega}^2)_1(1 \otimes v) = 0, \quad (\tilde{\omega}^2)_1(h_2(-\frac{1}{3}) \otimes v) = \frac{2}{3}h_2(-\frac{1}{3}) \otimes v.\]

Thus we see that $M^0_k$ and $W^0_l$ appear as direct summands. Since $V^0_L(\tau)$ is simple as a $\tau$-twisted $V_L$-module, (3-5) and the fusion rule $W^0_j \times W^0_l = M^0_k + W^0_l$ (see (3-7)) imply that any simple $M^0_k$-submodule of $V^0_L(\tau)$ is isomorphic to $M^0_k$ or $W^0_l$. Hence

\[(3-18) \quad V^0_L(\tau) \cong (M^0_k \otimes M^0_l) \oplus (W^0_l(\tau) \otimes W^0_l),\]

as $\tau$-twisted $M^0_k \otimes M^0_l$-modules, where

\[M^0_k(\tau) = \{ u \in V^0_L(\tau) | (\tilde{\omega}^2)_1 u = 0 \},\]
\[W^0_l(\tau) = \{ u \in V^0_L(\tau) | (\tilde{\omega}^2)_1 u = \frac{2}{3} u \}.\]

The $\tau$-twisted $M^0_k$-modules $M^0_k(\tau)$ and $W^0_l(\tau)$ are simple. Indeed, if $N$ is a $\tau$-twisted $M^0_k$-submodule of $M^0_k(\tau)$, then $N \otimes M^0_l$ is a $\tau$-twisted $M^0_k \otimes M^0_l$-submodule of $M^0_k(\tau) \otimes M^0_l$. By (2-6), $V_L \cdot (N \otimes M^0_l) = \text{span} \{ a_n(N \otimes M^0_l) | a \in V_L, n \in \mathbb{Q} \}$ is a $\tau$-twisted $V_L$-submodule of $V^0_L(\tau)$. The fusion rule $W^0_j \times M^0_l = W^0_j$ and (3-5) imply that $V_L \cdot (N \otimes M^0_l)$ is contained in $(N \otimes M^0_l) \oplus (W^0_l(\tau) \otimes W^0_l)$. Since $V^0_L(\tau)$ is a simple $\tau$-twisted $V_L$-module, we conclude that $M^0_k(\tau)$ is a simple $\tau$-twisted $M^0_k$-module.

Because of the fusion rule $W^0_j \times W^0_l = M^0_k + W^0_l$, we can not apply a similar argument to $W^0_l(\tau)$. Note that there are at most two inequivalent simple $\tau$-twisted $M^0_k$-modules by [Dong et al. 2004, Lemma 4.1] and [Dong et al. 2000, Theorem 10.2]. Note also that a weight in $M^0_k(\tau)$ or in $W^0_l(\tau)$ means an eigenvalue of $(\tilde{\omega}^1)_1$. First several terms of the characters of $M^0_k(\tau)$ and $W^0_l(\tau)$ can be calculated easily from (3-18) (see [Dong et al. 2004]).

\[\text{ch} \ M^0_k(\tau) = q^{1/9} + q^{1/9+2/3} + q^{1/9+1} + q^{1/9+4/3} + \cdots,\]
\[\text{ch} \ W^0_l(\tau) = q^{2/45} + q^{2/45+1/3} + q^{2/45+2/3} + q^{2/45+1} + \cdots.\]

Suppose $W^0_l(\tau)$ is not a simple $\tau$-twisted $M^0_k$-module. Let $N$ be the $\tau$-twisted $M^0_k$-submodule of $W^0_l(\tau)$ generated by the top level of $W^0_l(\tau)$. Then the top level of $N$ is a one dimensional space of weight $2/45$. If $N$ is not a simple $\tau$-twisted $M^0_k$-module, then the sum $U$ of all proper $\tau$-twisted $M^0_k$-submodules of $N$ is a unique
maximal $\tau$-twisted $M_k^0$-submodule of $N$. The quotient $N/U$ is a simple $\tau$-twisted $M_k^0$-module whose top level is of weight $2/45$. Denote the top level of $U$ by $U_\lambda$, where the weight $\lambda$ is $2/45 + n/3$ for some $1 \leq n \in \mathbb{Z}$. Consider the $\tau$-twisted Zhu algebra $A_\tau(M_k^0)$ of $M_k^0$. Since $U_\lambda$ is a finite dimensional $A_\tau(M_k^0)$-module, we can choose a simple $A_\tau(M_k^0)$-submodule $S$ of $U_\lambda$. By [Dong et al. 1998a, Proposition 5.4 and Theorem 7.2], there is a simple $\mathbb{N}$-graded weak $\tau$-twisted $M_k^0$-module $R$ with top level $R_\lambda$ being isomorphic to $S$ as an $A_\tau(M_k^0)$-module. It follows from [Yamauchi 2004, Corollary 3.8] that $R$ is in fact a simple $\tau$-twisted $M_k^0$-module. Here we note that $M_0^0$ is $\mathbb{C}_2$-cofinite and of CFT type by its structure (3-2). Since the top levels of $M_k^0(\tau)$, $N/U$, and $R$ have different weight, they are inequivalent simple $\tau$-twisted $M_k^0$-modules. If $N$ is a simple $\tau$-twisted $M_k^0$-module, then it is not equal to $W_0^0(\tau)$ by our assumption. The quotient $W_0^0(\tau)/N$ is a $\tau$-twisted $M_k^0$-module and the weight of its top level, say $\mu$, is $2/45 + m/3$ for some $1 \leq m \in \mathbb{Z}$. By a similar argument as above, we see that there is a simple $\tau$-twisted $M_k^0$-module whose top level is of weight $\mu$. Hence we have three inequivalent simple $\tau$-twisted $M_k^0$-modules in both cases. This contradicts the fact that there are at most two inequivalent simple $\tau$-twisted $M_k^0$-modules. Thus $W_0^0(\tau)$ is a simple $\tau$-twisted $M_k^0$-module.

Next, let $0 \neq v \in T_{\chi_j}$, $j = 1, 2$. From the definition of $V_L^j(\tau)$ in [Dong et al. 2004; Kitazume et al. 2003], we can calculate that

$$(\tilde{\omega}^2)_{1}(1 \otimes v) = \frac{1}{15} (1 \otimes v), \quad (\tilde{\omega}^2)_{1} u^j = \frac{2}{3} u^j,$$

where $u^j = h_{1/3}(-\frac{2}{3}) \otimes v - (-1)^j (3/2) \otimes v$. Thus $M_1^1$ or $M_2^2$ and $W_1^1$ or $W_1^2$ appear as $M_k^0$-submodules of $V_L^j(\tau)$. In order to distinguish $M_1^1$ and $M_2^2$ (resp. $W_1^1$ and $W_1^2$), we need to know the action of $K_2$ on these vectors (see [Kitazume et al. 2000b]). By a direct calculation, we can verify that

$$K_2(1 \otimes v) = (-1)^j \frac{2}{9} (1 \otimes v), \quad K_2 u^j = (-1)^j \frac{52}{9} u^j.$$

Hence $M_1^1$ and $W_1^1$ appear in $V_L^j(\tau)$ for $j = 1, 2$. Let

$$M_L^j(\tau) = \{ u \in V_L^j(\tau) \mid (\tilde{\omega}^2)_{1} u = \frac{2}{3} u \},$$

$$W_L^j(\tau) = \{ u \in V_L^j(\tau) \mid (\tilde{\omega}^2)_{1} u = \frac{1}{15} u \}, \quad j = 1, 2.$$

Then, $V_L^j(\tau) \cong (M_L^j(\tau) \otimes M_1^1) \oplus (W_L^j(\tau) \otimes W_1^1)$ as $\tau$-twisted $M_k^0 \otimes M_k^0$-modules for $j = 1, 2$. Moreover, $M_L^j(\tau)$ and $W_L^j(\tau)$, $j = 1, 2$ are simple $\tau$-twisted $M_k^0$-modules.

Recall that there are at most two inequivalent simple $\tau$-twisted $M_k^0$-modules. Looking at the smallest weight of $M_L^j(\tau)$ and $W_L^j(\tau)$, we see that the $M_L^j(\tau)$, $j = 0, 1, 2$ are equivalent, and the $W_L^j(\tau)$, $j = 0, 1, 2$ are equivalent, but $M_L^0(\tau)$ and
$W^0_T(\tau)$ are not equivalent. For simplicity, set $M_T(\tau) = M^0_T(\tau)$ and $W_T(\tau) = W^0_T(\tau)$. Then

$$
V^0_L(\tau) \cong (M_T(\tau) \otimes M^0_T) \oplus (W_T(\tau) \otimes W^0_T),
$$

and so

$$
V^j_L(\tau) \cong (M_T(\tau) \otimes M^{3-j}_T) \oplus (W_T(\tau) \otimes W^{3-j}_T), \quad j = 1, 2,
$$
as $\tau$-twisted $M^0_k \otimes M^0_T$-modules.

The structure of the simple $\tau^2$-twisted $V_L$-module $V^j_L(\tau^2)$, $j = 0, 1, 2$ as a $\tau^2$-twisted $M^0_k \otimes M^0_T$-module is similar to that of the case for $V^j_L(\tau)$. Let $0 \neq v \in T_{x_0}$ and let $1$ be the identity of $S[\tau^2]$. Then

$$(\tilde{\omega}^2)_1(1 \otimes v) = 0, \quad (\tilde{\omega}^2)_1(h'_2(-\frac{1}{3}) \otimes v) = \frac{2}{3}h'_2(-\frac{1}{3}) \otimes v$$

and so

$$
V^0_L(\tau^2) \cong (M^0_T(\tau^2) \otimes M^0_T) \oplus (W^0_T(\tau^2) \otimes W^0_T)
$$
as $\tau^2$-twisted $M^0_k \otimes M^0_T$-modules, where

$$
M^0_T(\tau^2) = \{ u \in V^0_L(\tau^2) \mid (\tilde{\omega}^2)_1u = 0 \},
$$

and

$$
W^0_T(\tau^2) = \{ u \in V^0_L(\tau^2) \mid (\tilde{\omega}^2)_1u = \frac{2}{3}u \}.
$$

By a similar argument as in the $\tau$-twisted case, we can show that $M^0_T(\tau^2)$ and $W^0_T(\tau^2)$ are inequivalent simple $\tau^2$-twisted $M^0_k$-modules.

Take a nonzero $v$ in $T_{x_1}$, $j = 1, 2$. Then

$$(\tilde{\omega}^2)_1(1 \otimes v) = \frac{1}{15}(1 \otimes v), \quad (\tilde{\omega}^2)_1v^j = \frac{2}{3}v^j,$$

where $v^j = h'_2(-\frac{2}{3}) \otimes v - (-1)^j \sqrt{-3}h'_2(-\frac{1}{3}) \otimes v$. Furthermore,

$$
K_2(1 \otimes v) = (-1)^j \frac{2}{9}(1 \otimes v), \quad K_2v^j = (-1)^j \frac{52}{9}v^j.
$$

Hence $V^j_L(\tau^2) \cong (M^j_T(\tau^2) \otimes M^j_T) \oplus (W^j_T(\tau^2) \otimes W^j_T)$ as $\tau^2$-twisted $M^0_k \otimes M^0_T$-modules for $j = 1, 2$, where

$$
M^j_T(\tau^2) = \{ u \in V^j_L(\tau^2) \mid (\tilde{\omega}^2)_1u = \frac{2}{3}u \},
$$

and

$$
W^j_T(\tau^2) = \{ u \in V^j_L(\tau^2) \mid (\tilde{\omega}^2)_1u = \frac{1}{15}u \}, \quad j = 1, 2.
$$

As in the $\tau$-twisted case, the $M^j_T(\tau^2)$, $j = 0, 1, 2$ are equivalent and the $W^j_T(\tau^2)$, $j = 0, 1, 2$ are equivalent. Set $M_T(\tau^2) = M^0_T(\tau^2)$ and $W_T(\tau^2) = W^0_T(\tau^2)$. Then

$$
V^j_L(\tau^2) \cong (M_T(\tau^2) \otimes M^j_T) \oplus (W_T(\tau^2) \otimes W^j_T), \quad j = 0, 1, 2,
$$
as $\tau^2$-twisted $M^0_k \otimes M^0_T$-modules.
Remark 3.5. The weight-three vector $K$ was denoted by different symbols in previous papers, namely, $v_1$, $v^3$, and $q$ were used in [Dong et al. 2004], [Kitazume et al. 2003], and [Kitazume et al. 2000b], respectively. They are related as follows: $K = -2\sqrt{2}v_1 = -2\sqrt{2}v^3 = 2\sqrt{2}q$. Thus, in the proof of [Kitazume et al. 2003, Proposition 6.8] $(v^3)_2$ should act on the top level of $V^j_L(\tau)$ as a scalar multiple of $(-1)^j/9\sqrt{2}$ for $j = 1, 2$. Moreover, (6.46) of [Kitazume et al. 2003] and the equation for $V^j_L(\tau)$ on page 265 of [Dong et al. 2004] should be replaced with Equation (3-19). This correction does not affect the results in the latter paper. However, certain changes are necessary in [Kitazume et al. 2003] along with the correction.

Note that

$$M_T (\tau^i)(\epsilon) = \{ u \in M_T (\tau^i) \mid \tau^i u = \xi^i u \},$$

$$W_T (\tau^i)(\epsilon) = \{ u \in W_T (\tau^i) \mid \tau^i u = \xi^i u \}$$

for $i = 1, 2, \epsilon = 0, 1, 2$. Another notation was used in [Kitazume et al. 2003], namely,

$$M_T (\tau^i)^\epsilon = \bigoplus_{n \in 1/9+\epsilon/3+\mathbb{Z}} (M_T (\tau^i))_n, \quad W_T (\tau^i)^\epsilon = \bigoplus_{n \in 2/4\epsilon/3+\mathbb{Z}} (W_T (\tau^i))_n,$$

where $U_n$ denotes the eigenspace of $U$ with eigenvalue $n$ for $(\tilde{\omega})_1$. The two sets of notation are related by

$$M_T (\tau^i)^\epsilon = M_T (\tau^i)(2\epsilon), \quad W_T (\tau^i)^\epsilon = W_T (\tau^i)(2\epsilon - 1).$$

Likewise,

$$\left( V^j_L (\tau) \right)^\epsilon = \bigoplus_{n \in 1/9+\epsilon/3+\mathbb{Z}} \left( V^j_L (\tau) \right)_n, \quad \left( V^j_L (\tau^2) \right)^\epsilon = \bigoplus_{n \in 1/9+\epsilon/3+\mathbb{Z}} \left( V^j_L (\tau^2) \right)_n$$

of [Kitazume et al. 2003, (7.16)] are denoted here by

$$\left( V^j_L (\tau) \right)^\epsilon = V^j_L (\tau)(2\epsilon), \quad \left( V^j_L (\tau^2) \right)^\epsilon = V^j_L (\tau^2)(2\epsilon)$$

for $j = 0, 1, 2$ and $\epsilon = 0, 1, 2$, where $U_n$ is the eigenspace of $U$ with eigenvalue $n$ for $\omega_1$.

By (3-3), the minimal eigenvalues of $(\tilde{\omega})_1$ on $M^0_t$ and $W^0_t$ are 0 and 2/5, respectively, while those on $M^j_t$ and $W^j_t$, $j = 1, 2$, are 2/3 and 1/15, respectively. Hence it follows from (3-19) that

$$\left( V^0_L (\tau) \right)^\epsilon \cong (M_T (\tau)^\epsilon \otimes M^0_t) \oplus (W_T (\tau)^\epsilon - 1 \otimes W^0_t),$$

$$\left( V^j_L (\tau) \right)^\epsilon \cong (M_T (\tau)^\epsilon + 1 \otimes M^3-j_t) \oplus (W_T (\tau)^\epsilon \otimes W^3-j_t), \quad j = 1, 2,$$
as $M^0$-modules for $\varepsilon = 0, 1, 2$, where $M^0 = M(0) \otimes M^0_i$. Similarly,

\begin{equation}
\begin{aligned}
(V^0_L(\tau^2))^\varepsilon & \cong (M_T(\tau^2)^\varepsilon \otimes M^0_i) \oplus (W_T(\tau^2)^{\varepsilon-1} \otimes W^0_i), \\
(V^j_L(\tau^2))^\varepsilon & \cong (M_T(\tau^2)^{\varepsilon+1} \otimes M^1_i) \oplus (W_T(\tau^2)^\varepsilon \otimes W^1_i), \\
& \quad j = 1, 2,
\end{aligned}
\end{equation}

as $M^0$-modules for $\varepsilon = 0, 1, 2$ (see [Kitazume et al. 2003, (7.17)]).

The following fusion rules of simple $M(0)$-modules will be necessary for the study of simple $V^0_L$-modules.

\begin{equation}
\begin{aligned}
W(0) \times M^0_k & = W^0_k, \\
W(0) \times M^1_k & = W^1_k + W^0_k, \\
W(0) \times M(\varepsilon) & = W(\varepsilon), \\
W(0) \times W(\varepsilon) & = M(\varepsilon) + W(\varepsilon), \\
W(0) \times M_T(\tau^i)(\varepsilon) & = W_T(\tau^i)(\varepsilon), \\
W(0) \times W_T(\tau^i)(\varepsilon) & = M_T(\tau^i)(\varepsilon) + W_T(\tau^i)(\varepsilon)
\end{aligned}
\end{equation}

for $i = 1, 2$ and $\varepsilon = 0, 1, 2$. In fact, the first four fusion rules, that is, the fusion rules among simple $M(0)$-modules appearing in untwisted simple $V^0_L$-modules, can be found in [Tanabe 2005]. The last two fusion rules involve simple $M(0)$-modules that appear in $\tau^i$-twisted simple $V^0_L$-modules. Their proofs can be found in the Appendix.

Fusion rules possess certain symmetries. Let $M^i$, $i = 1, 2, 3$ be modules for a vertex operator algebra $V$. Then by [Frenkel et al. 1993, Propositions 5.4.7 and 5.5.2]

\[ \dim I_V \left( \frac{M^3}{M^1 M^2} \right) = \dim I_V \left( \frac{M^3}{M^2 M^1} \right) = \dim I_V \left( \frac{(M^2)^\prime}{M^1 (M^3)^\prime} \right), \]

where $(M^i)^\prime$ is the contragredient module of $M^i$. Recall that the contragredient module $(U', Y_{U'})$ of a $V$-module $(U, Y_U)$ is defined as follows. As a vector space $U' = \bigoplus_n (U_n)^*$ is the restricted dual of $U$ and $Y_{U'}(\cdot, z)$ is determined by

\[ \langle Y_{U'}(a, z)v, u \rangle = \langle v, Y_U(e^{\tau L(1)}(-z^{-2})L(0)a, z^{-1})u \rangle \]

for $a \in V$, $u \in U$, and $v \in U'$.

In our case $M(0)$ is generated by the Virasoro element $\tilde{\omega}^1$ and the weight-three vector $J$. Moreover, $\langle L(1)u, v \rangle = \langle v, L(1)u \rangle$ and $\langle J(0)v, u \rangle = -\langle v, J(0)u \rangle$. Since the 20 simple $M(0)$-modules are distinguished by the action of $L(1)$ and $J(0)$ on their top levels, we know from [Dong et al. 2004, Tables 1, 3, and 4] that
the contragredient modules of the simple $M(0)$-modules are as follows.

$$M(\epsilon)' \cong M(2\epsilon), \quad W(\epsilon)' \cong W(2\epsilon), \quad \epsilon = 0, 1, 2,$$

$$M_T(\tau)(\epsilon)' \cong M_T(\tau^2)(\epsilon), \quad W_T(\tau)(\epsilon)' \cong W_T(\tau^2)(\epsilon), \quad \epsilon = 0, 1, 2$$

(see also [Dong et al. 1998a, Lemma 3.7] and [Tanabe 2005, Section 4.2]).

4. Structure of simple modules

Recall that $V^*_L = V_L(0) = M^0 \oplus W^0$ with $M^0 = M(0) \otimes M_0^0$ and $W^0 = W(0) \otimes W_0^0$. In this section we study the structure of the 30 known simple $V^*_L$-modules listed in Lemma 3.2. We discuss decompositions of these simple modules as modules for $M^0$. Those decompositions have been obtained in [Kitazume et al. 2003]. We review them briefly. (Some corrections are needed in that paper; see Remark 3.5.)

A vector in a $V^*_L$-module is said to be of weight $h$ if it is an eigenvector for $L(0) = \omega_1$ with eigenvalue $h$. We calculate the action of $(\tilde{\omega}^1)_1, (\tilde{\omega}^2)_1, J_2, K_2, P_1, (J_1 P)_2$, and $(K_1 P)_2$ on the top levels of the 30 known simple $V^*_L$-modules. Recall that the top level of a module means the homogeneous subspace of the module of smallest weight. The calculation is accomplished directly from the definition of untwisted or twisted vertex operators associated with the lattice $L$ and the automorphisms $\tau$ and $\tau^2$ (see [Dong and Lepowsky 1996; Frenkel et al. 1988; Lepowsky and Li 2004]). The results in this section will be used to determine the Zhu algebra $A(V^*_L)$ of $V^*_L$ in Section 5.

The vectors $J_1 P$ and $K_1 P$ are of weight 3. Their precise form in terms of the lattice vertex operator algebra $V_L$ is as follows.

$$J_1 P = 2\beta_1(-1)^3 + 3\beta_1(-1)^2\beta_2(-1) - 3\beta_1(-1)\beta_2(-1)^2 - 2\beta_2(-1)^3$$

$$- 4(\beta_2 - \beta_0)(-1)x(\alpha_1) + (\beta_0 - \beta_1)(-1)x(\alpha_2) + (\beta_1 - \beta_2)(-1)x(\alpha_0))$$

$$= \frac{13}{2} (2\beta_1(-1)^3 + 3\beta_1(-1)^2\beta_2(-1) - 3\beta_1(-1)\beta_2(-1)^2 - 2\beta_2(-1)^3) - 4K,$$

$$K_1 P = 3(\beta_1(-2)\beta_2(-1) - \beta_2(-2)\beta_1(-1))$$

$$- (\beta_2 - \beta_0)(-1)y(\alpha_1) + (\beta_0 - \beta_1)(-1)y(\alpha_2) + (\beta_1 - \beta_2)(-1)y(\alpha_0))$$

$$= \frac{7}{2} (\beta_1(-2)\beta_2(-1) - \beta_2(-2)\beta_1(-1)) + J.$$

The simple module $V_L(0)$. $V_L(0) = M^0 \oplus W^0$ as $M^0$-modules. The top level of $V_L(0)$ is $\mathbb{C}1$. By a property of the vacuum vector, all of $(\tilde{\omega}^1)_1, (\tilde{\omega}^2)_1, J_2, K_2, P_1, (J_1 P)_2$, and $(K_1 P)_2$ act as 0 on $\mathbb{C}1$.

The simple module $V_L(\epsilon), \epsilon = 1, 2$. By (3-16), we have

$$V_L(\epsilon) \cong (M(\epsilon) \otimes M_0^0) \oplus (W(\epsilon) \otimes W_0^0)$$
as $M^0$-modules for $\varepsilon = 1, 2$. The top level of $V_L(\varepsilon)$ is $\mathbb{C}v^{2,\varepsilon}$, where $v^{2,\varepsilon} = \alpha_1(-1) - \xi^\varepsilon \alpha_2(-1) \in W(\varepsilon) \otimes W_1^0$. We have

$$(\tilde{\omega}^1)_1v^{2,\varepsilon} = \frac{1}{3}v^{2,\varepsilon}, \quad (\tilde{\omega}^2)_1v^{2,\varepsilon} = \frac{2}{3}v^{2,\varepsilon}, \quad J_2v^{2,\varepsilon} = -(-1)^{\varepsilon^2}2\sqrt{-3}v^{2,\varepsilon},$$

$K_2v^{2,\varepsilon} = 0, \quad P_1v^{2,\varepsilon} = 0, \quad (J_1 P)_2v^{2,\varepsilon} = 0, \quad (K_1 P)_2v^{2,\varepsilon} = (-1)^\varepsilon 12\sqrt{-3}v^{2,\varepsilon}.$

**The simple module $V_{L,(a,j)}(0)$, $j = 1, 2$.** For $j = 1, 2$, (3-4) implies that $V_{L,(a,j)}(0)$ is a direct sum of simple $M^0$-modules of the form $A \otimes B$, where $A$ is a simple $M(0)$-module and $B$ is a simple $M^0$-module isomorphic to $M_1^j$ or $W_1^j$. For convenience, set $U^j(\varepsilon) = V_{L,(a,j)}(\varepsilon)$, $j = 1, 2$, $\varepsilon = 0, 1, 2$. Let

$$v^{3,j} = e^{(-1)^j(\beta_1-\beta_2)/3} + e^{(-1)^j(\beta_2-\beta_0)/3} + e^{(-1)^j(\beta_0-\beta_1)/3}.$$ 

Then $v^{3,j} \in U^j(0)$. Moreover, $(\omega^1)_1v^{3,j} = (\omega^2)_1v^{3,j} = 0$ and $(\tilde{\omega}^2)_1v^{3,j} = (2/3)v^{3,j}$. Hence $v^{3,j} \in M_1^j$ and $U^j(0)$ contains an $M^0_1$-submodule isomorphic to $M_1^j$. By the fusion rule $M_1^j \times W_1^0 = W_1^j$ of $M^0$-modules and [Dong and Lepowsky 1996, Proposition 11.9], $U^j(0)$ contains an $M^0_1$-submodule isomorphic to $W_1^j$ also. Thus $U^j(0)$ contains simple $M^0$-submodules of the form $A \otimes M_1^j$ and $A' \otimes W_1^j$ for some simple $M(0)$-modules $A$ and $A'$.

The minimal weight of $V_{L,(0,j)}(0)$ is 2/3. Its weight subspace is of dimension 3 and spanned by $e^{(1/3)(\beta_1-\beta_2)/3}$, $e^{(1/3)(\beta_2-\beta_0)/3}$, and $e^{(1/3)(\beta_0-\beta_1)/3}$. Thus the weight-2/3 subspace of $U^j(0)$ is $\mathbb{C}v^{3,j}$. Since $(\tilde{\omega}^1)_1v^{3,j} = 0$ and since only $M(0)$ is the simple $M(0)$-module whose minimal weight (eigenvalue of $(\tilde{\omega}^1)_1$) is 0 by [Dong et al. 2004], we conclude that $U^j(0)$ contains a simple $M^0$-submodule isomorphic to $M(0) \otimes M_1^j$.

The minimal eigenvalue of $(\tilde{\omega}^2)_1$ in $W_1^j$ is 1/15. Thus the eigenvalues of $(\tilde{\omega}^1)_1$ on $A'$ must be of the form $3/5 + n$, $n \in \mathbb{Z}$. By [Dong et al. 2004], only $W(0)$, $W(1)$, $W(2)$ are the simple $M(0)$-modules whose weights are of this form. The minimal weight of these simple modules are 8/5, 3/5 and 3/5, respectively. Since the weight-2/3 subspace of $U^j(0)$ is one dimensional, we see that $U^j(0)$ contains a simple $M^0$-submodule isomorphic to $W(0) \otimes W_1^j$.

From the fusion rules for $M^0_1$-modules, we obtain the fusion rules

$$(M(\varepsilon) \otimes M^0_1) \times (M(0) \otimes M_1^j) = M(\varepsilon) \otimes M_1^1,$$

$$(W(\varepsilon) \otimes W_1^0) \times (M(0) \otimes M_1^j) = W(\varepsilon) \otimes W_1^j,$$

for $M^0$-modules. Hence $U^j(\varepsilon) \cong (M(\varepsilon) \otimes M_1^j) \oplus (W(\varepsilon) \otimes W_1^j)$ for $j = 1, 2$ and $\varepsilon = 0, 1, 2$ by (3-4) and (3-16). In particular, $V_{L,(0,j)}(0) \cong (M(0) \otimes M_1^j) \oplus (W(0) \otimes W_1^j)$ as $M^0$-modules, $j = 1, 2$. The top level of $V_{L,(0,j)}(0)$ is $\mathbb{C}v^{3,j} \subset M(0) \otimes M_1^j$. We have
The simple module $V_{L,0,j}(\varepsilon)$, $j = 1, 2$, $\varepsilon = 1, 2$. We have shown above that $V_{L,0,j}(\varepsilon) \cong (M(\varepsilon) \otimes M_j^\varepsilon) \oplus (W(\varepsilon) \otimes W_j^\varepsilon)$ as $M^0$-modules, $j = 1, 2$, $\varepsilon = 1, 2$. The top level of $V_{L,0,j}(\varepsilon)$ is $\mathbb{C}v^{4,j,\varepsilon}$, where
\[
v^{4,j,\varepsilon} = e^{-(1/3)(\beta_1-\beta_2)/3} + \xi e^{-(1/3)(\beta_2-\beta_0)/3} + \xi^2 e^{-(1/3)(\beta_0-\beta_1)/3} \in W(\varepsilon) \otimes W_j^\varepsilon.
\]
We have
\[
\begin{align*}
(\tilde{\omega}^1)_{1}v^{4,j,\varepsilon} &= \frac{3}{5}v^{4,j,\varepsilon}, \\
(\tilde{\omega}^2)_{1}v^{4,j,\varepsilon} &= \frac{1}{15}v^{4,j,\varepsilon}, \\
J_2v^{4,j,\varepsilon} &= -(1)^j\sqrt{-3}v^{4,j,\varepsilon}, \\
P_1v^{4,j,\varepsilon} &= -(1)^{j+1}\sqrt{-3}v^{4,j,\varepsilon}, \\
(J_1P)_2v^{4,j,\varepsilon} &= -(1)^j24v^{4,j,\varepsilon}, \\
(K_1P)_2v^{4,j,\varepsilon} &= -(1)^j2\sqrt{-3}v^{4,j,\varepsilon}.
\end{align*}
\]
We have shown above that $V_{L,0,j}(\varepsilon) \cong (M_j^\varepsilon \otimes M_j^0) \oplus (W_j^\varepsilon \otimes W_j^0)$ as $M^0$-modules. The top level of $V_{L,0,j}(\varepsilon)$ is $\mathbb{C}v^{5,j,\varepsilon}$, where $v^{5,1} = e^{\beta_1/2} - e^{-\beta_2/2} \in M_k^\varepsilon \otimes M_j^0$, $v^{5,2} = e^{\beta_1/2} + e^{-\beta_2/2} \in W_k^\varepsilon \otimes W_j^0$. We have
\[
\begin{align*}
(\tilde{\omega}^1)_1v^{5,1} &= \frac{1}{2}v^{5,1}, \\
(\tilde{\omega}^2)_1v^{5,2} &= \frac{1}{10}v^{5,2}, \\
J_2v^{5,j} &= 0, \\
K_2v^{5,j} &= 0, \\
P_1v^{5,1} &= -\frac{1}{2}v^{5,2}, \\
P_1v^{5,2} &= v^{5,1}, \\
(J_1P)_2v^{5,j} &= 0, \\
(K_1P)_2v^{5,j} &= 0, \\
&j = 1, 2.
\end{align*}
\]
The simple module $V_{L,(c),j}(\varepsilon)$. By (3-4), $V_{L,0,0} \cong (M_k^\varepsilon \otimes M_j^0) \oplus (W_k^\varepsilon \otimes W_j^0)$ as $M^0$-modules, $j = 1, 2$. The top level of $V_{L,0,j}(\varepsilon)$ is $\mathbb{C}v^{6,j}$, where $v^{6,j} = e^{-(1/3)(\beta_2-\beta_0)/6} \in W_k^\varepsilon \otimes W_j^\varepsilon$. We have
\[
\begin{align*}
(\tilde{\omega}^1)_1v^{6,j} &= \frac{1}{10}v^{6,j}, \\
(\tilde{\omega}^2)_1v^{6,j} &= \frac{1}{15}v^{6,j}, \\
J_2v^{6,j} &= 0, \\
K_2v^{6,j} &= -(1)^j\frac{2}{5}v^{6,j}, \\
P_1v^{6,j} &= 0, \\
(J_1P)_2v^{6,j} &= -(1)^j2v^{6,j}, \\
(K_1P)_2v^{6,j} &= 0.
\end{align*}
\]
The simple module $V_{L}^0(\tau)(0)$. By (3-23), we have the isomorphism $V_{L}^0(\tau)(0) \cong (M_T(\tau)(0) \otimes M_j^0) \oplus (W_T(\tau)(0) \otimes W_j^0)$ as $M^0$-modules. The top level of $V_{L}^0(\tau)(0)$ is $\mathbb{C}v^7$, where $v^7 = 1 \otimes v \in M_T(\tau)(0) \otimes M_j^0$ and $0 \neq v \in T_0$. We have
\[
\begin{align*}
(\tilde{\omega}^1)_1v^7 &= \frac{1}{8}v^7, \\
(\tilde{\omega}^2)_1v^7 &= 0, \\
J_2v^7 &= \frac{14}{87}\sqrt{-3}v^7, \\
K_2v^7 &= 0, \\
P_1v^7 &= 0, \\
(J_1P)_2v^7 &= 0, \\
(K_1P)_2v^7 &= 0.
\end{align*}
\]
The simple module $V_{L}^0(\tau)(1)$. By (3-23), we have the isomorphism $V_{L}^0(\tau)(1) \cong (M_T(\tau)(1) \otimes M_j^0) \oplus (W_T(\tau)(1) \otimes W_j^0)$ as $M^0$-modules. The top level of $V_{L}^0(\tau)(1)$ is of dimension 2 with basis $\{v^{8,1}, v^{8,2}\}$, where
\[
\begin{align*}
v^{8,1} &= h_2(-1/3)^2 \otimes v \in M_T(\tau)(1) \otimes M_j^0, \\
v^{8,2} &= h_1(-2/3) \otimes v \in W_T(\tau)(1) \otimes W_j^0.
\end{align*}
\]
and $0 \neq v \in T_{x_0}$. We have

$$(\bar{\omega}^1)_1 v^{8.1} = \left( \frac{1}{9} + \frac{2}{3} \right) v^{8.1}, \quad (\bar{\omega}^2)_1 v^{8.1} = 0, \quad J_2 v^{8.1} = -\frac{238}{81} \sqrt{-3} v^{8.1}, \quad K_2 v^{8.1} = 0,$$

$$(\bar{\alpha}^1)_1 v^{8.1} = -\frac{4}{5} v^{8.2}, \quad (J_1 P)_2 v^{8.1} = \frac{1}{9} \sqrt{-3} v^{8.2}, \quad (K P)_2 v^{8.1} = 0,$$

$$(\bar{\omega}^1)_1 v^{8.2} = \left( \frac{2}{3} + \frac{1}{2} \right) v^{8.2}, \quad (\bar{\omega}^2)_1 v^{8.2} = \frac{3}{2} v^{8.2}, \quad J_2 v^{8.2} = -\frac{22}{81} \sqrt{-3} v^{8.2}, \quad K_2 v^{8.2} = 0,$$

$$(\bar{\alpha}^1)_1 v^{8.2} = 2 v^{8.1}, \quad (J_1 P)_2 v^{8.2} = -\frac{20}{3} \sqrt{-3} v^{8.1}, \quad (K P)_2 v^{8.2} = -\frac{20}{3} \sqrt{-3} v^{8.2}.$$

The simple module $V^0_L(\tau)(2)$. By (3-23), we have the isomorphism $V^0_L(\tau)(2) \cong (M_T(\tau)(2) \otimes M^0_T) \oplus (W_T(\tau)(2) \otimes W_0)$ as $M^0$-modules. The top level of $V^0_L(\tau)(2)$ is $C v^9$, where $v^9 = h_2(-1/3) \otimes v \in W_T(\tau)(2) \otimes W_0$ and $0 \neq v \in T_{x_0}$. We have

$$(\bar{\omega}^1)_1 v^9 = \frac{2}{3} v^9, \quad (\bar{\omega}^2)_1 v^9 = \frac{2}{3} v^9, \quad J_2 v^9 = -\frac{4}{81} \sqrt{-3} v^9, \quad K_2 v^9 = 0,$$

$$(\bar{\alpha}^1)_1 v^9 = 0, \quad (J_1 P)_2 v^9 = 0, \quad (K P)_2 v^9 = \frac{2}{3} \sqrt{-3} v^9.$$

The simple module $V^j_L(\tau)(0)$, $j = 1, 2$. By (3-23), we have the isomorphism $V^j_L(\tau)(0) \cong (M_T(\tau)(2) \otimes M^3_T) \oplus (W_T(\tau)(2) \otimes W^3_T)$ as $M^0$-modules for $j = 1, 2$. The top level of $V^j_L(\tau)(0)$ is $C v^{10,j}$, where $v^{10,j} = 1 \otimes v \in W_T(\tau)(2) \otimes W^3_T$ and $0 \neq v \in T_{x_j}$. We have

$$(\bar{\omega}^1)_1 v^{10,j} = \frac{2}{3} v^{10,j}, \quad (\bar{\omega}^2)_1 v^{10,j} = \frac{1}{15} v^{10,j}, \quad J_2 v^{10,j} = -\frac{4}{81} \sqrt{-3} v^{10,j},$$

$$K_2 v^{10,j} = -(-1)^j \frac{3}{5} v^{10,j}, \quad P_1 v^{10,j} = (-1)^j \frac{1}{9} \sqrt{-3} v^{10,j},$$

$$(J_1 P)_2 v^{10,j} = -(-1)^j \frac{8}{9} v^{10,j}, \quad (K P)_2 v^{10,j} = -\frac{2}{9} \sqrt{-3} v^{10,j}.$$

The simple module $V^j_L(\tau)(1)$, $j = 1, 2$. By (3-23), we have the isomorphism $V^j_L(\tau)(1) \cong (M_T(\tau)(0) \otimes M^3_T) \oplus (W_T(\tau)(0) \otimes W^3_T)$ as $M^0$-modules for $j = 1, 2$. The top level of $V^j_L(\tau)(1)$ is of dimension 2 with basis $\{v^{11,j,1}, v^{11,j,2}\}$, where

$$v^{11,j,1} = h_1(-2/3) \otimes v - (-1)^j \sqrt{-3} h_2(-1/3)^2 \otimes v \in M_T(\tau)(0) \otimes M^3_T,$$

$$v^{11,j,2} = 2h_1(-2/3) \otimes v + (-1)^j \sqrt{-3} h_2(-1/3)^2 \otimes v \in W_T(\tau)(0) \otimes W^3_T$$

and $0 \neq v \in T_{x_j}$. We have

$$(\bar{\omega}^1)_1 v^{11,j,1} = \frac{1}{5} v^{11,j,1}, \quad (\bar{\omega}^2)_1 v^{11,j,1} = \frac{2}{5} v^{11,j,1}, \quad J_2 v^{11,j,1} = \frac{14}{81} \sqrt{-3} v^{11,j,1},$$

$$K_2 v^{11,j,1} = -(-1)^j \frac{52}{9} v^{11,j,1}, \quad P_1 v^{11,j,1} = (-1)^j \frac{4}{9} \sqrt{-3} v^{11,j,1},$$

$$(J_1 P)_2 v^{11,j,1} = -(-1)^j \frac{52}{9} v^{11,j,2}, \quad (K P)_2 v^{11,j,1} = -\frac{28}{9} \sqrt{-3} v^{11,j,2},$$

$$(\bar{\omega}^1)_1 v^{11,j,2} = \frac{2}{3} + \frac{1}{2} v^{11,j,2}, \quad (\bar{\omega}^2)_1 v^{11,j,2} = \frac{1}{15} v^{11,j,2}.$$
The simple module $V^j_L(\tau)(2)$, $j = 1, 2$. By (3.23), we have the isomorphism $V^j_L(\tau)(2) \cong (M_T(\tau)(1) \otimes M^3_{i}j) \oplus (W_T(\tau)(1) \otimes W^3_{i})$ as $M^0$-modules for $j = 1, 2$. The top level of $V^j_L(\tau)(2)$ is $Cv^{12,j}$, where $v^{12,j} = h_2(-1/3) \otimes v \in W_T(\tau)(1) \otimes W^3_{i}$ and $0 \neq v \in T_{x_j}$. We have

$$(\bar{\omega}^1)_1 v^{12,j} = \left(\frac{2}{35} + \frac{1}{2}\right)v^{12,j}, \quad (\bar{\omega}^2)_1 v^{12,j} = \frac{1}{15}v^{12,j}, \quad J_2 v^{12,j} = -\frac{12}{91} \sqrt{-3} v^{12,j},$$

$$K_2 v^{12,j} = -(-1)^j \frac{1}{9} \sqrt{-3} v^{12,j}, \quad P_1 v^{12,j} = -(-1)^j \frac{5}{9} \sqrt{-3} v^{12,j},$$

$$(J_1 P)_2 v^{12,j} = -(-1)^j \frac{1}{9} \sqrt{-3} v^{12,j}, \quad (K_1 P)_2 v^{12,j} = \frac{10}{9} \sqrt{-3} v^{12,j}.$$ 

The simple module $V^0_L(\tau^2)(0)$. By (3.24), we have the isomorphism $V^0_L(\tau^2)(0) \cong (M_T(\tau^2)(0) \otimes M^0) \oplus (W_T(\tau^2)(0) \otimes W^0)$ as $M^0$-modules. The top level of $V^0_L(\tau^2)(0)$ is $Cv^{13}$, where $v^{13} = 1 \otimes v \in M_T(\tau^2)(0) \otimes M^0$ and $0 \neq v \in T_{x_0}$. We have

$$(\bar{\omega}^1)_1 v^{13} = \frac{1}{5} v^{13}, \quad (\bar{\omega}^2)_1 v^{13} = 0, \quad J_2 v^{13} = -\frac{14}{81} \sqrt{-3} v^{13}, \quad K_2 v^{13} = 0,$$

$$P_1 v^{13} = 0, \quad (J_1 P)_2 v^{13} = 0, \quad (K_1 P)_2 v^{13} = 0.$$ 

The simple module $V^0_L(\tau^2)(1)$. By (3.24), we have the isomorphism $V^0_L(\tau^2)(1) \cong (M_T(\tau^2)(1) \otimes M^0) \oplus (W_T(\tau^2)(1) \otimes W^0)$ as $M^0$-modules. The top level of $V^0_L(\tau^2)(1)$ is of dimension 2 with basis $\{v^{14,1}, v^{14,2}\}$, where

$$v^{14,1} = h_2^2(-1/3)^2 \otimes v \in M_T(\tau^2)(1) \otimes M^0,$$

$$v^{14,2} = h_1^2(-2/3) \otimes v \in W_T(\tau^2)(1) \otimes W^0,$$

and $0 \neq v \in T_{x_0}$. We have

$$(\bar{\omega}^1)_1 v^{14,1} = \left(\frac{1}{5} + \frac{2}{3}\right)v^{14,1}, \quad (\bar{\omega}^2)_1 v^{14,1} = 0,$$

$$J_2 v^{14,1} = \frac{238}{81} \sqrt{-3} v^{14,1}, \quad K_2 v^{14,1} = 0,$$

$$P_1 v^{14,1} = -\frac{4}{5} v^{14,2}, \quad (J_1 P)_2 v^{14,1} = -\frac{104}{9} \sqrt{-3} v^{14,2}, \quad (K_1 P)_2 v^{14,1} = 0,$$

$$J_2 v^{14,2} = \frac{22}{81} \sqrt{-3} v^{14,2}, \quad K_2 v^{14,2} = 0,$$

$$P_1 v^{14,2} = 2 v^{14,1}, \quad (J_1 P)_2 v^{14,2} = \frac{52}{9} \sqrt{-3} v^{14,1}, \quad (K_1 P)_2 v^{14,2} = \frac{20}{9} \sqrt{-3} v^{14,2}.$$
The simple module $V^0_L(\tau^2)(2)$. By Equation (3-24), we have the isomorphism $V^0_L(\tau^2)(2) \cong (M_T(\tau^2)(2) \otimes M^0_I) \oplus (W_T(\tau^2)(2) \otimes W^0_I)$ as $M^0$-modules. The top level of $V^0_L(\tau^2)(2)$ is $\mathbb{C}v^{15}$, where $v^{15} = h'_2(-1/3) \otimes v \in W_T(\tau^2)(2) \otimes W^0_I$ and $0 \neq v \in T_{x'_0}$. We have

$$
(\tilde{\omega}_1)^1v^{15} = \frac{2}{3}v^{15}, \quad (\tilde{\omega}_2)^1v^{15} = \frac{2}{3}v^{15}, \quad J_2v^{15} = \frac{4}{31}\sqrt{-3}v^{15}, \quad K_2v^{15} = 0,
$$

$$
P_1v^{15} = 0, \quad (J_1P)^2v^{15} = 0, \quad (K_1P)^2v^{15} = -\frac{4}{3}\sqrt{-3}v^{15}.
$$

The simple module $V^j_L(\tau^2)(0)$, $j = 1, 2$. By (3-24), we have the isomorphism $V^j_L(\tau^2)(0) \cong (M_T(\tau^2)(2) \otimes M^j_I) \oplus (W_T(\tau^2)(2) \otimes W^j_I)$ as $M^0$-modules for $j = 1, 2$. The top level of $V^j_L(\tau^2)(0)$ is $\mathbb{C}v^{16,j}$, where $v^{16,j} = 1 \otimes v \in W_T(\tau^2)(2) \otimes W^j_I$ and $0 \neq v \in T_{x'_j}$. We have

$$
(\tilde{\omega}_1)^1v^{16,j} = \frac{2}{35}v^{16,j}, \quad (\tilde{\omega}_2)^1v^{16,j} = \frac{1}{15}v^{16,j}, \quad J_2v^{16,j} = \frac{4}{31}\sqrt{-3}v^{16,j},
$$

$$
K_2v^{16,j} = (-1)^j\frac{1}{9}v^{16,j}, \quad P_1v^{16,j} = (-1)^j\frac{1}{9}\sqrt{-3}v^{16,j},
$$

$$
(J_1P)^2v^{16,j} = (-1)^j\frac{8}{9}v^{16,j}, \quad (K_1P)^2v^{16,j} = \frac{2}{9}\sqrt{-3}v^{16,j}.
$$

The simple module $V^j_L(\tau^2)(1)$, $j = 1, 2$. By (3-24), we have the isomorphism $V^j_L(\tau^2)(1) \cong (M_T(\tau^2)(0) \otimes M^j_I) \oplus (W_T(\tau^2)(0) \otimes W^j_I)$ as $M^0$-modules for $j = 1, 2$. The top level of $V^j_L(\tau^2)(1)$ is of dimension 2 with basis $\{v^{17,j,1}, v^{17,j,2}\}$, where

$$
v^{17,j,1} = h'_1(-2/3) \otimes v - (-1)^j\sqrt{-3}h'_2(-1/3)^2 \otimes v \in M_T(\tau^2)(0) \otimes M^j_I,$$

$$
v^{17,j,2} = 2h'_1(-2/3) \otimes v + (-1)^j\sqrt{-3}h'_2(-1/3)^2 \otimes v \in W_T(\tau^2)(0) \otimes W^j_I$$

and $0 \neq v \in T_{x'_j}$. We have

$$
(\tilde{\omega}_1)^1v^{17,j,1} = \frac{1}{9}v^{17,j,1}, \quad (\tilde{\omega}_2)^1v^{17,j,1} = \frac{3}{35}v^{17,j,1}, \quad J_2v^{17,j,1} = -\frac{14}{31}\sqrt{-3}v^{17,j,1},
$$

$$
K_2v^{17,j,1} = (-1)^j\frac{52}{9}v^{17,j,1}, \quad P_1v^{17,j,1} = (-1)^j\frac{4}{9}\sqrt{-3}v^{17,j,2},
$$

$$
(J_1P)^2v^{17,j,1} = (-1)^j\frac{8}{9}v^{17,j,2}, \quad (K_1P)^2v^{17,j,1} = \frac{28}{9}v^{17,j,2},
$$

$$
(\tilde{\omega}_1)^1v^{17,j,2} = \frac{2}{35}v^{17,j,2}, \quad (\tilde{\omega}_2)^1v^{17,j,2} = \frac{1}{15}v^{17,j,2}, \quad J_2v^{17,j,2} = \frac{1}{31}\sqrt{-3}v^{17,j,2},
$$

$$
K_2v^{17,j,2} = (-1)^j\frac{176}{9}\sqrt{-3}v^{17,j,2}, \quad P_1v^{17,j,2} = (-1)^j\frac{8}{9}\sqrt{-3}v^{17,j,2},
$$

$$
(J_1P)^2v^{17,j,2} = (-1)^j\frac{104}{9}v^{17,j,1} + (-1)^j\frac{200}{9}v^{17,j,2}, \quad (K_1P)^2v^{17,j,2} = \frac{56}{9}\sqrt{-3}v^{17,j,1} + \frac{10}{9}\sqrt{-3}v^{17,j,2}.
$$

The simple module $V^j_L(\tau^2)(2)$, $j = 1, 2$. By (3-24), we have the isomorphism $V^j_L(\tau^2)(2) \cong (M_T(\tau^2)(1) \otimes M^j_I) \oplus (W_T(\tau^2)(1) \otimes W^j_I)$ as $M^0$-modules for $j = 1, 2$. 

The top level of $V_L^j(\tau^2)(2)$ is $\mathbb{C}v^{18,j}$, where $v^{18,j} = h_j'(-1/3)\otimes v \in W_\tau^j(\tau^2)(1) \otimes W_i^j$ and $0 \neq v \in T_{\lambda_j}$. We have
\[
(\tilde{\sigma}^1)v^{18,j} = \left(\frac{2}{45} + \frac{1}{3}\right)v^{18,j}, \quad (\tilde{\sigma}^2)v^{18,j} = \frac{1}{15}v^{18,j}, \quad J_2v^{18,j} = \frac{22}{81}\sqrt{-3}v^{18,j}, \quad K_2v^{18,j} = (-1)^j\frac{2}{9}v^{18,j}, \quad P_1v^{18,j} = -(-1)^j\frac{5}{9}\sqrt{-3}v^{18,j}, \quad (J_1P)v^{18,j} = -(1)^j\frac{8}{9}\sqrt{-3}v^{18,j}, \quad (K_1P)v^{18,j} = -\frac{10}{9}\sqrt{-3}v^{18,j}.
\]

**Symmetries by $\sigma$.** We now consider the automorphisms $\sigma$ and $\theta$ of $V_L$ that are lifts of the isometries $\sigma$ and $\theta$ of the lattice $L$ defined by (3-1). Clearly, $\sigma \tau \sigma = \tau^2$, $\sigma \theta = \theta \sigma$, and $\tau \theta = \theta \tau$. Thus $\sigma$ and $\theta$ induce automorphisms of $V_L^j$ of order 2. We have $\sigma J = -J$, $\sigma K = -K$, $\sigma P = P$, $\theta J = J$, $\theta K = -K$, and $\theta P = -P$. Hence $\sigma$ and $\theta$ induce the same automorphism of $M_0^j$ and $\theta$ is the identity on $M(0)$. Note also that $\sigma (J_1 P) = -J_1 P$ and $\sigma (K_1 P) = -K_1 P$.

From the action of $\sigma$ on the top level of the 30 known simple $V_L^j$-modules or the action of $J_2$, $K_2$, $(J_1 P)_2$, and $(K_1 P)_2$, we know how $\sigma$ permutes those simple $V_L^j$-modules. In fact, $\sigma$ transforms $V_L^{(0,0)}$ into an equivalent simple $V_L^j$-module and interchanges the remaining simple $V_L^j$-modules as follows.

\[
V_L(1) \leftrightarrow V_L(2), \quad V_L^{(0,1)}(\varepsilon) \leftrightarrow V_L^{(0,2)}(2\varepsilon), \quad \varepsilon = 0, 1, 2,
\]
\[
V_L^{(v,1)} \leftrightarrow V_L^{(v,2)}, \quad V_L^j(\tau)(\varepsilon) \leftrightarrow V_L^j(\tau^2)(\varepsilon), \quad j, \varepsilon = 0, 1, 2.
\]

Note that $\sigma h_j = \xi^{-i}h_j'$, $i = 1, 2$. The top level of $V_L^j(\tau^2)(\varepsilon)$ can be obtained by replacing $h_j(i/3 + n)$ with $h_j'(i/3 + n)$ in the top level of $V_L^j(\tau)(\varepsilon)$ for $j, \varepsilon = 0, 1, 2$. The corresponding action of $\sigma$ on the simple $M(0)$-modules was discussed in [Dong et al. 2004, Section 4.4].

### 5. Classification of simple modules

We keep the notation in the preceding section. Thus $V_L^j = M^0 \oplus W^0$ with $M^0 = M(0) \otimes M_0^j$ and $W^0 = W(0) \otimes W_i^0$. In this section we show that any simple $V_L^j$-module is equivalent to one of the 30 simple $V_L^j$-modules listed in **Lemma 3.2**. The result will be established by considering the Zhu algebra $A(V_L)$ of $V_L^j$.

First, we review some notation and basic formulas for the Zhu algebra $A(V)$ of a vertex operator algebra $(V, Y, 1, \omega)$. Define two binary operations
\[
(5.1) \quad u \ast v = \sum_{i=0}^{\infty} \left(\frac{\text{wt } u}{i}\right) u_{i-1} v, \quad u \circ v = \sum_{i=0}^{\infty} \left(\frac{\text{wt } u}{i}\right) u_{i-2} v
\]
for $u, v \in V$ with $u$ being homogeneous and extend $\ast$ and $\circ$ for arbitrary $u \in V$ by linearity. Let $O(V)$ be the subspace of $V$ spanned by all $u \circ v$ for $u, v \in V$. Set $A(V) = V/O(V)$. By [Zhu 1996, Theorem 2.1.1], $O(V)$ is a two-sided ideal with
respect to the operation \(*\). Thus \(*\) induces an operation in \(A(V)\). Denote by \([v]\) the image of \(v \in V\) in \(A(V)\). Then \([u] \ast [v] = [u \ast v]\) and \(A(V)\) is an associative algebra by this operation. Moreover, \([1]\) is the identity and \([\omega]\) is in the center of \(A(V)\). For \(u, v \in V\), we write \(u \sim v\) if \([u] = [v]\). For \(\varphi, \psi \in \text{End} V\), we write \(\varphi \sim \psi\) if \(\varphi v \sim \psi v\) for all \(v \in V\). We need some basic formulas from [Zhu 1996].

\[
\begin{align*}
(5-2) \quad v \ast u & \sim \sum_{i=0}^{\infty} \left( \begin{array}{c}
\text{wt}(u) - 1 \\
i
\end{array} \right) u_{i-1} v, \\
(5-3) \quad \sum_{i=0}^{\infty} \left( \begin{array}{c}
\text{wt}(u) + m \\
i
\end{array} \right) u_{i-n-2} v & \in O(V), \quad n \geq m \geq 0.
\end{align*}
\]

Moreover (see [Wang 1993]),

\[
\begin{align*}
(5-4) \quad L(-n) & \sim (-1)^n \{ (n - 1)(L(-2) + L(-1)) + L(0) \}, \quad n \geq 1, \\
(5-5) \quad [\omega] \ast [u] & = [(L(-2) + L(-1))u],
\end{align*}
\]

where \(L(n) = \omega_{n+1}\). From (5-4) and (5-5) we have

\[
(5-6) \quad [L(-n)u] = (-1)^n(n - 1)[\omega] \ast [u] + (-1)^n[L(0)u], \quad n \geq 1.
\]

If \(u \in V\) is of weight 2, then \(u(-n - 3) + 2u(-n - 2) + u(-n - 1) \sim 0\) by (5-3), where \(u(n) = u_{n+1}\). Hence

\[
(5-7) \quad u(-n) \sim (-1)^n((n - 1)u(-2) + (n - 2)u(-1))
\]

for \(n \geq 1\). Then it follows from (5-1) and (5-2) that

\[
(5-8) \quad u(-n)w \sim (-1)^n(-u \ast w + nw \ast u + u(0)w)
\]

for \(n \geq 1, w \in V\). Likewise, if \(u\) is of weight 3 and \(u(n) = u_{n+2}\), then

\[
(5-9) \quad u(-n) \sim (-1)^{n+1}
\]

\[
\{ \frac{1}{2}(n - 1)(n - 2)u(-3) + (n - 1)(n - 3)u(-2) + \frac{1}{2}(n - 2)(n - 3)u(-1) \},
\]

\[
(5-10) \quad u(-n)w \sim (-1)^{n+1}(nu(-1)w + (n - 1)u(0)w - (n - 1)u \ast w + \frac{1}{2}n(n - 1)w \ast u),
\]

for \(n \geq 1, w \in V\).

For a homogeneous vector \(u \in V\), \(o(u) = u_{\text{wt}(u) - 1}\) is the weight zero component operator of \(Y(u, z)\). Extend \(o(u)\) for arbitrary \(u \in V\) by linearity. Note that we call a module in the sense of [Zhu 1996] an \(\mathbb{N}\)-graded weak module here. If \(M = \bigoplus_{n=0}^{\infty} M_{(n)}\) is an \(\mathbb{N}\)-graded weak \(V\)-module with \(M_{(0)} \neq 0\), then \(o(u)\) acts on its top level \(M_{(0)}\). Zhu’s theory [1996] says: (1) \(o(u) o(v) = o(u \ast v)\) as operators on the top level \(M_{(0)}\) and \(o(u)\) acts as 0 on \(M_{(0)}\) if \(u \in O(V)\). Thus \(M_{(0)}\) is an \(A(V)\)-module, where \([u]\) acts on \(M_{(0)}\) as \(o(u)\). (2) The map \(M \mapsto M_{(0)}\) is a bijection
between the set of isomorphism classes of simple \(n\)-graded weak \(V\)-modules and the set of isomorphism classes of simple \(A(V)\)-modules.

We return to \(V_k^L\). As in Section 3, we write \(L_i(n) = (\tilde{\omega})_{n+1}\), \(i = 1, 2\), \(J(n) = J_{n+2}\), and \(K(n) = K_{n+2}\). The Zhu algebras \(A(M(0))\) and \(A(M^0)\) were determined in [Dong et al. 2004] and [Kitazume et al. 2000b], respectively. Since \(O(M^0) \subset O(V_k^L)\), the image of \(M(0)\) (resp. \(M^0\)) in \(A(V_k^L)\) is a homomorphic image of \(A(M(0))\) (resp. \(A(M^0)\)). It is generated by \([\tilde{\omega}], [J], [K]\).

By a direct calculation, we have

\[
P_1 P = -16 \tilde{\omega} - 6 \tilde{\omega}^2, \\
P_0 P = -8(\tilde{\omega}^1)_{-2} - 8(\tilde{\omega}^2)_{-1}, \\
P_{-1} P = \frac{5}{273} J_1 K_1 P - \frac{12}{7} (\tilde{\omega}^1)_{-1} - \frac{18}{13}(\tilde{\omega}^2)_{-1} - 16(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-1}, \\
(5-11) \\
P_{-2} P = \frac{1}{156} J_0 K_1 P + \frac{1}{156} J_1 K_0 P - \frac{8}{7}(\tilde{\omega}^1)_{-1} - \frac{12}{13}(\tilde{\omega}^2)_{-1} - \frac{36}{13}(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-1} - 8(\tilde{\omega}^1)_{-1}(\tilde{\omega}^2)_{-1}. \\
\]

Moreover, \(J_2 P = K_2 P = 0\). Then, using formulas (5-4)–(5-10), we obtain

\[
(5-12) \quad [P] \star [P] = \frac{5}{273} [J_1 K_1 P] - \frac{36}{7} [\tilde{\omega}] \star [\tilde{\omega}] - \frac{9}{13}[\tilde{\omega}^2] \star [\tilde{\omega}^2] - 16[\tilde{\omega}] \star [\tilde{\omega}^2] + \frac{4}{7}[\tilde{\omega}] + \frac{6}{13}[\tilde{\omega}^2], \\
(5-13) \quad [P \circ P] = \frac{1}{156} [J] \star [K_1 P] - \frac{1}{156} [K_1 P] \star [J] + \frac{1}{156} [K] \star [J_1 P] - \frac{1}{156} [J_1 P] \star [K] = 0. \\
\]

It turns out that \(A(V_k^L)\) is generated by \([\tilde{\omega}], [\tilde{\omega}^2], [J], [K]\), and \([P]\) (Corollary 5.11). However, we first prove the following intermediate assertion.

**Proposition 5.1.** The Zhu algebra \(A(V_k^L)\) is generated by \([\tilde{\omega}], [\tilde{\omega}^2], [J], [K], [P], [J_1 P], \) and \([K_1 P]\).

**Proof.** Recall that \(L_i(n) P = 0\) for \(i = 1, 2, n \geq 1\), \(L_1(0) P = \frac{8}{2} P\), \(L_2(0) P = \frac{2}{5} P\), and \(J(n) P = K(n) P = 0\) for \(n \geq 0\). Thus from the commutation relations (3-9)–(3-11) and (3-13)–(3-15) we see that \(W^0\) is spanned by the vectors of the form

\[
(5-14) \quad L^1(-j_1) \cdots L^1(-j_r) L^2(-k_1) \cdots L^2(-k_s) J(-m_1) \cdots J(-m_p) K(-n_1) \cdots K(-n_q) P \\
\]

with \(j_1 \geq \cdots \geq j_r \geq 1, k_1 \geq \cdots \geq k_s \geq 1, m_1 \geq \cdots \geq m_p \geq 1, n_1 \geq \cdots \geq n_q \geq 1\).

Let \(v\) be a vector of this form. Its weight is

\[j_1 + \cdots + j_r + k_1 + \cdots + k_s + m_1 + \cdots + m_p + n_1 + \cdots + n_q + 2.\]
Since \( V^+_L = M^0 \oplus W^0 \) and since the image of \( M^0 \) in \( A(V^+_L) \) is generated by \([\tilde{\omega}^1]\), \([\tilde{\omega}^2]\), \([J]\), and \([K]\), it suffices to show that the image \([v]\) of \( v \) in \( A(V^+_L) \) is contained in the subalgebra generated by \([\tilde{\omega}^1]\), \([\tilde{\omega}^2]\), \([J]\), \([K]\), \([P]\), \([J_1 P]\), and \([K_1 P]\). We proceed by induction on the weight of \( v \). By formula (5-8) with \( u = \tilde{\omega}^i \), \( i = 1, 2 \) and the induction on the weight, we may assume that \( r = s = 0 \), that is,

\[
v = J(-m_1) \cdots J(-m_p) K(-n_1) \cdots K(-n_q) P.
\]

Moreover, by formula (5-10) with \( u = J \), we may assume that \( m_1 = \cdots = m_p = 1 \). Since \( J(m) \) and \( K(n) \) commute, we may also assume that \( n_1 = \cdots = n_q = 1 \) by a similar argument. Then \( v = J(-1)^p K(-1)^q P \).

Next, we reduce \( v \) to the case \( p \leq 1 \). For this purpose, we use a singular vector in \( W(0) \). Suppose \( p \geq 2 \). Then, since \( K(-1) \) commutes with \( J(m) \) and \( L^1(n) \), (5-15) implies that \( v = J(-1)^p K(-1)^q P \) is a linear combination of

\[
J(-1)^{p-2} L^1(-2) K(-1)^q P \quad \text{and} \quad J(-1)^{p-2} L^1(-1) K(-1)^q P.
\]

By (3-10), these two vectors can be written in the form \( L^1(-2) H K(-1)^q P \) and \( L^1(-1)^2 H' K(-1)^q P \), where \( H \) (resp. \( H' \)) is a polynomial in \( J(-1) \) and \( J(-3) \) (resp. \( J(-1) \), \( J(-2) \), and \( J(-3) \)). Then by (5-8) with \( u = \tilde{\omega}^1 \) and the induction on the weight, the assertion holds for \( v \). Hence we may assume that \( p \leq 1 \).

There is a singular vector \( \omega \)

\[
K(-1)^2 P - 210 L^2(-2) P = 0
\]

in \( W^0_i \). Thus, by a similar argument as above, we may assume that \( q \leq 1 \). Finally, it follows from (5-12) that \( [J(-1) K(-1) P] \) can be written by \([\tilde{\omega}^1]\), \([\tilde{\omega}^2]\), and \([P]\) in \( A(V^+_L) \). The proof is complete.

We will classify the simple \( V^+_L \)-modules using our knowledge of simple modules for \( M(0) \) and \( M^0_i \) together with fusion rules (3-25) and (3-7). Set

\[
\mathcal{M}_1 = \{ M(\varepsilon), M^e_k, M_T^e(\tau^i)(\varepsilon) \mid i = 1, 2, \varepsilon = 0, 1, 2 \},
\]

\[
\mathcal{W}_1 = \{ W(\varepsilon), W^e_k, W_T^e(\tau^i)(\varepsilon) \mid i = 1, 2, \varepsilon = 0, 1, 2 \},
\]

\[
\mathcal{M}_2 = \{ M^j_i \mid j = 0, 1, 2 \}, \quad \mathcal{W}_2 = \{ W^j_i \mid j = 0, 1, 2 \}.
\]

Then \( \mathcal{M}_1 \cup \mathcal{W}_1 \) (resp. \( \mathcal{M}_2 \cup \mathcal{W}_2 \)) is a complete set of representatives of isomorphism classes of simple \( M(0) \)-modules (resp. simple \( M^0_i \)-modules). A main point is that the fusion rules of the following form hold.

\[
W(0) \times M^1 = W^1, \quad W(0) \times W^1 = M^1 + W^1,
\]

\[
W^0_i \times M^2 = W^2, \quad W^0_i \times W^2 = M^2 + W^2.
\]
where $M^i \in \mathcal{M}_i$, $i = 1, 2$, and $W^i \in \mathcal{W}_i$ is determined by $M^i$ through the fusion rule $W(0) \times M^1 = W^1$ or $W^0 \times M^2 = W^2$.

Recall that $M^0$ is rational, $C_2$-cofinite, and of CFT type. Thus every $\mathbb{N}$-graded weak $M^0$-module is a direct sum of simple $M^0$-modules. As a result, every $\mathbb{N}$-graded weak $V_L^r$-module is decomposed into a direct sum of simple $M^0$-modules, and in particular $L(0) = \omega_1$ acts semisimply on it. Each weight subspace, that is, each eigenspace for $L(0)$ is not necessarily a finite dimensional space. However, any simple weak $V_L^r$-module is a simple ordinary $V_L^r$-module by [Abe et al. 2004, Corollary 5.8], since $V_L^r$ is $C_2$-cofinite and of CFT type.

We note that

\[(5-18) \quad W^0 \cdot W^0 = V_L^r.\]

Indeed, $W^0, W^0 = \text{span}\{a_n b \mid a, b \in W^0, n \in \mathbb{Z}\}$ is an $M^0$-submodule of $V_L^r$ by (2-6). Since $P, J_i K_i J_i P \in W^0$ and $\omega^0, \tilde{\omega}^0 \in M^0$, (5-11) implies that $W^0 \cap W^0 = M^0 \oplus W^0$. Each simple $M^0$-module is isomorphic to a tensor product $A \otimes B$ of a simple $M^0$-module $A$ and a simple $M^0$-module $B$. We show that only restricted simple $M^0$-modules can appear in $\mathbb{N}$-graded weak $V_L^r$-modules.

**Lemma 5.2.** Let $U$ be an $\mathbb{N}$-graded weak $V_L^r$-module. Then any simple $M^0$-submodule of $U$ is isomorphic to $M^1 \otimes M^2$ or $W^1 \otimes W^2$ for some $M^i \in \mathcal{M}_i$ and $W^i \in \mathcal{W}_i$, $i = 1, 2$.

**Proof.** Suppose $U$ contains a simple $M^0$-submodule $S^0 \cong M^1 \otimes M^2$ or $W^1 \otimes W^2$ for some $M^i \in \mathcal{M}_i$ and $W^i \in \mathcal{W}_i$. Let $S = V_L^r \cdot S^0 = \text{span}\{a_n w \mid a \in V_L^r, w \in S^0, n \in \mathbb{Z}\}$. Then (2-6) implies that $S$ is the $\mathbb{N}$-graded weak $V_L^r$-submodule of $U$ generated by $S^0$. By the construction of $S$, the difference of any two eigenvalues of $L(0)$ in $S$ is an integer. In fact, $S$ is an ordinary $V_L^r$-module by Remark 2.16.

If $v$ is a nonzero vector in $V_L^r$, then $v_n S^0 \neq 0$ for some $n \in \mathbb{Z}$. Indeed, Lemma 2.6 implies that the set $\{v \mid v_n S^0 = 0\}$ is an ideal of $V_L^r$. It is in fact 0, since $V_L^r$ is a simple vertex operator algebra and $S^0$ is a simple $M^0$-module. Then by the fusion rules (5-17), a simple $M^0$-module isomorphic to $W^1 \otimes W^2$ or $W^1 \otimes W^2$ must appear in $S$. However, the difference of the minimal eigenvalues of $L(0)$ in $M^1 \otimes M^2$ and $W^1 \otimes M^2$, or in $M^1 \otimes W^2$ and $W^1 \otimes W^2$ is not an integer. This is a contradiction. Thus $U$ does not contain a simple $M^0$-submodule isomorphic to $M^1 \otimes W^2$. By a similar argument, we can also show that there is no simple $M^0$-submodule isomorphic to $W^1 \otimes M^2$ in $U$. Hence the assertion holds.

Set $\mathcal{M} = \{M^i \otimes M^2 \mid M^i \in \mathcal{M}_i, i = 1, 2\}$ and $\mathcal{W} = \{W^1 \otimes W^2 \mid W^i \in \mathcal{W}_i, i = 1, 2\}$. Then each of $\mathcal{M}$ and $\mathcal{W}$ consists of 30 inequivalent simple $M^0$-modules. The top level of every simple $M^0$-module is of dimension one.

**Lemma 5.3.** If $U$ is a simple $\mathbb{N}$-graded weak $V_L^r$-module whose top level is of dimension one, then $U$ is isomorphic to one of the 23 known simple $V_L^r$-modules.
with one dimensional top level, namely, \( V_{L(0,j)}(\epsilon) \), \( j = 0, 1, 2, \epsilon = 0, 1, 2 \), \( V_{L(\tau,j)} \), \( j = 1, 2 \), \( V_{L}(\tau^2)(\epsilon) \), \( j = 0, 1, 2, \epsilon = 0, 2 \), and \( V_{L}(\tau^3)(\epsilon) \), \( j = 0, 1, 2, \epsilon = 0, 2 \).

**Proof.** Since \( U \) is a direct sum of simple \( M^0 \)-modules and since the top level, say \( U_\lambda \), of \( U \) is assumed to be of dimension one, it follows from Lemma 5.2 that \( U_\lambda \) is isomorphic to the top level of \( M^1 \otimes M^2 \) or the top level of \( W^1 \otimes W^2 \) as an \( A(M^0) \)-module for some \( M^i \in M_i, i = 1, 2 \). The Zhu algebra

\[
A(M^0) \cong A(M(0)) \times A(M^0)
\]

is commutative and the action of \( A(M^0) \) on the top level of \( M^1 \otimes M^2 \) and the top level of \( W^1 \otimes W^2 \) are known. Indeed, we know all possible action of the elements \([\bar{\omega}^1], [\bar{\omega}^2], [J], \) and \([K]\) of \( A(V^i_L) \) on \( U_\lambda \). Let \([\bar{\omega}^1], [\bar{\omega}^2], [J], \) and \([K]\) act on \( U_\lambda \) as scalars \( a_1, a_2, b_1, \) and \( b_2, \) respectively. There are 60 possible such quadruplets \((a_1, a_2, b_1, b_2)\).

Let \([P], [J_1 P], \) and \([K_1 P]\) act on \( U_\lambda \) as scalars \( x_1, x_2, \) and \( x_3, \) respectively. Then it follows from (5-12) that \([J_1 K_1 P]\) acts on \( U_\lambda \) as a scalar

\[
(5-19) \quad \frac{273}{5} x_1^2 + \frac{1404}{5} a_1^2 + \frac{189}{5} a_2^2 + \frac{4368}{5} a_1 a_2 - \frac{156}{5} a_1 - \frac{126}{5} a_2.
\]

From computer calculations, whose results are presented in an online supplement to this paper,1 and from formulas (5-4)–(5-10), we conclude that the vanishing of \([P \circ (J_1 P)]\) and \([P \circ (K_1 P)]\) imply, respectively,

\[
(5-20) \quad 15b_2x_1 + 5a_2x_2 - 2x_3 = 0, \quad (15a_2 - 1)x_2 = 0.
\]

Using (5-19), we can calculate

\[
[(J_1 P) \ast (J_1 P)], \quad [(K_1 P) \ast (K_1 P)], \quad [(J_1 P) \ast (K_1 P)]
\]

in a similar way and verify that the following equations hold.

\[
(5-21) \quad x_2^2 = \left( \frac{429164}{575} - \frac{37856}{425} a_2 + \frac{1669382}{48875} \right) \left( \frac{56}{85} b_2 x_2 - \frac{4056}{115} b_1 x_3 + \frac{348994464}{107525} a_1 \right) + \frac{137149584}{9775} a_1 a_2 - \frac{1030224}{1375} a_1 a_2 + \frac{137149584}{9775} a_1 a_2 + \frac{16160456}{537625} a_1 - \frac{419184}{9775} a_2 - \frac{209994}{48875} a_2 + \frac{1065516}{48875} a_2 - \frac{304}{187} b_1^2,
\]

\[
(5-22) \quad x_3^2 = \left( -\frac{37044}{575} a_1 - \frac{5684}{85} a_2 + \frac{741731}{9775 a_1} \right) + \frac{26308184}{48875} a_1 a_2 - \frac{8127098}{537625} a_1 - \frac{4775148}{25415} a_2 + \frac{18838017}{127050} a_2 - \frac{9722139}{653575} a_2 - \frac{180}{187} b_1^2,
\]

\[
(5-23) \quad x_2 x_3 = \left( -\frac{864}{5} a_1^2 + \frac{1248}{25} a_1 a_2 + \frac{1152}{5} a_2^2 + \frac{5904}{125} a_1 + \frac{184176}{125} a_2 - \frac{62112}{625} \right) x_1 - 36b_1 b_2.
\]

---

1 The authors can supply these expressions in machine readable form upon request.
We have obtained a system of equations (5-20)–(5-23) for $x_1$, $x_2$, $x_3$. We can solve this system of equations with respect to the 60 possible quadruplets $(a_1, a_2, b_1, b_2)$. Actually, there is no solution for 37 quadruplets of $(a_1, a_2, b_1, b_2)$. For each of the remaining 23 quadruplets $(a_1, a_2, b_1, b_2)$, the system of equations possesses a unique solution $(x_1, x_2, x_3)$. Furthermore, the 23 sets $(a_1, a_2, b_1, b_2, x_1, x_2, x_3)$ of values determined in this way coincide with the action of $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, $[K]$, $[P]$, $[J_1P]$, and $[K_1P]$ on the top level of the 23 known simple $V^r_L$-modules with one dimensional top level described in Section 4. Since $A(V^r_L)$ is generated by these seven elements, this implies that $U_\lambda$ is isomorphic to the top level of one of the 23 simple $V^r_L$-modules listed in the assertion as an $A(V^r_L)$-module. Thus the lemma holds by Zhu’s theorem. □

**Remark 5.4.** We also obtain some equations for $x_1x_2$ and $x_1x_3$ from $[P \ast (J_1P)]$ and $[P \ast (K_1P)]$. However, they are not sufficient to determine $x_1$, $x_2$, and $x_3$.

**Lemma 5.5.** Every $\mathbb{N}$-graded weak $V^r_L$-module contains a simple $M^0$-submodule isomorphic to a member of $\mathcal{M}$.

**Proof.** Suppose false and let $U$ be an $\mathbb{N}$-graded weak $V^r_L$-module which contains no simple $M^0$-submodule isomorphic to a member of $\mathcal{M}$. Then by Lemma 5.2, there is a simple $M^0$-submodule $W$ in $U$ such that $W \cong W^1 \otimes W^2$ for some $W^i \in \mathcal{W}$, $i = 1, 2$. The top level of $W$, say $W_\lambda$ for some $\lambda \in \mathcal{Q}$, is a one dimensional space. Take $0 \neq w \in W_\lambda$ and let $S = V^r_L \cdot w = \text{span}\{a_n w \mid a \in V^r_L, n \in \mathbb{Z}\}$, which is an ordinary $V^r_L$-module by (2-6) and Remark 2.16. Since $V^r_L = M^0 \oplus W^0$, it follows from our assumption and the fusion rules (5-17) that $S$ is isomorphic to a direct sum of finite number of copies of $W$ as an $M^0$-module. Thus $[\tilde{\omega}^1]$, $[\tilde{\omega}^2]$, $[J]$, and $[K]$ act on the top level $S_\lambda$ of $S$ as scalars, say $a_1, a_2, b_1$, and $b_2$, respectively. Then by a similar calculation as in the proof of Lemma 5.3, we see that $[P \circ (K_1P)] = 0$ implies

$$
(15a_2 - 1) o(J_1P) = 0
$$

as an operator on the top level $S_\lambda$. Recall that $[u] \in A(V^r_L)$ acts on $S_\lambda$ as $o(u) = u_{wt(u)} - 1$ for a homogeneous vector $u$ of $V^r_L$. Furthermore, we can calculate that

$$
o(J_1P) o(P) - o(P) o(J_1P) = 0,
$$

(5-25) \quad \begin{align*}
o(K_1P) o(P) - o(P) o(K_1P) &= \frac{2}{15} (15a_2 - 1) o(J_1P), \\
o(J_1P) o(K_1P) - o(K_1P) o(J_1P) &= \frac{96}{175} (15a_2 - 1)(65a_1 + 100a_2 + 441) o(P)
\end{align*}

as operators on $S_\lambda$.

By (5-24), $15a_2 - 1 = 0$ or $o(J_1P) = 0$ and so $o(P)$, $o(J_1P)$, and $o(K_1P)$ commute each other. Thus the action of $A(V^r_L)$ on $S_\lambda$ is commutative. Hence we can choose a one dimensional $A(V^r_L)$-submodule $T$ of $S_\lambda$. Zhu’s theory tells us that
there is a simple \( \mathbb{N}\)-graded weak \( V_L^r \)-module \( R \) whose top level \( R_h \) is isomorphic to \( T \) as an \( A(V_L^r) \)-module. Since \( \dim R_h = 1 \), \( R \) is isomorphic to one of the 23 simple \( V_L^r \)-modules listed in Lemma 5.3. In particular, \( R \) contains a simple \( M^0 \)-submodule \( M \) isomorphic to a member of \( \mathcal{M} \). Now, consider the \( V_L^r \)-submodule \( V_L^r \cdot T \) of \( S \) generated by \( T \). By Lemma 2.10, there is a surjective homomorphism of \( V_L^r \)-modules from \( V_L^r \cdot T \) onto \( R \). Then \( V_L^r \cdot T \) must contain a simple \( M^0 \)-submodule isomorphic to \( M \). This contradicts our assumption. The proof is complete. \( \square \)

**Lemma 5.6.** Let \( U \) be an \( \mathbb{N}\)-graded weak \( V_L^r \)-module and \( M \) be a simple \( M^0 \)-submodule of \( U \) such that \( M \cong M^1 \otimes M^2 \) as \( M^0 \)-modules for some \( M^i \in \mathcal{M} \), \( i = 1, 2 \). Then \( V_L^r \cdot M = \text{span} \{ a u \mid a \in V_L^r, u \in M, n \in \mathbb{Z} \} \) is a simple \( V_L^r \)-module. Moreover, \( V_L^r \cdot M = M \oplus W \), where \( W \) is a simple \( M^0 \)-module isomorphic to \( W^1 \otimes W^2 \) and \( W^i \), \( i = 1, 2 \) are determined from \( M^i \) by the fusion rules \( W(0) \times M^1 = W^1 \) and \( W^0 \times M^2 = W^2 \) of (5-17).

**Proof.** By Remark 2.16, \( V_L^r \cdot M \) is an ordinary \( V_L^r \)-module. Note that \( V_L^r \cdot M = (M^0 + W^0) \cdot M = M + W^0 \cdot M \). We see that \( W^0 \cdot M \neq 0 \) by a similar argument as in the proof of Lemma 5.2. Actually, \( W^0 \cdot (W^0 \cdot M) \subset (W^0 \cdot W^0) \cdot M = V_L^r \cdot M \) (see Lemma 2.6 and (5-18)) implies \( W^0 \cdot M \neq 0 \) also. Moreover, \( W^0 \cdot M \) is an \( M^0 \)-module by (2-6). Since \( M^0 \) is rational, \( W^0 \cdot M \) is decomposed into a direct sum of simple \( M^0 \)-modules, say \( W^0 \cdot M = \bigoplus_{\gamma \in \Gamma} S^\gamma \). Let \( W = W^1 \otimes W^2 \), where \( W^i \in W_i \), \( i = 1, 2 \) are determined by the fusion rules \( W(0) \times M^1 = W^1 \) and \( W^0 \times M^2 = W^2 \). The space \( I_{M^0(\frac{W}{W^0} M)} \) of intertwining operators of type \( \frac{W}{W^0} M \) is of dimension one and each \( S^\gamma \) is isomorphic to \( W \).

We want to show that \( |\Gamma| = 1 \). Suppose \( \Gamma \) contains at least two elements and take \( \gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2 \). Let \( \psi : S^{\gamma_2} \to S^{\gamma_1} \) be an isomorphism of \( M^0 \)-modules and \( p_{\gamma_2} : W^0 \cdot M \to S^{\gamma_2} \) be a projection. For \( a \in W^0 \) and \( u \in M \), set

\[
\langle \mathbf{g}_{\gamma_1}(a, z) u = p_{\gamma_1} Y_U(a, z) u, \quad \langle \mathbf{g}_{\gamma_2}(a, z) u = \psi p_{\gamma_2} Y_U(a, z) u,
\]

where \( Y_U(a, z) \) is the vertex operator of the \( \mathbb{N}\)-graded weak \( V_L^r \)-module \( U \). Then \( \mathbf{g}_{\gamma_i}(\cdot, z) \), \( i = 1, 2 \) are nonzero members in the one dimensional space \( I_{M^0(\frac{W}{W^0} M)} \), so that \( \mu \mathbf{g}_{\gamma_1}(\cdot, z) = \mathbf{g}_{\gamma_2}(\cdot, z) \) for some \( \mu \neq 0 \in \mathbb{C} \). Let \( 0 \neq v \in S^{\gamma_1} \). Then \( v \in W^0 \cdot M \) and so \( v = \sum_j (a_j)_{\gamma_1} u^j \) for some \( a_j \in W^0, u^j \in M, n_j \in \mathbb{Z} \). Take the coefficients of \( z^{-n_j-1} \) in both sides of \( \mu \mathbf{g}_{\gamma_1}(a_j, z) u^j = \mathbf{g}_{\gamma_2}(a_j, z) u^j \). Then \( \mu p_{\gamma_1} ((a_j)_{\gamma_1} u^j) = \psi p_{\gamma_2} ((a_j)_{\gamma_1} u^j) \). Summing up both sides of the equation with respect to \( j \), we have \( \mu u v = \psi v u \). However, \( v \in S^{\gamma_1} \) implies that \( p_{\gamma_1} v = v \) and \( p_{\gamma_2} v = 0 \). This is a contradiction since \( \mu \neq 0 \) and \( v \neq 0 \). Thus \( |\Gamma| = 1 \) and \( W^0 \cdot M \cong W \) as required.

If \( V_L^r \cdot M \) is not a simple \( V_L^r \)-module, then there is a proper \( V_L^r \)-submodule \( N \) of \( V_L^r \cdot M \). Since \( M \) and \( W \) are simple \( M^0 \)-modules, \( N \) must be isomorphic to \( M \) or \( W \) as an \( M^0 \)-module. Then the top level of \( N \) is of dimension one. The simple
Lemma 5.3. Each of $M^1 \otimes M^2$ and $W \otimes W^2$ for some $M^i \in \mathcal{M}$ is the given $M^0$-module structure. We denote the vertex operator of $\mathcal{V}_L^i$ by $\tilde{Y}(v, z)$ for $v \in \mathcal{V}_L^i$. Let $p_M : \mathcal{V}_L^i \rightarrow M^0$ and $p_W : \mathcal{V}_L^i \rightarrow W^0$ be projections and define $\mathcal{Y}(\cdot, z)$ and $\mathcal{Y}(\cdot, z)$ by

$$\mathcal{Y}(a, z)b = p_M \tilde{Y}(a, z)b, \quad \mathcal{Y}(a, z)b = p_W \tilde{Y}(a, z)b$$

for $a, b \in W^0$. Then by (5-18), $\mathcal{Y}(\cdot, z)$ and $\mathcal{Y}(\cdot, z)$ are nonzero intertwining operators of respective types $\mathcal{M}^0$ and $\mathcal{W}^0$. By the fusion rules (5-17), the space $I_{M^0}(\mathcal{M}^0)$ of $M^0$-intertwining operators of type $\mathcal{M}^0$ is of dimension one. Likewise, $\dim I_{M^0}(\mathcal{W}^0) = 1$. Note that $W^0 \cdot M^0 \subset W^0$ and that $\mathcal{Y}(a, z)b + \mathcal{Y}(a, z)b = \tilde{Y}(a, z)b$.

Let $p_M : U \rightarrow M$ and $p_W : U \rightarrow W$ be projections. Define $\mathcal{F}_i^M(\cdot, z)$ and $\mathcal{F}_i^W(\cdot, z)$, $i = 1, 2$ by

$$\mathcal{F}_i^M(a, z)w = p_M Y_i(a, z)w, \quad \mathcal{F}_i^W(a, z)w = p_W Y_i(a, z)w$$

for $a \in W^0$ and $w \in W$. Then $\mathcal{F}_i^M(\cdot, z)$ and $\mathcal{F}_i^W(\cdot, z)$ are intertwining operators of type $\mathcal{M}^0$ and $\mathcal{W}^0$, respectively. Clearly, $\mathcal{F}_i^M(a, z)w + \mathcal{F}_i^W(a, z)w = Y_i(a, z)w$. If $\mathcal{F}_i^M(\cdot, z) = 0$, then $W^0 \cdot W \in W$ and so $\mathcal{V}_L^i \cdot W = M^0 \cdot W + W^0 \cdot W \subset W$. This is a contradiction, since $U$ is a simple $\mathcal{V}_L^i$-module. Hence $\mathcal{F}_i^M(\cdot, z) \neq 0$. Let

$$\xi_i^W(a, z)v = Y_i(a, z)v$$

for $a \in W^0, v \in M$. Then $\xi_i^W(\cdot, z)$ is a nonzero intertwining operator of type $\mathcal{W}^0$ by (5-17). The space of $M^0$-intertwining operators $I_{M^0}(\mathcal{M}^0)$ of type $\mathcal{W}^0$ is of dimension one by (5-17). Similarly, $\dim I_{M^0}(\mathcal{W}^0) = \dim I_{M^0}(\mathcal{W}^0) = 1$. Therefore, $\mathcal{F}_2^M(\cdot, z) = \lambda \mathcal{F}_1^M(\cdot, z), \mathcal{F}_2^W(\cdot, z) = \mu \mathcal{F}_1^W(\cdot, z)$, and $\mathcal{F}_2^W(\cdot, z) = \gamma \xi_1^W(\cdot, z)$ for some $\lambda, \mu, \gamma \in \mathbb{C}$ with $\lambda \neq 0$ and $\gamma \neq 0$.

Now,

$$Y_i(a, z_1)Y_i(b, z_2)v = (\mathcal{F}_1^M(a, z_1) + \mathcal{F}_1^W(a, z_1))\xi_i^W(b, z_2)v,$$

$$Y_i(b, z_2)Y_i(a, z_1)v = (\mathcal{F}_1^M(b, z_2) + \mathcal{F}_1^W(b, z_2))\xi_i^W(a, z_1)v,$$

$$Y_i(Y(a, z_0)b, z_2)v = Y_i(\mathcal{Y}(a, z_0)b, z_2)v + \xi_i^W(\mathcal{Y}(a, z_0)b, z_2)v$$
for \( a, b \in W^0 \) and \( v \in M \). Taking the image of both sides of the Jacobi identity
\[
(5-26) \quad z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_i(a, z_1) Y_i(b, z_2) v - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_i(b, z_2) Y_i(a, z_1) v = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_i(\tilde{\gamma}(a, z_0)b, z_2) v
\]
under the projection \( p_M \), we obtain
\[
(5-27) \quad z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) \bar{\mathcal{F}}_i^M(a, z_1) \bar{g}_i^W(b, z_2) v - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) \bar{\mathcal{F}}_i^M(b, z_2) \bar{g}_i^W(a, z_1) v = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_i(\tilde{\phi}(a, z_0)b, z_2) v.
\]
Comparison of Equation (5-28) for \( i = 1 \) and \( i = 2 \), we have
\[
\gamma(\mu - 1) z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \bar{g}_1^W(\bar{\phi}(a, z_0)b, z_2) v = 0,
\]
since \( \bar{\mathcal{F}}_i^M(\cdot, z) = \lambda \mathcal{F}_i^M(\cdot, z) \), \( \bar{\mathcal{F}}_2^W(\cdot, z) = \mu \mathcal{F}_1^W(\cdot, z) \), and \( \bar{g}_2^W(\cdot, z) = \gamma \bar{g}_1^W(\cdot, z) \).

Now, \( z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) = z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \) by [Frenkel et al. 1988, Proposition 8.8.5] and so the above equation is equivalent to the following assertion.
\[
\gamma(\mu - 1)(z_2 + z_0)^k \bar{g}_1^W(\bar{\phi}(a, z_0)b, z_2) v = 0 \quad \text{for all} \quad k \in \mathbb{Z}.
\]
This implies that
\[
\gamma(\mu - 1) \bar{g}_1^W(\bar{\phi}(a, z_0)b, z_2) v = 0,
\]
since \( \bar{g}_1^W(\bar{\phi}(a, z_0)b, z_2) v \in M^0((z_0)) \|z_2, z_2^{-1}\| W((z_0)) \] for all \( \gamma(\mu - 1) \) and \( \bar{g}_1^W(\cdot, z) \) are nonzero, we conclude that \( \mu = 1 \).

Next, we use Equation (5-27). Since \( \bar{\phi}(a, z_0)b \in M^0((z_0)) \), we have
\[
Y_1(\bar{\phi}(a, z_0)b, z_2) v = Y_2(\bar{\phi}(a, z_0)b, z_2) v
\]
by our assumption. Then it follows from (5-27) for \( i = 1, 2 \) that
\[
(\lambda \gamma - 1) z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_1(\bar{\phi}(a, z_0)b, z_2) v = 0.
\]
Since \( \bar{\phi}(\cdot, z) \neq 0 \) and \( M \) is a simple \((M^0, Y_1)\)-module, a similar argument as above gives that \( \lambda \gamma = 1 \).
For $a \in M^0$, $b \in W^0$, $v \in M$, and $w \in W$,

$$Y_i(a+b, z)(v+w) = Y_i(a, z)v + Y_i(a, z)w + q_i^W(b, z)v + (\mathcal{F}^M_i(b, z) + \mathcal{F}_i^W(b, z))w.$$ 

Note that $Y_i(a, z)v, \mathcal{F}^M_i(b, z)w \in M((z))$ and $Y_i(a, z)w, q_i^W(b, z)v, \mathcal{F}_i^W(b, z)w \in W((z))$. Define $\varphi : U \rightarrow U$ by $\varphi(u) = \lambda u$ if $u \in M$ and $\varphi(u) = u$ if $u \in W$. Since $\mu = 1$ and $\lambda \gamma = 1$, we can verify that

$$Y_2(a+b, z)\varphi(v+w) = \varphi(Y_1(a+b, z)(v+w)).$$

Thus $\varphi$ is an isomorphism of $V_L^\tau$-modules from $(U, Y_1)$ onto $(U, Y_2)$. This completes the proof. \hfill \square

**Remark 5.8.** The proof of the above lemma is essentially the same as that of [Lam et al. 2005, Lemma C.3]. Consider the Jacobi identity for $a, b \in W^0$ and $w \in W$ and take the images of both sides of the identity under the projections $p_M$ and $p_W$, respectively. Then

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\mathcal{F}^M_i(a, z_1)\mathcal{F}_i^W(b, z_2)w - z_0^{-1}\delta\left(\frac{z_2-z_1}{z_0}\right)\mathcal{F}^M_i(b, z_2)\mathcal{F}_i^W(a, z_1)w$$

$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{F}_i^M(\mathcal{F}(a, z_0)b, z_2)w,$$

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)(q_i^W(a, z_1)\mathcal{F}^M_i(b, z_2) + \mathcal{F}^W_i(a, z_1)\mathcal{F}_i^W(b, z_2))w$$

$$- z_0^{-1}\delta\left(\frac{z_2-z_1}{z_0}\right)(q_i^W(b, z_2)\mathcal{F}^M_i(a, z_1) + \mathcal{F}^W_i(b, z_2)\mathcal{F}_i^W(a, z_1))w$$

$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)(Y_i(\mathcal{F}(a, z_0)b, z_2) + \mathcal{F}_i^W(\mathcal{F}(a, z_0)b, z_2))w.$$

Each of these two equations gives the identical equations in case of $i = 1$ and $i = 2$ provided that $\mu = 1$ and $\lambda \gamma = 1$.

**Theorem 5.9.** There are exactly 30 inequivalent simple $V_L^\tau$-modules. They are represented by the 30 simple $V_L^\tau$-modules listed in Lemma 3.2.

**Proof.** Let $U$ be a simple $V_L^\tau$-module. Then by Lemma 5.5, $U$ contains a simple $M^0$-submodule $M$ isomorphic to a member of $\mathcal{M}$. Since $U$ is a simple $V_L^\tau$-module, Lemma 5.6 implies that $U = M \oplus W$ for some simple $M^0$-submodule $W$ isomorphic to a member of $W$. In fact, the isomorphism class of $W$ is uniquely determined by $M$. By Lemma 5.7, $U$ admits a unique $V_L^\tau$-module structure. Since $\mathcal{M}$ consists of 30 members, it follows that there are at most 30 inequivalent simple $V_L^\tau$-modules. Hence the assertion holds. \hfill \square

**Theorem 5.10.** $V_L^\tau$ is a rational vertex operator algebra.
Lemma 5.2

Lemma 5.5

Set \( (3-23) \)

Zhu 1996

that

Dong et al. 1998a

Lemma 5.6

Theorem 2.2.3]. We know \( (3-25) \) and \( \). Hence we can determine the structure of Section 4

The Zhu algebra \( A \)

\[ i = \]

\[ (\gamma) \Gamma \]

\[ \gamma \]

\[ \] 51 dimensional semisimple associative algebra isomorphic to a direct sum of 23 copies of the one dimensional algebra \( \) and 7 copies of the algebra \( \text{Mat}_2(\mathbb{C}) \) of \( 2 \times 2 \) matrices. Moreover, \( A(V)^\) is generated by \( [\tilde{\omega}^1], [\tilde{\omega}^2], [J], [K], \) and \( [P] \).

Proof. Since \( V \) is rational, \( A(V)^\) is a finite dimensional semisimple associative algebra [Dong et al. 1998a, Theorem 8.1; Zhu 1996, Theorem 2.2.3]. We know all the simple \( V\) modules and the action of \( [\tilde{\omega}^1], [\tilde{\omega}^2], [J], [K], \) and \( [P] \) on their top levels in Section 4. Hence we can determine the structure of \( A(V) \) as in the assertion.

Appendix: Some fusion rules for \( M(0) \)

We give a proof of the fusion rules

\[ W(0) \times M_T(\tau^i)(\varepsilon) = W_T(\tau^i)(\varepsilon), \]

\[ W(0) \times W_T(\tau^i)(\varepsilon) = M_T(\tau^i)(\varepsilon) + W_T(\tau^i)(\varepsilon), \]

\[ i = 1, 2, \varepsilon = 0, 1, 2 \]

simple \( M(0)\) modules in \( (3-25) \).

Recall that \( V_L \cong M_0 \oplus W_0 \), where \( M_0 = M(0) \otimes M_i^0 \) and \( W_0 = W(0) \otimes W_i^0 \).

Set \( \hat{M}_T(\tau^i)(\varepsilon) = M_T(\tau^i)(\varepsilon) \otimes M_i^0 \) and \( \hat{W}_T(\tau^i)(\varepsilon) = W_T(\tau^i)(\varepsilon) \otimes W_i^0 \), which are simple \( M_0\) modules. Then

\[ V_L^0(\tau)(\varepsilon) \cong \hat{M}_T(\tau)(\varepsilon) \oplus \hat{W}_T(\tau)(\varepsilon), \]

\[ V_L^0(\tau^2)(\varepsilon) \cong \hat{M}_T(\tau^2)(\varepsilon) \oplus \hat{W}_T(\tau^2)(\varepsilon) \]

as \( M_0\) modules by \( (3-23) \) and \( (3-24) \). Denote by \( Y_1(\cdot, z) \) (resp. \( Y_2(\cdot, z) \)) the vertex operator of the simple \( V_L^0\) module \( V_L^0(\tau)(\varepsilon) \) (resp. \( V_L^0(\tau^2)(\varepsilon) \)). Let \( p_M : V_L^0(\tau)(\varepsilon) \to \hat{M}_T(\tau)(\varepsilon) \) and \( p_W : V_L^0(\tau^2)(\varepsilon) \to \hat{W}_T(\tau)(\varepsilon) \) be projections. We also use the same symbol \( p_M \) or \( p_W \) to denote a projection from \( V_L^0(\tau^2)(\varepsilon) \) onto \( \hat{M}_T(\tau^2)(\varepsilon) \) or onto \( \hat{W}_T(\tau^2)(\varepsilon) \). We fix \( i = 1, 2 \) and \( \varepsilon = 0, 1, 2 \). For simplicity of notation, set \( \hat{M} = \hat{M}_T(\tau^i)(\varepsilon) \) and \( \hat{W} = \hat{W}_T(\tau^i)(\varepsilon) \).

Let \( \mathcal{F}_i^M(a, z)w = p_M Y_1(a, z)w \) and \( \mathcal{F}_i^W(a, z)w = p_W Y_2(a, z)w \) for \( a \in W_0 \) and \( w \in \hat{W} \). Then \( \mathcal{F}_i^M(\cdot, z) \) and \( \mathcal{F}_i^W(\cdot, z) \) are intertwining operators of type \( (\hat{M}, \hat{W}) \) and...
Then \( g_i^W(\cdot, z) \) is an intertwining operator of type \((\hat{W}, \hat{W})\), since the fusion rule \( W^0_i \times M_0^0 = W^0_i \) of \( M_0^0 \)-modules implies that \( W^0_i \cdot \hat{M} = \text{span} \{ a_n \theta_a | a \in W^0, n \in \mathbb{Z} \} \) is contained in \( \hat{W} \). If \( g_i^W(\cdot, z) = 0 \), then \( V^0_i \cdot \hat{M} = (M^0 + W^0) \cdot \hat{M} \subset \hat{W} \). This is a contradiction, since \( V^0_i(\tau)(\varepsilon) \) and \( V^0_i(\tau^2)(\varepsilon) \) are simple \( V^0_i \)-modules. Thus \( g_i^W(\cdot, z) \neq 0 \). Similarly, \( g_i^M(\cdot, z) \neq 0 \). Indeed, if \( \mathcal{P}_i^M(\cdot, z) = 0 \), then \( V^0_i \cdot \hat{W} \subset \hat{W} \), which is a contradiction. Assume that \( \mathcal{P}_i^M(\cdot, z) = 0 \). Then \( W^0_i \cdot \hat{W} \subset \hat{M} \) and so \( W^0_i \cdot (W^0_i \cdot \hat{W}) \subset \hat{W} \). However, \( W^0_i \cdot (W^0_i \cdot \hat{W}) \supset (W^0_i \cdot W^0_i) \cdot \hat{W} = V^0_i \cdot \hat{W} \) by Lemma 2.6 and (5-18). This contradiction implies that \( g_i^W(\cdot, z) \neq 0 \).

Restricting the three nonzero intertwining operators \( \mathcal{P}_i^M(\cdot, z) \), \( \mathcal{P}_i^W(\cdot, z) \), and \( g_i^W(\cdot, z) \) to the first component of each of the tensor products \( W^0_i \otimes W^0_i \), \( M_i = M_i \)(\( \tau^i \))(\( \varepsilon \)) \( \otimes M_0^0 \), and \( \hat{W} = W^0_i \otimes W^0_i \), we obtain nonzero intertwining operators of type

\[
\begin{pmatrix}
\text{M}_i(\tau^i)(\varepsilon) \\
\text{W}(0) \text{W}(\tau^i)(\varepsilon)
\end{pmatrix}, \quad \begin{pmatrix}
\text{W}(\tau^i)(\varepsilon) \\
\text{W}(0) \text{W}(\tau^i)(\varepsilon)
\end{pmatrix}, \quad \begin{pmatrix}
\text{W}(\tau^i)(\varepsilon) \\
\text{W}(0) \text{M}(\tau^i)(\varepsilon)
\end{pmatrix}
\]

for \( M(0) \)-modules, respectively.

Let \( N^2 \) be one of \( M_i(\tau^i)(\varepsilon) \), \( W^0_i(\tau^i)(\varepsilon) \), \( i = 1, 2, \varepsilon = 0, 1, 2 \) and let \( N^3 \) be any of the 20 simple \( M(0) \)-modules. Then the top level \( N^j_i \) of \( N^j \) is of dimension one. By [Dong et al. 2004], the Zhu algebra \( A(M(0)) \) of \( M(0) \) is generated by \( [\tilde{\omega}^j] \) and \([J]\). Moreover, we know the action of \( o(\tilde{\omega}^j) \) and \( o(J) \) on \( N^j_i \). Thus, by an argument as in [Tanabe 2005, pp. 192–193], we can calculate that the dimension of

\[
\text{Hom}_{A(M(0))}(A(W(0)), \bigotimes_{A(M(0))} N^2_i, N^3_i)
\]

is at most one and it is equal to one if and only if the pair \((N^2, N^3)\) is one of

\[
(M_i(\tau^i)(\varepsilon), W^0_i(\tau^i)(\varepsilon)), \quad (W^0_i(\tau^i)(\varepsilon), M_i(\tau^i)(\varepsilon)), \quad (W^0_i(\tau^i)(\varepsilon), W^0_i(\tau^i)(\varepsilon))
\]

for \( i = 1, 2, \varepsilon = 0, 1, 2 \). Note that \( W(0) \) was denoted by \( W^0_k \) in [Tanabe 2005]. Now, the desired fusion rules are obtained by [Li 1999a, Proposition 2.10 and Corollary 2.13].

References


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