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NONEXISTENCE RESULTS AND CONVEX HULL PROPERTY FOR MAXIMAL SURFACES IN MINKOWSKI THREE-SPACE

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We study properly immersed maximal surfaces with nonempty boundary and singularities in three-dimensional Minkowski space. We use the maximum principle and scaling arguments to obtain nonexistence results for these surfaces when the boundary is planar. We also give sufficient conditions for such surfaces to satisfy the convex hull property.

1. Introduction

In recent years, maximal hypersurfaces in a Lorentzian manifold — that is, space-like submanifolds of codimension one with zero mean curvature — have been the object of considerable interest. Such hypersurfaces, and in general those having constant mean curvature, have a special significance in classical relativity [Marsden and Tipler 1980].

When the ambient space is the flat Minkowski space \mathbb{L}^{n+1} , Calabi [1970] (for $n \leq 3$) and Cheng and Yau [1976] (for arbitrary dimension) proved that a complete maximal hypersurface is necessarily a spacelike hyperplane. This result remains valid if we replace the completeness hypothesis by properness; see [Fernández and López 2004b]. Therefore, it does not make sense to consider global problems on regular maximal hypersurfaces in \mathbb{L}^{n+1} . Interesting problems are then those that deal with hypersurfaces with nonempty boundary or having certain type of singularities. In this line, Bartnik and Simon [1982/83] obtained results on the existence and regularity of spacelike solutions to the boundary value problem for the mean curvature operator in \mathbb{L}^{n+1} , and Kobayashi [1984] investigated surfaces with cone-like singularities. Estudillo and Romero [1992] defined a class of maximal surfaces with singularities of other types and studied criteria for such a surface to be a plane. On the other hand, Klyachin and Mikyukov [1993] have tackled the problem of existence of solutions to the maximal hypersurface equation in \mathbb{L}^{n+1} with prescribed boundary conditions and a finite number of singularities. Fernández, López and Souam [Fernández et al. 2005] proved that a complete embedded maximal surface

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with a finite set of singularities is an entire graph over any spacelike plane and that this family of maximal graphs has a structure of moduli space. We also mention the work of Umehara and Yamada [2006] where topological obstructions to the existence of this type of surfaces are given.

Maximal surfaces in \mathbb{L}^3 and minimal surfaces in Euclidean space are closely related. Both are solutions of variational problems, namely they are local maxima (minima) for the area functional. Both admit a Weierstrass representation (see [Kobayashi 1983] for maximal surfaces). The maximal surface equation and the minimal surface equation are both quasilinear elliptic equations and therefore enjoy a maximum principle. But contrary to the minimal case, solutions to the maximal surface equation can have isolated singularities, that is to say, points where the solution is not differentiable. Such points correspond to possible degeneracy of the ellipticity of the maximal surface equation. Geometrically at these singular points the Gauss curvature blows up, the Gauss map has no well-defined limit and the surface is asymptotic to the light cone.

In the minimal case, the maximum principle has been used by Schoen [1983], Hoffman and Meeks [1990], Meeks and Rosenberg [1993], López and Martín [2001], and others to derive remarkable results. In this paper we apply the maximum principle and scaling arguments to properly immersed maximal surfaces with nonempty boundary and isolated singularities in \mathbb{L}^3 . We get two types of results: nonexistence results for properly immersed maximal surfaces with singularities and planar boundary contained in a timelike or lightlike plane, and results generalizing the convex hull property for such surfaces. Recall that a surface satisfies the convex hull property if it lies in the convex hull of its boundary. Although compact maximal surfaces in \mathbb{L}^3 satisfy this property, since they have nonpositive euclidean Gauss curvature (see [Osserman 1971/72]), this is not true if compactness is not assumed. We give sufficient conditions for a properly immersed maximal surface (not necessarily compact and with singularities) to satisfy the convex hull property.

Organization of paper. Section 2 contains the necessary notations and definitions, a description of the behavior of maximal surfaces around an isolated singularity, and a discussion of the maximal surfaces we use as barriers: Lorentzian catenoids, maximal surfaces of Riemann and Scherk type, and spacelike planes. We finish the section giving a first generalization of the convex hull property to compact maximal surfaces with singularities.

In Section 3 we obtain nonexistence results for properly immersed maximal surfaces with singularities and boundary contained in a timelike plane. Letting

$$C^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 \leq 0, x_3 \geq 0\}$$

be the positive solid half-cone, we show:

Theorem A. *There exists no connected properly immersed maximal surface M such that $M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \geq 0, -ax_2 + x_3 \geq 0\}$ and $\partial M \subset C^+ \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 = 0\}$, for $a \in]-1, 1[$.*

This theorem holds even if we allow certain singularities (see Theorem 3.5).

Section 4 is devoted to the study of properly immersed maximal surfaces whose boundary is contained in a spacelike plane. Consider any region V of the form

$$V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq 0, -ax_2 + x_3 \leq 0, x_1 + bx_2 + c \geq 0\},$$

with $a \in]0, 1[$ and $b, c \in]-\infty, \infty[$.

Theorem B. *Let M be a connected properly immersed maximal surface contained in V and such that ∂M lies in a spacelike plane. Then M is a planar region.*

This result, too, holds even if we allow certain singularities (see Theorem 4.2 and Corollary 4.3). In the proof we construct a barrier surface *ad hoc* using the aforementioned Bartnik and Simon existence result. Theorem B is still valid if we replace V by C^+ (see Proposition 4.4).

Finally, in Section 5 we exploit the results of the preceding sections to give nonexistence results for properly immersed maximal surfaces with the boundary on a lightlike plane. We also prove:

Theorem C. *Any connected properly immersed maximal surface with singularities contained either in V or C^+ lies in the convex hull of its boundary and some of its singularities.*

Propositions 5.3 and 5.4 provide a precise formulation of this result.

2. Preliminaries

We denote by \mathbb{L}^3 the three dimensional Lorentz–Minkowski space $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, where the inner product corresponds to the form $dx_1^2 + dx_2^2 - dx_3^2$. A nonzero vector $v \in \mathbb{R}^3$ is called *spacelike*, *timelike* or *lightlike* if $\langle v, v \rangle$ is positive, negative or zero, respectively. The vector $(0, 0, 0)$ is considered spacelike. We say that a plane in \mathbb{L}^3 is *spacelike*, *timelike* or *lightlike* if the induced metric is Riemannian, nondegenerate and indefinite or degenerate, respectively. We also say that an affine plane in \mathbb{L}^3 is *spacelike*, *timelike* or *lightlike* if it is parallel to a spacelike, timelike or lightlike vector plane.

The *light cone* at $y = (y_1, y_2, y_3) \in \mathbb{L}^3$ is defined as

$$C(y) = \{x \in \mathbb{L}^3 \mid \langle x - y, x - y \rangle = 0\}.$$

We also set $C^+(y) = C(y) \cap \{x_3 \geq y_3\}$ and $C^-(y) = C(y) \cap \{x_3 \leq y_3\}$. Observe that lightlike vectors in \mathbb{L}^3 lie in the light cone $C((0, 0, 0))$.

Further, set $\mathbb{H}^2 = \mathbb{H}_+^2 \cup \mathbb{H}_-^2$, where $\mathbb{H}_+^2 = \{x \in \mathbb{L}^3 \mid \langle x, x \rangle = -1\} \cap \{x_3 \geq 0\}$ and $\mathbb{H}_-^2 = \{x \in \mathbb{L}^3 \mid \langle x, x \rangle = -1\} \cap \{x_3 \leq 0\}$.

Consider the stereographic projection $\sigma : \bar{\mathbb{C}} - \{|z| = 1\} \rightarrow \mathbb{H}^2$ for \mathbb{H}^2 , given by

$$(2-1) \quad \sigma(z) = \left(\frac{2 \operatorname{Im} z}{|z|^2 - 1}, \frac{2 \operatorname{Re} z}{|z|^2 - 1}, \frac{|z|^2 + 1}{|z|^2 - 1} \right),$$

where $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\sigma(\infty) = (0, 0, 1)$.

An immersion $X : M \rightarrow \mathbb{L}^3$ is *spacelike* if the tangent plane at any point is spacelike. In this case M must be orientable, that is to say, the Gauss map N is globally well defined and $N(M)$ lies in one of the components of \mathbb{H}^2 .

A *maximal immersion* is a spacelike immersion $X : M \rightarrow \mathbb{L}^3$ whose mean curvature vanishes. In this case $X(M)$ is said to be a *maximal surface* in \mathbb{L}^3 . Using isothermal parameters compatible with a fixed orientation $N : M \rightarrow \mathbb{H}^2$, M acquires a natural conformal structure, and the map $g = \sigma^{-1} \circ N$ is meromorphic. Moreover, there exists a holomorphic 1-form Φ_3 on M such that the 1-forms

$$(2-2) \quad \Phi_1 = \frac{i}{2} \left(\frac{1}{g} - g \right) \Phi_3, \quad \Phi_2 = -\frac{1}{2} \left(\frac{1}{g} + g \right) \Phi_3$$

are holomorphic, and together with Φ_3 , have no real periods on M and no common zeros. Up to a translation, the immersion is given by

$$(2-3) \quad X = \operatorname{Re} \int (\Phi_1, \Phi_2, \Phi_3).$$

The induced Riemannian metric ds^2 on M is given by $ds^2 = \lambda(du^2 + dv^2)$, where $z = u + iv$ is a conformal parameter and

$$\lambda = \frac{1}{2} (|\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2) = \left(\frac{|\Phi_3|}{2} \left(\frac{1}{|g|} - |g| \right) \right)^2.$$

Since M is spacelike, we have $|g| \neq 1$ on M and we can assume $|g| < 1$.

Conversely, let M , g and Φ_3 be a Riemann surface, a meromorphic map on M and a holomorphic 1-form on M . If $|g(p)| \neq 1$ for all $p \in M$, and if the 1-forms Φ_1 , Φ_2 , Φ_3 defined as above are holomorphic, have no real periods and no common zeros, then the conformal immersion X defined in (2-3) is maximal and its Gauss map is $\sigma \circ g$. We call (M, g, Φ_3) the Weierstrass representation of X . For more details see [Kobayashi 1983].

A maximal surface in \mathbb{L}^3 can be represented locally as a graph $x_3 = u(x_1, x_2)$ of a smooth function u such that $u_{x_1}^2 + u_{x_2}^2 < 1$ and

$$(2-4) \quad (1 - u_{x_1}^2)u_{x_2x_2} + 2u_{x_1}u_{x_2}u_{x_1x_2} + (1 - u_{x_2}^2)u_{x_1x_1} = 0.$$

The maximum principle for elliptic quasilinear equations then gives rise to:

Maximum principle for maximal surfaces. *Let S_1 and S_2 be two maximal surfaces in \mathbb{L}^3 which intersect tangentially at a point $p \in S_1 \cap S_2$. Suppose that u_i , for $i = 1, 2$ denotes the function defining S_i around p and that $u_1 \geq u_2$ (we say S_1 is above S_2 or S_2 is below S_1). Then $S_1 = S_2$ locally around p .*

Maximal surfaces with singularities. If in a maximal immersion $X : M \rightarrow \mathbb{L}^3$ we allow points $q \in M$ where the induced metric is not Riemannian we say that X (respectively, $X(M)$) has singularities and q (respectively, $X(q)$) is called a *singular point*. The different kinds of isolated singularities of maximal surfaces and the behavior of maximal surfaces around these points are well known; see [Kobayashi 1984; Ecker 1986; Miklyukov 1992; Fernández et al. 2005]. We recall the necessary material.

Let D be an open disc and $X : D \rightarrow \mathbb{L}^3$ a maximal immersion with a singular point in $q \in D$. There are two possibilities: either N extends continuously to q (q is a *spacelike* singular point) or not (q is a *lightlike* singular point).

In the second case $D - \{q\}$ with the induced metric is conformally equivalent to $\{z \in \mathbb{C}, 0 < r < |z| < 1\}$ and X extends to a conformal map $X : A_r \rightarrow \mathbb{L}^3$ with $X(\mathbb{S}^1) = X(q) = p$, where $A_r = \{z \in \mathbb{C}, r < |z| \leq 1\}$ and $\mathbb{S}^1 = \{z \in \mathbb{C}, |z| = 1\}$. Denote by $J(z) = 1/\bar{z}$ the inversion about \mathbb{S}^1 . Then Schwarz reflection allows us to assert that X extends analytically to $B_r = \{z \in \mathbb{C}, r < |z| < 1/r\}$ and satisfies $X \circ J = -X + 2p$. Therefore if (g, Φ_3) are the Weierstrass data of the extended immersion we have $J^*(\Phi_k) = -\bar{\Phi}_k$ for $k = 1, 2, 3$, where $J^*(\Phi_k)$ denotes the pullback of Φ_k under J : if $\Phi_k = f_k dz$ then $J^*(\Phi_k) = -\bar{z}^{-2}(f_k \circ J) d\bar{z}$. Thus $g \circ J = 1/\bar{g}$ and consequently $|g| = 1$ on \mathbb{S}^1 . Let Π be a spacelike plane containing $p = X(\mathbb{S}^1)$ and label $\pi : \mathbb{L}^3 \rightarrow \Pi$ as the Lorentzian orthogonal projection. If n (always even) is the number of zeros of Φ_3 on \mathbb{S}^1 and m denotes the degree of the map $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, we have:

Lemma 2.1 [Fernández et al. 2005]. *There exists a small closed disc U in Π centered at p such that $(\pi \circ X)^{-1}(p) \cap V = \mathbb{S}^1$ and $(\pi \circ X) : V - \mathbb{S}^1 \rightarrow U - \{p\}$ is a covering of $m + \frac{1}{2}n$ sheets, where V is the annular connected component of $(\pi \circ X)^{-1}(U)$ containing \mathbb{S}^1 .*

As a consequence, X is an embedding around q if and only if $m = 1$ and $n = 0$. In this case the point $p = X(\mathbb{S}^1)$ is said to be a *conelike singularity* of the maximal surface $X(D)$. Moreover, for r_0 close enough to 1, $X(A_{r_0})$ is the graph of a function u over Π . Locally, conelike singularities are points where the function defining the graph is not differentiable and correspond to possible degeneracy of the equation (2-4). Moreover, the graph of u is either above Π and asymptotic to $C^+(p)$ or below Π and asymptotic to $C^-(p)$, and the point p is called a *downward* or *upward pointing conelike singularity*, respectively.

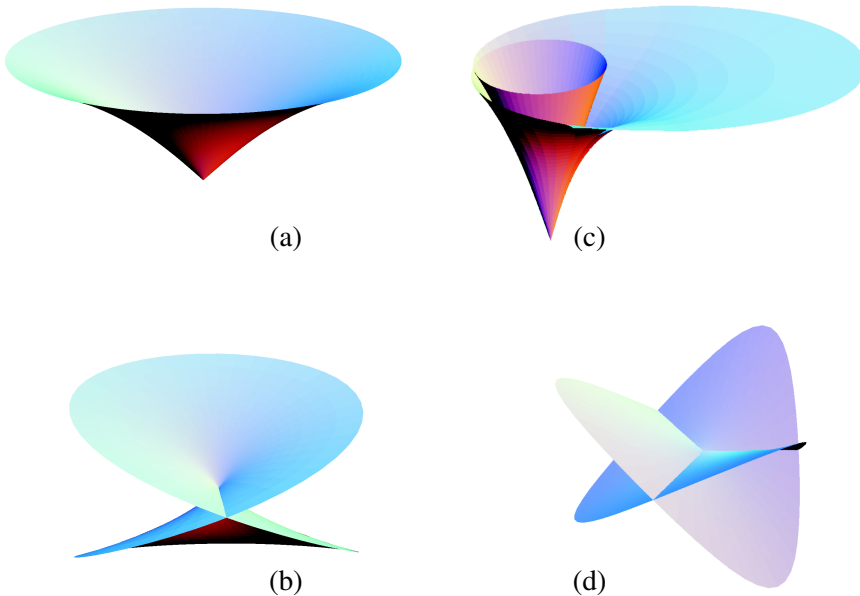


Figure 1. Different types of isolated singularities. (a) A downward pointing conelike singularity ($m = 1, n = 0$). (b) a downward pointing lightlike singularity with $m = 2, n = 0$. (c) a lightlike singularity with $m = 1, n = 2$. (d) a spacelike singularity with $n = 2$.

Lemma 2.2. *Let D be an open disc and $X : D \rightarrow \mathbb{L}^3$ a maximal immersion with a lightlike singular point in $q \in D$. Set $p = X(q)$. The neighborhoods U and V of Lemma 2.1 can be chosen so that:*

- (i) *If p is a lightlike singularity with $n = 0$, then $X(V)$ is either over Π and asymptotic to $C^+(p)$ or below Π and asymptotic to $C^-(p)$ (see Figure 1a,b).*
- (ii) *If, on the contrary, p is a lightlike singularity with $n > 0$, there exist points of $X(V)$ in both sides of the plane Π . In particular there exist a pair of curves α, β in V starting at q such that $X(\alpha) - \{p\}$ is over Π and asymptotic to $C^+(p)$ and $X(\beta) - \{p\}$ is below Π and asymptotic to $C^-(p)$ (see Figure 1c).*

Proof. Up to a Lorentzian isometry we can assume $\Pi = \{x_3 = 0\}$ and $p = (0, 0, 0)$. Let $X : A_r \rightarrow \mathbb{R}^3$ be a conformal reparametrization of the maximal immersion with $X(\mathbb{S}^1) = p$ and consider U, V as in Lemma 2.1. A thoughtful reading of the proof of Lemma 2.1 in [Fernández et al. 2005] will convince the reader that the same arguments prove (i).

For the proof of (ii) we use again the ideas of the same lemma. The Weierstrass data can be written in a neighborhood of \mathbb{S}^1 as

$$(2-5) \quad g(z) = z^m, \quad \Phi_3(z) = i \frac{\prod_{j=1}^n (z - a_j)}{z^{n/2+1}} f(z) dz,$$

where a_1, \dots, a_n are the zeros of Φ_3 on \mathbb{S}^1 (with multiplicity) and f is a nonvanishing holomorphic function. Recall that the multiplicity of the zero of Φ_3 at a_i coincides with the number of nodal curves of the harmonic function x_3 meeting at a_i minus one. By the maximum principle there are no domains bounded by nodal curves and x_3 changes sign when crossing a nodal curve. Since $n \geq 2$ there are points of $X(V)$ in both sides of Π and there exist at least a pair of domains $\Gamma, \Gamma' \subset V$ bounded by a pair of nodal curves of x_3 , a piece of $\partial V - \mathbb{S}^1$ and a point or a piece of \mathbb{S}^1 , such that $x_3(X(\Gamma)) > 0$ and $x_3(X(\Gamma')) < 0$.

To conclude we prove that the image of all the curves $\rho_\theta(s) = se^{i\theta}$, for $\theta \in K = [0, 2\pi] - \{\arg(a_1), \dots, \arg(a_n)\}$, is asymptotic to the cone $C(p)$. Taking into account (2-5) we can write

$$\begin{aligned} X(\rho_\theta(s)) = \operatorname{Re} \int_1^s i \frac{\prod_{j=1}^n (te^{i\theta} - a_j)}{t^{n/2+1} (e^{i\theta})^{n/2}} f(te^{i\theta}) \\ \times \left(\frac{i}{2} \left(\frac{e^{-im\theta}}{t^m} - t^m e^{im\theta} \right), -\frac{1}{2} \left(\frac{e^{-im\theta}}{t^m} + t^m e^{im\theta} \right), 1 \right) dt. \end{aligned}$$

Since $J^*(\Phi_3) = -\overline{\Phi_3}$, we deduce that

$$\operatorname{Im} \left(\frac{i \prod_{j=1}^n (e^{i\theta} - a_j)}{(e^{i\theta})^{\frac{n}{2}}} f(e^{i\theta}) \right) = 0.$$

Using this it is straightforward to see that

$$\lim_{s \rightarrow 1} \left\| \frac{X(\rho_\theta(s))}{x_3(X(\rho_\theta(s)))} - (\sin(m\theta), -\cos(m\theta), 1) \right\|_1 = 0,$$

where $\|\cdot\|_1$ is the \mathcal{C}^1 norm in $\mathcal{C}^1(K, \mathbb{R}^3)$. Therefore, we can consider a pair of curves $\alpha \in \Gamma$ and $\beta \in \Gamma'$ satisfying the requirements of statement (ii). \square

Definition 2.3. A point p as in Lemma 2.2(i) is called a *downward* or *upward pointing lightlike singularity*, as the case may be. We also call it a *general conelike singularity*.

If D is an open disc and $X : D \rightarrow \mathbb{L}^3$ is a maximal immersion with a spacelike singular point in $q \in D$, the local behavior at the singularity is similar to the case of minimal surfaces in \mathbb{R}^3 (see [Dierkes et al. 1992; Estudillo and Romero 1992; Fernández et al. 2005]): X is not a topological embedding, $D - \{q\}$ with the induced metric is conformally equivalent to $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$, the Weierstrass data (g, Φ_3)

extend analytically to q , $|g(q)| < 1$ and $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ has a zero at q . Up to a Lorentzian isometry we can assume that the tangent plane of $X(D)$ at $p = X(q)$ is $\Pi = \{x_3 = 0\}$ and $p = (0, 0, 0)$. The Weierstrass data of the immersion can be written as

$$g(z) = z^m f(z), \quad \Phi_3(z) = z^{m+n} dz,$$

where $m > 0$, n is the order of the zero of Φ at q and f is a holomorphic function with $f(0) \neq 0$. Up to a rotation around the axis x_3 , we can assume $\text{Im}(f(0)) = 0$. From here it is easy to derive that the asymptotic behavior of the immersion around the singularity is in polar coordinates

$$X(se^{i\theta}) = \left(\begin{array}{l} \frac{-s^{n+1}}{2f(0)(n+1)} \sin((n+1)\theta) + O(s^{n+2}), \\ \frac{-s^{n+1}}{2f(0)(n+1)} \cos((n+1)\theta) + O(s^{n+2}), \frac{s^{m+n+1}}{m+n+1} \cos((m+n+1)\theta) \end{array} \right),$$

where by $O(s^{n+2})$ we denote a function such that $s^{-n-2}O(s^{n+2})$ is bounded as $s \rightarrow 0$. Therefore, it is clear that X has a branch point at q of order n in the sense of [Gulliver et al. 1973].

Lemma 2.4 [Gulliver et al. 1973, Lemma 2.12]. *Let $X : D \rightarrow \mathbb{L}^3$ be a maximal immersion with a spacelike singular point at $q \in D$. Set $p = X(q)$ and let be S an embedded surface in \mathbb{L}^3 with $p \in S$. Suppose that for a neighborhood V of q , $X(V)$ lies on one side of S . Then the tangent plane to S at p coincides with the tangent plane to $X(D)$ at p .*

Remark 2.5. In the case of spacelike singularities, we always assume that the immersion $X : D \rightarrow \mathbb{L}^3$ is not a branched covering of an embedded surface; that is to say, q is not a false branch point. See [Gulliver et al. 1973].

Finally, we mention a property of maximal surfaces with singularities (see for example [Fernández and López 2004a]).

Lemma 2.6. *Let $X : M \rightarrow \mathbb{L}^3$ be a maximal immersion with isolated singularities. Then for all $q \in M$ there exists a neighborhood V , such that $X(V) - \{X(q)\}$ is contained in the exterior of $C(X(q))$.*

Remark 2.7. Let S be an embedded surface and $p \in S$. If the tangent plane of S at p is spacelike then S can be written in a neighborhood of p as the graph of a function h on a domain Ω of the plane $\{x_3 = 0\}$. Let M be another surface (possibly with singularities) and denote by π the orthogonal projection on $\{x_3 = 0\}$. In this context, we say that M lies *above* S in a neighborhood of p if $x_3(p') \geq h(x_1, x_2)$ for all $(x_1, x_2) \in \Omega$ and $p' \in M \cap \pi^{-1}(x_1, x_2)$. Naturally, M lies *below* S if $x_3(p') \leq h(x_1, x_2)$ instead.

Maximal surfaces with boundary. Let S' be a maximal surface, possibly with isolated singularities. Consider $S \subset S'$ such that the topological boundary of S in S' is nonempty and piecewise \mathcal{C}^1 . Then S is called a *maximal surface with boundary*; the topological boundary of S in S' is called the *boundary* of S , written ∂S . The *interior* of S is $\text{Int } S = S - \partial S$. Our definition allows singularities on the boundary of S .

Since the components of a maximal immersion are harmonic functions, the intersection of such an S with any plane Π having ∂S entirely to one side is a union of piecewise analytic curves, and each connected component of $S - (S \cap \Pi)$ is itself a maximal surface with boundary.

We say that S is a *properly immersed maximal surface with boundary* if, in the preceding situation, S' is a maximal surface properly immersed in \mathbb{L}^3 .

Theorem [Fernández and López 2004b]. *Let M be a properly immersed maximal surface with boundary such that, except for a compact set, it is contained in the region $\{x \in \mathbb{L}^3 \mid \langle x, x \rangle \geq \varepsilon\}$, for $\varepsilon > 0$. Then M is relative parabolic, it is to say, bounded harmonic functions on M are determined uniquely by their values at the boundary and the interior isolated singularities.*

(Note that in [Fernández and López 2004b] the definition of a maximal surface with boundary is more general than in this paper.)

Corollary 2.8. *Let M be a connected properly immersed maximal surface with boundary such that $M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq k\}$ and the boundary and the singularities are contained in $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = k\}$, for $k > 0$. Then M is a planar region.*

Barrier surfaces. For any $v \in \mathbb{R}^3 - \{(0, 0, 0)\}$ and $y \in \mathbb{R}^3$, define

$$\begin{aligned} H(y, v) &= \{x \in \mathbb{R}^3 \mid \langle v, x - y \rangle_e = 0\}, \\ H^+(y, v) &= \{x \in \mathbb{R}^3 \mid \langle v, x - y \rangle_e \geq 0\}, \\ H^-(y, v) &= \{x \in \mathbb{R}^3 \mid \langle v, x - y \rangle_e \leq 0\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_e$ is the Euclidean metric on \mathbb{R}^3 . Next, for $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $t \in \mathbb{R}$, set

$$\begin{aligned} \Pi_{\theta t} &= H((0, 0, t), (0, -\tan \theta, 1)), \\ \Pi_{\theta t}^+ &= H^+((0, 0, t), (0, -\tan \theta, 1)), \\ \Pi_{\theta t}^- &= H^-((0, 0, t), (0, -\tan \theta, 1)). \end{aligned}$$

In the case of $t = 0$ we write simply Π_θ instead of $\Pi_{\theta 0}$, and so on.

We also consider, for $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$,

$$\begin{aligned}\Sigma_\alpha &= H((0, 0, 0), (0, 1, -\tan \alpha)), \\ \Sigma_\alpha^+ &= H^+((0, 0, 0), (0, 1, -\tan \alpha)), \\ \Sigma_\alpha^- &= H^-((0, 0, 0), (0, 1, -\tan \alpha)).\end{aligned}$$

Observe that $\Pi_{\pi/4,t}$ and $\Pi_{-\pi/4,t}$ are lightlike planes, while the $\Pi_{\theta t}$ are spacelike planes for $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$. Also, for any $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$, there is an orthochronous hyperbolic rotation f_s of \mathbb{L}^3 of the form

$$f_s \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

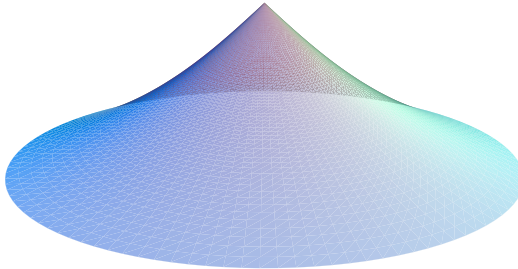
which preserves (individually) the light half-cones $C^+((0, 0, 0))$ and $C^-((0, 0, 0))$ and satisfies $f_s(\Pi_\theta) = \Pi_0$. Analogously, for $\theta \in]-\frac{\pi}{4}, \frac{\pi}{4}[$, there is an orthochronous isometry \tilde{f}_s of \mathbb{L}^3 composed of an orthochronous hyperbolic rotation f_s and a vertical translation, and such that $\tilde{f}_s(\Pi_{\theta t}^+) = \Pi_0^+$ and $\tilde{f}_s(\Pi_{\theta t}^-) = \Pi_0^-$, so also $\tilde{f}_s(\Pi_{\theta t}) = \Pi_0$. As for the Σ_α , they are timelike planes and there is an orthochronous hyperbolic rotation f_s of \mathbb{L}^3 that preserves the light half-cones and satisfies $f_s(\Sigma_\alpha) = \Sigma_0$. For more details about these isometries of \mathbb{L}^3 , see [Fernández and López 2004a].

Now we present the maximal surfaces that we use as barriers.

Lorentzian catenoids. The (vertical) *Lorentzian catenoid* \mathcal{C}_a is the maximal surface given on $\overline{\mathbb{D}} - \{0\} = \{z \in \mathbb{C} \mid 0 < |z| \leq 1\}$ by the Weierstrass data $g = z$ and $\Phi_3 = a dz/z$ (see figure for the case $a = 1$). We can express \mathcal{C}_a as the graph of the radially symmetric function

$$u(r) = - \int_0^r \frac{a}{\sqrt{t^2 + a^2}} dt, \quad r > 0.$$

Let $\mathcal{C} = \{\mathcal{C}_a, a \in]0, \infty[\}$ be the family of such catenoids. Lorentzian catenoids have been used as barriers for applications of the maximum principle in [Bartnik and Simon 1982/83] and [Ecker 1986].



Maximal surfaces of Riemann type. R. López, F. J. López and R. Souam studied in [López et al. 2000] the set of maximal surfaces in \mathbb{L}^3 that are foliated by pieces of circles. From among them, we take the one-parameter family of Riemann-type maximal surfaces. This is a family of singly periodic maximal surfaces that plays the same role that Riemann's minimal examples play in Euclidean space, and whose fundamental piece is a graph over a spacelike plane, having one planar end and two conelike singularities:



We recall the Weierstrass representation of one-half of a fundamental piece of such surfaces. For $r \in]1, \infty[$, consider the four-punctured torus

$$\mathcal{N} = \{(z, w) \in \mathbb{C}^* \times \mathbb{C} \mid w^2 = z(z^2 + 2rz + 1)\}$$

and define in the z -plane

$$s_0 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad s_1 = [r_1, 0[\times \{0\}, \quad s_2 =]-\infty, r_2] \times \{0\},$$

where $r_1 = -r + \sqrt{r^2 - 1}$ and $r_2 = -r - \sqrt{r^2 - 1}$. Observe that $r_2 < -1 < r_1 < 0$. Define $N \subset \mathcal{N}$ as the connected component of $z^{-1}(\mathbb{C} - \bigcup_{i=0}^2 s_i)$ containing the point

$$\left(\frac{1}{2}, \sqrt{\frac{5}{8} + \frac{1}{2}r}\right).$$

Finally set $M = \bar{N}$, the closure of N in \mathcal{N} .

For brevity, when $z(z^2 + 2rz + 1) \in \mathbb{R}^+$, we set

$$z_+ = (z, +\sqrt{z(z^2 + 2rz + 1)}), \quad z_- = (z, -\sqrt{z(z^2 + 2rz + 1)}).$$

On M we consider the Weierstrass data $g = z$ and $\Phi_3 = dz/w$ and the 1-forms Φ_j , $j = 1, 2$ given by (2-2). The lift γ of s_0 to M generates $\mathcal{H}_1(M, \mathbb{Z})$. It is not difficult to see that Φ_1 is exact and that Φ_2, Φ_3 have no real periods on γ , so we can consider the maximal immersion $X = (X_1, X_2, X_3) = \operatorname{Re} \int_{z_0}^z (\Phi_1, \Phi_2, \Phi_3)$.

Denote by γ_1 the lift to M of s_1 . It is not hard to prove that $X(\gamma_1)$ is a line parallel to $\{x_2 = x_3 = 0\}$. The set of singularities of the immersion is the trace of γ , and the image of these points under the immersion X is a single point, which we label P^r . We can choose z_0 so that $X(\gamma_1) = \{x_2 = x_3 = 0\}$ and $P^r = (0, P_2^r, P_3^r)$.

Let $\Theta(r) \in [-\pi, \pi[$ be the angle between $(0, 1, 0)$ and the vector P^r , given by

$$\cos \Theta(r) = \frac{P_2^r}{\sqrt{(P_2^r)^2 + (P_3^r)^2}}, \quad \sin \Theta(r) = \frac{P_3^r}{\sqrt{(P_2^r)^2 + (P_3^r)^2}}.$$

To use the surfaces of this family as barriers, we study the function $\Theta(r)$. We have $\Theta(r) = \arctan(h(r)/d(r))$, where

$$h(r) = X_3(-1_-) - X_3(r_1) = X_3(1_+) - X_3(0), \quad d(r) = X_2(-1_-) - X_2(r_1).$$

Hence

$$(2-6) \quad \begin{aligned} h(r) &= \operatorname{Re} \int_{r_1}^{-1_-} \Phi_3 = \int_{-1}^{r_1} \frac{dt}{\sqrt{t(t^2 + 2rt + 1)}}, \\ h(r) &= \operatorname{Re} \int_0^{1_+} \Phi_3 = \int_0^1 \frac{dt}{\sqrt{t(t^2 + 2rt + 1)}}, \\ d(r) &= \operatorname{Re} \int_{r_1}^{-1_-} \Phi_2 = -\frac{1}{2} \int_{-1}^{r_1} \frac{(1+t^2) dt}{t\sqrt{t(t^2 + 2rt + 1)}}. \end{aligned}$$

Since h and d are positive functions, $\Theta(r)$ lies in $]0, \frac{\pi}{2}[$. Moreover,

$$(2-7) \quad d(r) = rh(r) + I(r),$$

where

$$(2-8) \quad I(r) = \frac{1}{2} \int_{-1}^{r_1} \frac{\sqrt{t(t^2 + 2rt + 1)} dt}{t^2}.$$

From (2-7) and (2-8) we see that $\lim_{r \rightarrow 1} \Theta(r) = \frac{\pi}{4}$ and $\lim_{r \rightarrow +\infty} \Theta(r) = 0$. From (2-6) we observe that

$$(2-9) \quad h'(r) = \int_0^1 \frac{-t^2 dt}{(t(t^2 + 2rt + 1))^{3/2}}.$$

On the other hand, from (2-7) and (2-8) the derivative of d respect to r is

$$(2-10) \quad d'(r) = rh'(r) + \frac{3}{2}h(r).$$

According to (2-7) and (2-10) we have

$$\Theta'(r) = \frac{h'(r)d(r) - h(r)d'(r)}{h(r)^2 + d(r)^2} = \frac{I(r)h'(r) - \frac{3}{2}h(r)^2}{h(r)^2 + d(r)^2}.$$

Taking into account (2-8) and (2-9) we get $\Theta'(r) < 0$, so Θ is a one-to-one function $\Theta :]1, \infty[\rightarrow]0, \frac{\pi}{4}[$.

For $\delta \in]0, \frac{\pi}{4}[$ we shall denote by R_δ the maximal surface with boundary defined in \mathbb{L}^3 by the above immersion for $r = \Theta^{-1}(\delta)$ (see figure at the top of next page). We also set

$$\mathcal{R} = \{R_\delta \mid \delta \in]0, \frac{\pi}{4}[\}$$

Finally, we need to prove that $R_\delta \subset \Pi_\delta^- \cap \{x_3 \geq 0\}$. It is not difficult to see that the point $\{0\}$ is a planar end of the surface asymptotic to the plane $\{x_3 = 0\}$. Therefore,

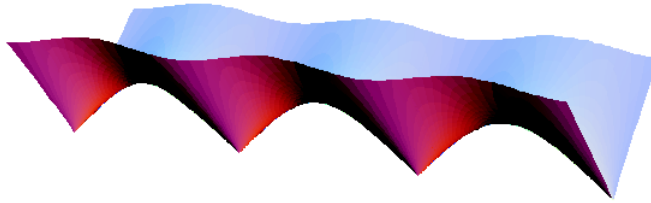


R_δ for $\delta = 0.595881$.

X_3 is bounded on M . From Corollary 2.8 we deduce that $R_\delta \subset \{x_3 \geq 0\}$. Moreover, from the above facts there exists $t > 0$ such that $R_\delta \subset \Pi_{\delta t}^-$. The maximum principle allows us to assert that $R_\delta \subset \Pi_\delta^-$.

Maximal surfaces of Scherk type. This family of singly periodic maximal surfaces of Scherk type was studied in depth in [Fernández and López 2004a], although an example had already appeared in [Kobayashi 1984]. For $b \in]0, 1[$, consider the maximal surface given on $\mathbb{D} - \{b, -b\}$ by the Weierstrass data $g(z) = iz$ and

$$\Phi_3(z) = \frac{z dz}{(z^2 - b^2)(b^2 z^2 - 1)}.$$



The surface is a graph over a spacelike plane, it is invariant under translation by $(0, \pi/(2b(b^2 + 1)), 0)$, and each fundamental piece of it has a conelike singularity. Up to translation we can assume that one of these singularities is at $(0, 0, 0)$, and then all the conelike singularities lie on the line $\{x_1 = x_3 = 0\}$. The ends are asymptotic to the totally geodesic horizontal half-cylinder ∂W_δ , where

$$W_\delta = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -\tan \delta x_1 + x_3 \geq 0, \tan \delta x_1 + x_3 \geq 0\},$$

for $\delta = \arctan(2b/(1 + b^2)) \in]0, \frac{\pi}{4}[$; for this reason we denote this Scherk-type surface by S_δ . By Corollary 2.8, S_δ lies entirely in W_δ .

The convex hull property. We now prove that a compact maximal surface, even one with isolated singularities, satisfies the convex hull property, that is, it lies in the convex hull of its boundary plus singularities. We will need the following version of the maximum principle for maximal surfaces with singularities. We would like to point out that the proof is inspired in the work [Gulliver et al. 1973].

Proposition 2.9. *Let $X : D \rightarrow \mathbb{L}^3$ be a maximal immersion with an isolated singular point in $q \in D$. Set $p = X(q)$ and let S be an embedded maximal surface (without singularities) in \mathbb{L}^3 with $p \in S$.*

- (i) If $X(D)$ is above S , then p is a downward pointing lightlike singularity.
- (ii) If $X(D)$ is below S , then p is an upward pointing lightlike singularity.

Proof. We prove (i); the proof of part (ii) is similar. Suppose first that p is a lightlike singularity but not downward pointing. Denote by Π the tangent plane to S at p that is a spacelike plane. From Lemma 2.2 we obtain a curve in $X(D)$ asymptotic to $C^-(p)$. Since S is asymptotic to Π in a neighborhood of p we deduce that there are points of $X(D)$ below S and this contradicts our assumptions.

Now assume p is a spacelike singularity. By Lemma 2.4, the tangent plane to $X(D)$ at p coincides with the tangent plane to S at p . Denote this plane by Π and by π the Lorentzian orthogonal projection on Π . Up to a Lorentzian isometry we can suppose that $p = (0, 0, 0)$ and $\Pi = \Pi_0$. Consider a disk Δ in Π centered at $(0, 0)$ such that S is the graph of a function h on Δ and $\Delta \subset \pi(X(D))$. Set $M = X(V)$, where V is the connected component of $(\pi \circ X)^{-1}(\Delta)$ containing q . If $\partial(M) \cap h(\Delta) \neq \emptyset$, we have an interior regular point in $X(D) \cap S$. By applying the maximum principle we obtain $M = h(\Delta)$ and then $h(\Delta)$ must contain a spacelike singularity. Taking into account Remark 2.5 we get a contradiction.

Suppose then that ∂M lies strictly above $h(\Delta)$. Then there exists $\theta \in]-\frac{\pi}{4}, 0[$ sufficiently small and f an hyperbolic rotation in \mathbb{L}^3 such that $f(\Pi) = \Pi_\theta$ and $f(\partial M)$ remains strictly above $h(\Delta)$. Note that the tangent plane to the maximal surface $f(M)$ at $(0, 0, 0)$ is Π_θ and thus we can assert that there are points of $f(M)$ below $h(\Delta)$. Translating in the positive x_3 -direction, we find a last contact point with $h(\Delta)$ which must be an interior regular point. As in the previous case, by using the maximum principle we obtain that $h(\Delta)$ coincides with the translate of $f(M)$ by some vector $(0, 0, t_0)$, for $t_0 > 0$. From Lemma 2.6 we see that $\pi^{-1}(0, 0, 0)$ intersects this translate at $(0, 0, t_0)$. But this implies $(0, 0, t_0) = (0, 0, 0)$, again a contradiction. \square

Proposition 2.10. *Let M be a compact maximal surface with isolated singularities. Then M lies in the convex hull of ∂M and its general conelike singularities.*

Proof. Let A be the set of general conelike singularities. If M is contained in a plane the result is obvious. Assume M is not flat and consider $v \in \mathbb{S}^2$ and $y \in \mathbb{R}^3$ such that $(\partial(M) \cup A) \subset H^+(y, v)$. We prove that $M \subset H^+(y, v)$.

We proceed by contradiction. Suppose that $M \cap (H^-(y, v) - H(y, v)) \neq \emptyset$. Let M' be a connected component of $M \cap H^-(y, v)$; then M' does not contain general conelike singularities.

First, assume v is a timelike vector, that is, $H(y, v)$ is a spacelike plane. There exists an interior point $p \in M'$ such that M' is contained in the slab determined by the parallel planes $H(p, v)$ and $H(y, v)$. Therefore we can use Proposition 2.9 to infer that p is a regular point of M' . Using the maximum principle we find $M' = H(p, v)$, a contradiction.

Analogously, if v is either spacelike or lightlike, we can deduce the existence of an interior point $p \in M'$ such that M' is contained in the slab determined by the parallel planes $H(p, v)$ and $H(y, v)$. If p were a spacelike singularity, Lemma 2.4 implies $H(p, v)$ is the tangent plane to M' at p , contradicting $|g(q)| < 1$. Assume p is a lightlike singularity. Up to a Lorentzian isometry we can assume that $p = (0, 0, 0)$ and

- $H(p, v) = \Sigma_\theta$ and $M' \subset \Sigma_\alpha^-$ if v is spacelike,
- $H(p, v) = \Pi_{\pi/4}$ and $M' \subset \Pi_{\pi/4}^+$ if v is lightlike.

By Lemma 2.6, M' is in the exterior of $C((0, 0, 0))$. Consider π , the Lorentzian orthogonal projection onto Π_0 . It is easy to prove that the preceding conditions imply that $\pi(M') \subset (\Pi_0 - \{(0, y, 0) \mid y \in \mathbb{R}\})$ in a neighborhood of $(0, 0, 0)$. This contradicts Lemma 2.1. Therefore, since p is not a singular point we infer that $H(y, v)$ is the tangent plane to M' at p , in contradiction with the fact that M is spacelike. \square

Remark 2.11. Proposition 2.10 holds even if M cannot be extended to an open maximal surface M' .

3. Maximal surfaces whose boundary is contained in a timelike plane

Recall that we defined

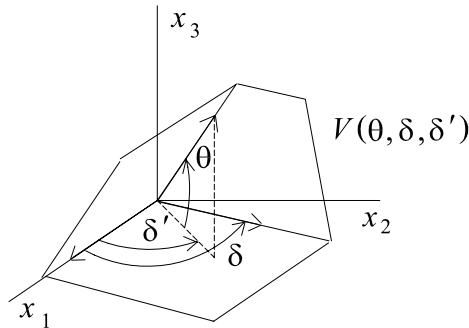
$$C^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 - x_3^2 \leq 0, x_3 \geq 0\}.$$

For $\theta \in]0, \frac{\pi}{4}[$ and $\delta, \delta' \in]0, \pi[$, we also define the convex region

$$V(\theta, \delta, \delta') = \Pi_0^+ \cap \Pi_\theta^- \cap H^+((0, 0, 0), (1, -\cot \delta, \cot \theta(\cot \delta - \cot \delta'))).$$

This is the convex hull of the half-lines with origin in $(0, 0, 0)$ and directions $(1, 0, 0)$, $(\cot \delta, 1, 0)$ and $(\cot \delta', 1, \tan \theta)$ (see figure).

Let τ_t denote the translation along the vector $(0, 0, t)$, where $t \in \mathbb{R}$.



Proposition 3.1. *Let $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$ be arbitrary. Suppose M is a connected, properly immersed maximal surface contained in $\Pi_{\pi/4}^+ \cap \Sigma_\alpha^+$, such that $\partial M \subset \Sigma_\alpha$. If there is a point $p_0 \in \partial M$ minimizing x_3 on ∂M , there must be some downward pointing lightlike singularity in the interior of M .*

Proof. After applying an orthochronous hyperbolic rotation f_s of \mathbb{L}^3 , we can assume $\alpha = 0$.

Let M be as in the hypothesis of the proposition and define $\hat{t} = x_3(p_0)$. For any $\theta \in [0, \frac{\pi}{4}[$, consider the set

$$\mathcal{F}_\theta = \{t \in [0, \hat{t}] \mid M \subset \Pi_{\theta t}^+\}.$$

Since \mathcal{F}_θ contains 0, it is nonempty. Suppose there is no singularity in $\text{Int } M$ as in the conclusion; we shall prove that $\mathcal{F}_\theta = [0, \hat{t}]$, and from there we will derive a contradiction.

We may assume $\hat{t} > 0$, the case $\hat{t} = 0$ being obvious. Clearly \mathcal{F}_θ is closed; we show that it is open. If $t \in \mathcal{F}_\theta$ then $[0, t] \subset \mathcal{F}_\theta$. We claim that if $t \in \mathcal{F}_\theta \cap [0, \hat{t}[$, there exists $\varepsilon > 0$ such that $[t, t+\varepsilon[\subset \mathcal{F}_\theta$. If not, we have two possibilities: either

- (a) there is an interior point p of M in the plane $\Pi_{\theta t}$, or
- (b) M is asymptotic to $\Pi_{\theta t}$ at infinity.

In case (a), p is not a singularity, for if it were, it would be downward pointing by Proposition 2.9(i), contrary to assumption. But if p is not a singularity, the interior maximum principle implies that M and $\Pi_{\theta t}$ coincide, in contradiction with the inequality $x_3(p_0) = \hat{t} > t$.

In case (b), we can assume $M \cap \Pi_{\theta t} = \emptyset$; otherwise there exists an interior point of M in $\Pi_{\theta t}$ and we may apply the previous argument. Consider an orthochronous isometry f of \mathbb{L}^3 such that $f(\Pi_{\theta t}) = \Pi_0$ and $f(\Pi_{\theta t}^+) = \Pi_0^+$. Set $\tilde{M} = f(M)$. Then we have a properly immersed maximal surface $\tilde{M} \subset \Pi_0^+$ asymptotic to Π_0 and disjoint from it.

Since the immersion is proper and $(0, 0, 0) \notin \tilde{M}$ we can find $\epsilon > 0$ sufficiently small so that the ball $B(\epsilon)$ of radius ϵ around $(0, 0, 0)$ is disjoint from \tilde{M} . Hence, there exists $\epsilon' \in]0, \epsilon[$ and $a_0 > 0$ small enough such that $\tau_{\epsilon'}(C_{a_0}) \subset B(\epsilon) \cup \Pi_0^-$. Now define

$$A = \{a \in]0, a_0] \mid \tau_{\epsilon'}(C_a) \cap \tilde{M} = \emptyset\}.$$

Clearly, $a_0 \in A$ and we can consider the infimum a' of A . We claim that $a' = 0$. Assume on the contrary that $a' > 0$. Since $\tau_{\epsilon'}(C_a)$ and \tilde{M} do not have a contact at infinity, there exists an interior point p of \tilde{M} in $\tau_{\epsilon'}(C_{a'})$. Taking into account Proposition 2.9(i) and the assumed absence of singularities, we see that p is a regular point of \tilde{M} . Applying the maximum principle, we obtain $\tilde{M} = \tau_{\epsilon'}(C_{a'})$, in contradiction with $\partial \tilde{M} \subset \Sigma_0 \cap \Pi_0^+$.

We have shown that if the conclusion of the theorem fails, then \mathcal{F}_θ is open and $\mathcal{F}_\theta = [0, \hat{t}]$. Hence $M \subset \Pi_{\theta \hat{t}}^+$ for all $\theta \in [0, \frac{\pi}{4}[$ and thus $M \subset \Pi_{\pi/4, \hat{t}}^+$. Now consider p_0 : if it is *not* a singular point in ∂M , the fact that $p_0 \in \partial(M) \cap \Pi_{\pi/4, \hat{t}}$ and $M \subset \Pi_{\pi/4, \hat{t}}^+ \cap \Sigma_0^+$ implies that the tangent plane p_0 is lightlike or timelike, which is a contradiction. If instead p_0 is a singular point, Lemma 2.6 implies that around p_0 the surface is in the exterior of $C(p_0)$, which again contradicts $M \subset \Pi_{\pi/4, \hat{t}}^+$. \square

Corollary 3.2. *There exists no connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior and such that $M \subset C^+ \cap \Sigma_\alpha^+$ and $\partial M \subset \Sigma_\alpha$, for any $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.*

Proof. This follows immediately from Proposition 3.1. \square

Corollary 3.3. *There exists no connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior and such that $M \subset C^+$ and ∂M lies in the intersection of C^+ with a timelike plane P .*

Proof. Assume that there exists such a maximal surface and consider a connected component M' of $M - (M \cap P)$. Up to an elliptic rotation and a translation we can assume the timelike plane is the plane Σ_α , for $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$, $M' \subset C^+ \cap \Sigma_\alpha^+$ and $\partial M' \subset \Sigma_\alpha$. An immediate application of Corollary 3.2 to M' leads to a contradiction. \square

Theorem 3.4. *There exists no connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior and such that $M \subset W_\delta \cap \Sigma_\alpha^+$ and $\partial M \subset \Sigma_\alpha \cap C^+$, for $\delta \in]0, \frac{\pi}{4}[$, $\alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.*

Proof. Consider the isometry f_ε of \mathbb{L}^3 with $\tanh \varepsilon = \tan \frac{\pi}{8}$. It is not difficult to see that $f_\varepsilon(M) \subset \Pi_{\pi/8}^+ \cap W_{\delta'} \cap \Sigma_{\alpha'}^+$ and $\partial f_\varepsilon(M) \subset \Sigma_{\alpha'} \cap C^+$, where

$$\tan \alpha' = \frac{\tan \alpha + \tan \frac{\pi}{8}}{\tan \frac{\pi}{8} \tan \alpha + 1},$$

$$\tan \delta' = \min\{\tan \frac{\pi}{8}, \cosh \varepsilon \tan \delta (\tan \frac{\pi}{8} \tan \alpha + 1)\}.$$

For simplicity of notation we consider $M \subset \Pi_{\pi/8}^+ \cap W_\delta \cap \Sigma_\alpha^+$ and $\partial M \subset \Sigma_\alpha \cap C^+$.

We claim that $(0, 0, 0) \notin \partial M$ and so $\partial M \subset \Pi_0^+ - \Pi_0$. If not, we deduce from Lemma 2.6 that around $(0, 0, 0)$ the maximal surface M is in the exterior of $C((0, 0, 0))$, but this contradicts $\partial M \subset C^+$.

Now consider the Scherk-type maximal surface $S_{\delta/2}$ (page 13) asymptotic to the boundary of the region $W_{\delta/2}$. We will prove that $S_{\delta/2} \cap M = \emptyset$. It is clear that there exist $t_0 \in]-\infty, 0]$ and $t_1 \in]0, \infty[$ such that $\tau_{t_0}(S_{\delta/2}) \cap M = \emptyset$ and $\tau_{t_1}(S_{\delta/2}) \cap M \neq \emptyset$. Therefore, we can define

$$\hat{t} = \infimum\{t \in]t_0, \infty[\mid \tau_t(S_{\delta/2}) \cap M \neq \emptyset\}.$$

Suppose $\hat{t} \leq 0$. Observe that, since $M \subset \Pi_{\pi/8}^+ \cap W_\delta \cap \Sigma_\alpha^+$, $\partial M \subset C^+ - \{(0, 0, 0)\}$ and $S_{\delta/2} \cap C^+ = \{(0, 0, 0)\}$, then $\tau_{\hat{t}}(S_{\delta/2})$ and M can have a contact point neither at infinity nor at the boundary. Hence there exists an interior point of M in $\tau_{\hat{t}}(S_{\delta/2})$. Taking into account our assumptions on the singularities and Proposition 2.9(i) we deduce that this point is not a singularity. Then, by applying the maximum principle we get that M and $\tau_{\hat{t}}(S_{\delta/2})$ coincide which contradicts the hypothesis on ∂M . Thus $\hat{t} > 0$ and $S_{\delta/2} \cap M = \emptyset$.

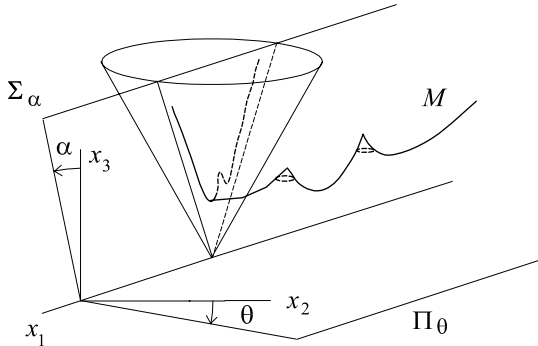
Now consider $S_{\delta/2}^\lambda$, the homothetic shrinking of $S_{\delta/2}$ by λ , $\lambda \geq 1$. We shall prove that $S_{\delta/2}^\lambda \cap M = \emptyset$ for all $\lambda \geq 1$. Suppose on the contrary that there exists $\lambda' \geq 1$ such that $S_{\delta/2}^{\lambda'} \cap M \neq \emptyset$. We set

$$\hat{\lambda} = \text{infimum}\{\lambda \in]1, \infty[\mid S_{\delta/2}^\lambda \cap M \neq \emptyset\}.$$

Clearly $S_{\delta/2}^{\hat{\lambda}}$ and M do not touch either at infinity or at the boundary. Therefore there must exist an interior point of M in $S_{\delta/2}^{\hat{\lambda}}$. Again using Proposition 2.9(i) and our hypothesis on the singularities we deduce that this point is not a singularity and so by applying the maximum principle we obtain that $S_{\delta/2}^{\hat{\lambda}}$ and M coincide. But this contradicts our assumptions on ∂M .

Thus $S_{\delta/2}^\lambda \cap M = \emptyset$ for all $\lambda \geq 1$. Taking into account that $S_{\delta/2}$ is asymptotic to $C^+(\{(0, 0, 0)\})$ near the conelike singularity $(0, 0, 0)$, we deduce that $M \subset C^+$ and the Corollary 3.2 finishes the proof. \square

Theorem 3.5. *There exist no connected properly immersed maximal surface M without downward pointing lightlike singularities in the interior and such that $M \subset \Pi_\theta^+ \cap \Sigma_\alpha^+$ and $\partial M \subset \Sigma_\alpha \cap C^+$, for $\theta, \alpha \in]-\frac{\pi}{4}, \frac{\pi}{4}[$.*



Proof. Suppose there exists such an M . We observe that if $\theta \leq 0$ or $\alpha < 0$ we can consider an orthochronous hyperbolic rotation f_s such that $f_s(M) \subset \Pi_{\theta'}^+ \cap \Sigma_{\alpha'}^+$ and $\partial f_s(M) \subset \Sigma_{\alpha'} \cap C^+$ for some $\theta' \in]0, \frac{\pi}{4}[$, $\alpha' \in [0, \frac{\pi}{4}[$. As in the previous theorem, for the sake of simplicity of notation we assume $M \subset \Pi_\theta^+ \cap \Sigma_\alpha^+$ and $\partial M \subset \Sigma_\alpha \cap C^+$, for $\theta \in]0, \frac{\pi}{4}[$, $\alpha \in [0, \frac{\pi}{4}[$.

Since $\partial M \subset \Sigma_\alpha \cap C^+$ we have that there exists $p_0 \in \partial M$ such that $x_3(p_0) \leq x_3(p)$ for all $p \in \partial M$. As in the preceding theorem it is easy to see that $p_0 \neq (0, 0, 0)$ and so $\lambda = x_3(p_0) > 0$. Then, reasoning as in Proposition 3.1 we can conclude $M \subset \Pi_{\theta\lambda}^+ \cap \Sigma_\alpha^+$.

Denote by \tilde{R}_δ the Riemann type maximal example that results after applying an elliptic rotation of $\frac{\pi}{2}$ along the axis x_3 on R_δ for any $\delta \in]0, \frac{\pi}{4}[$. We assert that $M \cap \tilde{R}_\delta = \emptyset$. Observe that we can consider $t_0 \leq 0$ and $t_1 \in \mathbb{R}$ such that $\tau_{t_0}(\tilde{R}_\delta) \cap M = \emptyset$ and $\tau_{t_1}(\tilde{R}_\delta) \cap M \neq \emptyset$. Now define

$$\hat{t} = \infimum\{t \in]t_0, \infty[\mid \tau_t(\tilde{R}_\delta) \cap M \neq \emptyset\}.$$

Suppose $\hat{t} \leq 0$. Note that $\tau_{\hat{t}}(\tilde{R}_\delta)$ and M can have a contact point neither at infinity nor at the boundary. Hence there exists an interior point of M in $\tau_{\hat{t}}(\tilde{R}_\delta)$. Making use of Proposition 2.9(i) and taking into account our assumptions on singularities, we deduce that the point is not a singularity. Therefore, by applying the maximum principle we get that $\tau_{\hat{t}}(\tilde{R}_\delta)$ and M coincide. But this contradicts our hypothesis on ∂M . Thus $\hat{t} > 0$ and $\tilde{R}_\delta \cap M = \emptyset$.

Consider now \tilde{R}_δ^λ the homothetic shrinking of \tilde{R}_δ by λ , $\lambda > 0$. Next we prove that $\tilde{R}_\delta^\lambda \cap M = \emptyset$ for all $\lambda \geq 1$. Assume that there exists $\lambda' > 1$ such that $\tilde{R}_\delta^{\lambda'} \cap M \neq \emptyset$. We define

$$\hat{\lambda} = \infimum\{\lambda \in]1, \lambda'[\mid \tilde{R}_\delta^\lambda \cap M \neq \emptyset\}.$$

Observe that $\tilde{R}_\delta^{\lambda'}$ and M do not touch either at infinity or at the boundary for all $\lambda \geq 1$. Therefore there is an interior point of M in $\tilde{R}_\delta^{\lambda'}$. Using again our assumptions on singularities and Proposition 2.9(i) we deduce that the point is not a singularity. Then by applying the maximum principle we obtain that $\tilde{R}_\delta^{\lambda'}$ and M coincide, which contradicts our hypothesis on ∂M . The same argument proves that $\tilde{R}_\delta^\lambda \cap M = \emptyset$ for all $\lambda \leq 1$.

Analogously, considering \hat{R}_δ the Riemann type maximal example that results after applying a rotation of $-\frac{\pi}{2}$ along the axis x_3 on R_δ for any $\delta \in]0, \frac{\pi}{4}[$, we obtain $\hat{R}_\delta^\lambda \cap M = \emptyset$ for all $\lambda \in \mathbb{R}$.

Furthermore, it is not difficult to prove that

$$\begin{aligned} (\Pi_\theta^+ \cap \Sigma_\alpha^+) - \left(\bigcup_{\lambda \in \mathbb{R}} \tilde{R}_\delta^\lambda \cup \bigcup_{\lambda \in \mathbb{R}} \hat{R}_\delta^\lambda \right) &\subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -\tan \delta x_1 + x_2 + x_3 \geq 0\} \\ &\cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \tan \delta x_1 + x_2 + x_3 \geq 0\}. \end{aligned}$$

Taking this into account, we can assert

$$M \subset (\Pi_\theta^+ \cap \Sigma_\alpha^+) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -\tan \delta x_1 + x_2 + x_3 \geq 0, \tan \delta x_1 + x_2 + x_3 \geq 0\}.$$

A direct computation shows that

$$\Pi_\theta^+ \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -\tan \delta x_1 + x_2 + x_3 \geq 0, \tan \delta x_1 + x_2 + x_3 \geq 0\} \subset W_{\delta'},$$

where $\delta' \in]0, \frac{\pi}{4}[$ is given by $\tan \delta' = \tan \delta \tan \theta / (1 + \tan \theta)$. Then Theorem 3.4 concludes the proof. \square

To finish this section, we analyze the case of maximal surfaces whose boundary is contained in a timelike plane but not necessarily in C^+ .

Proposition 3.6. *There exists no connected properly immersed maximal surface M with at least one connected component of ∂M contained in the intersection $\Sigma_\alpha^+ \cap \Sigma_{-\alpha}^+ \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$, where $\alpha \in]0, \frac{\pi}{4}[$.*

Proof. Let B a connected component of ∂M satisfying

$$B \subset \Sigma_\alpha^+ \cap \Sigma_{-\alpha}^+ \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\},$$

where $\alpha \in]0, \frac{\pi}{4}[$. The function x_2 has at least one minimum on B , and this minimum cannot be a singularity. Then, the tangent vector to the boundary at this point is vertical and therefore the tangent plane of the maximal surface at this point is timelike, which is contrary to our assumptions. \square

Corollary 3.7. *There exists no connected properly immersed maximal surface M contained in $V(\theta, \delta, \delta')$ with ∂M contained in a timelike plane.*

Proof. From the hypothesis it is not difficult to see that there exists an isometry of \mathbb{L}^3 that sends the timelike plane to the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$ and in particular the image of ∂M lies in $\Sigma_\alpha^+ \cap \Sigma_{-\alpha}^+ \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0\}$ for some $\alpha \in]0, \frac{\pi}{4}[$. The result is then a consequence of Proposition 3.6. \square

4. Maximal surfaces whose boundary is contained in a spacelike plane

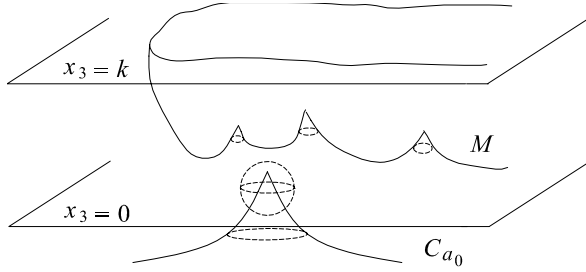
We now obtain, using the maximum principle, other results about maximal surfaces whose boundary is contained in a spacelike plane but that cannot be inferred from the theorem of Fernández and López quoted on page 9. We start with a result similar to Corollary 2.8.

Proposition 4.1. *Let M be a connected properly immersed maximal surface without downward pointing lightlike singularities in the interior such that*

$$M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_3 \leq k\} \quad \text{and} \quad \partial M \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = k\},$$

for $k > 0$. Then M is a planar region.

Proof. We proceed by contradiction. Assume that there exists $t \geq 0$ such that $M \subset \Pi_{0t}^+$ but $M \not\subset \Pi_{0t'}^+$ for any $t < t'$. We claim that $M \cap \Pi_{0t} = \emptyset$. If not, there exists an interior point p of M in Π_{0t} . Then, from Proposition 2.9(i) and our assumptions on singularities, we deduce that p is a regular point. It then follows from the maximum principle that $M = \Pi_{0t}$, contradicting the hypothesis on ∂M .



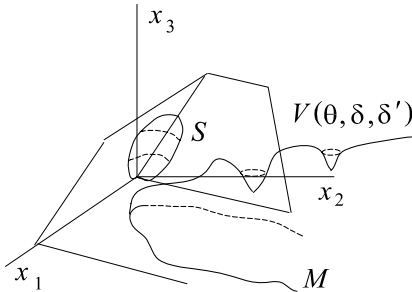
Since M is properly immersed and $M \cap \Pi_{0t} = \emptyset$ we can find $\varepsilon > 0$ such that the ball of radius ε about $(0, 0, t)$ does not meet M . Hence, there are constants $\varepsilon' \in]0, \varepsilon[$ and $a_0 > 0$ sufficiently small such that $\tau_{t+\varepsilon'}(C_{a_0})$ is contained in Π_{0t}^- and also in the ball of radius ε around $(0, 0, t)$. Now define

$$A = \{a \in]0, a_0] \mid \tau_{t+\varepsilon'}(C_a) \cap M = \emptyset\}.$$

Clearly, $a_0 \in A$ and we can consider the infimum a' of A . We claim $a' = 0$. Assume on the contrary that $a' > 0$. Then as $\tau_{t+\varepsilon'}(C_a)$ and M do not have a contact either at infinity or at the boundary, we infer that there is an interior point of M in $\tau_{t+\varepsilon'}(C_{a'})$. Taking into account Proposition 2.9(i) and our assumptions on the singularities we infer that the interior point is not a singularity and then, by applying the interior maximum principle we obtain $M = \tau_{t+\varepsilon'}(C_{a'})$ which contradicts the hypothesis on ∂M .

Therefore, $a' = 0$ and so $M \subset \Pi_{0t+\varepsilon'}^+$ contradicting our assumption at the beginning of the proof. \square

Theorem 4.2. *Let M be a connected properly immersed maximal surface without upward pointing lightlike singularities in the interior such that $M \subset V(\theta, \delta, \delta')$ and $\partial M \subset \Pi_0$. Then M is a planar region.*



Proof. Up a translation we can assume that

$$M \cap (\Pi_\theta \cup H((0, 0, 0), (1, -\cot \delta, \cot \theta(\cot \delta - \cot \delta')))) = \emptyset.$$

Now we observe that

$$\begin{aligned}\tau_{-1}(\mathbb{H}_+^2) \cap \Pi_\theta \cap V(\theta, \delta, \delta') &= \alpha_1, \\ \tau_{-1}(\mathbb{H}_+^2) \cap H((0, 0, 0), (1, -\cot \delta, \cot \theta(\cot \delta - \cot \delta'))) \cap V(\theta, \delta, \delta') &= \alpha_2,\end{aligned}$$

where α_1 and α_2 are regular curves. The union of these curves is a continuous curve of $\tau_{-1}(\mathbb{H}_+^2)$ and the tangent vectors to these curves at the point $(0, 0, 0)$ are contained in the plane Π_0 and are linearly independent.

Since $\tau_{-1}(\mathbb{H}_+^2)$ is a spacelike surface, it is well-known (see Theorem 4.1 in [Bartnik and Simon 1982/83]) that there exists S a maximal surface (it is even a graph on the x_3 -plane) spanned by the curve $\alpha_1 \cup \alpha_2$. Note that the tangent plane of S at $(0, 0, 0)$ is the plane Π_0 . On the other hand, using Proposition 2.10 and Remark 2.11, we see that S is contained in the convex hull of its boundary and thus $S \subset V(\theta, \delta, \delta')$.

Now, we denote by S^λ the homothetic shrinking of S by λ , $\lambda > 0$. As M is properly immersed it is possible to find $\lambda_0 > 0$ such that $S^{\lambda_0} \cap M = \emptyset$. Next we prove that $S^\lambda \cap M = \emptyset$ for all $\lambda > 0$. Assume that there exists $\lambda' > 0$ such that $S^{\lambda'} \cap M \neq \emptyset$. We denote by

$$\hat{\lambda} = \infimum\{\lambda \in]\lambda_0, \lambda'] \mid S^\lambda \cap M \neq \emptyset\}.$$

Observe that S^λ and M do not contact at the boundary for all $\lambda > 0$. Therefore there is an interior point of M in $S^{\hat{\lambda}}$. It follows from Proposition 2.9(ii) and the conditions on the singularities that the contact point is not a singularity. Then, by applying the maximum principle we obtain $S^{\hat{\lambda}} = M$ which contradicts the assumptions on ∂M .

Hence, taking into account that the tangent plane of S at $(0, 0, 0)$ is Π_0 we obtain

$$V(\theta, \delta, \delta') - \bigcup_{\lambda \in \mathbb{R}} S^\lambda \subset \Pi_0,$$

from which we deduce that $M \subset \Pi_0$. □

Corollary 4.3. *Let M be a connected properly immersed maximal surface without general conelike singularities in the interior such that $M \subset V(\theta, \delta, \delta')$ and ∂M is contained in a spacelike plane. Then M is a planar region.*

Proof. Let Π be the spacelike plane such that $\partial M \subset \Pi$ and M' a connected component of $M - (M \cap \Pi)$. Denote by Π^+ the half-space determined by Π such that $M' \subset \Pi^+$. Then, it is not difficult to see that there exists an isometry of \mathbb{L}^3 , f , that verifies $f(\Pi) = \Pi_0$ and $f(V(\theta, \delta, \delta') \cap \Pi^+) \subset V(\hat{\theta}, \hat{\delta}, \hat{\delta}')$, for some $\hat{\theta}$, $\hat{\delta}$ and $\hat{\delta}'$. Therefore, the corollary follows from Theorem 4.2. □

By contrast:

Proposition 4.4. *Let M be a connected properly immersed maximal surface without general conelike singularities in the interior such that $M \subset C^+$ and ∂M is contained in a spacelike plane. Then M is a planar region.*

Proof. Our hypotheses imply that ∂M is compact. We consider the intersection of M with all the timelike planes $H(y, v)$ such that $\partial M \subset H^+(y, v)$. By applying Corollary 3.3 to the connected components of M contained in $H^-(y, v)$ we obtain that $M \subset H^+(y, v)$ for all the timelike planes described above. Then M is also compact and Proposition 2.10 proves that M is a planar region. \square

5. Maximal surfaces whose boundary is contained in a lightlike plane and the convex hull property

As a consequence of the previous sections we deduce the following results for maximal surfaces whose boundary is contained in a lightlike plane.

Proposition 5.1. *There exists no connected properly immersed maximal surface M without general conelike singularities in the interior and such that $M \subset V(\theta, \delta, \delta')$ and ∂M is contained in a lightlike plane.*

Proof. Suppose there exists such an M . Let Π be the lightlike plane such that $\partial M \subset \Pi$. Then, we can consider the pencil of planes through the line $L = \Pi \cap \Pi_0$, that is, the set of planes sharing the line L . Since M cannot be flat, it is possible to find a spacelike or timelike plane in the pencil that intersects M transversally. But Corollaries 4.3 and 3.7 lead to a contradiction in each case. \square

Proposition 5.2. *There exists no connected properly immersed maximal surface M without general conelike singularities in the interior and such that $M \subset C^+$ and ∂M is contained in a lightlike plane.*

Proof. This can be demonstrated like Proposition 5.1, but using Proposition 4.4 and Corollary 3.3. \square

As we saw in Section 2, a compact maximal surface lies in the convex hull of its boundary and the set of its general conelike singularities. This is not true for noncompact maximal surfaces in general. However, Theorem 4.2 and Proposition 4.4 can be seen as a convex hull type property. We have proved that if certain conditions are satisfied then the surfaces lie in the convex hull of their boundary. In the remainder of the section, we use the results obtained in the previous sections to give a generalization of these results. More precisely:

Proposition 5.3. *Any connected properly immersed maximal surface contained in $V(\theta, \delta, \delta')$ lies in the convex hull of its boundary and its general conelike singularities.*

Proof. Let M be a minimal surface satisfying the hypotheses of the proposition and denote by A the set of general conelike singularities of M . If M is contained in a plane the result is obvious. Assume then that M is not flat and consider $v \in \mathbb{S}^2$ and $y \in \mathbb{R}^3$ such that $(\partial(M) \cup A) \subset H^+(y, v)$. We have to prove that $M \subset H^+(y, v)$ too.

We proceed by contradiction, and suppose that $M \cap (H^-(y, v) - H(y, v)) \neq \emptyset$. Let M' be a connected component of $M \cap H^-(y, v)$.

If v is spacelike, so $H(y, v)$ is a timelike plane, Corollary 3.7 leads to a contradiction.

If v is timelike, so $H(y, v)$ is spacelike, the assumption that M' is not flat contradicts Corollary 4.3.

Finally, if v is lightlike, so is the plane $H(y, v)$, and Proposition 5.1 gives a contradiction. \square

Proposition 5.4. *Any connected properly immersed maximal surface contained in C^+ lies in the convex hull of its boundary and its general conelike singularities.*

Proof. The proof is obtained as for the preceding proposition, using Corollary 3.3 and Propositions 5.2 and 4.4. \square

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