IRREDUCIBLE REPRESENTATIONS FOR THE ABELIAN EXTENSION OF THE LIE ALGEBRA OF DIFFEOMORPHISMS OF TORI IN DIMENSIONS GREATER THAN 1

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We classify the irreducible weight modules of the abelian extension of the Lie algebra of diffeomorphisms of tori of dimension greater than 1, with finite-dimensional weight spaces.

1. Introduction

Let $W_{\nu+1}$ be the Lie algebra of diffeomorphisms of the $(\nu+1)$-dimensional torus. If $\nu = 0$, the universal central extension of the complex Lie algebra $W_1$ is the Virasoro algebra, which, together with its representations, plays a very important role in many areas of mathematics and physics [Belavin et al. 1984; Dotsenko and Fateev 1984; Di Francesco et al. 1997]. The representation theory of the Virasoro algebra has been studied extensively; see, for example, [Kac 1982; Kaplansky and Santharoubane 1985; Chari and Pressley 1988; Mathieu 1992].

If $\nu \geq 1$, however, the Lie algebra $W_{\nu+1}$ has no nontrivial central extension [Ramos et al. 1990]. But $W_{\nu+1}$ has abelian extensions whose abelian ideals are the central parts of the corresponding toroidal Lie algebras; see [Berman and Billig 1999], for example. There is a close connection between irreducible integrable modules of the toroidal Lie algebra and irreducible modules of the abelian extension $\mathcal{L}$; see [Berman and Billig 1999; Eswara Rao and Moody 1994; Jiang and Meng 2003], for instance. In fact, the classification of integrable modules of toroidal Lie algebras and their subalgebras depends heavily on the classification of irreducible representations of $\mathcal{L}$ and its subalgebras. See [Billig 2003] for the constructions of the abelian extensions for the group of diffeomorphisms of a torus.

In this paper we study the irreducible weight modules of $\mathcal{L}$, for $\nu \geq 1$. If $V$ is an irreducible weight module of $\mathcal{L}$, some of whose central charges $c_0, \ldots, c_{\nu}$ are nonzero, one can assume that $c_0, \ldots, c_{N}$ are $\mathbb{Z}$-linearly independent and $c_{N+1} = \cdots = c_{\nu} = 0$, where $N \geq 0$. We prove that if $N \geq 1$, then $V$ must have weight

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spaces which are infinite-dimensional. So if all the weight spaces of \( V \) are finite-dimensional, \( N \) vanishes. We classify the irreducible modules of \( \mathcal{L} \) with finite-dimensional weight spaces and some nonzero central charges. We prove that such a module \( V \) is isomorphic to a highest weight module. The highest weight space \( T \) is isomorphic to an irreducible \( (\mathcal{A}_v+W_v) \)-module all of whose weight spaces have the same dimension, where \( \mathcal{A}_v \) is the ring of Laurent polynomials in \( v \) commuting variables, regarded as a commutative Lie algebra. An important step is to characterize the \( \mathcal{A}_v \)-module structure of \( T \). It turns out that the action of \( \mathcal{A}_v \) on \( T \) is essentially multiplication by polynomials in \( \mathcal{A}_v \). Therefore \( T \) can be identified with Larsson’s construction [1992] by a result in [Eswara Rao 2004]. That is, \( T \) is a tensor product of \( gl_v \)-module with \( \mathcal{A}_v \).

When all the central charges of \( V \) are zero, we prove that the abelian part acts on \( V \) as zero if \( V \) is a uniformly bounded \( \mathcal{L} \)-module. So the result in this case is not complete.

Throughout the paper, \( \mathbb{C}, \mathbb{Z}_+ \) and \( \mathbb{Z}_- \) denote the sets of complex numbers, positive integers and negative integers.

## 2. Basic concepts and results

Let \( \mathcal{A}_{v+1} = \mathbb{C}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \) \((v \geq 1)\) be the ring of Laurent polynomials in commuting variables \( t_0, t_1, \ldots, t_v \). For \( n = (n_1, n_2, \ldots, n_v) \in \mathbb{Z}^v, n_0 \in \mathbb{Z} \), we denote \( t_0^{n_0} t_1^{n_1} \cdots t_v^{n_v} \) by \( t_0^n t^v \). Let \( \mathcal{K} \) be the free \( \mathcal{A}_{v+1} \)-module with basis \( \{ k_0, k_1, \ldots, k_v \} \) and let \( d\mathcal{K} \) be the subspace spanned by all elements of the form

\[
\sum_{i=0}^v r_i t_0^{r_0} t^i k_i, \quad \text{for } (r_0, r) = (r_0, r_1, \ldots, r_v) \in \mathbb{Z}^{v+1}.
\]

Set \( \mathcal{H} = \mathcal{K} / d\mathcal{K} \) and denote the image of \( t_0^{r_0} t^i k_i \) itself. Then \( \mathcal{H} \) is spanned by the elements \( \{ t_0^{r_0} t^r k_p \mid p = 0, 1, \ldots, v, r_0 \in \mathbb{Z}, r \in \mathbb{Z}^v \} \) with relations

\[
\sum_{p=0}^v r_p t_0^{r_0} t^r k_p = 0.
\]

(2-1)

Let \( \mathcal{D} \) be the Lie algebra of derivations on \( \mathcal{A}_{v+1} \). Then

\[
\mathcal{D} = \left\{ \sum_{p=0}^v f_p(t_0, t_1, \ldots, t_v) d_p \mid f_p(t_0, t_1, \ldots, t_v) \in \mathcal{A}_{v+1} \right\},
\]

where \( d_p = t_p \partial / \partial t_p \), \( p = 0, 1, \ldots, v \). From [Berman and Billig 1999] we know that the algebra \( \mathcal{D} \) admits two nontrivial 2-cocycles with values in \( \mathcal{K} \):

\[
\tau_1(t_0^{m_0} t^m d_u, t_0^{m_0} t^m d_b) = -n_u m_b \sum_{p=0}^v m_p t_0^{m_0+n_0} t^{m+n} k_p.
\]
ABELIAN EXTENSION OF LIE ALGEBRA OF DIFFEOMORPHISMS OF $T^n$

Let $\tau = \mu_1 \tau_1 + \mu_2 \tau_2$ be an arbitrary linear combination of $\tau_1$ and $\tau_2$. Then the corresponding abelian extension of $\mathcal{D}$ is

$$\mathcal{L} = \mathcal{D} \oplus \mathcal{H},$$

with the Lie bracket

(2-2)  

$$\begin{align*}
[t^{m_0} a, t^{n_0} b] &= n_a t^{m_0 + n_0} k_p + \delta_{ab} \sum_{p=0}^{v} m_p t^{m_0 + n_0 + n_p} k_p, \\
[t^{m_0} d, t^{n_0} b] &= n_d t^{m_0 + n_0} b_p + n_{kd} t^{m_0 + n_0} d_p + \tau (t^{m_0} d, t^{n_0} b).
\end{align*}$$

The sum

$$\mathfrak{h} = \left( \bigoplus_{i=0}^{v} \mathbb{C} k_i \right) \oplus \left( \bigoplus_{i=0}^{v} \mathbb{C} d_i \right)$$

is an abelian Lie subalgebra of $\mathcal{L}$. An $\mathcal{L}$-module $V$ is called a weight module if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda},$$

where $V_{\lambda} = \{v \in V \mid h \cdot v = \lambda(h) v \text{ for all } h \in \mathfrak{h}\}$. Denote by $P(V)$ the set of all weights. Throughout the paper, we assume that $V$ is an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces. Since $V$ is irreducible, we have

$$k_i | V = c_i,$$

where the constants $c_i$, for $i = 0, 1, \ldots, v$, are called the central charges of $V$.

**Lemma 2.1.** Let $A = (a_{ij})$ $(0 \leq i, j \leq v)$ be a $(v+1) \times (v+1)$ matrix such that $\det A = 1$ and $a_{ij} \in \mathbb{Z}$. There exists an automorphism $\sigma$ of $\mathcal{L}$ such that

$$\begin{align*}
\sigma (t^{m} k_j) &= \sum_{p=0}^{v} a_{pj} t^{m A^T} k_p, \\
\sigma (t^{m} d_j) &= \sum_{p=0}^{v} b_{jp} t^{m A^T} d_p, \quad 0 \leq j \leq v,
\end{align*}$$

where $t^{m} = t^{m_0} t^{m}$, $B = (b_{ij}) = A^{-1}$.

3. The structure of $V$ with nonzero central charges

In this section, we discuss the weight module $V$ which has nonzero central charges. It follows from Lemma 2.1 that we can assume that $c_0, c_1, \ldots, c_N$ are $\mathbb{Z}$-linearly independent, i.e., if $\sum_{i=0}^{N} a_i c_i = 0$, $a_i \in \mathbb{Z}$, then all $a_i (i = 0, \ldots, N)$ must be zero,
and $c_{N+1} = c_{N+2} = \cdots = c_v = 0$, where $N \geq 0$. For $\tilde{m} = (m_0, m)$, denote $t_{0}^{m_0}t_{m}^{}$ by $t_{\tilde{m}}^{}$ as in Lemma 2.1. It is easy to see that $V$ has the decomposition

$$V = \bigoplus_{\tilde{m} \in \mathbb{Z}^{v+1}} V_{\tilde{m}},$$

where $V_{\tilde{m}} = \{ v \in V \mid d_i(v) = (\gamma_0(d_i) + m_i)v, i = 0, 1, \ldots, v \}$, with $\gamma_0 \in P(V)$ a fixed weight, and $\tilde{m} = (m_0, m_1, \ldots, m_v) \in \mathbb{Z}^{v+1}$. If $V$ has finite-dimensional weight spaces, the $V_{\tilde{m}}$ are finite-dimensional, for $\tilde{m} \in \mathbb{Z}^{v+1}$.

In Lemmas 3.1–3.6 we assume that $V$ has finite-dimensional weight spaces.

**Lemma 3.1.** For $p \in \{0, 1, \ldots, v\}$ and $0 \neq t_{\tilde{m}}^{} k_p \in \mathcal{L}$, if there is a nonzero element $v$ in $V$ such that $t_{\tilde{m}}^{} k_p v = 0$, then $t_{\tilde{m}}^{} k_p$ is locally nilpotent on $V$.

**Lemma 3.2.** Let $t_{0}^{m_0}t_{m}^{} k_p \in \mathcal{L}$ be such that $\tilde{m} = (m_0, m) \neq \tilde{0}$, and there exists $0 \leq a \leq N$ such that $m_a \neq 0$ if $N < p \leq v$. If $t_{0}^{m_0}t_{m}^{} k_p$ is locally nilpotent on $V$, then $\dim V_{\tilde{a}} > \dim V_{\tilde{a} + \tilde{m}}$ for all $\tilde{n} \in \mathbb{Z}^{v+1}$.

**Proof.** Case 1: $p \in \{0, 1, \ldots, N\}$. We first prove that $\dim V_{\tilde{a}} \geq \dim V_{\tilde{a} + \tilde{m}}$ for all $\tilde{n} \in \mathbb{Z}^{v+1}$. Suppose $\dim V_{\tilde{a}} = m$, $\dim V_{\tilde{a} + \tilde{m}} = n$. Let $\{w_1, w_2, \ldots, w_n\}$ be a basis of $V_{\tilde{a} + \tilde{m}}$ and $\{w'_1, w'_2, \ldots, w'_m\}$ a basis of $V_{\tilde{a}}$. We can assume that $m_a \neq 0$ for some $0 \leq a \leq v$ distinct from $p$, where $\tilde{m} = (m_0, m) = (m_0, m_1, \ldots, m_v)$. Since $t_{\tilde{m}}^{} k_p$ is locally nilpotent on $V$ and $V_{\tilde{a} + \tilde{m}}$ is finite-dimensional, there exists $k > 0$ such that $(t_{\tilde{m}}^{} k_p)^k V_{\tilde{a} + \tilde{m}} = 0$. Therefore

$$(t_{-\tilde{m}}^{} d_a)^k (t_{\tilde{m}}^{} k_p)^k (w_1, w_2, \ldots, w_n) = 0.$$

On the other hand, by induction on $k$, we can deduce that

$$(t_{-\tilde{m}}^{} d_a)^k (t_{\tilde{m}}^{} k_p)^k = \sum_{i=0}^{k} \frac{k!}{i! (k-i)! (k-i)!} m_a^i c_p^i (t_{\tilde{m}}^{} k_p)^{k-1-i} (t_{-\tilde{m}}^{} d_a)^{k-i}.$$

Therefore

$$t_{\tilde{m}}^{} k_p \left( \sum_{i=0}^{k-1} \frac{k!}{i! (k-i)! (k-i)!} m_a^i c_p^i (t_{\tilde{m}}^{} k_p)^{k-1-i} (t_{-\tilde{m}}^{} d_a)^{k-1-i} \right) t_{-\tilde{m}}^{} d_a (w_1, w_2, \ldots, w_n) = -k! m_a^k c_p^k (w_1, w_2, \ldots, w_n).$$

Assume that

$$\left( \sum_{i=0}^{k-1} \frac{k!}{i! (k-i)! (k-i)!} m_a^i c_p^i (t_{\tilde{m}}^{} k_p)^{k-1-i} (t_{-\tilde{m}}^{} d_a)^{k-1-i} \right) t_{-\tilde{m}}^{} d_a (w_1, w_2, \ldots, w_n) = (w'_1, w'_2, \ldots, w'_m) C,$$

with $C \in \mathbb{C}^{m \times n}$, and that

$$t_{\tilde{m}}^{} k_p (w'_1, w'_2, \ldots, w'_m) = (w_1, w_2, \ldots, w_n) B,$$

where $B$ is an $n \times n$ matrix.
with $B \in \mathbb{C}^{n \times m}$. Then

$$BC = -k!m^k_0c^k_p f.$$ 

This implies that $m \geq n$. So dim $V_\bar{n} \geq$ dim $V_{\bar{n}+\bar{m}}$ for all $\bar{n} \in \mathbb{Z}^{v+1}$. Also, by (3-1) and the fact that $r(B) = n$, we know that $m > n$ if and only if there exists $v \in V_\bar{n}$ such that $t^{\bar{m}}k_p \cdot v = 0$. Since $t^{\bar{m}}k_p$ is locally nilpotent on $V$, there exist an integer $s \geq 0$ and $w \in V_{\bar{n}+s\bar{m}}$ such that

$$(t^{\bar{m}}k_p) \cdot w = 0.$$ 

Therefore $(t^{\bar{m}}k_p) t^{\bar{m}}k_p \cdot w = t^{\bar{m}}k_p(t^{\bar{m}}k_p \cdot w) = 0$. If $t^{\bar{m}}k_p \cdot w = 0$, by the proof above, dim $V_{\bar{n}+s\bar{m}−\bar{m}} \leq$ dim $V_{\bar{n}+s\bar{m}}$, contradicting the fact that dim $V_{\bar{n}+s\bar{m}−\bar{m}}$ $\geq$ dim $V_{\bar{n}+s\bar{m}}$. Therefore $(t^{\bar{m}}k_p)^r \cdot w \neq 0$ for all $r \in \mathbb{N}$. Since

$$(t^{\bar{m}}k_p)^s t^{\bar{m}}k_p \cdot w = t^{\bar{m}}k_p(t^{\bar{m}}k_p)^s \cdot w = 0$$

and $(t^{\bar{m}}k_p)^s \cdot w \in V_\bar{n}$, it follows that there is a nonzero element $v$ in $V_\bar{n}$ such that $t^{\bar{m}}k_p \cdot v = 0$. Thus $n < m$.

**Case 2:** $N < p \leq v$. The proof is similar to that of case 1, but we have to consider $t^{\bar{m}}d_p$ and $t^{\bar{m}}k_p$ instead and use the $\mathbb{Z}$-linear independence of $c_1, \ldots, c_N$. \hfill \Box

**Lemma 3.3.** Let $0 \neq t^{\bar{m}}k_p \in \mathcal{L}$ and $0 \neq t^{\bar{m}}k_p \in \mathcal{L}$ be such that $(m_0, \ldots, m_N) \neq 0$, $(n_0, \ldots, n_N) \neq 0$ if $N < p \leq v$, where $\bar{m} = (m_0, m_1, \ldots, m_v)$.

1. If $t^{\bar{m}}k_p$ is locally nilpotent on $V$, $t^{\bar{m}}k_q$ is locally nilpotent for $q = 0, 1, \ldots, v$.
2. If both $0 \neq t^{\bar{m}}k_p$ and $0 \neq t^{\bar{m}}k_q$ are locally nilpotent on $V$, then $t^{\bar{m}+\bar{m}}k_p$ is locally nilpotent.
3. If $0 \neq t^{\bar{m}+\bar{m}}k_p$ is locally nilpotent on $V$ and $(m_0+n_0, \ldots, m_N+n_N) \neq 0$ if $N < p \leq v$, then $t^{\bar{m}}k_p$ or $t^{\bar{m}}k_p$ is locally nilpotent.

**Lemma 3.4.** For $0 \leq p \leq v$, let $0 \neq t^{\bar{m}}k_p \in \mathcal{L}$ be such that $(m_0, \ldots, m_N) \neq 0$, where $\bar{m} = (m_0, m_1, \ldots, m_v)$. Then $t^{\bar{m}}k_p$ or $t^{\bar{m}}k_p$ is locally nilpotent on $V$.

**Proof.** The proof occupies the next few pages. We first deal with the case $0 \leq p \leq N$. Without losing generality, we can take $p = 0$.

Suppose the lemma is false. By Lemma 3.2, for any $\bar{r} \in \mathbb{Z}^{v+1}$ we have

$$\dim V_{\bar{r}+\bar{m}} = \dim V_{\bar{r}} = \dim V_{\bar{r}−\bar{m}}, \quad t^{\bar{m}}k_0V_{\bar{r}} = V_{\bar{r}+\bar{m}}, \quad t^{−\bar{m}}k_0V_{\bar{r}} = V_{\bar{r}−\bar{m}}.$$ 

Fix $\bar{r} = (r_0, \bar{r}) \in \mathbb{Z}^{v+1}$ such that $V_{\bar{r}} \neq 0$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V_{\bar{r}}$ and set

$$v_i(k\bar{m}) = \frac{1}{c_0} t^{\bar{m}}k_0 \cdot v_i, \quad i = 1, 2, \ldots, n.$$
where $k \in \mathbb{Z} \setminus \{0\}$. Then \(\{v_1(k\tilde{m}), v_2(k\tilde{m}), \ldots, v_n(k\tilde{m})\}\) is a basis of \(V_{r+k\tilde{m}}\). Let 
\(B_{-\tilde{m},\tilde{m}}^{(0)}, B_{\tilde{m},-\tilde{m}}^{(0)} \in \mathbb{C}^{n \times n}\) be such that 
\[
\frac{1}{c_0} t^{\tilde{m}} k_0 (v_1(-\tilde{m}), v_2(-\tilde{m}), \ldots, v_n(-\tilde{m})) = (v_1, v_2, \ldots, v_n) B_{-\tilde{m},\tilde{m}}^{(0)},
\]
\[
\frac{1}{c_0} t^{-\tilde{m}} k_0 (v_1(\tilde{m}), v_2(\tilde{m}), \ldots, v_n(\tilde{m})) = (v_1, v_2, \ldots, v_n) B_{\tilde{m},-\tilde{m}}^{(0)}.
\]
Since \(t^{\tilde{m}} k_0 \) and \(t^{-\tilde{m}} k_0\) are commutative, it is easy to deduce that 
\[
B_{-\tilde{m},\tilde{m}}^{(0)} = B_{\tilde{m},-\tilde{m}}^{(0)}.
\]
By Lemma 3.1, \(B_{-\tilde{m},\tilde{m}}^{(0)}\) is an \(n \times n\) invertible matrix.

Claim. \(B_{-\tilde{m},\tilde{m}}^{(0)}\) does not have distinct eigenvalues.

Proof. Set \(c = 1/c_0\). To prove the claim, we need to consider \(ct^{\tilde{m}} k_0 ct^{-\tilde{m}} k_0 - \lambda \text{id}\), where \(\lambda \in \mathbb{C}^*\). As in the proof of Lemma 3.1, we can deduce that if there is a nonzero element \(v\) in \(V\) such that \((ct^{\tilde{m}} k_0 ct^{-\tilde{m}} k_0 - \lambda \text{id}) v = 0\), then \(ct^{\tilde{m}} k_0 ct^{-\tilde{m}} k_0 - \lambda \text{id}\) is locally nilpotent on \(V\). On the other hand, we have 
\[
(ct^{\tilde{m}} k_0 ct^{-\tilde{m}} k_0 - \lambda \text{id})^l (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) (B_{-\tilde{m},\tilde{m}}^{(0)} - \lambda \text{id})^l.
\]
Therefore the claim holds. \(\square\)

For \(p \in \{1, 2, \ldots, v\}\), let \(C_{-\tilde{m},\tilde{m}}^{(p)}; C_{\tilde{m},-\tilde{m}}^{(p)} \in \mathbb{C}^{n \times n}\) be such that 
\[
t^{\tilde{m}} k_p (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) C_{-\tilde{m},\tilde{m}}^{(p)},
\]
\[
t^{\tilde{m}} k_p (v_1(-\tilde{m}), v_2(-\tilde{m}), \ldots, v_n(-\tilde{m})) = (v_1, v_2, \ldots, v_n) C_{\tilde{m},-\tilde{m}}^{(p)}.
\]
Since 
\[
\frac{1}{c_0} t^{-\tilde{m}} k_0 t^{\tilde{m}} k_p (v_1, v_2, \ldots, v_n) = t^{\tilde{m}} k_p \frac{1}{c_0} t^{-\tilde{m}} k_0 (v_1, v_2, \ldots, v_n),
\]
we have 
\[
C_{-\tilde{m},\tilde{m}}^{(p)} = B_{-\tilde{m},\tilde{m}}^{(0)} C_{-\tilde{m},\tilde{m}}^{(p)}.
\]
Furthermore, by the fact that 
\[
\frac{1}{c_0} t^{\tilde{m}} k_0 \frac{1}{c_0} t^{-\tilde{m}} k_0 t^{\tilde{m}} k_p (v_1, v_2, \ldots, v_n) = t^{\tilde{m}} k_p \frac{1}{c_0} t^{-\tilde{m}} k_0 (v_1, v_2, \ldots, v_n)
\]
and 
\[
t^{\tilde{m}} k_q \frac{1}{c_0} t^{-\tilde{m}} k_0 t^{\tilde{m}} k_p = t^{\tilde{m}} k_p \frac{1}{c_0} t^{-\tilde{m}} k_0 t^{\tilde{m}} k_q,
\]
we deduce that

\begin{equation}
B^{(p)}_{\bar{m}, \bar{m}} C^{(p)}_{\bar{m}, \bar{0}} = C^{(p)}_{\bar{m}, \bar{m}} B^{(p)}_{\bar{m}, \bar{m}}, \quad C^{(p)}_{\bar{m}, \bar{0}} C^{(q)}_{\bar{m}, \bar{0}} = C^{(q)}_{\bar{m}, \bar{m}} C^{(p)}_{\bar{m}, \bar{0}} \quad 1 \leq p, q \leq v.
\end{equation}

Hence there exists \( D \in \mathbb{C}^{n \times n} \) such that \( \{ D^{-1} B^{(p)}_{\bar{m}, \bar{m}} D, D^{-1} C^{(p)}_{\bar{m}, \bar{0}} D \mid 1 \leq p \leq v \} \) are all upper triangular matrices. If we set

\[
(w_1, w_2, \ldots, w_n) = (v_1, v_2, \ldots, v_n) D
\]

and

\[
w_i(k\bar{m}) = \frac{1}{c_0} t^{k\bar{m}} k_0 w_i, 1 \leq i \leq n, k \in \mathbb{Z} \setminus \{0\},
\]

then

\[
\frac{1}{c_0} t^{k\bar{m}} k_0 (w_1(-\bar{m}), w_2(-\bar{m}), \ldots, w_n(-\bar{m})) = (w_1, \ldots, w_n) D^{-1} B^{(0)}_{\bar{m}, \bar{m}} D,
\]

\[
t^{\bar{m}} k_p (w_1, w_2, \ldots, w_n) = (w_1(\bar{m}), \ldots, w_n(\bar{m})) D^{-1} C^{(p)}_{\bar{m}, \bar{0}} D.
\]

So we can assume that \( B^{(0)}_{\bar{m}, \bar{m}}, C^{(p)}_{\bar{m}, \bar{0}}, \) and \( C^{(p)}_{\bar{m}, -\bar{m}}, \) for \( 1 \leq p \leq v \) are all invertible upper triangular matrices. Furthermore, because

\[
\left(t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 - \lambda \text{id}\right)^l (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) (C^{(p)}_{\bar{m}, -\bar{m}} - \lambda \text{id})^l,
\]

the argument used in the proof of the claim shows that \( C^{(p)}_{\bar{m}, -\bar{m}} \) also does not have distinct eigenvalues. For \( 1 \leq p \leq N, \) set

\[
B^{(p)}_{\bar{m}, -\bar{m}} = \frac{1}{c_0} C^{(p)}_{\bar{m}, -\bar{m}}
\]

and for \( 0 \leq p \leq N \) denote by \( \lambda_p \) the eigenvalue of \( B^{(p)}_{\bar{m}, -\bar{m}}. \)

Let \( A^{(a)}_{\bar{k}\bar{m}, \bar{0}} \) and \( A^{(a)}_{\bar{k}\bar{m}, \bar{k}\bar{m}} \), for \( 0 \leq a \leq v \) and \( k, k_1, k_2 \in \mathbb{Z} \setminus \{0\}, \) be such that

\[
t^{k\bar{m}} d_a (v_1, v_2, \ldots, v_n) = (v_1(k\bar{m}), v_2(k\bar{m}), \ldots, v_n(k\bar{m})) A^{(a)}_{\bar{k}\bar{m}, \bar{0}},
\]

\[
t^{k\bar{m}} d_a (v_1(k\bar{m}), v_2(k\bar{m}), \ldots, v_n(k\bar{m})) = (v_1(k_1\bar{m} + k_2\bar{m}), \ldots, v_n(k_1\bar{m} + k_2\bar{m})) A^{(a)}_{\bar{k}\bar{m}, \bar{k}\bar{m}}.
\]

**Case 1:** \( v > 1. \) Since \( t^{\bar{m}} k_0 = t^{\bar{m}} k_0 \neq 0, \) it follows that there exists \( 1 \leq a \leq v \) such that \( m_a \neq 0, \) where \( m = (m_1, m_2, \ldots, m_v). \) Let \( b \in \{1, \ldots, v\} \) be such that \( a \neq b. \) Consider

\begin{equation}
[t^{-\bar{m}} d_a, \frac{1}{c_0} t^{\bar{m}} k_0] = m_a \frac{1}{c_0} k_0, \quad [t^{-\bar{m}} d_a, t^{\bar{m}} k_b] = m_a k_b.
\end{equation}

**Case 1.1:** There exists \( b \in \{0, 1, \ldots, v\} \) such that \( b \neq 0, a \) and \( c_b = 0. \) Then

\[
A^{(a)}_{\bar{m}, \bar{k}\bar{m}} = B^{(0)}_{\bar{m}, \bar{m}} A^{(a)}_{\bar{m}, \bar{0}} + m_a I, \quad A^{(a)}_{\bar{m}, \bar{m}} C^{(b)}_{\bar{m}, \bar{0}} = C^{(b)}_{\bar{m}, -\bar{m}} A^{(a)}_{\bar{m}, \bar{0}}.
\]
By (3-2) and (3-3),

\[ A^{(a)}_{\bar{m}, \tilde{0}} + m_a B^{(0)}_{\bar{m}, \tilde{m}}^{-1} = C^{(b)}_{\tilde{m}, \bar{0}} A^{(a)}_{\bar{m}, \bar{0}} C^{(b)}_{\tilde{m}, \bar{0}}^{-1}. \]

But the sum on the left-hand side cannot be similar to \( A^{(a)}_{\bar{m}, \tilde{0}} \), since \( m_a \neq 0 \) and \( B^{(0)}_{\bar{m}, \tilde{m}}^{-1} \) is an invertible upper triangular matrix and does not have different eigenvalues. Thus this case is excluded.

**Case 1.2:** \( c_b \neq 0 \) for all \( b \in \{0, 1, \ldots, v\}, b \neq 0, a \). By (3-4) and (3-2), we have

\[
B^{(0)}_{\bar{m}, \tilde{m}} A^{(a)}_{\bar{m}, \tilde{0}} B^{(0)}_{\tilde{m}, \bar{m}}^{-1} + m_a B^{(0)}_{\bar{m}, \tilde{m}}^{-1} - m_a B^{(b)}_{\bar{m}, \tilde{m}}^{-1} = B^{(0)}_{\bar{m}, \tilde{m}} C^{(b)}_{\tilde{m}, \bar{0}} A^{(a)}_{\bar{m}, \bar{0}} C^{(b)}_{\tilde{m}, \bar{0}}^{-1} B^{(0)}_{\tilde{m}, \bar{m}}^{-1}. \]

(1) There exists \( b \neq 0 \) and \( a \) such that \( \lambda_0 \neq \lambda_b \). Then \( m_a B^{(0)}_{\bar{m}, \tilde{m}}^{-1} - m_a B^{(b)}_{\bar{m}, \tilde{m}}^{-1} \) is an invertible upper triangular matrix and does not have different eigenvalues. As in case 1.1, we deduce a contradiction.

(II) \( \lambda_0 = \lambda_b \) for all \( b \in \{1, \ldots, v\} \) distinct from \( a \).

(II.1) Suppose first that \( c_a = 0 \) (in this case \( N = v - 1, a = v \)) or \( c_a \neq 0 \) and \( \lambda_a = \lambda_0 \) (in this case \( N = v \)). Since \( \sum_{p=0}^{v} m_p t^m k_p = 0 \), we have

\[
\sum_{p=0}^{v} m_p t^m k_p = 0.
\]

So \( \sum_{p=0}^{v} m_p C^{(p)}_{\tilde{m}, \bar{m}} = 0 \), and therefore

\[
\sum_{p=0}^{v} m_p c_p = 0,
\]

which contradicts the assumption that \( c_0, \ldots, c_N \) are \( \mathbb{Z}\)-linearly independent.

(II.2) Now suppose \( c_a \neq 0 \), \( \lambda_a \neq \lambda_0 \) and there exists \( b \neq 0 \) and \( a \) such that \( m_b \neq 0 \). We deduce a contradiction as in case 1.2(1) by interchanging \( a \) by \( b \).

(II.3) Suppose \( c_a \neq 0 \), \( \lambda_a \neq \lambda_0 \) and \( m_b = 0 \) for all \( b \in \{1, \ldots, v\} \) distinct from \( a \). Then \( m_0 c_0 \lambda_0 + m_a c_a \lambda_a = 0 \). The proof of this case is the same as in case 2.2 below.

**Case 2:** \( v = 1 \). In this case \( a = 1 \).

**Case 2.1:** \( c_a = 0 \). Since \([t^{-\tilde{m}} d_0, t^\tilde{m} k_0] = [t^{-\tilde{m}} k_0, t^\tilde{m} d_0] = 0\), we have

\[
A^{(0)}_{\bar{m}, \bar{m}} = B^{(0)}_{\bar{m}, \tilde{m}} A^{(0)}_{\tilde{m}, \bar{m}}, \quad A^{(0)}_{\bar{m}, \tilde{m}} = B^{(0)}_{\bar{m}, \tilde{m}} A^{(0)}_{\tilde{m}, \bar{m}}.
\]

Therefore

\[
[t^{-\tilde{m}} d_0, t^\tilde{m} d_0](v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) B^{(0)}_{\tilde{m}, \bar{m}}\left[A^{(0)}_{\tilde{m}, \bar{m}}, A^{(0)}_{\bar{m}, \bar{m}}\right].
\]
At the same time, we have
\[ [t^{-\vec{m}}d_0, t^{\vec{m}}d_0] = 2m_0d_0 + m_0^2(-\mu_1 + \mu_2)(m_0c_0 + m_1c_1), \]
where \( \tau = \mu_1 \tau_1 + \mu_2 \tau_2 \) as above. So
\[
\tag{3-5} B_{-\vec{m}, \vec{m}}^{(0)}[A_{-\vec{m}, \vec{0}}^{(0)}, A_{\vec{m}, \vec{0}}^{(0)}] = (2m_0(\gamma_0(d_0) + r_0) + m_0^2(-\mu_1 + \mu_2)(m_0c_0 + m_1c_1))I,
\]
where \( \gamma_0 \) is the weight fixed above. Since \( \gamma_0 \) is arbitrary, we can choose it such that
\[ 2m_0(\gamma_0(d_0) + r_0) + m_0^2(-\mu_1 + \mu_2)(m_0c_0 + m_1c_1) \neq 0. \]
But \( B_{-\vec{m}, \vec{m}}^{(0)} \) is an invertible triangular matrix and does not have different eigenvalues, in contradiction with (3-5).

**Case 2.2:** \( c_0 \neq 0. \) Since
\[
[t^{-\vec{m}}d_0, t^{\vec{m}}k_0] = -m_1k_1, [t^{-\vec{m}}d_1, t^{\vec{m}}k_0] = m_1k_0 \text{ and }
[t^{\vec{m}}d_0, t^{-\vec{m}}k_0] = m_1k_1, [t^{\vec{m}}d_1, t^{-\vec{m}}k_0] = -m_1k_0,
\]
we have
\[
[k_0t^{-\vec{m}}d_0 + k_1t^{-\vec{m}}d_1, t^{\vec{m}}k_0] = [k_0t^{\vec{m}}d_0 + k_1t^{\vec{m}}d_1, t^{-\vec{m}}k_0] = 0.
\]
Therefore
\[
k_0A_{-\vec{m}, \vec{m}}^{(0)} + k_1A_{-\vec{m}, \vec{m}}^{(1)} = B_{-\vec{m}, \vec{m}}^{(0)}(k_0A_{-\vec{m}, \vec{0}}^{(0)} + k_1A_{-\vec{m}, \vec{0}}^{(1)}),
k_0A_{\vec{m}, -\vec{m}}^{(0)} + k_1A_{\vec{m}, -\vec{m}}^{(1)} = B_{\vec{m}, -\vec{m}}^{(0)}(k_0A_{\vec{m}, \vec{0}}^{(0)} + k_1A_{\vec{m}, \vec{0}}^{(1)}),
\]
and
\[
[k_0t^{-\vec{m}}d_0 + k_1t^{-\vec{m}}d_1, k_0t^{\vec{m}}d_0 + k_1t^{\vec{m}}d_1](v_1, \ldots, v_n)
= (v_1, \ldots, v_n)B_{-\vec{m}, \vec{m}}^{(0)}[k_0A_{-\vec{m}, \vec{0}}^{(0)} + k_1A_{-\vec{m}, \vec{0}}^{(1)}], k_0A_{\vec{m}, \vec{0}}^{(0)} + k_1A_{\vec{m}, \vec{0}}^{(1)}].
\]
At the same time, we have
\[
[k_0t^{-\vec{m}}d_0 + k_1t^{-\vec{m}}d_1, k_0t^{\vec{m}}d_0 + k_1t^{\vec{m}}d_1]
= 2(m_0c_0 + m_1c_1)(c_0d_0 + c_1d_1) - (m_0c_0 + m_1c_1)^3(\mu_1 - \mu_2) \text{ id}.
\]
Since \( c_0 \) and \( c_1 \) are \( \mathbb{Z} \)-linearly independent, we know that \( m_0c_0 + m_1c_1 \neq 0. \) As in case 2.1, we deduce a contradiction.

This concludes the first part of the proof. We next turn to the second major case, \( N < p \leq \nu. \)

If \( N \geq 1 \) or \( N = 0, \) we have \( (m_1, \ldots, m_\nu) \neq 0, \) and the lemma follows from the first part and Lemma 3.3. Otherwise, let \( t^{\vec{m}}k_\nu = t_0^{m_0}k_\nu. \) Set \( \mathcal{J}_0 = \bigoplus_{m_0 \in \mathbb{Z}} \mathbb{C}t_0^{m_0}d_0 \oplus \mathbb{C}k_0 \) and \( W = U(\mathcal{J}_0)v, \) where \( v \in V_\tau \) is a homogeneous element. Since \( c_0 \neq 0, \) the sets \( \{ \dim W_{(m_0, 0)+i} | n_0 \in \mathbb{Z} \} \) are not uniformly bounded. But if neither \( t_0^{m_0}k_\nu \)
nor $t_0^{-m_0}k_p$ is locally nilpotent, then $t_0k_p$ and $t_0^{-1}k_p$ are not locally nilpotent. So by Lemmas 3.2 and 3.1, $\dim V(n_0,0)+\bar{t} = \dim V\bar{t}$ for all $n_0 \in \mathbb{Z}$, which is impossible since $\dim V(n_0,0)+\bar{t} \geq \dim W(n_0,0)+\bar{t}$. This proves Lemma 3.4 $\square$

For $0 \leq p \leq N$, consider the direct sum

$$\bigoplus_{m_p \in \mathbb{Z}} C t_p^m d_p \oplus C k_p,$$

which is a Virasoro Lie subalgebra of $\mathcal{L}$. Since $c_p \neq 0$, it follows from [Mathieu 1992] that there is a nonzero $v_p \in V\bar{t}$ for some $\bar{r} \in \mathbb{Z}^{v+1}$ such that

$$t_p^m d_p v_p = 0 \quad \text{for all } m_p \in \mathbb{Z}_+$$

or

$$t_p^m d_p v_p = 0 \quad \text{for all } m_p \in \mathbb{Z}_-. $$

**Lemma 3.5.** If $v_p \in V\bar{t}$ satisfies (3-6), the sets

$$\{t_p^m k_q \mid m_p \in \mathbb{Z}_+, q = 0, 1, 2, \ldots, v, q \neq p\}$$

are all locally nilpotent on $V$. Likewise for (3-7), with $\mathbb{Z}_+$ replaced by $\mathbb{Z}_-$. 

**Proof.** We only prove the first statement. Suppose it is false; then by Lemma 3.3 $t_p k_q$ is not locally nilpotent on $V$ for some $q \in \{0, 1, \ldots, v\}$, $q \neq p$. By Lemma 3.4, $t_p^{-1}k_q$ is locally nilpotent. Therefore there exists $k \in \mathbb{Z}_+$ such that

$$(t_p^{-1}k_q)^{k-1}v_p \neq 0, \quad (t_p^{-1}k_q)^{k}v_p = 0.$$ 

So

$$t_p^2 d_p (t_p^{-1}k_q)^{k}v_p = -kt_p k_q (t_p^{-1}k_q)^{k-1}v_p + (t_p^{-1}k_q)^{k}t_p^2 d_p v_p$$

$$= -kt_p k_q (t_p^{-1}k_q)^{k-1}v_p = 0.$$ 

This implies that $t_p k_q$ is locally nilpotent, a contradiction. $\square$

**Lemma 3.6.** If $v_p \in V\bar{t}$ satisfies (3-6), the sets

$$\{t^\bar{m} k_p \mid \bar{m} = (m_0, \ldots, m_v) \in \mathbb{Z}^{v+1}, m_p \in \mathbb{Z}_+\}$$

are all locally nilpotent on $V$. Likewise for (3-7), with $\mathbb{Z}_+$ replaced by $\mathbb{Z}_-$. 

**Proof.** Again we only prove the first statement. Without loss of generality, we assume that $p = 0$. Let $\mathcal{K}$ be the subspace of $\mathcal{L}$ spanned by elements of $\mathcal{K}$ which are locally nilpotent on $V$. If $t^{\bar{m}} k_0$, for any $\bar{m} \in \mathbb{Z}^{v} \setminus \{0\}$, is not locally nilpotent on $V$, the lemma holds thanks to Lemmas 3.3 and 3.5. Suppose $\mathcal{K} \cap \{t^{\bar{m}} k_0 \mid \bar{m} \in \mathbb{Z}^v\} \neq \{0\}$. By Lemmas 3.2, 3.3 and 3.5, if $t^{\bar{m}} k_0 \in \mathcal{K}$, then $t^{-\bar{m}} k_0 \notin \mathcal{K}$, and $t^{m_0} t^{\bar{m}} k_0 \in \mathcal{K}$ for all $m_0 > 0$.

**Case 1:** Suppose $t_0^{-m_0} t^{-\bar{m}} k_0 \in \mathcal{K}$ for any $t^{\bar{m}} k_0 \in \mathcal{K}$. Then the lemma is proved.
Case 2: Suppose there exists 0 ≠ \( t^m k_0 \in \mathcal{K} \) such that \( t_0 t^{-m} k_0 \notin \mathcal{K} \). Since \( m = (m_1, \ldots, m_v) \neq 0 \), we can assume that \( m_a \neq 0 \) for some \( a \in \{1, 2, \ldots, v\} \). Let \( V_{t_0} \) be such that
\[
\dim V_{t_0} = \min\{\dim V_{\tilde{s}} \mid V_{\tilde{s}} \neq 0, \tilde{s} \in \mathbb{Z}^{v+1}\}.
\]

Case 2.1: Assume \( t_0^i t^{-m} k_0 \notin \mathcal{K} \) for any \( i > 0 \). Let \( l \in \mathbb{Z}_+ \) and consider
\[
(3-8) \quad \sum_{i=0}^{l} a_i t_0^{-i} t^{-m} k_0 t_0^i t^{-m} k_0 v = 0,
\]
where \( v \in V_{t_0} \setminus \{0\} \). By Lemma 3.4, \( \{t_0^i t^{-m} k_0, t_0^{-i} t^m k_0 \mid i \in \mathbb{Z}_+\} \subseteq \mathcal{K} \). So by Lemma 3.2, we have
\[
t_0^i t^{-m} k_0 V_{t_0} = t_0^{-i} t^m k_0 V_{t_0} = t_0^i t^{-m} d_p V_{t_0} = t_0^{-i} t^m d_p V_{t_0} = 0, \quad i \in \mathbb{Z}_+, 0 \leq p \leq v.
\]

Let \( j \in \{0, 1, \ldots, l\} \). From (3-8) we have
\[
t_0^{-j} t^m d_a t_0^{-i} t^m d_a \left( \sum_{i=0}^{l} a_i t_0^{-i} t^{-m} k_0 t_0^i t^{-m} k_0 \right) v = 0.
\]
Therefore
\[
\sum_{i=0}^{l} a_i (-m_a) t_0^{-i} k_0 (-m_a) t_0^{-j} k_0 v = a_j m^2 k_0^2 = 0.
\]
So \( a_j = 0, j = 0, 1, \ldots, l \). This means \( \{t_0^{-i} t^{-m} k_0 t_0^i t^{-m} k_0 \mid 0 \leq i \leq l\} \) are linearly independent. Since \( l \) can be any positive integer, it follows that \( V_{t_0-(0,2m)} \) is infinite-dimensional, a contradiction.

Case 2.2: Assume there exists \( l \in \mathbb{Z}_+ \) such that
\[
t_0^{-l} t^{-m} k_0 \notin \mathcal{K}, \quad t_0^l t^{-m} k_0 \in \mathcal{K}.
\]
(I) Assume that \( t_0^i t^{-m} k_0 \in \mathcal{K} \) for any \( i \in \mathbb{Z}_+ \). Let \( s > 0 \) and consider
\[
\sum_{i=1}^{s} a_i t_0^{-i} t^m k_0 t^{-i} t^m k_0 v = 0.
\]
Similar to the proof above, we can deduce that \( V_{t_0-(l,0)} \) is infinite-dimensional, in contradiction with the assumption that \( V \) has finite-dimensional weight spaces.

(II) Assume there exists \( s_1 \in \mathbb{Z}_+ \) such that
\[
t_0^l t^{-m} k_0 \in \mathcal{K}, \quad t_0^l t^{-2m} k_0 \in \mathcal{K}, \quad \ldots, \quad t_0^l t^{-s_1 m} k_0 \in \mathcal{K}, \quad t_0^l t^{-(s_1+1) m} k_0 \notin \mathcal{K}.
\]
Then there exist \( s_2, s_3, \ldots, s_k, \ldots \) such that \( s_i \geq s_1 \) for \( i = 2, 3, \ldots, k, \ldots \) and
\[
t_0^l t^{-(s_1-s_2-\cdots-s_{i-2}) m} k_0 \in \mathcal{K}, \quad t_0^l t^{-(s_1-s_2-\cdots-s_{i-2}-1) m} k_0 \in \mathcal{K}, \quad \ldots
\]

Let $V$ be an irreducible weight module of $\mathcal{L}$, \( t_0^{\mu} t^{(-s_1-s_2-\cdots-s_{i-1})m} k_0 \in \mathcal{H} \), \( t_0^{\mu} t^{(-s_1-s_2-\cdots-s_{i-1})m} k_0 \not\in \mathcal{H} \).

Assume that
$$
\sum_{i=1}^{s_1} a_i t_0^{-l} t_{i}^{m} k_0 t^{-l} t_{-i}^{m} k_0 + \sum_{i=1}^{s_2} a_{s_1+i} t_0^{-2l} t^{(s_1+i)m} k_0 t^{-(s_1+i)m} k_0 + \ldots
$$
$$
+ \sum_{i=1}^{s_3} a_{s_1+s_2+i} t_0^{-3l} t^{(s_1+s_2+i)m} k_0 t^{-(s_1+s_2+i)m} k_0 + \ldots
$$
$$
+ \sum_{i=1}^{s_4} a_{s_1+s_2+s_3+i} t_0^{-4l} t^{(s_1+s_2+s_3+i)m} k_0 t^{-(s_1+s_2+s_3+i)m} k_0 \bigg) v = 0.
$$

Let
$$
t_{-j}^m d_a t_{0}^l t_{-j}^m d_a, \quad 1 \leq j \leq s_1,
$$
$$
t_{-j}^{-l} t^{(s_1+j)m} d_a t_{0}^{2l} t^{-(s_1+j)m} d_a, \quad 1 \leq j \leq s_2,
$$
$$
\ldots,
$$
$$
t_{-j}^{-(k-1)l} t^{(s_1+s_2+\cdots+s_{j-1}+j)m} d_a t_{0}^{kl} t^{-(s_1+s_2+\cdots+s_{j-1}+j)m} d_a, \quad 1 \leq j \leq s_k
$$
act on the two sides of the above equation respectively. By Lemma 3.4, we deduce that \( a_i = 0 \), for \( i = 1, 2, \ldots, s_1 \), and that

$$
a_{s_1+s_2+\cdots+s_{j-1}+i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, s_j, \quad 2 \leq j \leq k.
$$

Since \( k \) can be any positive integer, it follows that \( V_{t_0-\{0\}} \) is infinite-dimensional, which contradicts our assumption. The lemma is proved. \( \square \)

Lemmas 3.1 through 3.6 immediately yield the following result.

**Theorem 3.7.** Let \( V \) be an irreducible weight module of \( \mathcal{L} \) such that \( c_0, \ldots, c_N \) are \( \mathbb{Z} \)-linearly independent and \( N \geq 1 \). Then \( V \) has weight spaces that are infinite-dimensional.

Let
$$
\mathcal{L}_+ = \sum_{p=0}^{\nu} t_0 C[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] k_p \oplus \sum_{p=0}^{\nu} t_0 C[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p,
$$
$$
\mathcal{L}_- = \sum_{p=0}^{\nu} t_0^{-1} C[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] k_p \oplus \sum_{p=0}^{\nu} t_0^{-1} C[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p,
$$
$$
\mathcal{L}_0 = \sum_{p=0}^{\nu} C[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] k_p \oplus \sum_{p=0}^{\nu} C[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p.
$$

Then
$$
\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-.
$$
Definition 3.8. Let $W$ be a weight module of $\mathcal{L}$. If there is a nonzero vector $v_0 \in W$ such that

$$\mathcal{L}_+ v_0 = 0, \ W = U(\mathcal{L}) v_0,$$

then $W$ is called a highest weight module of $\mathcal{L}$. If there is a nonzero vector $v_0 \in W$ such that

$$\mathcal{L}_- v_0 = 0, \ W = U(\mathcal{L}) v_0,$$

then $W$ is called a lowest weight module of $\mathcal{L}$.

From Lemmas 3.2 and 3.6, we obtain:

Theorem 3.9. Let $V$ be an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces and with central charges $c_0 \neq 0$, $c_1 = c_2 = \cdots = c_v = 0$. Then $V$ is a highest or lowest weight module of $\mathcal{L}$.

In the remainder of this section we assume that $V$ is an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces and with central charges $c_0 \neq 0$, $c_1 = \cdots = c_v = 0$.

Set

$$T = \begin{cases} \{ v \in V \mid \mathcal{L}_+ v = 0 \} & \text{if } V \text{ is a highest weight module of } \mathcal{L}, \\ \{ v \in V \mid \mathcal{L}_- v = 0 \} & \text{if } V \text{ is a lowest weight module of } \mathcal{L}. \end{cases}$$

Then $T$ is a $\mathcal{L}_0$-module and

$$V = U(\mathcal{L}_-) T \quad \text{or} \quad V = U(\mathcal{L}_+) T.$$ 

Since $V$ is an irreducible $\mathcal{L}$-module, $T$ is an irreducible $\mathcal{L}_0$-module. $T$ has the decomposition

$$T = \bigoplus_{m \in \mathbb{Z}^v} T_m,$$

where $m = (m_1, m_2, \ldots, m_v)$, $T_m = \{ v \in T \mid d_i v = (m_i + \mu(d_i))v, 1 \leq i \leq v \}$ and $\mu$ is a fixed weight of $T$. As in the proof in [Jiang and Meng 2003; Eswara Rao and Jiang 2005], we can deduce:

Theorem 3.10. (1) For all $m, n \in \mathbb{Z}^v$, $p = 1, 2, \ldots, v$, we have

$$\dim T_m = \dim T_n, t^{m} k_p \cdot T = 0, t^{m} k_0 (v_1(n), \ldots, v_m(n)) = c_0 (v_1(m + n), v_2(m + n), \ldots, v_n(m + n)), t^{m} d_0 (v_1(n), v_2(n), \ldots, v_n(n)) = \mu(d_0) (v_1(m + n), v_2(m + n), \ldots, v_n(m + n)),$$

where $\{v_1(0), \ldots, v_m(0)\}$ is a basis of $T_0$ and $v_i(m) = \frac{1}{c_0} t^{m} k_0 v_i(0)$, for $i = 1, 2, \ldots, m$. 
(2) As an \((\mathcal{A}_v \oplus \mathcal{D}_v)\)-module, \(T\) is isomorphic to

\[ F^\alpha (\psi, b) = V(\psi, b) \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] \]

for some \(\alpha = (\alpha_1, \ldots, \alpha_v)\), \(\psi\), and \(b\), where \(\mathcal{A}_v = \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}]\). \(\mathcal{D}_v\) is the derivation algebra of \(\mathcal{A}_v\), and \(V(\psi, b)\) is an \(m\)-dimensional, irreducible \(\mathcal{D}_v\)-module satisfying \(\psi(I) = b \text{id}_{V(\psi, b)}\).

\[ t^\ell d_p(w \otimes t^m) = (m_p + \alpha_p)w \otimes t^{\ell+m} + \sum r_i \psi(E_{ip})w \otimes t^{\ell+m} \]

for \(w \in V(\psi, b)\).

Let

\[ M = \text{Ind}_{\mathcal{L}_+ + \mathfrak{J}_0}^{\mathcal{L}} T \quad \text{or} \quad M = \text{Ind}_{\mathcal{L}_+ + \mathfrak{J}_0}^{\mathcal{L}} T. \]

**Theorem 3.11.** Among the submodules of \(M\) intersecting \(T\) trivially, there is a maximal one, which we denote by \(M^{\text{ind}}\). Moreover \(V \cong M / M^{\text{ind}}\).

4. The structure of \(V\) with \(c_0 = \cdots = c_v = 0\)

Assume that \(V\) is an irreducible weight module of \(\mathcal{L}\) with finite-dimensional weight spaces and \(c_0 = \cdots = c_v = 0\).

**Lemma 4.1.** For any \(\tilde{t}^\ell k_p \in \mathcal{L}\), \(\tilde{t}^\ell k_p\) or \(t^{\ell-1} k_p\) is locally nilpotent on \(V\).

**Lemma 4.2.** If \(V\) is uniformly bounded, \(\tilde{t}^\ell k_p\) is locally nilpotent on \(V\) for any \(\tilde{t}^\ell k_p \in \mathcal{L}\).

**Proof.** For \(\tilde{t}^\ell k_p \in \mathcal{L}\), by Lemma 4.1, \(\tilde{t}^\ell k_p\) or \(t^{\ell-1} k_p\) is nilpotent on \(V_m\) for all \(m \in \mathbb{Z}^{v+1}\). Since \(V\) is uniformly bounded, i.e., \(\max\{\dim V_m \mid m \in \mathbb{Z}^{v+1}\} < \infty\), there exists \(N \in \mathbb{Z}_+\) such that

\[ (\tilde{t}^\ell k_p t^{\ell-1} k_p)^N V = 0, \ (\tilde{t}^\ell k_p t^{\ell-1} k_p)^{N-1} V \neq 0 \]

If the lemma is false, we can assume that \(t^{\ell-1} k_p\) is not locally nilpotent on \(V\). Therefore for any \(0 \neq v \in V\), we have \(t^{\ell-1} k_p v \neq 0\). So

\[ (\tilde{t}^\ell k_p)^N V = 0. \]

Let \(t^{-2\ell} d_q \in \mathcal{L}\) be such that \(p \neq q\) and \(r_q \neq 0\). By the fact that \([t^{-2\ell} d_q, \tilde{t}^\ell k_p] = r_q t^{\ell-1} k_p\), we deduce that \(t^{\ell-1} k_p (t^{\ell-1} k_p)^{N-1} V = 0\), a contradiction. \(\Box\)

**Lemma 4.3.** If there exists \(0 \neq v \in V\) such that \(t^m k_p v = 0\) for all \(m \in \mathbb{Z}^{v+1}\) and \(0 \leq p \leq v\). Then \(\mathcal{L}(V) = 0\).

**Proof.** This follows from (2-2), since \(\mathcal{L}\) is commutative and \(V\) is an irreducible \(\mathcal{L}\)-module. \(\Box\)

**Theorem 4.4.** If \(V\) is uniformly bounded, \(\tilde{t}^\ell k_p V\) vanishes for any \(\tilde{t}^\ell k_p \in \mathcal{L}\).
Proof. Let $0 \neq t_i k_p \in \mathcal{K}$. If $t_i k_p V = 0$, it is easy to prove that $\mathcal{K}(V) = 0$. If $t_i k_p V \neq 0$. Since $V$ is uniformly bounded, by Lemma 4.2, there exists $l \in \mathbb{Z}_+$ such that
\begin{equation}
(t_i k_p t_i^{-1} k_p)^l V = 0, \quad (t_1 k_p t_1^{-1} k_p)^l \neq 0.
\end{equation}
If there exists $s \in \mathbb{Z}_+$ such that $(t_1^{-1} k_p)^s V = 0$, $(t_1^{-1} k_p)^s \neq 0$. By the fact that $[t^\mu d_i, t_i^{-1} k_p] = -t_i^{-1} r^\mu k_p$ and $[t^\mu d_p, t_i^{-1} k_p] = t_i^{-1} r^\mu k_i$, we have
\begin{equation}
t^\tilde{r} k_p (t_i^{-1} k_p)^s V = t^\tilde{r} k_i (t^{-\tilde{r}} k_p)^s V = 0 \quad \text{for all } \tilde{r} \in \mathbb{Z}^{v+1}.
\end{equation}

If $(t_i^{-1} k_p)^s V \neq 0$ for all $s \in \mathbb{Z}_+$. Then by (4-1) there is $r \geq 0$ such that $(t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r} V = 0$ for all $0 \leq i \leq r$, and $(t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V \neq 0$. So for any $\tilde{m} \in \mathbb{Z}^{v+1}$, we have
\begin{equation}
t^{-\tilde{m}} d_i (t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V = 0, \quad t^{-\tilde{m}} d_p (t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V = 0.
\end{equation}
Therefore
\begin{align*}
t^\tilde{r} k_p (t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V &= 0, \\
t^\tilde{r} k_i (t_i k_p)^{l-r} (t_i^{-1} k_p)^{l+r+1} V &= 0,
\end{align*}
for all $\tilde{r} \in \mathbb{Z}^{v+1}$.

Case 1: $\nu \in 2\mathbb{Z} + 1$. By the preceding discussion, there exist nonnegative integers $l_i$ and $r_i$, for $i = 0, 2, 4, \ldots, \nu - 1$, such that
\begin{equation}
(t_i k_{v-1})^{l_i} (t_i^{-1} k_{v-1})^{r_i} (t_{v-2} k_{v-3})^{l_{v-3}} (t_{v-2}^{-1} k_{v-3})^{r_{v-3}} \cdots (t_1 k_0)^{l_0} (t_1^{-1} k_0)^{r_0} V \neq 0
\end{equation}
and
\begin{equation}
t^{\tilde{m}} k_p (t_i k_{v-1})^{l_i} (t_i^{-1} k_{v-1})^{r_i} (t_{v-2} k_{v-3})^{l_{v-3}} (t_{v-2}^{-1} k_{v-3})^{r_{v-3}} \cdots (t_1 k_0)^{l_0} (t_1^{-1} k_0)^{r_0} V
\end{equation}
vanishes for all $0 \leq p \leq \nu$ and $\tilde{m} \in \mathbb{Z}^{v+1}$. By Lemma 4.3, the conclusion of the theorem holds.

Case 2: $\nu \in 2\mathbb{Z}$. Then there exist nonnegative integers $l_i$ and $r_i$, for $i = 0, 2, 4, \ldots, \nu - 2$, such that
\begin{equation}
W = (t_{v-1} k_{v-2})^{l_{v-2}} (t_{v-1}^{-1} k_{v-2})^{r_{v-2}} (t_{v-3} k_{v-4})^{l_{v-4}} (t_{v-3}^{-1} k_{v-4})^{r_{v-4}} \cdots (t_1 k_0)^{l_0} (t_1^{-1} k_0)^{r_0} V
\end{equation}
is nonzero and
\begin{equation}
(t^\tilde{m} k_p W = 0
\end{equation}
for all $0 \leq p \leq \nu - 1$ and $\tilde{m} \in \mathbb{Z}^{v+1}$. By (2-1), we know that
\begin{equation}
(t^\tilde{m} k_p W = 0,
\end{equation}
for $\tilde{m} \in \mathbb{Z}^{v+1}$ such that $m_v \neq 0$. If there exists $t^{\tilde{m}}_0 k_v$ satisfying $t^{\tilde{m}}_0 k_v W \neq 0$, let

$$\mathcal{L}_v = \text{span}\{ t^m d_i, t^m d_v, t^m k_v \mid t^m = t_0^{m_0} t_1^{m_1} \cdots t_{v-1}^{m_{v-1}}, 0 \leq i \leq v - 1, \}
= \langle m_0, \ldots, m_{v-1}, 1 \rangle, \quad \tilde{m} = (m_0, \ldots, m_{v-1}) \in \mathbb{Z}^v, \tilde{m} \in \mathbb{Z}^{v+1} \rangle,$$

$$W' = U(\mathcal{L}_v) W.$$ Then $W' \neq 0$ and

$$t^\tilde{m} k_p W' = 0, \quad t^\tilde{m} k_v W' = 0,$$ for all $0 \leq p \leq v - 1$, $\tilde{m} \in \mathbb{Z}^{v+1}$, and $\tilde{n} \in \mathbb{Z}^{v+1}$ such that $n_v \neq 0$. If there exists $0 \neq t^\tilde{n} k_v$ such that $t^\tilde{n} k_v W' \neq 0$, we have

$$(t^{-m} k_v)^l(t^m k_v)^l W' = 0 \quad \text{and} \quad (t^{-m} k_v)^l-1(t^m k_v)^l-1 W' \neq 0$$

for some $l \in \mathbb{Z}_+$. As in the preceding proof, we can deduce that there exists a nonzero $\nu \in W'$ such that

$$t^{\tilde{m}} k_v \nu = 0$$
for all $\tilde{n} \in \mathbb{Z}^v$. Therefore

$$t^{\tilde{m}} k_p \nu = 0$$
for all $\tilde{m} \in \mathbb{Z}^{v+1}$ and $0 \leq p \leq v$. We have proved that $\mathfrak{X}(V) = 0$. \hfill \Box

References


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