IRREDUCIBLE REPRESENTATIONS FOR THE ABELIAN EXTENSION OF THE LIE ALGEBRA OF DIFFEOMORPHISMS OF TORI IN DIMENSIONS GREATER THAN 1

CUIPO JIANG AND QIFEN JIANG
IRREDUCIBLE REPRESENTATIONS FOR THE ABELIAN EXTENSION OF THE LIE ALGEBRA OF Diffeomorphisms OF TORI IN DIMENSIONS GREATER THAN 1

Cuipo Jiang and Qifen Jiang

We classify the irreducible weight modules of the abelian extension of the Lie algebra of diffeomorphisms of tori of dimension greater than 1, with finite-dimensional weight spaces.

1. Introduction

Let \( W_{\nu+1} \) be the Lie algebra of diffeomorphisms of the \((\nu+1)\)-dimensional torus. If \( \nu = 0 \), the universal central extension of the complex Lie algebra \( W_1 \) is the Virasoro algebra, which, together with its representations, plays a very important role in many areas of mathematics and physics [Belavin et al. 1984; Dotsenko and Fateev 1984; Di Francesco et al. 1997]. The representation theory of the Virasoro algebra has been studied extensively; see, for example, [Kac 1982; Kaplansky and Santharoubane 1985; Chari and Pressley 1988; Mathieu 1992].

If \( \nu \geq 1 \), however, the Lie algebra \( W_{\nu+1} \) has no nontrivial central extension [Ramos et al. 1990]. But \( W_{\nu+1} \) has abelian extensions whose abelian ideals are the central parts of the corresponding toroidal Lie algebras; see [Berman and Billig 1999], for example. There is a close connection between irreducible integrable modules of the toroidal Lie algebra and irreducible modules of the abelian extension \( \mathcal{L} \); see [Berman and Billig 1999; Eswara Rao and Moody 1994; Jiang and Meng 2003], for instance. In fact, the classification of integrable modules of toroidal Lie algebras and their subalgebras depends heavily on the classification of irreducible representations of \( \mathcal{L} \) and its subalgebras. See [Billig 2003] for the constructions of the abelian extensions for the group of diffeomorphisms of a torus.

In this paper we study the irreducible weight modules of \( \mathcal{L} \), for \( \nu \geq 1 \). If \( V \) is an irreducible weight module of \( \mathcal{L} \) some of whose central charges \( c_0, \ldots, c_{\nu} \) are nonzero, one can assume that \( c_0, \ldots, c_N \) are \( \mathbb{Z} \)-linearly independent and \( c_{N+1} = \cdots = c_{\nu} = 0 \), where \( N \geq 0 \). We prove that if \( N \geq 1 \), then \( V \) must have weight

MSC2000: primary 17B67, 17B65; secondary 17B68.

Keywords: irreducible representation, abelian extension, central charge.

Work supported in part by NSF of China, grants No. 10271076 and No. 10571119.
spaces which are infinite-dimensional. So if all the weight spaces of $V$ are finite-dimensional, $N$ vanishes. We classify the irreducible modules of $L$ with finite-dimensional weight spaces and some nonzero central charges. We prove that such a module $V$ is isomorphic to a highest weight module. The highest weight space $T$ is isomorphic to an irreducible $(\mathfrak{A}_v + W_v)$-module all of whose weight spaces have the same dimension, where $\mathfrak{A}_v$ is the ring of Laurent polynomials in $v$ commuting variables, regarded as a commutative Lie algebra. An important step is to characterize the $\mathfrak{A}_v$-module structure of $T$. It turns out that the action of $\mathfrak{A}_v$ on $T$ is essentially multiplication by polynomials in $\mathfrak{A}_v$. Therefore $T$ can be identified with Larsson’s construction [1992] by a result in [Eswara Rao 2004]. That is, $T$ is a tensor product of $\mathfrak{gl}_v$-module with $\mathfrak{A}_v$.

When all the central charges of $V$ are zero, we prove that the abelian part acts on $V$ as zero if $V$ is a uniformly bounded $L$-module. So the result in this case is not complete.

Throughout the paper, $C$, $\mathbb{Z}_+$ and $\mathbb{Z}_-$ denote the sets of complex numbers, positive integers and negative integers.

2. Basic concepts and results
Let $\mathfrak{A}_{v+1} = C[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}]$ $(v \geq 1)$ be the ring of Laurent polynomials in commuting variables $t_0, t_1, \ldots, t_v$. For $\mathbf{n} = (n_1, n_2, \ldots, n_v) \in \mathbb{Z}^v$, $n_0 \in \mathbb{Z}$, we denote $t_0^{n_0} t_1^{n_1} \cdots t_v^{n_v}$ by $t_0^{n_0} t_1^{n_1}$. Let $\mathcal{X}$ be the free $\mathfrak{A}_{v+1}$-module with basis $\{k_0, k_1, \ldots, k_v\}$ and let $d\mathcal{X}$ be the subspace spanned by all elements of the form

$$\sum_{i=0}^{v} r_i t_0^{r_i} t^k i, \quad \text{for } (r_0, r_1, \ldots, r_v) \in \mathbb{Z}^{v+1}.$$

Set $\mathfrak{X} = \mathcal{X} / d\mathcal{X}$ and denote the image of $t_0^{r_0} t^k i$ still by itself. Then $\mathfrak{X}$ is spanned by the elements $\{t_0^{r_0} t^k i | p = 0, 1, \ldots, v, r_0 \in \mathbb{Z}, r \in \mathbb{Z}^v\}$ with relations

$$(2-1) \quad \sum_{p=0}^{v} r_p t_0^{r_p} t^k i = 0.$$

Let $\mathfrak{D}$ be the Lie algebra of derivations on $\mathfrak{A}_{v+1}$. Then

$$\mathfrak{D} = \left\{ \sum_{p=0}^{v} f_p(t_0, t_1, \ldots, t_v) d_p \mid f_p(t_0, t_1, \ldots, t_v) \in \mathfrak{A}_{v+1} \right\},$$

where $d_p = t_p \partial / \partial t_p$, $p = 0, 1, \ldots, v$. From [Berman and Billig 1999] we know that the algebra $\mathfrak{D}$ admits two nontrivial 2-cocycles with values in $\mathfrak{X}$:

$$\tau_1(t_0^{m_0} t_1^{m_1} d_{i_0} + t_0^{n_0} t_1^{n_1} b_k) = -n_0 m_p \sum_{p=0}^{v} m_p t_0^{m_0 + n_0} t_1^{n_1 + n_2} k,$$
ABELIAN EXTENSION OF LIE ALGEBRA OF DIFFEOMORPHISMS OF $T^n$

$\tau_2(t_0^{m_0}t^n d_a, t_0^{n_0}t^n d_b) = m_a n_b \sum_{p=0}^{v} m_p t_0^{m_0+n_0} t^n k_p$. 

Let $\tau = \mu_1 \tau_1 + \mu_2 \tau_2$ be an arbitrary linear combination of $\tau_1$ and $\tau_2$. Then the corresponding abelian extension of $\mathfrak{B}$ is

$\mathcal{L} = \mathfrak{B} \oplus \mathcal{K}$,

with the Lie bracket

(2-2) $[t_0^{m_0}t^n d_a, t_0^{n_0}t^n d_b] = n_a t_0^{m_0+n_0} t^n k_b + \delta_{ab} \sum_{p=0}^{v} m_p t_0^{m_0+n_0} t^n k_p,$

$[t_0^{m_0}t^n d_a, t_0^{n_0}t^n d_b] = n_a t_0^{m_0+n_0} t^n d_b - m_b t_0^{m_0+n_0} t^n d_a$

$+ \tau(t_0^{m_0}t^n d_a, t_0^{n_0}t^n d_b).$

The sum

$\mathfrak{h} = \left( \bigoplus_{i=0}^{v} \mathbb{C} k_i \right) \oplus \left( \bigoplus_{i=0}^{v} \mathbb{C} d_i \right)$

is an abelian Lie subalgebra of $\mathcal{L}$. An $\mathcal{L}$-module $V$ is called a weight module if

$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda},$

where $V_{\lambda} = \{ v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h} \}$. Denote by $P(V)$ the set of all weights. Throughout the paper, we assume that $V$ is an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces. Since $V$ is irreducible, we have

$k_i |_{V} = c_i,$

where the constants $c_i$, for $i = 0, 1, \ldots, v$, are called the central charges of $V$.

**Lemma 2.1.** Let $A = (a_{ij})$ ($0 \leq i, j \leq v$) be a $(v+1) \times (v+1)$ matrix such that $\det A = 1$ and $a_{ij} \in \mathbb{Z}$. There exists an automorphism $\sigma$ of $\mathcal{L}$ such that

$\sigma(t^n k_j) = \sum_{p=0}^{v} a_{pj} t^n A^T k_p,$

$\sigma(t^n d_j) = \sum_{p=0}^{v} b_{jp} t^n A^T d_p,$

$0 \leq j \leq v,$

where $t^n = t_0^{m_0} t^n$, $B = (b_{ij}) = A^{-1}$.

**3. The structure of $V$ with nonzero central charges**

In this section, we discuss the weight module $V$ which has nonzero central charges. It follows from Lemma 2.1 that we can assume that $c_0, c_1, \ldots, c_N$ are $\mathbb{Z}$-linearly independent, i.e., if $\sum_{i=0}^{N} a_i c_i = 0$, $a_i \in \mathbb{Z}$, then all $a_i (i = 0, \ldots, N)$ must be zero,
and \( c_{N+1} = c_{N+2} = \cdots = c_{\nu} = 0 \), where \( N \geq 0 \). For \( \bar{m} = (m_0, m) \), denote \( t_0^{m_0} t_1^{m_1} \) by \( t_{\bar{m}} \) as in Lemma 2.1. It is easy to see that \( V \) has the decomposition

\[
V = \bigoplus_{\bar{m} \in \mathbb{Z}^{v+1}} V_{\bar{m}},
\]

where \( V_{\bar{m}} = \{ v \in V \mid d_i(v) = (\gamma_0(d_i) + m_i)v, i = 0, 1, \ldots, v \} \), with \( \gamma_0 \in P(V) \) a fixed weight, and \( \bar{m} = (m_0, m_1, \ldots, m_v) \in \mathbb{Z}^{v+1} \). If \( V \) has finite-dimensional weight spaces, the \( V_{\bar{m}} \) are finite-dimensional, for \( \bar{m} \in \mathbb{Z}^{v+1} \).

**In Lemmas 3.1–3.6 we assume that \( V \) has finite-dimensional weight spaces.**

**Lemma 3.1.** For \( p \in \{0, 1, \ldots, v\} \) and \( 0 \neq t_{\bar{m}} k_p \in \mathcal{L} \), if there is a nonzero element \( v \) in \( V \) such that \( t_{\bar{m}} k_p v = 0 \), then \( t_{\bar{m}} k_p \) is locally nilpotent on \( V \).

**Lemma 3.2.** Let \( t_0^{m_0} t_1^{m_1} k_p \in \mathcal{L} \) be such that \( \bar{m} = (m_0, m) \neq \bar{0} \), and there exists \( 0 \leq a \leq N \) such that \( m_a \neq 0 \) if \( N < p \leq v \). If \( t_0^{m_0} t_1^{m_1} k_p \) is locally nilpotent on \( V \), then \( \dim V_{\bar{m}} = \dim V_{\bar{m} + \bar{n}} \) for all \( \bar{n} \in \mathbb{Z}^{v+1} \).

**Proof:** Case 1: \( p \in \{0, 1, \ldots, N\} \). We first prove that \( \dim V_{\bar{m}} \geq \dim V_{\bar{m} + \bar{n}} \) for all \( \bar{n} \in \mathbb{Z}^{v+1} \). Suppose \( \dim V_{\bar{m}} = m, \dim V_{\bar{m} + \bar{n}} = n \). Let \( \{w_1, w_2, \ldots, w_n\} \) be a basis of \( V_{\bar{m} + \bar{n}} \), and \( \{w'_1, w'_2, \ldots, w'_m\} \) a basis of \( V_{\bar{m}} \). We can assume that \( m_a \neq 0 \) for some \( 0 \leq a \leq v \) distinct from \( p \), where \( \bar{m} = (m_0, m) = (m_0, m_1, \ldots, m_v) \). Since \( t_{\bar{m}} k_p \) is locally nilpotent on \( V \) and \( V_{\bar{m} + \bar{n}} \) is finite-dimensional, there exists \( k > 0 \) such that \( (t_{\bar{m}} k_p)^k V_{\bar{m} + \bar{n}} = 0 \). Therefore

\[
(t_{\bar{m}} k_p)^k \sum_{\bar{m}} (t_{\bar{m}} k_p)^k (w_1, w_2, \ldots, w_n) = 0.
\]

On the other hand, by induction on \( k \), we can deduce that

\[
(t_{\bar{m}} k_p)^k (t_{\bar{m}} k_p)^k = \sum_{i=0}^{k} \frac{k!}{i! (k-i)!} (t_{\bar{m}} k_p)^k (t_{\bar{m}} k_p)^k (t_{\bar{m}} k_p)^k = -k! m_a^k (w_1, w_2, \ldots, w_n).
\]

Therefore

\[
t_{\bar{m}} k_p \left( \sum_{i=0}^{k-1} \frac{k!}{i! (k-i)!} (t_{\bar{m}} k_p)^k (t_{\bar{m}} k_p)^{k-i} (t_{\bar{m}} k_p)^{k-1-i} t_{\bar{m}} k_p (w_1, w_2, \ldots, w_n) \right) = -k! m_a^k (w_1, w_2, \ldots, w_n).
\]

Assume that

\[
\left( \sum_{i=0}^{k-1} \frac{k!}{i! (k-i)!} (t_{\bar{m}} k_p)^k (t_{\bar{m}} k_p)^{k-i} (t_{\bar{m}} k_p)^{k-1-i} \right) t_{\bar{m}} k_p (w_1, w_2, \ldots, w_n) = (w'_1, w'_2, \ldots, w'_m) C,
\]

with \( C \in \mathbb{C}^{m \times n} \), and that

\[
t_{\bar{m}} k_p (w'_1, w'_2, \ldots, w'_m) = (w_1, w_2, \ldots, w_n) B,
\]

where \( B \) is a basis of \( V_{\bar{m}} \), and \( C \) is the change of basis matrix from \( (w'_1, w'_2, \ldots, w'_m) \) to \( (w_1, w_2, \ldots, w_n) \).
with $B \in \mathbb{C}^{n \times m}$. Then

$$BC = -k! m^k a^k c_p I.$$ 

This implies that $m \geq n$. So $\dim V_{\tilde{n}} \geq \dim V_{\tilde{n} + \tilde{m}}$ for all $\tilde{n} \in \mathbb{Z}^{v+1}$. Also, by (3.1) and the fact that $r(B) = n$, we know that $m > n$ if and only if there exists $v \in V_{\tilde{n}}$ such that $t^{\tilde{m}} k_p \cdot v = 0$. Since $t^{\tilde{m}} k_p$ is locally nilpotent on $V$, there exist an integer $s \geq 0$ and $w \in V_{\tilde{n} + s \tilde{m}}$ such that

$$(t^{\tilde{m}} k_p) \cdot w = 0.$$ 

Therefore $(t^{-\tilde{m}} k_p) t^{\tilde{m}} k_p \cdot w = t^{\tilde{m}} k_p (t^{-\tilde{m}} k_p \cdot w) = 0$. If $t^{-\tilde{m}} k_p \cdot w = 0$, by the proof above, $\dim V_{\tilde{n} + s \tilde{m} - \tilde{m}} < \dim V_{\tilde{n} + s \tilde{m}}$, contradicting the fact that $\dim V_{\tilde{n} + s \tilde{m} - \tilde{m}} \geq \dim V_{\tilde{n} + s \tilde{m}}$. Therefore $(t^{-\tilde{m}} k_p)^s \cdot w \neq 0$ for all $r \in \mathbb{N}$. Since

$$(t^{-\tilde{m}} k_p)^s \cdot t^{\tilde{m}} k_p \cdot w = t^{\tilde{m}} k_p (t^{-\tilde{m}} k_p)^s \cdot w = 0$$ 

and $(t^{-\tilde{m}} k_p)^s \cdot w \in V_{\tilde{n}}$, it follows that there is a nonzero element $v$ in $V_{\tilde{n}}$ such that $t^{\tilde{m}} k_p \cdot v = 0$. Thus $n < m$.

**Case 2:** $N < p \leq v$. The proof is similar to that of case 1, but we have to consider $t^{-\tilde{m}} d_p$ and $t^{\tilde{m}} k_p$ instead and use the $\mathbb{Z}$-linear independence of $c_1, \ldots, c_N$. 

**Lemma 3.3.** Let $0 \neq t^{\tilde{m}} k_p \in \mathcal{L}$ and $0 \neq t^{\tilde{m}} k_p \in \mathcal{L}$ be such that $(m_0, \ldots, m_N) \neq 0$, $(n_0, \ldots, n_N) \neq 0$ if $N < p \leq v$, where $\tilde{m} = (m_0, m_1, \ldots, m_v)$.

1. If $t^{\tilde{m}} k_p$ is locally nilpotent on $V$, $t^{\tilde{m}} k_q$ is locally nilpotent for $q = 0, 1, \ldots, v$.
2. If both $0 \neq t^{\tilde{m}} k_p$ and $0 \neq t^{\tilde{m}} k_p$ are locally nilpotent on $V$, then $t^{\tilde{m} + \tilde{m}} k_p$ is locally nilpotent.
3. If $0 \neq t^{\tilde{m} + \tilde{m}} k_p$ is locally nilpotent on $V$ and $(m_0 + n_0, \ldots, m_N + n_N) \neq 0$ if $N < p \leq v$, then $t^{\tilde{m}} k_p$ or $t^{\tilde{m}} k_p$ is locally nilpotent.

**Lemma 3.4.** For $0 \leq p \leq v$, let $0 \neq t^{\tilde{m}} k_p \in \mathcal{L}$ be such that $(m_0, \ldots, m_N) \neq 0$, where $\tilde{m} = (m_0, m_1, \ldots, m_v)$. Then $t^{\tilde{m}} k_p$ or $t^{-\tilde{m}} k_p$ is locally nilpotent on $V$.

**Proof.** The proof occupies the next few pages. We first deal with the case $0 \leq p \leq N$. Without losing generality, we can take $p = 0$.

Suppose the lemma is false. By **Lemma 3.2**, for any $\tilde{r} \in \mathbb{Z}^{v+1}$ we have

$$\dim V_{\tilde{r} + \tilde{m}} = \dim V_{\tilde{r}} = \dim V_{\tilde{r} - \tilde{m}}, \quad t^{\tilde{m}} k_0 V_{\tilde{r}} = V_{\tilde{r} + \tilde{m}}, \quad t^{-\tilde{m}} k_0 V_{\tilde{r}} = V_{\tilde{r} - \tilde{m}}.$$ 

Fix $\tilde{r} = (r_0, \tilde{r}) \in \mathbb{Z}^{v+1}$ such that $V_{\tilde{r}} \neq 0$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V_{\tilde{r}}$ and set

$$v_i(k\tilde{m}) = \frac{1}{c_0} t^{k\tilde{m}} k_0 \cdot v_i, \quad i = 1, 2, \ldots, n,$$
Lemma 3.1

We can deduce that if there is a \( v_1(k\bar{m}), v_2(k\bar{m}), \ldots, v_n(k\bar{m}) \) is a basis of \( V_{r+k\bar{m}} \). Let \( B^{(0)}_{-\bar{m},\bar{m}}, B^{(0)}_{\bar{m},-\bar{m}} \in \mathbb{C}^{n \times n} \) be such that

\[
\frac{1}{c_0} t^{\bar{m}} k_0 (v_1 (-\bar{m}), v_2 (-\bar{m}), \ldots, v_n (-\bar{m})) = (v_1, v_2, \ldots, v_n) B^{(0)}_{\bar{m},-\bar{m}},
\]

\[
\frac{1}{c_0} t^{-\bar{m}} k_0 (v_1 (\bar{m}), v_2 (\bar{m}), \ldots, v_n (\bar{m})) = (v_1, v_2, \ldots, v_n) B^{(0)}_{\bar{m},-\bar{m}}.
\]

Since \( t^{\bar{m}} k_0 \) and \( t^{-\bar{m}} k_0 \) are commutative, it is easy to deduce that

\[
B^{(0)}_{\bar{m},-\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}}.
\]

By Lemma 3.1, \( B^{(0)}_{\bar{m},-\bar{m}} \) is an \( n \times n \) invertible matrix.

Claim. \( B^{(0)}_{\bar{m},-\bar{m}} \) does not have distinct eigenvalues.

Proof. Set \( c = 1/c_0 \). To prove the claim, we need to consider \( c t^{\bar{m}} k_0 c t^{-\bar{m}} k_0 - \lambda \ id \), where \( \lambda \in \mathbb{C}^* \). As in the proof of Lemma 3.1, we can deduce that if there is a nonzero element \( v \) in \( V \) such that \( (c t^{\bar{m}} k_0 c t^{-\bar{m}} k_0 - \lambda \ id) v = 0 \), then \( c t^{\bar{m}} k_0 c t^{-\bar{m}} k_0 - \lambda \ id \) is locally nilpotent on \( V \). On the other hand, we have

\[
(c t^{\bar{m}} k_0 c t^{-\bar{m}} k_0 - \lambda \ id)^l (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n)(B^{(0)}_{\bar{m},-\bar{m}} - \lambda \ id)^l.
\]

Therefore the claim holds. \( \Box \)

For \( p \in \{1, 2, \ldots, v\} \), let \( C^{(p)}_{\bar{m},\bar{m}}, C^{(p)}_{\bar{m},-\bar{m}} \in \mathbb{C}^{n \times n} \) be such that

\[
t^{\bar{m}} k_p (v_1, v_2, \ldots, v_n) = (v_1 (\bar{m}), \ldots, v_n (\bar{m})) C^{(p)}_{\bar{m},\bar{m}},
\]

\[
t^{\bar{m}} k_p (v_1 (-\bar{m}), \ldots, v_n (-\bar{m})) = (v_1, v_2, \ldots, v_n) C^{(p)}_{\bar{m},-\bar{m}}.
\]

Since

\[
\frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_p (v_1, v_2, \ldots, v_n) = t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 (v_1, v_2, \ldots, v_n),
\]

we have

\[
(3-2) \quad C^{(p)}_{\bar{m},-\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}} C^{(p)}_{\bar{m},\bar{m}}.
\]

Furthermore, by the fact that

\[
\frac{1}{c_0} t^{\bar{m}} k_0 \frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_p (v_1, v_2, \ldots, v_n) = t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 \frac{1}{c_0} t^{-\bar{m}} k_0 (v_1, v_2, \ldots, v_n)
\]

and

\[
t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_p = t^{\bar{m}} k_p \frac{1}{c_0} t^{-\bar{m}} k_0 t^{\bar{m}} k_q,
\]
we deduce that

\[(3-3) \quad B_{-\vec{m},\vec{m}}^{(0)} C^{(p)}_{\vec{m},\vec{m}} = C^{(p)}_{\vec{m},\vec{m}} B_{-\vec{m},\vec{m}}^{(0)}, \quad C^{(p)}_{\vec{m},\vec{m}} C^{(q)}_{\vec{m},\vec{m}} = C^{(q)}_{\vec{m},\vec{m}} C^{(p)}_{\vec{m},\vec{m}}, \quad 1 \leq p, q \leq v.\]

Hence there exists \( D \in \mathbb{C}^{n \times n} \) such that \( \{D^{-1} B_{-\vec{m},\vec{m}}^{(0)} D, D^{-1} C^{(p)}_{\vec{m},\vec{m}} D \mid 1 \leq p \leq v \} \) are all upper triangular matrices. If we set

\[(w_1, w_2, \ldots, w_n) = (v_1, v_2, \ldots, v_n) D\]

and

\[w_i(k\vec{m}) = \frac{1}{c_0} t^{k\vec{m}} k_0 w_i, \quad 1 \leq i \leq n, \quad k \in \mathbb{Z} \setminus \{0\},\]

then

\[\frac{1}{c_0} t^{k\vec{m}} k_0 (w_1(-\vec{m}), w_2(-\vec{m}), \ldots, w_n(-\vec{m})) = (w_1, \ldots, w_n) D^{-1} B_{-\vec{m},\vec{m}}^{(0)} D,\]

\[t^{k\vec{m}} k_p (w_1, w_2, \ldots, w_n) = (w_1(\vec{m}), \ldots, w_n(\vec{m})) D^{-1} C^{(p)}_{\vec{m},\vec{m}} D.\]

So we can assume that \( B_{-\vec{m},\vec{m}}^{(0)}, C^{(p)}_{\vec{m},\vec{m}}, \) and \( C^{(p)}_{\vec{m},-\vec{m}}, \) for \( 1 \leq p \leq v \) are all invertible upper triangular matrices. Furthermore, because

\[\left( t^{k\vec{m}} k_p \frac{1}{c_0} t^{-k\vec{m}} k_0 - \lambda \text{id} \right) (v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n)(C^{(p)}_{\vec{m},-\vec{m}} - \lambda \text{id}),\]

the argument used in the proof of the claim shows that \( C^{(p)}_{\vec{m},-\vec{m}} \) also does not have distinct eigenvalues. For \( 1 \leq p \leq N, \) set

\[B_{\vec{m},-\vec{m}}^{(p)} = \frac{1}{c_p} C^{(p)}_{\vec{m},-\vec{m}}\]

and for \( 0 \leq p \leq N \) denote by \( \lambda_p \) the eigenvalue of \( B_{\vec{m},-\vec{m}}^{(p)} .\)

Let \( A^{(a)}_{k\vec{m},0} \) and \( A^{(a)}_{k1\vec{m},k2\vec{m}}, \) for \( 0 \leq a \leq v \) and \( k, k_1, k_2 \in \mathbb{Z} \setminus \{0\}, \) be such that

\[t^{k\vec{m}} d_a (v_1, v_2, \ldots, v_n) = (v_1(k\vec{m}), v_2(k\vec{m}), \ldots, v_n(k\vec{m})) A^{(a)}_{k\vec{m},0},\]

\[t^{k\vec{m}} d_a (v_1(k2\vec{m}), v_2(k2\vec{m}), \ldots, v_n(k2\vec{m})) = (v_1(k1\vec{m} + k2\vec{m}), \ldots, v_n(k1\vec{m} + k2\vec{m})) A^{(a)}_{k1\vec{m},k2\vec{m}}.\]

**Case 1:** \( v > 1. \) Since \( t^{k\vec{m}} k_0 = t^{k\vec{m}} - t^{k\vec{m}} k_0 \neq 0, \) it follows that there exists \( 1 \leq a \leq v \) such that \( m_a \neq 0, \) where \( m = (m_1, m_2, \ldots, m_v). \) Let \( b \in \{1, \ldots, v\} \) be such that \( a \neq b. \) Consider

\[(3-4) \quad [t^{-k\vec{m}} d_a, \frac{1}{c_0} t^{k\vec{m}} k_0] = m_a \frac{1}{c_0} k_0, \quad [t^{-k\vec{m}} d_a, t^{k\vec{m}} k_b] = m_a k_b.\]

**Case 1.1:** There exists \( b \in \{0, 1, \ldots, v\} \) such that \( b \neq 0, a \) and \( c_b = 0. \) Then

\[A^{(a)}_{-\vec{m},\vec{m}} = B_{-\vec{m},\vec{m}}^{(0)} A^{(a)}_{-\vec{m},0} + m_a I, \quad A^{(a)}_{-\vec{m},\vec{m}} C^{(b)}_{\vec{m},\vec{0}} = C^{(b)}_{-\vec{m},0} A^{(a)}_{-\vec{m},\vec{0}}.\]
By (3.2) and (3.3),
\[
A^{(a)}_{-\tilde{m},0} + m_a B^{(0)}_{\tilde{m},-\tilde{m}} = C^{(b)}_{\tilde{m},0} A^{(a)}_{-\tilde{m},0} C^{(b)}_{\tilde{m},0}^{-1}.
\]
But the sum on the left-hand side cannot be similar to \(A^{(a)}_{-\tilde{m},0}\), since \(m_a \neq 0\) and \(B^{(0)}_{\tilde{m},-\tilde{m}}\) is an invertible upper triangular matrix and does not have different eigenvalues. Thus this case is excluded.

**Case 1.2**: \(c_b \neq 0\) for all \(b \in \{0, 1, \ldots, v\}, b \neq 0, a\). By (3.4) and (3.2), we have
\[
B^{(0)}_{\tilde{m},-\tilde{m}} A^{(a)}_{-\tilde{m},\tilde{m}} B^{(0)}_{\tilde{m},-\tilde{m}} + m_a B^{(0)}_{\tilde{m},-\tilde{m}} - m_a B^{(b)}_{\tilde{m},-\tilde{m}}^{-1} = B^{(0)}_{\tilde{m},-\tilde{m}} C^{(b)}_{\tilde{m},0} A^{(a)}_{-\tilde{m},0} C^{(b)}_{\tilde{m},0}^{-1} B^{(0)}_{\tilde{m},-\tilde{m}}^{-1}.
\]

(I) There exists \(b \neq 0\) and \(a\) such that \(\lambda_0 \neq \lambda_b\). Then \(m_a B^{(0)}_{\tilde{m},-\tilde{m}} - m_a B^{(b)}_{\tilde{m},-\tilde{m}}^{-1}\) is an invertible upper triangular matrix and does not have different eigenvalues. As in case 1.1, we deduce a contradiction.

(II) \(\lambda_0 = \lambda_b\) for all \(b \in \{1, \ldots, v\}\) distinct from \(a\).

(II.1) Suppose first that \(c_a = 0\) (in this case \(N = v - 1, a = v\)) or \(c_a \neq 0\) and \(\lambda_a = \lambda_0\) (in this case \(N = v\)). Since \(\sum_{p=0}^v m_p t^{\bar{m}} k_p = 0\), we have
\[
\sum_{p=0}^v m_p c^{(p)}_{\bar{m},-\bar{m}} = 0.
\]
So \(\sum_{p=0}^v m_p c^{(p)}_{\bar{m},-\bar{m}} = 0\), and therefore
\[
\sum_{p=0}^v m_p c_p = 0,
\]
which contradicts the assumption that \(c_0, \ldots, c_N\) are \(\mathbb{Z}\)-linearly independent.

(II.2) Now suppose \(c_a \neq 0\), \(\lambda_a \neq \lambda_0\) and there exists \(b \neq 0\) and \(a\) such that \(m_b \neq 0\). We deduce a contradiction as in case 1.2(I) by interchanging \(a\) by \(b\).

(II.3) Suppose \(c_a \neq 0\), \(\lambda_a \neq \lambda_0\) and \(m_b = 0\) for all \(b \in \{1, \ldots, v\}\) distinct from \(a\). Then \(m_0 c_0 \lambda_0 + m_a c_a \lambda_a = 0\). The proof of this case is the same as in case 2.2 below.

**Case 2.**: \(v = 1\). In this case \(a = 1\).

**Case 2.1**: \(c_a = 0\). Since \([t^{-\bar{m}} d_0, t^{\bar{m}} k_0] = [t^{-\bar{m}} k_0, t^{\bar{m}} d_0] = 0\), we have
\[
A^{(0)}_{-\bar{m},\bar{m}} = B^{(0)}_{\bar{m},-\bar{m}} A^{(0)}_{-\bar{m},\bar{m}}, \quad A^{(0)}_{\bar{m},-\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}} A^{(0)}_{-\bar{m},\bar{m}}.
\]
Therefore
\[
[t^{-\bar{m}} d_0, t^{\bar{m}} d_0](v_1, v_2, \ldots, v_n) = (v_1, v_2, \ldots, v_n) B^{(0)}_{-\bar{m},\bar{m}} [A^{(0)}_{-\bar{m},\bar{m}}, A^{(0)}_{\bar{m},\bar{m}}].
\]
At the same time, we have 
\[
[t^{-\bar{m}}d_0, t^{\bar{m}}d_0] = 2m_0d_0 + m_0^2(-\mu_1 + \mu_2)(m_0k_0 + m_1k_1),
\]
where \( \tau = \mu_1\tau_1 + \mu_2\tau_2 \) as above. So
\[
(3-5) \quad B^{(0)}_{-\bar{m},\bar{m}}[A^{(0)}_{-\bar{m},\bar{m}}, A^{(0)}_{-\bar{m},\bar{m}}] = (2m_0(\gamma_0(d_0) + r_0) + m_0^2(-\mu_1 + \mu_2)(m_0c_0 + m_1c_1))I,
\]
where \( \gamma_0 \) is the weight fixed above. Since \( \gamma_0 \) is arbitrary, we can choose it such that
\[
2m_0(\gamma_0(d_0) + r_0) + m_0^2(-\mu_1 + \mu_2)(m_0c_0 + m_1c_1) \neq 0.
\]
But \( B^{(0)}_{-\bar{m},\bar{m}} \) is an invertible triangular matrix and does not have different eigenvalues, in contradiction with (3-5).

**Case 2.2:** \( c_\alpha \neq 0 \). Since
\[
[t^{-\bar{m}}d_0, t^{\bar{m}}k_0] = -m_1k_1, [t^{-\bar{m}}d_1, t^{\bar{m}}k_0] = m_1k_0 \text{ and }
[t^{\bar{m}}d_0, t^{-\bar{m}}k_0] = m_1k_1, [t^{\bar{m}}d_1, t^{-\bar{m}}k_0] = -m_1k_0,
\]
we have
\[
[k_0t^{-\bar{m}}d_0 + k_1t^{-\bar{m}}d_1, t^{\bar{m}}k_0] = [k_0t^{\bar{m}}d_0 + k_1t^{\bar{m}}d_1, t^{-\bar{m}}k_0] = 0.
\]
Therefore
\[
k_0A^{(0)}_{-\bar{m},\bar{m}} + k_1A^{(1)}_{-\bar{m},\bar{m}} = B^{(0)}_{-\bar{m},\bar{m}}[k_0A^{(0)}_{-\bar{m},\bar{m}} + k_1A^{(1)}_{-\bar{m},\bar{m}}],
\]
and
\[
[k_0t^{-\bar{m}}d_0 + k_1t^{-\bar{m}}d_1, k_0t^{\bar{m}}d_0 + k_1t^{\bar{m}}d_1](v_1, \ldots, v_n)
\]
\[
= (v_1, \ldots, v_n)B^{(0)}_{-\bar{m},\bar{m}}[k_0A^{(0)}_{-\bar{m},\bar{m}} + k_1A^{(1)}_{-\bar{m},\bar{m}}, k_0A^{(0)}_{\bar{m},\bar{m}} + k_1A^{(1)}_{\bar{m},\bar{m}}].
\]
At the same time, we have
\[
[k_0t^{-\bar{m}}d_0 + k_1t^{-\bar{m}}d_1, k_0t^{\bar{m}}d_0 + k_1t^{\bar{m}}d_1]
\]
\[
= 2(m_0c_0 + m_1c_1)(c_0d_0 + c_1d_1) - (m_0c_0 + m_1c_1)^3(\mu_1 - \mu_2) \text{id}.
\]
Since \( c_0 \) and \( c_1 \) are \( \mathbb{Z} \)-linearly independent, we know that \( m_0c_0 + m_1c_1 \neq 0 \). As in case 2.1, we deduce a contradiction.

This concludes the first part of the proof. We next turn to the second major case, \( N < p \leq v \).

If \( N \geq 1 \) or \( N = 0 \), we have \( (m_1, \ldots, m_v) \neq 0 \), and the lemma follows from the first part and Lemma 3.3. Otherwise, let \( t^{\bar{m}}k_p = t^{m_0}k_p \). Set \( \mathcal{L}_0 = \bigoplus_{m_0 \in \mathbb{Z}} \mathbb{C}t^{m_0}d_0 \oplus \mathbb{C}k_0 \) and \( W = U(\mathcal{L}_0)v \), where \( v \in V_\mathbb{Z} \) is a homogeneous element. Since \( c_0 \neq 0 \), the sets \( \{\dim W_{(m_0,0)+\bar{r}} \mid n_0 \in \mathbb{Z} \} \) are not uniformly bounded. But if neither \( t^{m_0}k_p \)
or $t_0^{-m_0} k_p$ is locally nilpotent, then $t_0 k_p$ and $t_0^{-1} k_p$ are not locally nilpotent. So by Lemmas 3.2 and 3.1, \( \dim V_{(n_0,0)+\hat{\eta}} = \dim V_\hat{\nu} \) for all $n_0 \in \mathbb{Z}$, which is impossible since \( \dim V_{(n_0,0)+\hat{\eta}} \geq \dim W_{(n_0,0)+\hat{\eta}} \). This proves Lemma 3.4.

For $0 \leq p \leq N$, consider the direct sum
\[
\bigoplus_{m_p \in \mathbb{Z}} \mathbb{C} t_p^{m_p} d_p \oplus \mathbb{C} k_p,
\]
which is a Virasoro Lie subalgebra of \( \mathcal{L} \). Since $c_\rho \neq 0$, it follows from [Mathieu 1992] that there is a nonzero $v_p \in V_\hat{\nu}$ for some $\hat{\nu} \in \mathbb{Z}^{v+1}$ such that
\[
t_p^{m_p} d_p v_p = 0 \quad \text{for all } m_p \in \mathbb{Z}_+.
\]

Lemma 3.5. If $v_p \in V_\hat{\nu}$ satisfies (3-6), the sets
\[
\{ t_p^{m_p} k_q | m_p \in \mathbb{Z}_+, q = 0, 1, 2, \ldots, v, q \neq p \}
\]
are all locally nilpotent on $V$. Likewise for (3-7), with $\mathbb{Z}_+$ replaced by $\mathbb{Z}_-$.

Proof. We only prove the first statement. Suppose it is false; then by Lemma 3.3 $t_p k_q$ is not locally nilpotent on $V$ for some $q \in \{0, 1, \ldots, v\}$, $q \neq p$. By Lemma 3.4, $t_p^{-1} k_q$ is locally nilpotent. Therefore there exists $k \in \mathbb{Z}_+$ such that
\[
(t_p^{-1} k_q)^k v_p \neq 0, \quad (t_p^{-1} k_q)^k v_p = 0.
\]
So
\[
t_p^2 d_p (t_p^{-1} k_q)^k v_p = -kt_p k_q (t_p^{-1} k_q)^k v_p + (t_p^{-1} k_q)^k t_p^2 d_p v_p
\]
\[
= -kt_p k_q (t_p^{-1} k_q)^k v_p = 0.
\]
This implies that $t_p k_q$ is locally nilpotent, a contradiction.

Lemma 3.6. If $v_p \in V_\hat{\nu}$ satisfies (3-6), the sets
\[
\{ t^{m_p} k_p | \bar{m} = (m_0, \ldots, m_v) \in \mathbb{Z}^{v+1}, m_p \in \mathbb{Z}_+ \}
\]
are all locally nilpotent on $V$. Likewise for (3-7), with $\mathbb{Z}_+$ replaced by $\mathbb{Z}_-$.

Proof: Again we only prove the first statement. Without loss of generality, we assume that $p = 0$. Let $\mathcal{K}'$ be the subspace of $\mathcal{K}$ spanned by elements of $\mathcal{K}$ which are locally nilpotent on $V$. If $t^{m_0} k_0$, for any $m \in \mathbb{Z}^v \setminus \{0\}$, is not locally nilpotent on $V$, the lemma holds thanks to Lemmas 3.3 and 3.5. Suppose $\mathcal{K}' \cap \{ t^{m_0} k_0 | m \in \mathbb{Z}^v \} \neq \{0\}$. By Lemmas 3.2, 3.3 and 3.5, if $t^{m_0} k_0 \in \mathcal{K}'$, then $t^{-m_0} k_0 \not\in \mathcal{K}'$, and $t_0^{m_0} t^{m_0} k_0$ is $\mathcal{K}'$ for all $m_0 > 0$.

Case 1: Suppose $t_0^{m_0} t^{-m_0} k_0 \in \mathcal{K}'$ for any $t^{m_0} k_0 \in \mathcal{K}'$. Then the lemma is proved.
Lemma 3.4

Then there exist \( s \in \mathbb{Z}^v + 1 \).

**Case 2.1:** Assume \( t_0^it^{-m}k_0 \notin \mathcal{C} \) for any \( i > 0 \). Let \( l \in \mathbb{Z}_+ \) and consider

\[
\sum_{i=0}^{l} a_i t_0^{-i} t^{-m} k_0 t_0^l t^{-m} k_0 v = 0,
\]

where \( v \in V_{t_0} \setminus \{0\} \). By Lemma 3.4, \( \{t_0^i t^{-m}k_0 \mid i \in \mathbb{Z}_+ \} \subseteq \mathcal{C} \). So by Lemma 3.2, we have

\[
t_0^l t^{-m} k_0 V_{t_0} = t_0^{-i} t^{-m} k_0 V_{t_0} = t_0^l d_p V_{t_0} = 0, \quad i \in \mathbb{Z}_+, \quad 0 \leq p \leq v.
\]

Let \( j \in \{0, 1, \ldots, l\} \). From (3-8) we have

\[
t_0^{-j} t^{-m} d_0 t_0^j t^{-m} (\sum_{i=0}^{l} a_i t_0^{-i} t^{-m} k_0 t_0^l t^{-m} k_0) v = 0.
\]

Therefore

\[
\sum_{i=0}^{l} a_i (-m_a) t_0^{-j} k_0 (-m_a) t_0^{-j} k_0 v = a_j m_a^2 c^2 v = 0.
\]

So \( a_j = 0 \), \( j = 0, 1, \ldots, l \). This means \( \{t_0^{-i} t^{-m} k_0 t_0^l t^{-m} k_0 v \mid 0 \leq i \leq l \} \) are linearly independent. Since \( l \) can be any positive integer, it follows that \( V_{t_0-\{0,2m\}} \) is infinite-dimensional, a contradiction.

**Case 2.2:** Assume there exists \( l \in \mathbb{Z}_+ \) such that

\[
t_0^l t^{-m} k_0 \notin \mathcal{C} \prime, \quad t_0^l t^{-m} k_0 \in \mathcal{C} \prime.
\]

(I) Assume that \( t_0^l t^{-m} k_0 \in \mathcal{C} \prime \) for any \( i \in \mathbb{Z}_+ \). Let \( s > 0 \) and consider

\[
\sum_{i=1}^{s} a_i t_0^{-l} t^{-m} k_0 t^{-m} k_0 v = 0.
\]

Similar to the proof above, we can deduce that \( V_{t_0-\{0\}} \) is infinite-dimensional, in contradiction with the assumption that \( V \) has finite-dimensional weight spaces.

(II) Assume there exists \( s_1 \in \mathbb{Z}_+ \) such that

\[
t_0^l t^{-m} k_0 \in \mathcal{C} \prime, \quad t_0^l t^{-2m} k_0 \in \mathcal{C} \prime, \quad \ldots, \quad t_0^l t^{-s_1m} k_0 \in \mathcal{C} \prime, \quad t_0^l t^{-s_1m} k_0 \notin \mathcal{C} \prime.
\]

Then there exist \( s_2, s_3, \ldots, s_k, \ldots \) such that \( s_i \geq s_1 \) for \( i = 2, 3, \ldots, k, \ldots \) and

\[
t_0^l t^{-s_1, s_2, \ldots, s_{i-1}m} k_0 \in \mathcal{C} \prime, \quad t_0^l t^{-s_1, s_2, \ldots, s_{i-1}m} k_0 \notin \mathcal{C} \prime, \ldots
\]
Immediately yield the following result.  

Let $V$ be an irreducible weight module of Lemma 3.4, $t_0^m t(-s_1-s_2-\cdots-s_{i-1}-s_i) k_0 \not\in \mathcal{A}'$, $t_0^m t(-s_1-s_2-\cdots-s_{i-1}-s_i) k_0 \not\in \mathcal{A}'$.

Assume that
\[
\left(\sum_{i=1}^{s_1} a_i t_0^{-l} t^{i+m} k_0 t^{l-m} k_0 + \sum_{i=1}^{s_2} \alpha_{s_1+i} t_0^{2l} t^{(s_1+i)m} k_0 t_0^{l-(s_1+i)m} k_0 + \ldots + \sum_{i=1}^{s_k} \alpha_{s_1+s_2+i} t_0^{-(s_1+s_2+i)m} k_0 t_0^{(k-1)l} t^{-(s_1+s_2+\cdots+s_{i-1}+i)m} k_0 \right) v = 0.
\]

Let
\[
\begin{align*}
t^{m} d_{a_0} t^{l} &= d_{a_0}, & 1 \leq j \leq s_1, \\
t_0^{l} t^{(s_1+j)m} d_{a_0} &= d_{a_0}, & 1 \leq j \leq s_2, \\
\ldots, \\
t_0^{-(k-1)l} t^{(s_1+s_2+\cdots+s_{j-1}+j)m} d_{a_0} &= d_{a_0}, & 1 \leq j \leq s_k
\end{align*}
\]
act on the two sides of the above equation respectively. By Lemma 3.4, we deduce that $a_i = 0$, for $i = 1, 2, \ldots, s_1$, and that
\[
a_{s_1+s_2+\cdots+s_j+i} = 0 \quad \text{for } i = 1, 2, \ldots, s_j, 2 \leq j \leq k.
\]

Since $k$ can be any positive integer, it follows that $V_{t_0-(t,0)}$ is infinite-dimensional, which contradicts our assumption. The lemma is proved. \hfill \square

Lemmas 3.1 through 3.6 immediately yield the following result.

**Theorem 3.7.** Let $V$ be an irreducible weight module of $\mathcal{L}$ such that $c_0, \ldots, c_N$ are $\mathbb{Z}$-linearly independent and $N \geq 1$. Then $V$ has weight spaces that are infinite-dimensional.

Let
\[
\begin{align*}
\mathcal{L}_+ &= \sum_{p=0}^{v} t_0 c[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p + \sum_{p=0}^{v} t_0 c[t_0, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p, \\
\mathcal{L}_- &= \sum_{p=0}^{v} t_0^{-1} c[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p + \sum_{p=0}^{v} t_0^{-1} c[t_0^{-1}, t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p, \\
\mathcal{L}_0 &= \sum_{p=0}^{v} c[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p + \sum_{p=0}^{v} c[t_1^{\pm 1}, \ldots, t_v^{\pm 1}] d_p.
\end{align*}
\]

Then
\[
\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_0 \oplus \mathcal{L}_-.
\]
Definition 3.8. Let $W$ be a weight module of $\mathcal{L}$. If there is a nonzero vector $v_0 \in W$ such that

$$\mathcal{L}v_0 = 0, \quad W = U(\mathcal{L})v_0,$$

then $W$ is called a highest weight module of $\mathcal{L}$. If there is a nonzero vector $v_0 \in W$ such that

$$\mathcal{L}v_0 = 0, \quad W = U(\mathcal{L})v_0,$$

then $W$ is called a lowest weight module of $\mathcal{L}$.

From Lemmas 3.2 and 3.6, we obtain:

Theorem 3.9. Let $V$ be an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces and with central charges $c_0 \neq 0, c_1 = c_2 = \cdots = c_v = 0$. Then $V$ is a highest or lowest weight module of $\mathcal{L}$.

In the remainder of this section we assume that $V$ is an irreducible weight module of $\mathcal{L}$ with finite-dimensional weight spaces and with central charges $c_0 \neq 0, c_1 = c_2 = \cdots = c_v = 0$.

Set

$$T = \begin{cases} \{ v \in V \mid \mathcal{L}v = 0 \} & \text{if } V \text{ is a highest weight module of } \mathcal{L}, \\ \{ v \in V \mid \mathcal{L}v = 0 \} & \text{if } V \text{ is a lowest weight module of } \mathcal{L}. \end{cases}$$

Then $T$ is a $\mathcal{L}_0$-module and

$$V = U(\mathcal{L}_-)T \quad \text{or} \quad V = U(\mathcal{L}_+)T.$$ 

Since $V$ is an irreducible $\mathcal{L}$-module, $T$ is an irreducible $\mathcal{L}_0$-module. $T$ has the decomposition

$$T = \bigoplus_{m \in \mathbb{Z}^v} T_m,$$

where $m = (m_1, m_2, \ldots, m_v)$, $T_m = \{ v \in T \mid d_i v = (m_i + \mu(d_i))v, 1 \leq i \leq v \}$ and $\mu$ is a fixed weight of $T$. As in the proof in [Jiang and Meng 2003; Eswara Rao and Jiang 2005], we can deduce:

Theorem 3.10. (1) For all $m, n \in \mathbb{Z}^v$, $p = 1, 2, \ldots, v$, we have

$$\dim T_m = \dim T_n, \quad t^m k_p : T = 0,$$

$$t^m k_0 (v_1(n), \ldots, v_m(n)) = c_0(v_1(m+n), v_2(m+n), \ldots, v_n(m+n)),$$

$$t^m d_0 (v_1(n), v_2(n), \ldots, v_n(n)) = \mu(d_0)(v_1(m+n), v_2(m+n), \ldots, v_n(m+n)),$$

where $\{ v_1(0), \ldots, v_m(0) \}$ is a basis of $T_0$ and $v_i(m) = \frac{1}{c_0} t^m k_0 v_i(0)$, for $i = 1, 2, \ldots, m.$
(2) As an \((\mathcal{A}_v \oplus \mathcal{D}_v)\)-module, \(T\) is isomorphic to 
\[F^\alpha(\psi, b) = V(\psi, b) \otimes \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}]\]
for some \(\alpha = (\alpha_1, \ldots, \alpha_v), \psi,\) and \(b,\) where \(\mathcal{A}_v = \mathbb{C}[t_1^{\pm 1}, \ldots, t_v^{\pm 1}],\) \(\mathcal{D}_v\) is the derivation algebra of \(\mathcal{A}_v,\) and \(V(\psi, b)\) is an \(m\)-dimensional, irreducible \(\mathfrak{gl}_v(\mathbb{C})\)-module satisfying \(\psi(I) = b\ id_{V(\psi, b)}\) and 
\[t^\nu d_p(w \otimes t^m) = (m_p + \alpha_p)w \otimes t^{\nu+m} + \sum_{i=1}^{v} r_i \psi(E_{ip})w \otimes t^{\nu+m}\]
for \(w \in V(\psi, b).\)

Let 
\[M = \text{Ind}_{\mathcal{L}_- + \mathcal{L}_0}^{\mathcal{L}_+} T \quad \text{or} \quad M = \text{Ind}_{\mathcal{L}_- + \mathcal{L}_0}^{\mathcal{L}_+} T.\]

**Theorem 3.11.** Among the submodules of \(M\) intersecting \(T\) trivially, there is a maximal one, which we denote by \(M^{\text{rad}}.\) Moreover \(V \cong M/M^{\text{rad}}.\)

### 4. The structure of \(V\) with \(c_0 = \cdots = c_v = 0\)

Assume that \(V\) is an irreducible weight module of \(\mathcal{L}\) with finite-dimensional weight spaces and \(c_0 = \cdots = c_v = 0.\)

**Lemma 4.1.** For any \(t^\tilde{r}k_p \in \mathcal{H}, t^\tilde{r}k_p\) or \(t^{-\tilde{r}}k_p\) is locally nilpotent on \(V.\)

**Lemma 4.2.** If \(V\) is uniformly bounded, \(t^\tilde{r}k_p\) is locally nilpotent on \(V\) for any \(t^\tilde{r}k_p \in \mathcal{H}.\)

**Proof.** For \(t^\tilde{r}k_p \in \mathcal{H},\) by Lemma 4.1, \(t^\tilde{r}k_p\) or \(t^{-\tilde{r}}k_p\) is nilpotent on \(V_{\tilde{m}}\) for all \(\tilde{m} \in \mathbb{Z}^{v+1}.\) Since \(V\) is uniformly bounded, i.e., \(\max\{\dim V_{\tilde{m}} \mid \tilde{m} \in \mathbb{Z}^{v+1}\} < \infty,\) there exists \(N \in \mathbb{Z}_+\) such that 
\[(t^\tilde{r}k_p t^{-\tilde{r}}k_p)^N V = 0, (t^\tilde{r}k_p t^{-\tilde{r}}k_p)^{N-1} V \neq 0\]
If the lemma is false, we can assume that \(t^{-\tilde{r}}k_p\) is not locally nilpotent on \(V.\) Therefore for any \(0 \neq v \in V,\) we have \(t^{-\tilde{r}}k_pv \neq 0.\) So 
\[(t^\tilde{r}k_p)^N V = 0.\]

Let \(t^{-2\tilde{r}}d_q \in \mathcal{H}\) be such that \(p \neq q\) and \(r_q \neq 0.\) By the fact that \([t^{-2\tilde{r}}d_q, t^\tilde{r}k_p] = r_q t^{-\tilde{r}}k_p,\) we deduce that \(t^{-\tilde{r}}k_p(t^\tilde{r}k_p)^{N-1} V = 0,\) a contradiction. \(\square\)

**Lemma 4.3.** If there exists \(0 \neq v \in V\) such that \(t^{\tilde{m}}k_pv = 0\) for all \(\tilde{m} \in \mathbb{Z}^{v+1}\) and \(0 \leq p \leq v.\) Then \(\mathcal{H}(V) = 0.\)

**Proof.** This follows from (2-2), since \(\mathcal{H}\) is commutative and \(V\) is an irreducible \(\mathcal{L}\)-module. \(\square\)

**Theorem 4.4.** If \(V\) is uniformly bounded, \(t^\tilde{r}k_pV\) vanishes for any \(t^\tilde{r}k_p \in \mathcal{H}.\)
Proof: Let \(0 \neq t_i k_p \in \mathcal{I}\). If \(t_i k_p V = 0\), it is easy to prove that \(\mathcal{I}(V) = 0\). If \(t_i k_p V \neq 0\). Since \(V\) is uniformly bounded, by Lemma 4.2, there exists \(l \in \mathbb{Z}_+\) such that

\[
(4-1) \quad (t_i k_p t_i^{-1} k_p)^l V = 0, \quad (t_i k_p t_i^{-1} k_p)^{l-1} V \neq 0.
\]

If there exists \(s \in \mathbb{Z}_+\) such that \((t_i^{-1} k_p)^s V = 0\), \((t_i^{-1} k_p)^{s-1} V \neq 0\). By the fact that \([t_i^{\bar{m}} d_i, t_i^{-1} k_p] = -t_i^{-1} t_i^{\bar{m}} k_p\) and \([t_i^{\bar{m}} d_p, t_i^{-1} k_p] = t_i^{-1} t_i^{\bar{m}} k_i\), we have

\[
\tilde{r} k_p(t_i^{-1} k_p)^{s-1} V = \tilde{r} k_i(t_i^{-1} k_p)^{s-1} V = 0 \quad \text{for all } \tilde{r} \in \mathbb{Z}^{v+1}.
\]

If \((t_i^{-1} k_p)^s V \neq 0\) for all \(s \in \mathbb{Z}_+\). Then by (4-1) there is \(r \geq 0\) such that \((t_i k_p)^{l-r}(t_i^{-1} k_p)^{l+r+1} V = 0\) for all \(0 \leq i \leq r\), and \((t_i k_p)^{l-r-1}(t_i^{-1} k_p)^{l+r+1} V \neq 0\). So for any \(\bar{m} \in \mathbb{Z}^{v+1}\), we have

\[
(t_i^{\bar{m}} d_i(t_i k_p)^{l-r}(t_i^{-1} k_p)^{l+r+1} V = 0, \quad t_i^{\bar{m}} d_p(t_i k_p)^{l-r}(t_i^{-1} k_p)^{l+r+1} V = 0.
\]

Therefore

\[
\begin{align*}
\tilde{r} k_p(t_i k_p)^{l-r-1}(t_i^{-1} k_p)^{l+r+1} V &= 0, \\
\tilde{r} k_i(t_i k_p)^{l-r-1}(t_i^{-1} k_p)^{l+r+1} V &= 0,
\end{align*}
\]

for all \(\tilde{r} \in \mathbb{Z}^{v+1}\).

**Case 1:** \(v \in 2\mathbb{Z}_++1\). By the preceding discussion, there exist nonnegative integers \(l_i\) and \(r_i\), for \(i = 0, 2, 4, \ldots, v - 1\), such that

\[
(t_0 k_{v-1})^{l_0-1}(t_v^{-1} k_{v-1})^{r_0-1}(t_{v-2} k_{v-3})^{l_2-1}(t_{v-3} k_{v-3})^{r_2-1} \cdots (t_1 k_0)^{l_1-1}(t_1^{-1} k_0)^{r_1-1} V \neq 0
\]

and

\[
\begin{align*}
\tilde{m} k_p(t_v k_{v-1})^{l_0-1}(t_v^{-1} k_{v-1})^{r_0-1}(t_{v-2} k_{v-3})^{l_2-1}(t_{v-3} k_{v-3})^{r_2-1} \cdots (t_1 k_0)^{l_1-1}(t_1^{-1} k_0)^{r_1-1} V \\
&\quad \text{vanishes for all } 0 \leq p \leq v \text{ and } \tilde{m} \in \mathbb{Z}^{v+1}.
\end{align*}
\]

By Lemma 4.3, the conclusion of the theorem holds.

**Case 2:** \(v \in 2\mathbb{Z}\). Then there exist nonnegative integers \(l_i\) and \(r_i\), for \(i = 0, 2, 4, \ldots, v - 2\), such that

\[
W = (t_{v-1} k_{v-2})^{l_{v-2}}(t_{v-1}^{-1} k_{v-2})^{r_{v-2}}(t_{v-3} k_{v-4})^{l_{v-4}}(t_{v-3}^{-1} k_{v-4})^{r_{v-4}} \cdots (t_1 k_0)^{l_1}(t_1^{-1} k_0)^{r_1} V
\]

is nonzero and

\[
(4-2) \quad t_i^{\bar{m}} k_p W = 0
\]

for all \(0 \leq p \leq v - 1\) and \(\bar{m} \in \mathbb{Z}^{v+1}\). By (2-1), we know that

\[
(4-3) \quad t_i^{\bar{m}} k_i W = 0,
\]
for \( \mathbf{m} \in \mathbb{Z}^{v+1} \) such that \( m_v \neq 0 \). If there exists \( t^0 \mathbf{k}_v \), satisfying \( t^0 \mathbf{k}_v W \neq 0 \), let

\[
\mathcal{L}_v = \text{span} \{ t^m d_v, t^m d_v, t^m \mathbf{k}_v \mid t^m = t_0^m t_1^m \cdots t_{v-1}^m, 0 \leq i < v - 1, m = (m_0, \ldots, m_{v-1}) \in \mathbb{Z}^v, \mathbf{m} \in \mathbb{Z}^{v+1} \},
\]

\[
W' = U(\mathcal{L}_v) W.
\]

Then \( W' \neq 0 \) and

\[
t^\mathbf{m} k_p W' = 0, \quad t^n \mathbf{k}_v W' = 0,
\]

for all \( 0 \leq p \leq v - 1 \), \( \mathbf{m} \in \mathbb{Z}^{v+1} \), and \( \mathbf{n} \in \mathbb{Z}^{v+1} \) such that \( n_v \neq 0 \). If there exists

\[
0 \neq t^\mathbf{m} k_v \text{ such that } t^\mathbf{m} k_v W' \neq 0,
\]

we have

\[
(t^{-m} k_v)^l (t^m k_v)^l W' = 0 \quad \text{and} \quad (t^{-m} k_v)^{l-1} (t^m k_v)^{l-1} W' \neq 0
\]

for some \( l \in \mathbb{Z}_+ \). As in the preceding proof, we can deduce that there exists a nonzero \( v \in W' \) such that

\[
t^\mathbf{n} k_v v = 0
\]

for all \( \mathbf{n} \in \mathbb{Z}^v \). Therefore

\[
t^\mathbf{m} k_p v = 0
\]

for all \( \mathbf{m} \in \mathbb{Z}^{v+1} \) and \( 0 \leq p \leq v \). We have proved that \( \mathcal{Z}(V) = 0 \). \( \square \)

References


ABELIAN EXTENSION OF LIE ALGEBRA OF DIFFEORMORPHISMS OF $T^n$


Received December 3, 2005.

Cuipo Jiang
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAOTONG UNIVERSITY
SHANGHAI 200030
CHINA
cpjjiang@sjtu.edu.cn

Qifen Jiang
DEPARTMENT OF MATHEMATICS
SHANGHAI JIAOTONG UNIVERSITY
SHANGHAI 200030
CHINA
qfjiang@sjtu.edu.cn